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Obstacles for Sobolev-homeomorphisms with low rank: pointwise a.e. vs distributional Jacobians

Obstacles pour les homéomorphismes de Sobolev de faible rang : jacobiens ponctuels p.p. vs jacobiens distributionnels

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Abstract. We show that for any k and $s > \frac{k}{k+1}$ there exist neither $W^{s,\frac{k}{s}}$-Sobolev nor C^s-Hölder homeomorphisms from the disk $\overline{\mathbb{B}^n}$ into $\mathbb{R}^N$ whose gradient has rank $< k$ *in the distributional sense.* This complements known examples of such kind of homeomorphisms whose gradient has rank $< k$ *almost everywhere.*

Résumé. Nous montrons que pour tout k et $s > \frac{k}{k+1}$, il n'existe pas d'homéomorphismes $W^{s,\frac{k}{s}}$-Sobolev ou C^s-Hölder du disque $\overline{\mathbb{B}^n}$ dans $\mathbb{R}^N$ dont le gradient a un rang $< k$ *au sens distributionnel.* Ceci complète les exemples connus de ce type d'homéomorphismes dont le gradient a un rang $< k$ *presque partout.*

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1. Introduction

Throughout this paper let $n \geq 2$ and $s \in (0,1]$. In [5,14] it was shown that for any $s \in (0,1)$ and $k \in \{2,\dots,n\}$ there exists a C^s-homeomorphism $u\colon \overline{\mathbb{B}^n} \to \mathbb{R}^n$ such that $\nabla u \in L^1$ and

$$\operatorname{rank}(\nabla u) < k \quad \text{a.e. in } \mathbb{B}^n. \tag{1}$$

Here and throughout this note, when we say homeomorphism, we implicitly always mean homeomorphism *onto the image.*

As is well known, the pointwise a.e. derivative is a way less restrictive object than the distributional derivative which captures more fine geometric properties. The simplest example is the Heaviside function which has a.e. vanishing derivative, but is certainly not constant — because the *distributional* derivative does not vanish. Effects of this type are also known for distributional vs a.e. Jacobians, see for example [6,13].

The purpose of this note is to show that results similar to [5,14] are wrong if the pointwise a.e. notion in (1) is replaced with a distributional version.

Since a map $u \in C^s$ does not need to be differentiable, the notion of distributional $\operatorname{rank}(\nabla u)$ might not be immediate, but the idea is simple: take any k-form monomial in $\bigwedge^k \mathbb{R}^N$

$$\mathrm{d}p^{i_1} \wedge \ldots \wedge \mathrm{d}p^{i_k}.$$

For any smooth map $f\colon \mathbb{B}^n \to \mathbb{R}^N$, the pullback

$$f^*(\mathrm{d}p^{i_1} \wedge \cdots \wedge \mathrm{d}p^{i_k}) = \sum_{\alpha_1,\ldots,\alpha_k=1}^{n} \partial_{\alpha_1} f^{i_1}\, \partial_{\alpha_2} f^{i_2} \cdots \partial_{\alpha_k} f^{i_k}\, \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_k}$$

is a k-form whose components are the determinants of $k \times k$-submatrices of $(\nabla f^{i_1}, \ldots, \nabla f^{i_k}) \in \mathbb{R}^{n\times k}$. In particular, if f is differentiable then $\operatorname{rank}(\nabla f) < k$ is equivalent to

$$f^*(\mathrm{d}p^{i_1} \wedge \cdots \wedge \mathrm{d}p^{i_k}) = 0 \quad \forall\, 1 \le i_1 < \cdots < i_k \le N.$$

On the other hand, $k \times k$-determinants of submatrices of $(\nabla f^{i_1}, \ldots, \nabla f^{i_k}) \in \mathbb{R}^{n\times k}$ are Jacobians and those can be defined in a distributional sense for Hölder and Sobolev maps.

Consequently, it is reasonable to say that $f\colon \mathbb{B}^n \to \mathbb{R}^N$ and $k \in \{1, \ldots, N\}$ satisfy

$$\operatorname{rank}(\nabla f) < k \quad \text{in the distributional sense}$$

if for any $1 \le i_1 < \cdots < i_k \le N$ we have

$$f^*(\mathrm{d}p^{i_1} \wedge \cdots \wedge \mathrm{d}p^{i_k}) = 0$$

in the distributional sense — we shall recall the precise meaning of the latter in Section 2.

Here is our main result.

Theorem 1. *Fix any $s \ge \frac{k}{k+1}$, $k \ge 2$. Denote by $\mathbb{B}^n$ the unit ball in $\mathbb{R}^n$. There exists no homeomorphism $u\colon \overline{\mathbb{B}^n} \to \mathbb{R}^N$ belonging to $W^{s,\frac{k}{s}}(\mathbb{B}^n, \mathbb{R}^N)$ such that*

$$\operatorname{rank}(\nabla u) < k \quad \textit{in the distributional sense.}$$

Here and henceforth for $s \in (0,1)$ the fractional Sobolev space $W^{s,p}(\Omega)$ is the one induced by the Gagliardo seminorm

$$[f]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}}\, \mathrm{d}x\, \mathrm{d}y \right)^{\frac{1}{p}}.$$

By slicing arguments and Fubini's theorem, cf. Lemma 8, any $W^{s,\frac{k}{s}}(\mathbb{B}^n, \mathbb{R}^N)$-homeomorphism for $k < n$ induces a $W^{s,\frac{k}{s}}(\mathbb{B}^k, \mathbb{R}^N)$-homeomorphism on the k-dimensional ball $\mathbb{B}^k$ — and the above notion of distributional rank is stable under that slicing operation. Thus, when proving Theorem 1 one can assume w.l.o.g. $k = n$. Actually, in the case $k = n$, we don't even have to assume that $u\colon \mathbb{B}^n \to \mathbb{R}^N$ is a homeomorphism onto its target, it is only used that u restricted to the boundary $\partial \mathbb{B}^n$ is one-to-one. Precisely we have the following result.

Theorem 2. *Let*

$$f\colon \mathbb{S}^{n-1} \longrightarrow \mathbb{R}^N$$

be a homeomorphism.

Then for any $s \in (0,1]$, $s \ge \frac{n}{n+1}$, there exists no map $u \in W^{s,\frac{n}{s}}(\mathbb{B}^n, \mathbb{R}^N)$ (homeomorphism or not) with the following properties:

- $\operatorname{rank}(\nabla u) < n$ *in the distributional sense in $\mathbb{B}^n$;*
- $f = u\big|_{\partial \mathbb{B}^n}$ *in the sense of traces.*

From Theorem 2 and Sobolev embedding $C^{s+\varepsilon} \hookrightarrow W^{s,\frac{n}{s}}_{loc}$, we have in particular the following result.

Corollary 3. *Let $s > \frac{k}{k+1}$. The C^s-homeomorphism as the one constructed in [5,14] cannot exist if the assumption*

$$\operatorname{rank}(\nabla u) < k \quad \text{a.e.}$$

is substituted with

$$\operatorname{rank}(\nabla u) < k \quad \text{in the distributional sense.}$$

In particular, Corollary 3 implies the following: a homeomorphism such as the one constructed in [5] cannot exist if we additionally assume it belongs to $W^{1,k}$, see [5, Theorem 12]. Indeed, a simple approximation argument shows that for $u \in W^{1,k}$ the notion of $\operatorname{rank}(\nabla u) < k$ in the distributional sense is equivalent to the notion of $\operatorname{rank}(\nabla u) < k$ in the a.e. sense. Thus Corollary 3 implies [5, Theorem 12].

The assumption $s > \frac{k}{k+1}$ in Corollary 3 is notable. The notion of $\operatorname{rank}(\nabla f) < k$ in the distributional sense is well-defined for C^s-maps with $s > 1 - \frac{1}{k}$, see Section 2.

Question 4. *Does Corollary 3 hold for $s \in \left(1 - \frac{1}{k}, \frac{k}{k+1}\right)$?*

Indeed, Question 4 is reminiscent of a conjecture by Gromov that one cannot C^s-embed two-dimensional surfaces into the Heisenberg group $\mathbb{H}_1$ for $s > \frac{1}{2}$. This is known to be true for $s > \frac{2}{3}$, [7]. Notably [15,23] recently constructed examples of C^s-maps from two-dimensional surfaces into the Heisenberg group, whose *boundary map* is homeomorphic — which suggests that the threshold $s \geq \frac{n}{n+1}$ in Theorem 2 could be sharp at least in some dimensions. As a matter of fact, our main tool is a consequence of a technique developed for maps into the Heisenberg group in [9,10,18]: any homeomorphism can be nontrivially "linked" with a differential form, more precisely

Lemma 5 ([9,10,18]). *Let $f : \mathbb{S}^{n-1} \to \mathbb{R}^N$, $N \geq n+2$, be a smooth homeomorphism. There exists $\omega \in C^\infty_c\left(\bigwedge^{n-1} \mathbb{R}^N\right)$ such that $\mathrm{d}\omega \equiv 0$ in a neighborhood of $f(\mathbb{S}^{n-1}) \subset \mathbb{R}^N$ and*

$$\int_{\mathbb{S}^{n-1}} f^*(\omega) = 1. \tag{2}$$

See Lemma 9 for more explanation and a slightly sharper statement. The main idea to prove Theorem 2 is to show that the rank condition is not compatible with (2).

Lastly, let us mention that Corollary 3 can also be proven for the limiting Hölder case $C^{\frac{k}{k+1}+}$, the adaptations are left to the reader — the underlying technical arguments are discussed in [10].

2. The distributional rank condition

The theory of distributional Jacobians has a long tradition with celebrated contributions by Ball [1], Brezis, Nirenberg [3], Coifman, Lions [4], Müller [16], Reshetnyak [17], Wente [24], Tartar [21], among many others. We recall and adapt the notion to our setting, but the results of this section are likely well-known at least to experts.

Definition 6. *Let $s \in (0,1]$, $p \in (1,\infty)$. We say that $f \in W^{s,p}(\mathbb{B}^n, \mathbb{R}^N)$ satisfies*

$$f^*(\mathrm{d}p^{i_1} \wedge \cdots \wedge \mathrm{d}p^{i_k}) = 0$$

in the ($W^{s,p}$-)distributional sense if for any $\varphi \in C^\infty_c(\mathbb{B}^n)$ and any $f_\varepsilon \in C^\infty(\overline{\mathbb{B}^n}, \mathbb{R}^N)$ with

$$\|f_\varepsilon - f\|_{W^{s,p}} \xrightarrow{\varepsilon \to 0} 0$$

we have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{B}^n} f^*_\varepsilon(\mathrm{d}p^{i_1} \wedge \cdots \wedge \mathrm{d}p^{i_k})\varphi = 0.$$

The latter integral is to be understood vectorial, namely it means

$$\int_{\mathbb{B}^n} f_\varepsilon^*(\mathrm{d}p^{i_1}\wedge\cdots\wedge\mathrm{d}p^{i_k})\varphi := \left(\int_{\mathbb{B}^n} f_\varepsilon^*(\mathrm{d}p^{i_1}\wedge\cdots\wedge\mathrm{d}p^{i_k})\wedge\varphi\,\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_{n-k}}\right)_{1\le j_1\le\cdots\le j_{n-k}\le n}.$$

The following type of estimate is essentially known since [20], inspired by the results in [4]. In special cases an extremely elegant proof using harmonic extensions was given in [2], see also [11] for the relation to commutator estimates and harmonic extensions. See also [19] where this is revisited using the arguments of [11].

Lemma 7. *Let $s > 1-\frac{1}{k}$ and $p > k$. For any $f \in W^{s,p}(\mathbb{B}^n,\mathbb{R}^N)$ and f_ε as in Definition 6,*

$$f^*(\mathrm{d}p^{i_1}\wedge\cdots\wedge\mathrm{d}p^{i_k})[\varphi] \equiv f^*_{W^{s,p}}(\mathrm{d}p^{i_1}\wedge\cdots\wedge\mathrm{d}p^{i_k})[\varphi] := \lim_{\varepsilon\to 0}\int_{\mathbb{B}^n} f_\varepsilon^*(\mathrm{d}p^{i_1}\wedge\cdots\wedge\mathrm{d}p^{i_k})\varphi$$

is a linear functional on $\varphi \in C_c^\infty(\mathbb{B}^n)$, independent of the precise choice of f_ε.

Moreover we have

$$\left|f^*(\mathrm{d}p^{i_1}\wedge\cdots\wedge\mathrm{d}p^{i_k})[\varphi]\right| \precsim [f]^k_{W^{s,p}}[\varphi]_{W^{(1-s)k,\frac{p}{p-k}}}$$

and for $f_1, f_2 \in W^{s,p}(\mathbb{B}^n,\mathbb{R}^N)$ we have

$$\begin{aligned}&\left|f_1^*(\mathrm{d}p^{i_1}\wedge\cdots\wedge\mathrm{d}p^{i_k})[\varphi] - f_2^*(\mathrm{d}p^{i_1}\wedge\cdots\wedge\mathrm{d}p^{i_k})[\varphi]\right|\\ &\qquad\precsim [f_1-f_2]_{W^{s,p}(\mathbb{B}^n)}\left([f_1]^{k-1}_{W^{s,p}(\mathbb{B}^n)} + [f_2]^{k-1}_{W^{s,p}(\mathbb{B}^n)}\right)[\varphi]_{W^{(1-s)k,\frac{p}{p-k}}}.\end{aligned}$$

In particular if $f \in W^{s_1,p_1}(\mathbb{B}^n,\mathbb{R}^N)\cap W^{s_2,p_2}(\mathbb{B}^n,\mathbb{R}^N)$ then the linear functional as a W^{s_1,p_1}-limit or a W^{s_2,p_2}-limit coincides, i.e.

$$f^*_{W^{s_1,p_1}}(\mathrm{d}p^{i_1}\wedge\cdots\wedge\mathrm{d}p^{i_k})[\varphi] = f^*_{W^{s_2,p_2}}(\mathrm{d}p^{i_1}\wedge\cdots\wedge\mathrm{d}p^{i_k})[\varphi]$$

for all $\varphi \in C_c^\infty(\mathbb{B}^n,\mathbb{R}^N)$.

Observe that this gives naturally also a suitable notion of C^s-distributional rank, because C^s embeds into $W^{s-\varepsilon,q}$ for any $\varepsilon > 0$, $q \in (1,\infty)$.

Theorem 1 is a consequence of Theorem 2, once we observe the following restriction result.

Lemma 8. *Let $s > \frac{k+1}{k+2}$, $p > k$ for some $k \in \mathbb{N}$. Assume that $f \in W^{s,p}(\mathbb{S}^{k+1},\mathbb{R}^N)$ is continuous and for some $j \in \{1,\ldots,k+1\}$ we have*

$$\operatorname{rank}(\nabla f) < j \quad \textit{in the distributional sense in } \mathbb{S}^{k+1}.$$

If we slice

$$\mathbb{S}^{k+1} = \bigcup_{t\in[-1,1]} \{t\}\times\sqrt{1-t^2}\,\mathbb{S}^k$$

then for $\mathscr{L}^1$-a.e. $t \in (-1,1)$,

$$g_t := f\big|_{\{t\}\times\sqrt{1-t^2}\mathbb{S}^k} \in W^{s,p}\left(\{t\}\times\sqrt{1-t^2}\,\mathbb{S}^k,\mathbb{R}^N\right) \tag{3}$$

and

$$\operatorname{rank}(\nabla g_t) < j \quad \textit{in the distributional sense in } \mathbb{S}^k. \tag{4}$$

Proof. The fact that (3) holds is an application of Fubini's theorem, see e.g. [13, Lemma A.2]. For a more direct proof see [12, Section 6.2]: more precisely, let $f_\ell \in C^\infty(\mathbb{S}^{k+1},\mathbb{R}^N)$ be a smooth approximation of f in $W^{s,p}(\mathbb{S}^{k+1},\mathbb{R}^N)$, i.e.

$$\|f_\ell - f\|_{W^{s,p}(\mathbb{S}^{k+1},\mathbb{R}^N)} \xrightarrow{\ell\to\infty} 0.$$

Then (up to a not relabeled subsequence) for $\mathscr{L}^1$-a.e. $t \in (-1,1)$,

$$f_\ell\big|_{\{t\}\times\sqrt{1-t^2}\mathbb{S}^k} \text{converges to some } g_t \text{ in } W^{s,p}\left(\{t\}\times\sqrt{1-t^2}\,\mathbb{S}^k,\mathbb{R}^N\right), \tag{5}$$

and we have for any representative of f

$$g_t := f\big|_{\{t\}\times\sqrt{1-t^2}\mathbb{S}^k} \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in (-1,1). \tag{6}$$

Of course, for the above there is no restriction on s or p needed, i.e. this is true for any $s \in (0,1)$, $p \in (1,\infty)$.

Fix a representative f. We denote the set $\Gamma = \Gamma(f)$ by

$$\Gamma := \big\{t \in (-1,1) : (5) \text{ and } (6) \text{ are true for this } t\big\}.$$

Let $\psi \in C^\infty\big(\bigwedge^{k-j}\mathbb{S}^k\big)$ be arbitrary and set (in the sense of pullback)

$$\psi_t(\theta) := \psi\left(\frac{1}{\sqrt{1-t^2}}\theta\right) \quad \text{for } t \in (-1,1), \theta \in \sqrt{1-t^2}\mathbb{S}^k.$$

For $t \in \Gamma$ define

$$\begin{aligned} G(t) &:= \int_{\{t\}\times\sqrt{1-t^2}\mathbb{S}^k} g_t^*(\mathrm{d}p^{i_1}\wedge\cdots\wedge\mathrm{d}p^{i_j})\wedge\psi_t(\theta)\,\mathrm{d}\theta \\ &\equiv \lim_{\varepsilon\to 0}\int_{\{t\}\times\sqrt{1-t^2}\mathbb{S}^k} (g_t)_\varepsilon^*(\mathrm{d}p^{i_1}\wedge\cdots\wedge\mathrm{d}p^{i_j})\wedge\psi_t(\theta)\,\mathrm{d}\theta \end{aligned}$$

for any smooth $W^{s,p}$-approximation $(g_t)_\varepsilon$ of g_t on $\{t\}\times\sqrt{1-t^2}\mathbb{S}^k$. By assumptions on p and s and Lemma 7 this is well-defined for each $t \in \Gamma$.

We want to show

$$G(t) = 0 \quad \text{a.e. } t \in (-1,1). \tag{7}$$

Once we have (7), since $\psi \in C^\infty\big(\bigwedge^{k-j}\mathbb{S}^k\big)$ was arbitary, we would also have that for $\mathscr{L}^1$-a.e. $t \in (-1,1)$

$$\int_{\{t\}\times\sqrt{1-t^2}\mathbb{S}^k} f\big|_{\{t\}\times\sqrt{1-t^2}\mathbb{S}^k}{}^*(\mathrm{d}p^{i_1}\wedge\cdots\wedge\mathrm{d}p^{i_j})\phi(\theta)\,\mathrm{d}\theta = 0 \quad \forall\, \phi \in C^\infty\big(\{t\}\times\sqrt{1-t^2}\mathbb{S}^k\big).$$

That is, if we can prove (7) we have proven (4).

To establish (7), recall that since $s > \frac{k+1}{k+2}$ and $p > k$ and $\operatorname{rank}(\nabla f) < j$ in the distributional sense in $\mathbb{S}^{k+1}$, we have for any $\varphi \in C^\infty\big(\bigwedge^{k+1-j}\mathbb{S}^{k+1}\big)$,

$$0 = \lim_{\ell\to\infty}\int_{\mathbb{S}^{k+1}} f_\ell^*(\mathrm{d}p^{i_1}\wedge\cdots\wedge\mathrm{d}p^{i_j})\wedge\varphi.$$

Let $\eta \in C_c^\infty\big((-1,1)\big)$ arbitrary. We extend ψ to a map φ on $\mathbb{S}^{k+1}$, vanishing at the poles $t = \pm 1$, via

$$\varphi(t,\theta) := \eta(t)\,\psi\left(\frac{1}{\sqrt{1-t^2}}\theta\right)\wedge\mathrm{d}t \quad \text{for } t \in [-1,1], \theta \in \sqrt{1-t^2}\mathbb{S}^k.$$

Then $\varphi \in C^\infty\big(\bigwedge^{k-j+1}\mathbb{S}^{k+1}\big)$ and we have

$$\begin{aligned} 0 &= \lim_{\ell\to\infty}\int_{\mathbb{S}^{k+1}} f_\ell^*(\mathrm{d}p^{i_1}\wedge\cdots\wedge\mathrm{d}p^{i_j})\wedge\varphi \\ &= \lim_{\ell\to\infty}\int_{(-1,1)}\int_{\{t\}\times\sqrt{1-t^2}\mathbb{S}^k} f_\ell^*(\mathrm{d}p^{i_1}\wedge\cdots\wedge\mathrm{d}p^{i_j})\wedge\psi_t(\theta)\eta(t)\,\mathrm{d}\theta\,\mathrm{d}t \\ &= \lim_{\ell\to\infty}\int_{(-1,1)} \eta(t)G_\ell(t)\,\mathrm{d}\theta\,\mathrm{d}t \end{aligned} \tag{8}$$

where for a.e. $t \in \Gamma$ we set

$$G_\ell(t) := \int_{\{t\}\times\sqrt{1-t^2}\mathbb{S}^k} f_\ell^*(\mathrm{d}p^{i_1}\wedge\cdots\wedge\mathrm{d}p^{i_j})\wedge\psi_t(\theta)\,\mathrm{d}\theta.$$

Observe that by Lemma 7, for a.e. $t \in \operatorname{supp}\eta \subset (-1,1)$

$$\begin{aligned} &\big|G_\ell(t) - G_{\ell'}(t)\big| \\ &\precsim [f_\ell - f'_\ell]_{W^{s,p}(\{t\}\times\sqrt{1-t^2}\mathbb{S}^k)}\Big([f_\ell]_{W^{s,p}(\{t\}\times\sqrt{1-t^2}\mathbb{S}^k)} + [f'_\ell]_{W^{s,p}(\{t\}\times\sqrt{1-t^2}\mathbb{S}^k)}\Big)^{k-1}[\psi]_{W^{(1-s)k,\frac{p}{p-k}}(\mathbb{S}^k)} \end{aligned}$$

and thus by Fubini's theorem, cf. [12, Theorem 6.24] (using that $p > k$ and Hölder's inequality)

$$\int_{\operatorname{supp}\eta} \left|G_\ell(t) - G_{\ell'}(t)\right| \mathrm{d}t \precsim [f_\ell - f_{\ell'}]_{W^{s,p}(\mathbb{S}^{k+1})} \left([f_\ell]_{W^{s,p}(\mathbb{S}^{k+1})} + [f'_\ell]_{W^{s,p}(\mathbb{S}^{k+1})}\right)^{k-1} [\psi]_{W^{(1-s)k,\frac{p}{p-k}}(\mathbb{S}^k)}.$$

From this we conclude that G_ℓ is a Cauchy sequence in $L^1(\operatorname{supp}\eta, \mathrm{d}t)$. Thus, (8) implies

$$0 = \int_{(-1,1)} \eta(t) \lim_{\ell\to\infty} G_\ell(t)\, \mathrm{d}\theta\, \mathrm{d}t \tag{9}$$

Moreover, since f_ℓ is a smooth approximation of g_t for any $t \in \Gamma$ we have

$$G(t) = \lim_{\ell\to\infty} G_\ell(t) \quad \text{for a.e. } t \in \operatorname{supp}\eta.$$

Thus, (9) implies

$$\int \eta(t) G(t) = 0.$$

This holds for any $\eta \in C^\infty_c\big((-1,1)\big)$. Since $G(t)$ is in $L^1_{\mathrm{loc}}(-1,1)$ we conclude that (7) holds. We can conclude. □

3. Using the linking number: proof of Theorem 2

As discussed in the introduction, the proof of Theorem 2 is based on the recent arguments developed for maps into the Heisenberg group [9,10,18].

The main ingredient is the following lemma, which is essentially just a reformulation of the well-known fact from Algebraic Topology that the homology class $H_{N-(n-1)-1}(\mathbb{R}^N \setminus K)$ is nontrivial if K is a homeomorphic to the $\mathbb{S}^{n-1}$-sphere. Indeed this fact can be found in the early chapters of any algebraic topology book, see e.g. [22, Corollary 1.29]. However, this particular reformulation transforms this fact into an analytically easily usable tool.

Lemma 9. *Let $f\colon \mathbb{S}^{n-1} \to \mathbb{R}^N$, $N \geq n+2$, be a homeomorphism. There exist $\varepsilon > 0$ (depending on f) and $\omega \in C^\infty_c\big(\bigwedge^{n-1}\mathbb{R}^N\big)$ such that:*

- $\mathrm{d}\omega \equiv 0$ *in a neighborhood of $f(\mathbb{S}^{n-1}) \subset \mathbb{R}^N$;*
- *for any $\widetilde{f} \in C^\infty(\mathbb{S}^{n-1}, \mathbb{R}^N)$ with*

$$\|\widetilde{f} - f\|_{L^\infty(\mathbb{S}^{n-1})} < \varepsilon$$

we have

$$\int_{\mathbb{S}^{n-1}} \widetilde{f}^*(\omega) = 1.$$

For a proof we refer to [18, Proposition 9.2] or [10, Lemma 2.6]. $\mathrm{d}\omega$ essentially represents a "surface" that is linked with $f(\mathbb{S}^{n-1})$ — where "surface" is to be understood in a general sense, see [8].

Proof of Theorem 2. Assume on the contrary the existence of u as in Theorem 2.

By [13, Lemma A.4] there exist $u_\varepsilon \in C^\infty(\overline{\mathbb{B}^n})$ such that $[u_\varepsilon - u]_{W^{s,\frac{n}{s}}(\mathbb{B}^n)} \xrightarrow{\varepsilon\to 0} 0$ and $u_\varepsilon\big|_{\partial\mathbb{B}^n}$ uniformly converges to f as $\varepsilon \to 0$.

From Lemma 9 we find ω so that $\mathrm{d}\omega \equiv 0$ around $f(\mathbb{S}^{n-1})$ and

$$\int_{\mathbb{S}^{n-1}} u_\varepsilon^*(\omega) = 1 \quad \forall\, \varepsilon \ll 1.$$

By Stokes' theorem this is equivalent to

$$\int_{\mathbb{B}^n} u_\varepsilon^*(\mathrm{d}\omega) = 1 \quad \forall\, \varepsilon \ll 1. \tag{10}$$

If we write

$$\mathrm{d}\omega = \sum_I \kappa_I \mathrm{d}p^I$$

where each $I = (i_1, \dots, i_n)$ is a strictly ordered tuple in $\{1, \dots, N\}$, then we have by (10)

$$1 = \sum_I \int_{\mathbb{B}^n} u_\varepsilon^*(\mathrm{d}p^I)\kappa_I(u_\varepsilon) \quad \forall\, \varepsilon \ll 1.$$

Moreover, by the uniform Lipschitz continuity of κ

$$\kappa_I(u_\varepsilon) \xrightarrow{\varepsilon \to 0} \kappa_I(u) \quad \text{in } W^{s,\frac{n}{s}}(\mathbb{B}^n). \tag{11}$$

Since $\mathrm{d}\omega \equiv 0$ in a neighborhood of $f(\mathbb{S}^{n-1})$ we also have $\mathrm{d}\omega \equiv 0$ in a neighborhood of $u_\varepsilon(\mathbb{S}^{n-1})$ for all suitably small ε. For such ε we have $\kappa_I(u_\varepsilon) \equiv 0$ on $\partial\mathbb{B}^n$ (in the classical sense).

Since the trace operator is an extension of the continuous trace operator, we find that

$$\kappa_I(u_\varepsilon) \in W_0^{s,\frac{n}{s}}(\mathbb{B}^n).$$

By the convergence (11) we conclude

$$\kappa_I(u) \in W_0^{s,\frac{n}{s}}(\mathbb{B}^n).$$

Here $W_0^{s,p}(\mathbb{B}^n)$ is the closure of $C_c^\infty(\mathbb{B}^n)$-maps under the $W^{s,p}(\mathbb{B}^n)$-norm and we have used the trace identification with the continuous trace (since $s\frac{n}{s} = n > 1$), see [12, Chapter 9].

Thus we find

$$\varphi_\delta^I \in C_c^\infty(\mathbb{B}^n)$$

with

$$\left[\varphi_\delta^I - \kappa_I(u)\right]_{W^{s,\frac{n}{s}}(\mathbb{B}^n)} \xrightarrow{\delta \to 0} 0.$$

In particular for ε and δ suitably small we have

$$\left[\varphi_\delta^I - \kappa_I(u_\varepsilon)\right]_{W^{s,\frac{n}{s}}(\mathbb{B}^n)} \ll 1 \quad \forall\, \varepsilon \in (0,\varepsilon_0),\ \delta \in (0,\delta_0).$$

Recall that I is an n-tuple. By the continuity of distributional Jacobian, Lemma 7 for $k = n$, since $s \geq \frac{n}{n+1} > 1 - \frac{1}{n}$ and $\frac{n}{s} > n$, we have

$$\sum_I \int_{\mathbb{B}^n} u_\varepsilon^*(\mathrm{d}p^I)\left(\varphi_\delta^I - \kappa_I(u_\varepsilon)\right) \precsim [u]^n_{W^{s,\frac{n}{s}}(\mathbb{B}^n)} \left[\varphi_\delta^I - \kappa_I(u_\varepsilon)\right]_{W^{(1-s)n,\frac{1}{1-s}}}.$$

Since $s \geq \frac{n}{n+1}$ we see that $(1-s)n \leq s$, and thus by Sobolev embedding

$$\sum_I \int_{\mathbb{B}^n} u_\varepsilon^*(\mathrm{d}p^I)\left(\varphi_\delta^I - \kappa_I(u_\varepsilon)\right) \precsim [u]^n_{W^{s,\frac{n}{s}}(\mathbb{B}^n)} \left[\varphi_\delta^I - \kappa_I(u_\varepsilon)\right]_{W^{s,\frac{n}{s}}}.$$

We conclude that for ε_1 and δ_1 suitably small,

$$\frac{1}{2} \leq \sum_I \int_{\mathbb{B}^n} u_\varepsilon^*(\mathrm{d}p^I)\varphi_\delta^I \quad \forall\, \varepsilon \in (0,\varepsilon_1),\ \delta \in (0,\delta_1).$$

On the other hand, by the assumption of $\operatorname{rank}(\nabla u) \leq n-1$ in the distributional sense, we have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{B}^n} u_\varepsilon^*(\mathrm{d}p^I)\varphi_\delta^I = 0.$$

This is a contradiction, and we can conclude. □

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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