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On the width of σ-subnormality in finite groups

Sur la largeur de la sous-normalité σ dans les groupes finis

Adolfo Ballester-Bolinches [a], Sergey Kamornikov [b], Vicent Pérez-Calabuig [a] and Xiaolan Yi [c]

[a] Universitat de València, Dr. Moliner 50, 46100 Burjassot, València, Spain

[b] Francisk Skorina State Gomel University, 104 Sovetskaya Str., 246019, Gomel, Belarus

[c] Zhejiang Sci-Tech University, 310018, Hangzhou, P. R. China

E-mails: adolfo.ballester@uv.es, sfkamornikov@mail.ru, vicent.perez-calabuig@uv.es, yixiaolan2005@126.com

Abstract. Let $\sigma = \{\sigma_i \mid i \in I\}$ be a partition of the set of all primes with $\sigma \neq \{\mathbb{P}\}$. Problem 19.84 of the Kourovka Notebook shows that a σ-analogue of the classical Wielandt criterion of subnormality is not true in general. In this paper, we prove a Wielandt-type σ-subnormality criterion in terms of the Baer–Suzuki width of the class of all σ_0 groups, where σ_0 is the member of σ such that $2 \in \sigma_0$.

Résumé. Soit $\sigma = \{\sigma_i \mid i \in I\}$ une partition de l'ensemble des nombres premiers de $\sigma \neq \{\mathbb{P}\}$. Le problème 19.84 du cahier Kourovka montre qu'un σ-analogue du critère de sous-normalité classique de Wielandt n'est pas vrai en général. Dans cet article, nous prouvons un critère de sous-normalité de type Wielandt en termes de largeur de Baer–Suzuki de la classe de tous les groupes σ_0, où σ_0 est le membre de σ tel que $2 \in \sigma_0$.

Keywords. Finite group, σ-subnormal subgroup, subnormal subgroup, σ-nilpotent group, Baer–Suzuki theorem.

Mots-clés. Groupe fini, sous-groupe σ-sous-normal, sous-groupe sous-normal, groupe σ-nilpotent, théorème de Baer–Suzuki.

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1. Introduction

All groups considered in this paper are finite.

In his seminal paper [10], Skiba introduced the subgroup embedding property of σ-subnormality, a generalisation of subnormality for a partition $\sigma = \{\sigma_i \mid i \in I\}$ of the set $\mathbb{P}$ of all primes.

A subgroup H of a group G is called *σ-subnormal* in G if there is a chain of subgroups

$$H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$$

such that for every $1 \le j \le n$, either H_{j-1} is a normal subgroup of H_j, or $H_j / \mathrm{Core}_{H_j}(H_{j-1})$ is a σ_i-group for some $i \in I$.

Several σ-subnormality criteria have been extensively studied, yielding a prolific and inspiring body of results. In this context, one of the particularly noteworthy lines of research is the following σ-version of a classical result of Wielandt [7, Theorem 7.3.3].

Problem 1 (Problem 19.84 of the Kourovka Notebook [1]). Suppose that a subgroup A of a finite group G is σ-subnormal in $\langle A, A^x\rangle$ for each $x \in G$. Does it imply that A is σ-subnormal in G?

Although the answer is negative in general (see [3, Example 1]), positive answers have been proven for the class of all σ-soluble groups [3], and the class of all $3'$-groups [6].

Wielandt's subnormality criterion is closely related with a renowned result by Baer and Suzuki characterising the p-radical $\mathrm{O}_p(G)$ of a group G for a given prime p: for each $x \in G$, $\langle x, x^g\rangle$ is a p-subgroup of G for every $g \in G$ if and only if $x \in \mathrm{O}_p(G)$ [4, Theorem I.1.14]).

Baer–Suzuki theorem cannot be extended in general to the π-radical $\mathrm{O}_\pi(G)$ of a group G, where π is an arbitrary set of primes [9, Example 1], nor the soluble radical [5, Remark 1.5]. Then, the following definition in [5] naturally arises.

Recall that a Fitting class is a class of groups which is closed by taking subnormal subgroups and normal products.

Definition 2 ([5, Definition 1.15]). *Let $\mathfrak{X}$ be a Fitting class. The* Baer–Suzuki width $\mathrm{BS}(\mathfrak{X})$ *is defined as the smallest natural n such that for every group G, the $\mathfrak{X}$-radical $\mathfrak{X}(G)$ coincides with the set of elements $x \in G$ such that $\langle x^{g_1}, \dots, x^{g_n}\rangle$ belongs to $\mathfrak{X}$, for every $g_1, \dots, g_n \in G$. If such an n does not exist, we set* $\mathrm{BS}(\mathfrak{X}) := \infty$.

We write $\mathrm{BS}(\pi)$ for the Baer–Suzuki width of the Fitting class of all π-groups. Then $\mathrm{BS}(\pi) = 2$ for all subsets π of odd primes [9, Theorem 1]. Baer–Suzuki theorem yields $\mathrm{BS}(\mathfrak{N}) = 2$ for the Fitting class $\mathfrak{N}$ of all nilpotent groups, and $\mathrm{BS}(\mathfrak{S}) = 4$ for the Fitting class $\mathfrak{S}$ of all soluble groups [5, Theorem 1.1].

Problem 1 and Definition 2 motivate the following.

Definition 3. *Let σ be a partition of $\mathbb{P}$. The* Wielandt width $\mathrm{W}(\sigma)$ *of σ is defined as the smallest natural m such that, for every group G, the set of σ-subnormal subgroups of G coincides with the set of subgroups H of G such that H^{x_i} is σ-subnormal in $\langle H^{x_1}, H^{x_2}, \dots, H^{x_m}\rangle$, for all $x_1, x_2, \dots, x_m \in G$ and all $1 \le i \le m$.*

The following example in [9] shows that the Wielandt width of a partition can be arbitrarily large.

Example 4. Let $p \ge 5$ be a prime number and take $\pi = \{q \mid q \in \mathbb{P},\ q < p\}$ and $\sigma = \{\pi, \pi'\}$, a binary partition. Let G be the symmetric group of degree p, and let x be a transposition in G. For $m < p-1$, any m transpositions generate a π-subgroup, that is, the subgroup $\langle x^{g_1}, \dots, x^{g_m}\rangle$ is a π-subgroup for every $g_1, \dots, g_m \in G$. If $H = \langle x\rangle$, then the subgroups H^{g_i} are σ-subnormal in $\langle H^{g_1}, \dots, H^{g_m}\rangle$ for every $g_1, \dots, g_m \in G$ and each $1 \le i \le m$; however, H is not σ-subnormal in G.

We denote by σ_0 the member of a partition σ of $\mathbb{P}$ containing the prime 2. The main result of this paper provides a σ-subnormality criterion in terms of the Baer–Suzuki width of the class of all σ_0-groups.

Theorem 5. $\mathrm{W}(\sigma) = \mathrm{BS}(\sigma_0)$, *for every partition* $\sigma \neq \{\mathbb{P}\}$.

Corollary 6 ([8, Theorem 4]). *If* $\sigma_0 = \{2\}$, *then a subgroup* H *of a group* G *is* σ*-subnormal in* G *if and only if* H *is* σ*-subnormal in* $\langle H, H^x\rangle$ *for each* $x \in G$.

In [11, Theorem 2] it is proved that if π is a proper subset of $\mathbb{P}$ then $r-1 \le \mathrm{BS}(\pi) \le \max\{11, 2(r-2)\}$, where r is the minimal prime not in π. This yields lower and upper bounds of $\mathrm{W}(\sigma)$.

Corollary 7. *Let* $\sigma \neq \{\mathbb{P}\}$ *be a partition of* $\mathbb{P}$. *Then,* $r-1 \le \mathrm{W}(\sigma) \le \max\{11, 2(r-2)\}$, *where* r *is the minimal prime not in* σ_0.

In particular, $r \ge 3$ as $2 \in \sigma_0$. Moreover, if $r \ge 5$, then $\mathrm{W}(\sigma) \ge 4$. In [11] we also find the following conjecture.

Conjecture 8 ([11, Conjecture 1]). *Let* π *be a proper subset of* $\mathbb{P}$ *containing at least two elements, and let* r *be the minimal prime not in* π. *Then*

$$\mathrm{BS}(\pi) = \begin{cases} r, & \text{if } r \in \{2,3\}, \\ r-1, & \text{if } r \ge 5. \end{cases}$$

Corollary 9. *Let* $\sigma \neq \{\mathbb{P}\}$ *be a partition of the set of all primes such that* σ_0 *contains at least two prime numbers and let* r *be the minimal prime not in* σ_0. *If Conjecture 8 is true, then*

$$\mathrm{W}(\sigma) = \begin{cases} r, & \text{if } r \in \{2,3\}, \\ r-1, & \text{if } r \ge 5. \end{cases}$$

From Corollary 9, we see that if Conjecture 8 is true, then Problem 1 is true if, and only if, $\sigma_0 = \{2\}$.

2. Preliminary results

We shall adhere to [10] for terminology and basic results about the σ-theory of groups. Let $\sigma = \{\sigma_i \mid i \in I\}$ be a partition of the set of all primes, and $\mathfrak{N}_\sigma$ be the class of all groups that are direct products of σ_i-groups. $\mathcal{N}_\sigma$ is a subgroup-closed saturated Fitting formation [10, Corollary 2.4 and Lemma 2.5]. Furthermore, the σ-subnormal subgroups of a group G are precisely the $K-\mathcal{N}_\sigma$-subnormal subgroups of G (see [2, Chapter 6]).

The $\mathcal{N}_\sigma$-radical of a group G is denoted by $F_\sigma(G)$. Thus, $F_\sigma(G)$ is the largest normal σ-nilpotent subgroup of G. The $\mathcal{N}_\sigma$-residual of G is denoted by $G^{\mathfrak{N}_\sigma}$. Hence, $G^{\mathfrak{N}_\sigma}$ is the smallest normal subgroup of G with σ-nilpotent quotient.

Let $\mathrm{sn}_\sigma(G)$ be the set of σ-subnormal subgroups of a group G. Our first lemma collects some basic properties of σ-subnormal subgroups (see [10, Lemma 2.6]).

Lemma 10. *Let* H, K *and* N *be subgroups of a group* G. *Suppose that* $H \in \mathrm{sn}_\sigma(G)$ *and* N *is normal in* G. *Then, the following statements hold:*

(1) $H \cap K \in \mathrm{sn}_\sigma(K)$;
(2) *if* $K \subseteq H$ *and* $K \in \mathrm{sn}_\sigma(H)$, *then* $K \in \mathrm{sn}_\sigma(G)$;
(3) *if* $K \in \mathrm{sn}_\sigma(G)$, *then* $H \cap K \in \mathrm{sn}_\sigma(G)$;
(4) $HN/N \in \mathrm{sn}_\sigma(G/N)$;
(5) *if* $N \subseteq K$, *then* $K/N \in \mathrm{sn}_\sigma(G/N)$ *if, and only if,* $K \in \mathrm{sn}_\sigma(G)$;
(6) *if* $H \in \mathfrak{N}_\sigma$, *then* $H \subseteq F_\sigma(G)$; *in particular, if* H *is* σ_i*-group, then* $H \le \mathrm{O}_{\sigma_i}(G)$;

(7) *if H is σ-subnormal in G, then $H^{\mathcal{N}_\sigma}$ is subnormal in G (see [2, Lemma 6.1.9 and Proposition 6.1.10]);*
(8) *if G is σ-nilpotent, then every subgroup of G is σ-subnormal in G.*

The set of all σ-subnormal subgroups of a group G is a sublattice of the subgroup lattice of G. This result can be deduced from the results on lattice formations presented in [2, Chapter 6]. However, we give a proof for the sake of completeness.

Lemma 11. *The set of all σ-subnormal subgroups of a group G is a sublattice of the subgroup lattice of G.*

Proof. By Lemma 10, we only need to prove that if U and V are σ-subnormal in G, then $X = \langle U, V\rangle$ is σ-subnormal in G. Arguing by induction on $|G|$, we may assume that $U \neq 1$ and $V \neq 1$. Let N be a minimal normal subgroup of G. By Lemma 10, XN is σ-subnormal in G. If XN were a proper subgroup of G, then X would be σ-subnormal in XN and so in G by Lemma 10. Hence we may assume that $G = XN$ for every minimal normal subgroup N of G. Therefore we may assume that $\mathrm{Core}_G(X) = 1$. By Lemma 10, $A = U^{\mathfrak{N}_\sigma}$ and $B = V^{\mathfrak{N}_\sigma}$ are subnormal in G and so they are normalised by N [4, Lemma A.14.3]. In particular, the normal closure $\langle A, B\rangle^G$ of $\langle A, B\rangle$ in G is contained in $\mathrm{Core}_G(X) = 1$. Thus, U and V are nilpotent. By Lemma 10, X is contained in $F_\sigma(G)$. Then $1 \neq F_\sigma(G)$ and so we may assume that $N \subseteq F_\sigma(G)$. Hence $G = F_\sigma(G)$ and G is σ-nilpotent. By Lemma 10, X is σ-subnormal in G. □

Corollary 12. *Let H be a σ_i-subgroup of a group G for some $i \in I$ and let $x_1, x_2, \dots, x_m \in G$. If H^{x_j} is σ-subnormal in $\langle H^{x_1}, H^{x_2}, \dots, H^{x_m}\rangle$ for every $1 \le j \le m$, then $\langle H^{x_1}, H^{x_2}, \dots, H^{x_m}\rangle$ is a σ_i-group.*

Proof. Since H^{x_j} is a σ-subnormal σ_i-subgroup of $\langle H^{x_1}, H^{x_2}, \dots, H^{x_m}\rangle$, by Lemma 10,

$$H^{x_j} \subseteq \mathrm{O}_{\sigma_i}\big(\langle H^{x_1}, H^{x_2}, \dots, H^{x_m}\rangle\big),$$

for every $1 \le j \le m$. Thus,

$$\langle H^{x_1}, H^{x_2}, \dots, H^{x_m}\rangle = \mathrm{O}_{\sigma_i}\big(\langle H^{x_1}, H^{x_2}, \dots, H^{x_m}\rangle\big).$$

Hence, $\langle H^{x_1}, H^{x_2}, \dots, H^{x_m}\rangle$ is a σ_i-group. □

3. Proof of Theorem 5

Proof. Assume that $\mathrm{BS}(\sigma_0) = m$. It is clear that $m \ge 2$.

Suppose that $\mathrm{W}(\sigma) \le m$ is not true. Then there exists a group G with a subgroup H such that H^{x_j} is σ-subnormal in $\langle H^{x_1}, H^{x_2}, \dots, H^{x_m}\rangle$ for every $x_1, x_2, \dots, x_m \in G$, $1 \le j \le m$, but H is not σ-subnormal in G. Let us choose the pair (G, H) with $|G| + |H|$ minimal. Then $H \neq 1$ and, by [3, Theorem A], G is not σ-soluble.

Let N be a minimal normal subgroup of G. By Lemma 10, the subgroup $H^{x_j}N/N$ is σ-subnormal in

$$\big\langle H^{x_1}N/N, H^{x_2}N/N, \dots, H^{x_m}N/N\big\rangle = \langle H^{x_1}, H^{x_2}, \dots, H^{x_m}\rangle N/N$$

for every $x_1, x_2, \dots, x_m \in G$, $1 \le j \le m$. By the minimality of the pair (G, H), it holds $HN/N \in \mathrm{sn}(G/N)$. Hence $HN \in \mathrm{sn}_\sigma(G)$. Moreover, if $|HN| < |G|$, by the minimality of (G, H), it follows that H is σ-subnormal in HN. By Lemma 10, H is σ-subnormal in G, contrary to supposition. Therefore, $HN = G$ for every minimal normal subgroup N of G.

By Lemma 10, $(H^{\mathfrak{N}_\sigma})^{x_j}$ is subnormal in $\big\langle (H^{\mathfrak{N}_\sigma})^{x_1}, (H^{\mathfrak{N}_\sigma})^{x_2}, \dots, (H^{\mathfrak{N}_\sigma})^{x_m}\big\rangle$, for every $x_1, x_2, \dots, x_m \in G$ and each $1 \le j \le m$. In particular, for every $x \in G$ the subgroup $H^{\mathfrak{N}_\sigma}$ is subnormal in $\big\langle H^{\mathfrak{N}_\sigma}, (H^{\mathfrak{N}_\sigma})^x\big\rangle$. Then Wielandt's subnormality criterion yields $H^{\mathfrak{N}_\sigma}$ is subnormal in G.

Let M be a maximal subgroup of G containing H. By [7, Lemma 7.3.16], it holds that $H^{\mathfrak{N}_\sigma}$ is contained in $\mathrm{Core}_G(M)$. Assume that $H^{\mathfrak{N}_\sigma} \neq 1$. Then, $\mathrm{Core}_G(M) \neq 1$, and therefore, $G = H\,\mathrm{Core}_G(M) \subseteq M$, a contradiction. Thus, $H^{\mathfrak{N}_\sigma} = 1$, and H is σ-nilpotent.

Assume by a way of contradiction that H is not a cyclic group of prime power order. Then, H has at least two maximal subgroups H_1 and H_2 such that $H_1 \neq H_2$. Since H is σ-nilpotent, $H_1^{x_j}$ and $H_2^{x_j}$ are σ-subnormal in $\langle H^{x_1}, H^{x_2}, \dots, H^{x_m}\rangle$ for every $x_1, x_2, \dots, x_m \in G$ and each $1 \le j \le m$ by Lemma 10. Thus, by Lemma 10, $H_i^{x_j}$ is σ-subnormal in $\langle H_i^{x_1}, H_i^{x_2}, \dots, H_i^{x_m}\rangle$ for every $x_1, x_2, \dots, x_m \in G$, each $1 \le i \le m$, and $i = 1,2$. Since $|H_i| + |G| < |H| + |G|$, from the minimality of (G, H) it follows that H_i is σ-subnormal in G, $i = 1,2$. By Lemma 11, $H = \langle H_1, H_2\rangle$ is σ-subnormal in G, contrary to assumption. Therefore, $H = \langle h\rangle$ is a cyclic group and $|H| = p^n$ for some prime p and some positive integer n. Consider two possible cases.

Case 1. Assume that $p \in \sigma_0$. By Lemma 12,

$$\langle h^{x_1}, h^{x_2}, \dots, h^{x_m}\rangle$$

is a σ_0-group for every $x_1, x_2, \dots, x_m \in G$. Since $m = \mathrm{BS}(\sigma_0)$, $h \in \mathrm{O}_{\sigma_0}(G)$. Thus, $H = \langle h\rangle \subseteq \mathrm{O}_{\sigma_0}(G)$, and therefore, H is σ-subnormal in G, a contradiction.

Case 2. Assume that $p \notin \sigma_0$. Let σ_k be the member of σ containing p. Since $\sigma_k \neq \sigma_0$, σ_k is a set composed of odd primes. By Lemma 12 the subgroup

$$\langle h^{x_1}, h^{x_2}, \dots, h^{x_m}\rangle$$

is a σ_k-group for every $x_1, x_2, \dots, x_m \in G$. In particular, it follows that for every $g \in G$, $\langle h, h^g\rangle$ is a σ_k-group. Applying [9, Theorem 1], we conclude that $h \in \mathrm{O}_{\sigma_k}(G)$, and therefore, $H = \langle h\rangle$ is σ-subnormal in G. This final contradiction yields $\mathrm{W}(\sigma) \le \mathrm{BS}(\sigma_0)$.

We prove next that $\mathrm{BS}(\sigma_0) \le \mathrm{W}(\sigma)$. Call $\pi := \sigma_0$ and $k := \mathrm{W}(\sigma)$. Let $x \in G$ such that

$$\langle x^{g_1}, x^{g_2}, \dots, x^{g_k}\rangle$$

is a π-group for every $g_1, g_2, \dots, g_k \in G$. Then

$$\langle X^{g_1}, X^{g_2}, \dots, X^{g_k}\rangle$$

is a π-group, where $X = \langle x\rangle$. Hence X^{g_i} is σ-subnormal in $\langle X^{g_1}, X^{g_2}, \dots, X^{g_k}\rangle$ for every $g_1, g_2, \dots, g_k \in G$ and each $1 \le i \le k$.

Since $\mathrm{W}(\sigma) = k$, it follows that X is σ-subnormal in G. By Lemma 10, $X \subseteq F_\sigma(G)$, and hence $X \subseteq \mathrm{O}_\pi(G)$ as X is a π-group. Therefore, $x \in \mathrm{O}_\pi(G)$.

Hence, we conclude that $\mathrm{BS}(\pi) \le k$, and the proof of the theorem is complete. □

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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