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Isogeny and overconvergence

Isogénie et surconvergence

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Abstract. In this paper, we apply Tsuzuki's main theorem in [12] to establish a criterion for when two abelian varieties over a function field K of characteristic p are isogenous. Specifically, assuming that their endomorphism algebras tensored with $\mathbb{Q}_p$ are division algebras, we prove that if the maximal quotients of minimal slope (i.e., the unique maximal isoclinic quotient corresponding to the minimal slope, defined up to isogeny) of their associated p-divisible groups are isogenous, then the abelian varieties themselves are isogenous over K. We also extend this result to certain p-divisible groups, highlighting the deep connection between isogenies of abelian varieties and the structure of overconvergent F-isocrystals.

Résumé. Dans cet article, nous appliquons le théorème principal de Tsuzuki dans [12] afin d'établir un critère permettant de déterminer quand deux variétés abéliennes sur un corps de fonctions K de caractéristique p sont isogènes. Plus précisément, en supposant que leurs algèbres d'endomorphismes tensorisées par $\mathbb{Q}_p$ sont des algèbres à division, nous démontrons que si les quotients maximaux de pente minimale (c'est-à-dire le quotient isocline maximal unique correspondant à la pente minimale, défini à isogénie près) de leurs groupes p-divisibles associés sont isogènes, alors les variétés abéliennes elles-mêmes sont isogènes sur K. Nous étendons également ce résultat à certains groupes p-divisibles, mettant en évidence le lien profond entre les isogénies de variétés abéliennes et la structure des F-isocristaux surconvergents.

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The aim of this note is to establish a new criterion on detecting isogenies of abelian varieties over function fields of characteristic $p > 0$. Namely, abelian varieties A/K over function fields induce, via their Néron models, abelian schemes on an open subset of a connected proper smooth curve $C/\mathbb{F}_q$ (with function field K). By the theory of Oort–Zink (see [8]), the associated p-divisible group of an abelian scheme admits a slope filtration up to isogeny—that is, while the filtration itself is not unique, its isogeny class is uniquely determined by the Newton polygon. Our main result is that one can detect the isogeny class of the abelian variety by looking at only one of its slopes. More precisely, under the assumption that the endomorphism algebras tensored with $\mathbb{Q}_p$ are division algebras, if the maximal quotient of minimal slope (i.e., the unique maximal isoclinic quotient corresponding to the minimal slope, defined up to isogeny) of the p-divisible groups of A and B are isogenous, then the abelian varieties themselves are isogenous over K.

Our proof is using the "minimal slope conjecture" proved by [3,12] and the fully faithfulness of various functors between p-divisible groups and overconvergent F-isocrystals. Finally, we extend our result to the case of p-divisible groups over varieties over finite fields and to higher transcendence degree fields of characteristic p.

First, let us recall the "minimal slope conjecture" proved by [3,12]. We consider X a smooth connected scheme over a finite field and $M^\dagger$ an overconvergent F-isocrystal on X, and denote the convergent F-isocrystal on X associated to $M^\dagger$ by M. We say M admits a slope filtration if there exists an increasing filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M$$

of M as convergent F-isocrystals on X/K such that:

(i) M_i/M_{i-1} is nonzero and isoclinic of slope s_i;
(ii) $s_1 < s_2 < \cdots < s_r$.

We call s_1 (resp. s_r) the minimal slope (resp. the maximal slope) of M when $M \neq 0$. Such a slope filtration exists if the Newton polygons of the Frobenius structure of a convergent F-isocrystal M are constant, which is always true after a certain shrinking of X (see [12, 1.1]). Hence M always admits a unique slope filtration as convergent F-isocrystals after a certain shrinking of X.

We can now recall the theorem of [3,12]:

Theorem 1. *Assume that $\mathscr{M}^\dagger$ and $\mathscr{N}^\dagger$ are two irreducible overconvergent isocrystals over X with slope filtrations. After renumbering the slope filtration as follows:*

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_{r-1} \supset M_r = 0$$

with the sequence of slopes $s_0 > s_1 > \cdots > s_{r-1}$, suppose that there is a nontrivial morphism $h\colon N/N_1 \to M/M_1$ between the minimal slope sub-objects (or, by duality between the maximal slope quotients) as convergent F-isocrystals. Then there exists a unique isomorphism $g^\dagger\colon N^\dagger \to M^\dagger$ of overconvergent F-isocrystals compatible with the map g.

We will need the following lemma:

Lemma 2. *Let A be an abelian variety over a function field of characteristic p. We denote $\mathscr{A}$ its Néron model over the projective smooth connected curve C with function field K. Let U be an open subset of C where A/K has good reduction. Let $D^\dagger(A)$ be the overconvergent F-isocrystal over $U/\mathbb{Q}_p$ associated to the abelian scheme $\mathscr{A}\big|_U$. Then $D^\dagger(A)$ is semisimple and it will be simple if and only if* $\mathrm{End}(A) \otimes \mathbb{Q}_p$ *is a division algebra.*

Proof. Note that by [9, Theorem 1.2], $D^\dagger(A)$ is semisimple. Now a semisimple module is simple if and only if its endomorphism ring is a division ring. But again by [9, Theorem 1.1], $\mathrm{End}\big(D^\dagger(A)\big) \simeq \mathrm{End}\big(\mathscr{A}\big|_U\big)\otimes\mathbb{Q}_p \simeq \mathrm{End}(A)\otimes\mathbb{Q}_p$, where the second isomorphism is deduced by universal properties of the Néron model and smoothness of $\mathscr{A}\big|_U/U$. □

Remark 3.

(1) Assume $\mathrm{End}(A)\otimes\mathbb{Q}$ to be a division algebra. If it is equal to $\mathbb{Q}$, then obviously $\mathrm{End}(A)\otimes\mathbb{Q}_p$ is still a division algebra. If $F := \mathrm{End}(A)\otimes\mathbb{Q}$ is strictly larger than $\mathbb{Q}$ and a commutative field, then $F_{\mathbb{Q}_p}$ is a division algebra if and only if (p) is a primary ideal of O_F.
(2) Note that if $\mathrm{End}(A)\otimes\mathbb{Q}_p$ is a division algebra, then so is $\mathrm{End}(A)\otimes\mathbb{Q}$, in particular, since abelian varieties are up to isogeny semisimple, this implies that A/K is simple. However, this does not imply that the p-divisible group $\mathscr{A}\big|_U[p^\infty]$ is simple. In particular, it can have multiple slopes.

We can now state the main application to abelian varieties over function fields of characteristic p that we have in mind.

Theorem 4. *Let A and B be abelian varieties over a function field K of characteristic p. Suppose that:*

(1) $\mathrm{End}(A) \otimes \mathbb{Q}_p$ *and* $\mathrm{End}(B) \otimes \mathbb{Q}_p$ *are (not necessarily identical) division algebras;*
(2) *the maximal quotients of minimal slope (i.e., the unique maximal isoclinic quotients corresponding to the minimal slope, defined up to isogeny) of $A[p^\infty]$ and $B[p^\infty]$ are isogenous as p-divisible groups.*

Then A and B are isogenous over K.

Proof. Let $\mathscr{A}$ and $\mathscr{B}$ be the Néron models of A and B over a smooth projective curve C with function field K. Choose a nonempty open subset $U \subset C$ over which both A and B have good reduction and such that the associated p-divisible groups $\mathscr{A}\big|_U[p^\infty]$ and $\mathscr{B}\big|_U[p^\infty]$ have constant Newton polygons. In this setting, the canonical slope filtration exists on the p-divisible groups, and we denote by Q_A and Q_B the maximal quotients of minimal slope of $A[p^\infty]$ and $B[p^\infty]$, respectively. By the functoriality of the slope filtration and the fact that it is defined by a universal property, these quotients extend canonically to the p-divisible groups associated to the Néron models over U; denote these extensions by Q_A^U and Q_B^U. By hypothesis, there exists an isogeny

$$\phi_\eta \colon Q_A \longrightarrow Q_B,$$

defined on the generic fiber (over the field K). Then, by applying de Jong's local result [4, Theorem 1.1] to each $\mathrm{Spec}(\mathscr{O}_v)$ (v a closed point in U), one obtains local extensions of the isogeny ϕ_{η_v} to isogenies ϕ_v. These ϕ_v are compatible at the generic point $\mathrm{Spec}(K) \subset \mathrm{Spec}(\mathscr{O}_v)$, as they all restrict to ϕ_η. By normality of U and the fact that an isogeny is determined by its behavior on codimension-1 points (see the proof of [10, Proposition 4]), these local pieces $\{\phi_v\}$ glue to a single isogeny

$$\phi \colon Q_A^U \longrightarrow Q_B^U$$

over U.

The slope filtration at the level of p-divisible groups over U induces a slope filtration on both $D^\dagger(A)$ and $D^\dagger(B)$ as convergent F-isocrystals with therefore same minimal slope. Moreover, by our first hypothesis and Lemma 2, $D^\dagger(A)$ and $D^\dagger(B)$ are irreducible so that by the minimal slope conjecture (see [12, Theorem 1.3] and its final form [3, Corollary 1.1.4]), this implies that the overconvergent F-isocrystals themselves are isomorphic:

$$D^\dagger(A) \cong D^\dagger(B).$$

We could again use [9, Theorem 1.1] to deduce that A and B are isogenous, but let us consider another proof that can apply to p-divisible groups in the next corollary.

Since the functor from F-crystals up to isogeny to convergent F-isocrystals, and from convergent F-isocrystals to overconvergent F-isocrystals, is fully faithful (see [2, 2.4] and [6, Theorem 1.1]), we deduce that the Dieudonné modules up to isogeny associated with A and B are isomorphic.

By the fully faithfulness of the Dieudonné functor (established by [5]), we conclude that the p-divisible groups $\mathscr{A}[p^\infty]\big|_U$ and $\mathscr{B}[p^\infty]\big|_U$ are isomorphic up to isogeny. Consequently, their generic fibers $A[p^\infty]$ and $B[p^\infty]$ over K are isogenous.

Finally, by the fully faithfulness of the functor from abelian varieties over K to their p-divisible groups (proved by de Jong in [4, Theorem 2.6]), we deduce that A and B are isogenous over K. □

Remark 5. We note that, as the referee pointed out, our Theorem 4 can be extended to higher transcendence degree fields of characteristic p using Tsuzuki's general form of the minimal slope conjecture in higher dimensions (see [12, Theorem 1.4]). Indeed, the proof of Theorem 4 carries through: the overconvergent F-isocrystals $D^\dagger(A)$ and $D^\dagger(B)$ are semisimple, as noted in the proof of [3, Theorem 5.2.2] (and hence irreducible by Lemma 2 under our hypotheses), and the spreading of the isogeny can be established by reducing the question to [4, Corollary 1.2], following the same reasoning as in [10, Proposition 4].

Corollary 6. *Let G and H be two p-divisible groups over X, a smooth variety over a finite field. Assume that both have constant Newton polygon on X, and that their associated Dieudonné crystals are overconvergent and irreducible. Then if G and H have their slope filtration with same maximal slope quotients (or minimal slope quotients) isomorphic. Then G and H are isogenous.*

Proof. Under our hypothesis $D^\dagger(G) \simeq D^\dagger(H)$, implying as in the proof of the previous theorem $G \simeq H$ up to isogeny. □

Remark 7. Assume that X is a smooth, geometrically connected curve, and let G be a p-divisible group over X that is potentially semistable in the sense of [11, Theorem 4.5], so that its associated Dieudonné crystal $D(G)^\dagger$ is overconvergent. Suppose further that G is principally polarized in such a way that the associated Rosati involution is positive, which in turn implies that the Frobenius linear endomorphism on the Dieudonné module $D(G_x)$ is self-adjoint and hence semisimple for a closed point x of X. In addition, assume that $D(G)^\dagger$ is ι-pure. Then, by [9, Proposition 7.9], $D(G)^\dagger$ is semisimple in F-Isoc†. Finally, if we further assume that

$$\mathrm{End}(G) \otimes \mathbb{Q}_p$$

is a division algebra, it follows that

$$\mathrm{End}_{F\text{-Isoc}^\dagger}(D(G)^\dagger) = \mathrm{End}(G) \otimes \mathbb{Q}_p$$

is also a division algebra, and since $D(G)^\dagger$ is semisimple, it is in fact simple. Note that the condition on $\mathrm{End}(G) \otimes \mathbb{Q}_p$ does not imply that G itself is simple.

As a concluding remark, we hope that this result will contribute to a better understanding of moduli spaces of abelian varieties over function fields or more generally of finitely generated fields of characteristic p. It might also be interesting to compare our result with the notion of minimal p-divisible groups introduced by Oort [7]. Finally, we mention another application of the "minimal slope conjecture" by Ambrosi–D'Addezio [1] in deducing finiteness results on the torsion of abelian varieties.

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Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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