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Research article
Harmonic analysis

A generic threshold phenomenon in weighted ℓ^2

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Abstract. We study threshold phenomena in weighted ℓ^2-spaces. Our main result is a summable Baire category version of Körner's topological Ivashev-Musatov Theorem, which we show is optimal in several respects.

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1. Introduction

1.1. *Threshold phenomena in weighted ℓ^2*

Given positive numbers $w = (w_n)_{n=0}^{\infty}$, we denote by $\ell^2(w)$ the Hilbert space of distributions S on $\mathbb{T}$ satisfying

$$\|S\|_{\ell^2(w)} = \left(\sum_n \left|\widehat{S}(n)\right|^2 w_{|n|}\right)^{1/2} < \infty.$$

We uniquely identify a distribution S on $\mathbb{T}$ with its Fourier coefficients, thus we shall often use the notation $S \in \ell^2(\lambda)$, instead of the natural alternative $\left\{\widehat{S}(n)\right\}_n \in \ell^2(\lambda)$. It is straightforward to verify that $\ell^2(w)$ is separable, containing the trigonometric polynomials as a dense subset. If $w_n = 1$ is constant, then we retain the classical ℓ^2-space, which by Parseval's Theorem can be identified with $L^2(\mathbb{T}, \mathrm{d}m)$, the space of square integrable functions f on the unit-circle $\mathbb{T}$:

$$\sum_n \left|\widehat{f}(n)\right|^2 = \int_{\mathbb{T}} \left|f(\zeta)\right|^2 \mathrm{d}m(\zeta) < \infty,$$

where $\mathrm{d}m$ denotes the arc-length measure on $\mathbb{T}$. Now it is of interest to exhibit function theoretical properties of elements belonging to $\ell^2(w)$. A natural question that arises is: *when does $\ell^2(w)$ contain the continuous functions on $\mathbb{T}$?* The following observation clarifies that Parseval's Theorem gives rise to the following threshold phenomenon.

Proposition 1. *Let $(\lambda_n)_n$ be positive numbers with $\lambda_n \uparrow +\infty$. Then there exists a continuous function f on $\mathbb{T}$, such that*

$$\sum_n \left|\widehat{f}(n)\right|^2 \lambda_{|n|} = \infty.$$

This observation is certainly expected, and likely well-known to experts. In fact, the failure of the embedding of the continuous functions into $\ell^2(\lambda)$, as demonstrated by Proposition 1, has the immediate dual implication that $\ell^2(w)$ contains elements that are not Fourier coefficients of complex Borel measures on $\mathbb{T}$, whenever $w_n \downarrow 0$. For the reader's convenience, we outline short proofs in the appendix.

Our main purpose in this note is to find the precise threshold for when $\ell^2(w)$ always consists of continuous functions in $\mathbb{T}$, thus complementing the above observation.

1.2. *Topologically bad support*

Our main result is a weighted ℓ^2-analogue of Körner's topological Ivashev-Musatov Theorem in [7].

Theorem 2. *Let $(\lambda_n)_n$ be positive numbers satisfying the following hypotheses:*

(i) $\sum_n \frac{1}{\lambda_n} = +\infty$;

(ii) *there exists $C > 1$, such that for all $n \geq 1$:*

$$C^{-1}\lambda_n \leq \lambda_k \leq C\lambda_n, \quad n \leq k \leq 2n.$$

Then there exists a positive function $f \in L^\infty(\mathbb{T}, \mathrm{d}m)$ such that $\operatorname{supp}(f\,\mathrm{d}m)$ *has empty interior, and*

$$\sum_n \left|\widehat{f}(n)\right|^2 \lambda_{|n|} < \infty.$$

Our theorem is essentially sharp, since if the hypothesis in (i) is violated, then Cauchy–Schwartz inequality implies

$$\sum_n \left|\widehat{f}(n)\right| \leq \left(\sum_n \left|\widehat{f}(n)\right|^2 \lambda_n\right)^{1/2} \left(\sum_n \frac{1}{\lambda_n}\right)^{1/2} < \infty,$$

hence any element in $\ell^2(\lambda)$ belongs to the so-called Wiener algebra, and are thus continuous on $\mathbb{T}$. Primary examples of spaces for which our Theorem 2 applies to are Dirichlet-type space of elements $f \in L^2(\mathbb{T}, \mathrm{d}m)$ with

$$\sum_n \left|\widehat{f}(n)\right|^2 \left(1+|n|\right)^\gamma < \infty, \quad 0 < \gamma \leq 1.$$

In this classical framework, it is a well-known fact that one can exhibit elements which are not continuous, while if $\gamma > 1$, classical Sobolev embeddings ensure containment into Hölder-type spaces. For instance, see [2]. At the end of Section 2, we shall explain why Theorem 2 cannot simply be derived from methods involving sparse Fourier support, such as Riesz-type products and lacunary series.

A natural follow-up question is whether the regularity hypothesis (ii) is truly necessary, or if it can be replaced by a more natural monotonicity condition. Our next observation clarifies these points.

Theorem 3. *There exists a sequence of positive, increasing numbers $(\lambda_n)_n$ with $\lambda_n \uparrow +\infty$ and $\sum_n \frac{1}{\lambda_n} = +\infty$, such that whenever S is a non-zero distribution in $\mathbb{T}$ with*

$$\sum_n \left|\widehat{S}(n)\right|^2 \lambda_{|n|} < \infty,$$

then $\operatorname{supp}(S) = \mathbb{T}$.

This result is essentially similar to Körner's Theorem 1.2 in [5], and our proof will take inspiration from it. It remains unclear what a "critical" version of Theorem 2 would entail, which aligns well with other uncertainty principles in harmonic analysis, such as the Ivashev-Musatov Theorem and the Beurling–Malliavin Multiplier Theorem. For further discussions on related topics, see [6] and [3].

1.3. *A Baire category version*

It turns out that the elements appearing in the statement of Theorem 2 are generic. To this end, let $\mathscr{C}$ denote the collection of non-empty compact subsets of $\mathbb{T}$ equipped with the so-called *Hausdorff metric*

$$d_{\mathscr{C}}(E,K) = \sup_{\zeta\in E} \operatorname{dist}(\zeta,K) + \sup_{\xi\in K} \operatorname{dist}(\xi,E).$$

It is straightforward to verify that $(\mathscr{C}, d_{\mathscr{C}})$ is a complete metric space. We can actually offer the following Baire category version of Theorem 2.

Theorem 4. *Let* $(\lambda_n)_n$ *be positive numbers satisfying the hypotheses* (i)–(ii) *of Theorem 2. Consider the collection* $\mathscr{L}_{\mathscr{C}}(\lambda) \subset \ell^2(\lambda) \times \mathscr{C}$ *of ordered pairs* (f,E) *satisfying the properties:*

(i) $\operatorname{supp}(f) \subseteq E$;
(ii) $0 \le f(\zeta) \le 2$, *for* $\zeta \in \mathbb{T}$;

equipped with the metric

$$d_\lambda\big((f,E),(g,K)\big) := d_{\mathscr{C}}(E,K) + \|f-g\|_{\ell^2(\lambda)}.$$

Then the sub-collection of pairs $(f,E) \in \mathscr{L}_{\mathscr{C}}(\lambda)$ *with* E *having no interior is generic in the complete metric space* $\big(\mathscr{L}_{\mathscr{C}}(\lambda), d_\lambda\big)$.

Our principal emphasis will be to prove Theorem 4, from which Theorem 2 follows as an immediate corollary.

1.4. *Organization and notation*

The paper is organized as follows: Section 2 is devoted to proving Theorem 4, where we take inspiration from work of T. W. Körner in [7]. In Section 3, we deduce Theorem 3 from a slightly stronger result, which utilizes a simple but useful lemma of Körner and Meyer. In the appendix, we give short sketches of proofs for Proposition 1.

For two positive numbers $A, B > 0$, we will frequently use the notation $A \lesssim B$ to mean that $A \le cB$ for some positive constant $c > 0$. If both $A \lesssim B$ and $B \lesssim A$ hold, we will write $A \asymp B$.

2. Bounded functions with bad support

2.1. *The doubling condition*

The following lemma summarizes how the hypothesis (ii) of Theorem 2 will be used throughout. By means of substituting λ_n with $\max(\lambda_n, 1)$, we may without loss of generality always assume that $\lambda_n \ge 1$ for all n.

Lemma 5. *Let* $(\lambda_n)_n$ *be positive numbers with the hypothesis* (ii) *of Theorem 2. Then there exists* $M(\lambda) > 1$ *such that for any integer* $M > M(\lambda)$, *the following statements hold:*

$$(1+n)^{-M}\lambda_n \le (1+m)^{-M}\lambda_m, \quad 1 \le m < n, \tag{1}$$

$$\sum_{j=1}^{n} \frac{j^{M-1}}{\lambda_j} \le 10M(\lambda)\frac{n^M}{\lambda_n}, \quad n = 1,2,3,\dots, \tag{2}$$

$$\sum_{j>n} \frac{1}{\lambda_j j^{M+1}} \le M(\lambda)\frac{1}{n^M \lambda_n}, \quad n = 1,2,3,\dots \tag{3}$$

Note that the second condition means that $\lambda_n(1+n)^{-M}$ is non-increasing whenever $M > M(\lambda)$, which we shall use frequently. In particular, this implies that λ_n has at most polynomial growth.

Proof. The proof is simple. Let $C(\lambda) > 1$ be the constant appearing in the hypothesis (ii) of Theorem 2. Fix two arbitrary integers $1 \leq m \leq n$ and let $l \asymp \log_2(n/m)$ be the integer with $n \leq 2^l m$. Repeatedly, using the assumption (ii) on $(\lambda_n)_n$, we readily get

$$\lambda_n \leq \lambda_m C(\lambda)^{l+1} \leq \lambda_m \frac{(1+n)^M}{(1+m)^M}, \quad 1 \leq m < n,$$

whenever $M > M(\lambda)$ large enough. The claim in (2) readily follows from (1): j^{M-1}/λ_j being non-decreasing, by means of further enlarging $M(\lambda)$, if necessary. In order to prove (3), we first note that

$$\lambda_{2^j} 2^{jA} \leq C(\lambda) 2^{-A} \lambda_{2^{j+1}} 2^{A(j+1)} \leq \lambda_{2^{j+1}} 2^{A(j+1)}, \quad j = 0, 1, 2, \dots$$

whenever, say $A > 100 \log C(\lambda)$. With this at hand, we get that for any $n \geq 1$:

$$\sum_{j>n} \frac{1}{\lambda_j j^{M+1}} = \sum_{j > \log n} \sum_{k \asymp 2^j} \frac{1}{\lambda_k k^{M+1}} \asymp \sum_{j > \log n} \frac{1}{\lambda_{2^j} 2^{jM}} \lesssim \frac{1}{\lambda_n n^A} \sum_{j > \log n} 2^{-j(M-A)} \lesssim \frac{1}{\lambda_n n^M},$$

whenever $M > A$. □

2.2. *A smooth localizing function*

The proof of our result hinges on the following principal lemma on uniformly bounded functions with small amplitudes.

Lemma 6. *Let* $(\lambda_n)_n$ *be positive numbers satisfying the hypotheses* (i)–(ii) *of Theorem 2. Then there exists an integer* $N(\lambda) > 0$ *such that for any* $N \geq N(\lambda)$ *the following statement holds: for any* $0 < \varepsilon < 1$, *there exists* $\psi_\varepsilon \in C^\infty(\mathbb{T})$, *such that:*

- (i) $0 \leq \psi_\varepsilon \leq 1 + \varepsilon$ *on* $\mathbb{T}$;
- (ii) $\psi_\varepsilon = 0$ *in a neighborhood of* 1, *whose length tends to zero as* $\varepsilon \to 0$;
- (iii) $\int_{\mathbb{T}} \psi_\varepsilon \, dm = 1$;
- (iv) $\sup_{n \neq 0} |\widehat{\psi}_\varepsilon(n)| \leq \min(\varepsilon, c(N) \varepsilon^{1-N} |n|^{-N})$, *where* $c(N) > 0$ *is a constant only depending on* N;
- (v) $\sum_{n \neq 0} |\widehat{\psi}_\varepsilon(n)|^2 \lambda_{|n|} \leq \varepsilon^2$.

In order to prove our lemma we shall need the following building block from the work of T. W. Körner, see [7, Lemma 20].

Lemma 7 (T. W. Körner). *Given positive integers* $M, S > 0$, *there exists constants* $A(M,S) > 0$ *and* $\delta(M,S) > 0$ *such that the following statement holds: for any* $0 < \eta < 1/2$, *one can find smooth real-valued functions* $g_{\eta,M,S}$ *satisfying the following conditions:*

- (i) $g_{\eta,M,S}(\zeta) = 1$, *for* $|\zeta - 1| \leq \delta(M,S)\eta$;
- (ii) $-S^{-1} \leq g_{\eta,M,S} \leq 1$ *on* $\mathbb{T}$;
- (iii) $\widehat{g}_{\eta,M,S}(0) = 0$;
- (iv)

$$|\widehat{g}_{\eta,M,S}(n)| \leq \eta A(M,S) \min\left(1, (\eta|n|)^M, (\eta|n|)^{-M}\right), \quad n \neq 0.$$

Furthermore, the parameters $A(M,S), \delta(M,S) \to 0$ as $S \to \infty$.

Proof of Lemma 6. Fix a number $M > M(\lambda)$ as in the statement of Lemma 5. Let $0 < \varepsilon < 1$, pick an integer $S = S(\varepsilon) > 2/\varepsilon$, such that

$$L(\varepsilon) := \sum_{1/\varepsilon \leq j \leq S(\varepsilon)} \frac{1}{\lambda_j} \geq 1/\varepsilon. \tag{4}$$

This is possible due to hypothesis (i) of Theorem 2. Now applying Lemma 7, we consider functions of the form

$$\psi_\varepsilon(\zeta) := 1 - \frac{1}{L(\varepsilon)} \sum_{1/\varepsilon \le j \le S(\varepsilon)} \frac{1}{\lambda_j} g_{1/j,M,S(\varepsilon)}(\zeta), \quad \zeta \in \mathbb{T},$$

where the functions $g_{1/j,M,S(\varepsilon)}$ are as in Körner's Lemma. Now the properties (i)–(iii) are immediate consequences of the corresponding properties of the g's from Körner's Lemma. Note that the statement in (iv) follows from item (iv) of Körner's Lemma, since our sum only involves terms $j > 1/\varepsilon$. It therefore remains only to verify (v). Since all functions involved are real-valued, we only need to estimate the positive Fourier coefficients. The Fourier terms will be decomposed into the following three terms:

$$\sum_{n>0} \left|\widehat{\psi_\varepsilon}(n)\right|^2 \lambda_n = \left(\sum_{1\le n<1/\varepsilon} + \sum_{1/\varepsilon \le n \le S(\varepsilon)} + \sum_{n>S(\varepsilon)} \right) \left|\widehat{\psi_\varepsilon}(n)\right|^2 \lambda_n.$$

In order to estimate the first term, we shall utilize the estimate in (iv) of Körner's Lemma, in conjunction with (3) of Lemma 5, as follows:

$$\begin{aligned}
\sum_{1\le n<1/\varepsilon} \left|\widehat{\psi_\varepsilon}(n)\right|^2 \lambda_n &\le \frac{1}{L(\varepsilon)^2} \sum_{1\le n<1/\varepsilon} \frac{1}{\lambda_n} \left(\sum_{1/\varepsilon \le j \le S(\varepsilon)} \frac{1}{j} \left|\widehat{g}_{1/j,M,S(\varepsilon)}(n)\right| \right)^2 \\
&\lesssim \frac{A(M,S)^2}{L(\varepsilon)^2} \sum_{1\le n<1/\varepsilon} \lambda_n n^{2M} \left(\sum_{1/\varepsilon \le j \le S(\varepsilon)} \frac{1}{\lambda_j j^{M+1}} \right)^2 \\
&\lesssim \frac{A(M,S)^2}{L(\varepsilon)^2} \frac{\varepsilon^{2M}}{\lambda^2_{[1/\varepsilon]}} \sum_{1\le n<1/\varepsilon} \lambda_n n^{2M} \\
&\lesssim \frac{A(M,S)^2}{L(\varepsilon)^2} \frac{1}{\varepsilon \lambda_{[1/\varepsilon]}} \\
&\lesssim \frac{A(M,S)^2}{L(\varepsilon)}.
\end{aligned}$$

In the last step, we again utilized the condition (ii) of $(\lambda_n)_n$, but in the following different way:

$$\frac{1}{\varepsilon \lambda_{[1/\varepsilon]}} \asymp \sum_{1/\varepsilon \le j \le 2/\varepsilon} \frac{1}{\lambda_j} \le L(\varepsilon). \tag{5}$$

The third term is estimated in similar way as the first, using (iv) of Körner's Lemma, but now in conjunction with (2) of Lemma 5:

$$\begin{aligned}
\sum_{n>S} \left|\widehat{\psi_\varepsilon}(n)\right|^2 \lambda_n &\le \frac{A^2(M,S)}{L^2(\varepsilon)} \cdot \left(\sum_{n>S} \frac{\lambda_n}{n^{2M}} \right) \cdot \left(\sum_{1/\varepsilon \le j \le S} \frac{1}{\lambda_j} j^{M-1} \right)^2 \\
&\le \frac{A^2(M,S)}{L^2(\varepsilon)} \left(\frac{\lambda_S}{S^M} \sum_{n>S} \frac{1}{n^M} \right) \frac{S^{2M}}{\lambda_S^2} \\
&\lesssim \frac{A^2(M,S)}{L^2(\varepsilon)} \frac{S(\varepsilon)}{\lambda_{S(\varepsilon)}} \\
&\lesssim \frac{A^2(M,S)}{L(\varepsilon)}.
\end{aligned}$$

In the last step, again an estimate as (5) involving $S(\varepsilon)$. In order to estimate the first sum, we shall need to further decompose the Fourier coefficients of ψ_ε into the following two terms:

$$\widehat{\psi_\varepsilon}(n) = \frac{1}{L(\varepsilon)} \sum_{1/\varepsilon \le j \le n} \frac{1}{\lambda_j} \widehat{g}_{1/j,M,S(\varepsilon)}(n) + \frac{1}{L(\varepsilon)} \sum_{n<j\le S(\varepsilon)} \frac{1}{\lambda_j} \widehat{g}_{1/j,M,S(\varepsilon)}(n) =: \widehat{\psi_1}(n) + \widehat{\psi_2}(n),$$

for $1/\varepsilon \le n \le S(\varepsilon)$. Applying (iv) of Körner's Lemma in conjunction with (2) of Lemma 5, we get

$$\begin{aligned}\sum_{1/\varepsilon\le n\le S}\left|\widehat{\psi_1}(n)\right|^2\lambda_n &\le \frac{A(M,S)^2}{L(\varepsilon)^2}\sum_{1/\varepsilon\le n\le S}\frac{\lambda_n}{n^{2M}}\left(\sum_{1/\varepsilon\le j\le n}\frac{j^{M-1}}{\lambda_j}\right)^2\\ &\lesssim \frac{A(M,S)^2}{L(\varepsilon)^2}\sum_{1/\varepsilon\le n\le S}\frac{1}{\lambda_n}\\ &= \frac{A(M,S)^2}{L(\varepsilon)}.\end{aligned}$$

Arguing similarly for the second term, instead using (3) of Lemma 5, implies

$$\sum_{1/\varepsilon\le n\le S}\left|\widehat{\psi_2}(n)\right|^2\lambda_n \le \frac{A(M,S)^2}{L(\varepsilon)^2}\sum_{1/\varepsilon\le n\le S}\lambda_n n^{2M}\left(\sum_{j>n}\frac{1}{\lambda_j j^{M+1}}\right)^2 \lesssim \frac{A(M,S)^2}{L(\varepsilon)}.$$

Combining, we arrive at $\sum_{n>0}\left|\widehat{\psi_\varepsilon}(n)\right|^2\lambda_n \lesssim A\big(M,S(\varepsilon)\big)^2/L(\varepsilon) \lesssim A\big(M,S(\varepsilon)\big)^2\varepsilon$. The proof follows by a simple re-scaling argument in the parameter $\varepsilon > 0$. □

2.3. *A Baire category argument*

In order to carry out the Baire category argument, we need to set up an appropriate functional theoretical framework.

Lemma 8. *The subset $\mathscr{S}_\lambda \subset \ell^2(\lambda)$ consisting of elements $f \in L^\infty(\mathbb{T}, \mathrm{d}m)$ with*

$$0 \le f(\zeta) \le 2, \quad \mathrm{d}m\text{-a.e. } \zeta \in \mathbb{T},$$

forms a closed subset of $\ell^2(\lambda)$.

Proof. The proof is simple, hence we only sketch it. Since the set $\mathscr{S}_\lambda$ is convex, it suffices to show that the set is weakly closed. To this end, we note that for any positive $\varphi \in C^\infty(\mathbb{T})$, one has

$$0 \le \int_{\mathbb{T}} f\varphi\,\mathrm{d}m \le 2\int_{\mathbb{T}}\varphi\,\mathrm{d}m.$$

Now choosing φ to be Poisson kernels w.r.t. the unit-disc $\{|z| < 1\}$ and using standard properties of their boundary behavior, one easily concludes the proof. □

Note that $\mathscr{S}_\lambda$ only defines a cone in $\ell^2(\lambda)$. We denote by $\mathscr{L}_\lambda \subset \mathscr{S}_\lambda \times \mathscr{C}$ the collection of ordered pairs (f, E) with $f \in \mathscr{S}_\lambda$ and E a compact set, such that

$$\operatorname{supp}(f) \subseteq E.$$

We now record the following simple lemma, whose proof is immediate from Lemma 8 and standard properties of support.

Lemma 9. *The set $\mathscr{L}_\lambda$ of ordered pair (f, E) equipped with the metric*

$$d_\lambda\big((f,E),(g,K)\big) := \|f-g\|_{\ell^2(\lambda)} + d_{\mathscr{C}}(E,K),$$

becomes a complete metric space.

We outline the principal result in this subsection.

Proposition 10. *Let $(\lambda_n)_n$ be positive numbers satisfying the hypotheses* (i)–(ii) *from Theorem 2. For any $a \in \mathbb{T}$, consider the set*

$$\mathscr{E}_a = \big\{(f,E) \in \mathscr{L}_\lambda : E \text{ does not meet an open arc containing } a\big\}.$$

Then $\mathscr{E}_a$ is an open and dense subset in the metric space $(\mathscr{L}_\lambda, d_\lambda)$.

Proof. To see why $\mathscr{E}_a$ open, let $(f,E) \in \mathscr{E}_a$ and pick $\delta > 0$ such that the arc $I_{2\delta}(a)$ centered at a of length 4δ does not meet E. Now if $d_\lambda\big((f,E),(g,K)\big) \leq \delta/2$, then we in particular have that $d_{\mathscr{C}}(E,K) \leq \delta/2$, hence we can infer that $I_{\delta/2}(a) \cap K = \emptyset$. This shows that $\mathscr{E}_a$ is indeed open.

In order to verify that $\mathscr{E}_a$ is dense, it suffices to show that for any $(f,E) \in \mathscr{L}_\lambda$ and any $\delta > 0$, there exists $(g,K) \in \mathscr{E}_a$ such that

$$d_\lambda\big((f,E),(g,K)\big) < \delta.$$

By means of re-scaling f and employing a simple argument involving convolution with a smooth approximate of the identity, we may actually assume that $f \in C^\infty(\mathbb{T})$ with $0 \leq f(\zeta) \leq 2-\delta$ for all $\zeta \in \mathbb{T}$. We now invoke Lemma 6. There exists an integer $N(\lambda) > 0$, such that for any $N > 100N(\lambda)$, the following statement holds. For any $0 < \varepsilon < 1$, there exists $\psi_\varepsilon \in C^\infty(\mathbb{T})$ such that:

(i) $0 \leq \psi_\varepsilon \leq 1+\varepsilon$ on $\mathbb{T}$;
(ii) $\psi_\varepsilon = 0$ in a neighborhood $J_\varepsilon(1)$ of $\zeta = 1$, whose length tends to zero as $\varepsilon \to 0$;
(iii) $\int_{\mathbb{T}} \psi_\varepsilon \, \mathrm{d}m = 1$;
(iv) $\sup_{n\neq 0} \big|\widehat{\psi_\varepsilon}(n)\big| \leq \min\big(\varepsilon, c(N)\varepsilon^{1-N}|n|^{-N}\big)$, where $c(N) > 0$ is a constant only depending on N;
(v) $\sum_{n\neq 0} \big|\widehat{\psi_\varepsilon}(n)\big|^2 \lambda_{|n|} \leq \varepsilon^2$.

Now set $f_\varepsilon(\zeta) = f(\zeta)\cdot\psi_\varepsilon(\zeta\overline{a})$ and $E_\varepsilon = E \setminus J_\varepsilon(a)$. It easily follows that the pair $(f_\varepsilon, E_\varepsilon)$ belongs to $\mathscr{L}_\lambda$ and we have $d_{\mathscr{C}}(E,E_\varepsilon) \to 0$ as $\varepsilon \to 0+$. Therefore, it only remains to show that

$$\lim_{\varepsilon\to 0+} \sum_n \big|\widehat{f_\varepsilon}(n) - \widehat{f}(n)\big|^2 \lambda_{|n|} = 0.$$

Since all functions involved are real-valued, we only need to estimate the sum for $n \geq 0$. First we write

$$\widehat{f_\varepsilon}(n) - \widehat{f}(n) = \sum_{m\neq n} \widehat{f}(m) e^{ia(n-m)} \widehat{\psi_\varepsilon}(n-m) = \Bigg(\sum_{|m|\leq n/2} + \sum_{\substack{|m|>n/2\\ m\neq n}}\Bigg) \widehat{f}(m) e^{ia(n-m)} \widehat{\psi_\varepsilon}(n-m) =: \Sigma_1 + \Sigma_2.$$

In order to make the second sum small, we use property (iv) and the smoothness of f. Hence for any desirable value of $A > 0$, there exists $C(A) > 0$, such that

$$\Bigg|\sum_{\substack{|m|>n/2\\ m\neq n}} \widehat{f}(m) e^{ia(n-m)} \widehat{\psi_\varepsilon}(n-m)\Bigg| \leq \varepsilon \sum_{|m|>n/2} \big|\widehat{f}(m)\big| \leq \frac{C(A)\varepsilon}{(1+n)^A}.$$

This in conjunction with the property (1) of $(\lambda_n)_n$ in Lemma 5 gives

$$\sum_{n\geq 0} \Bigg|\sum_{\substack{|m|>n/2\\ m\neq n}} \widehat{f}(m) e^{ia(n-m)} \widehat{\psi_\varepsilon}(n-m)\Bigg|^2 \lambda_n \leq C(A)^2 \varepsilon^2 \sum_{n\geq 0} \frac{\lambda_n}{(1+n)^{2A}} \leq C'(A)\varepsilon^2. \tag{6}$$

It remains only to estimate the Fourier coefficients of Σ_1, which we shall split into two further parts. Let $M(\varepsilon) \to +\infty$ as $\varepsilon \to 0+$ be a parameter to be specified later. For $0 \leq n < M(\varepsilon)$, we make use of the property (v) of ψ_ε, which gives

$$\begin{aligned}
&\sum_{0\leq n<M(\varepsilon)} \Bigg|\sum_{|m|\leq n/2} \widehat{f}(m) e^{ia(n-m)} \widehat{\psi_\varepsilon}(n-m)\Bigg|^2 \lambda_n \\
&\qquad\leq \sum_{0\leq n<M(\varepsilon)} \sum_{|m|\leq n/2} \big|\widehat{f}(m)\big|^2 \sum_{|m|\leq n/2} \big|\widehat{\psi_\varepsilon}(n-m)\big|^2 \lambda_n \\
&\qquad\asymp \sum_{0\leq n<M(\varepsilon)} \sum_{|m|\leq n/2} \big|\widehat{f}(m)\big|^2 \sum_{|m|\leq n/2} \big|\widehat{\psi_\varepsilon}(n-m)\big|^2 \lambda_{|n-m|} \qquad (7)\\
&\qquad\leq \varepsilon^2 \sum_{0\leq n<M(\varepsilon)} \sum_{|m|\leq n/2} \big|\widehat{f}(m)\big|^2 \leq \varepsilon^2 M(\varepsilon) \|f\|_{L^2}^2.
\end{aligned}$$

In the first step, we used Cauchy–Schwartz inequality for the inner sum, while in the second step we used the assumption $\lambda_n \asymp \lambda_{|n-m|}$ whenever $|m| \leq n/2$. Moving forward, it remains only to estimate the sum for $n \geq M(\varepsilon)$. This time, we instead make use of property (iv), in the following way:

$$\left|\sum_{|m|\leq n/2} \widehat{f}(m)e^{ia(n-m)}\widehat{\psi_\varepsilon}(n-m)\right|^2 \leq C(N)^2\varepsilon^{2-2N}\left(\sum_{|m|\leq n/2}\left|\widehat{f}(m)\right||n-m|^{-N}\right)^2$$
$$\lesssim C(N)^2\varepsilon^{2-2N}n^{-2N}\|f\|^2_{\ell^1}.$$

With this estimate at hand, we get

$$\sum_{n\geq M(\varepsilon)}\left|\sum_{|m|\leq n/2} \widehat{f}(m)e^{ia(n-m)}\widehat{\psi_\varepsilon}(n-m)\right|^2\lambda_n \lesssim C(N)^2\|f\|^2_{\ell^1}\varepsilon^{2-2N}\sum_{n\geq M(\varepsilon)}\frac{\lambda_n}{n^{2N}}. \tag{8}$$

Utilizing part (1) of Lemma 5, we obtain

$$\varepsilon^{2-2N}\sum_{n\geq M(\varepsilon)}\frac{\lambda_n}{n^{2N}} \lesssim \varepsilon^{2-2N}\frac{\lambda_{M(\varepsilon)}}{M(\varepsilon)^{N(\lambda)}}\sum_{n\geq M(\varepsilon)}\frac{1}{n^{2N-N(\lambda)}} \lesssim \varepsilon^{2-2N}\frac{\lambda_{M(\varepsilon)}}{M(\varepsilon)^{2N-1}}.$$

Now choosing $M(\varepsilon) \asymp \varepsilon^{-3/2}$ and returning back to (8), we arrive at

$$\sum_{n\geq M(\varepsilon)}\left|\sum_{|m|\leq n/2} \widehat{f}(m)e^{ia(n-m)}\widehat{\psi_\varepsilon}(n-m)\right|^2\lambda_n \lesssim \lambda_{\varepsilon^{-3/2}}(\varepsilon^{3/2})^{(2N+1)/3} \longrightarrow 0, \quad \varepsilon \to 0+$$

since $N > 100N(\lambda)$ in view of (1) of Lemma 5. Now since (6) and (7) can also be made arbitrarily small, we conclude

$$\lim_{\varepsilon\to 0+}\sum_{n\geq 0}\left|\widehat{f_\varepsilon}(n)-\widehat{f}(n)\right|^2\lambda_{|n|} = 0.$$

This proves that $\mathscr{E}_a$ is dense in the metric space $(\mathscr{L}_\lambda, d_\lambda)$. □

With this at hand, we easily complete the proof Theorem 4.

Proof of Theorem 4. Pick a countable dense subset $\{a_j\}_j \subset \mathbb{T}$ and note that according to Proposition 10, and the Baire category theorem, the set $\mathscr{E} = \cap_j \mathscr{E}_{a_j}$ is dense in $\mathscr{L}_\lambda$. Now whenever $(f, E) \in \mathscr{E}$, then $E \cap \{a_j\}_j = \emptyset$ by construction, hence E cannot have any interior point. Since $\operatorname{supp}(f) \subseteq E$, we conclude that f satisfies the required properties. □

2.4. *Functions with sparse Fourier support*

Here, we briefly explain why techniques involving sparse Fourier support cannot simply prove Theorem 2. To this end, given a subset of integers Λ, we denote by $C_\Lambda(\mathbb{T})$ the closed subspace of continuous functions f on $\mathbb{T}$ with the property that $\operatorname{supp}(\widehat{f}) \subseteq \Lambda$. Now a set Λ is said to be a *Sidon set* if there exists $C(\Lambda) > 0$, such that

$$\sum_n\left|\widehat{f}(n)\right| \leq C(\lambda)\sup_{\zeta\in\mathbb{T}}\left|f(\zeta)\right|, \quad \forall f \in C_\Lambda(\mathbb{T}).$$

Examples of Sidon sets include lacunary sequence $\Lambda = (N_k)_k$ with $\inf_k N_{k+1}/N_k > 1$, which is a classical result due to Zygmund. For instance, see [10, Chapter 5], or [4, Lemma 1.4 in Chapter V]. A remarkable arithmetic characterization of Sidon sets was given by G. Pisier in [9]. Given an integer n, we denote by $R(n,\Lambda)$ the number of ways to write $n = \sum_j \varepsilon_j\lambda_j$ as finite linear combinations of $\lambda_j \in \Lambda$ and with $\varepsilon_j \in \{-1, 0, 1\}$ for all j. G. Pisier proved that Λ is a Sidon set if and only if there exists a number $0 < \gamma < 1$, such that for any finite subset $\Gamma \subset \Lambda$, we have

$$\sup_{n\in\mathbb{Z}} R(n,\Gamma) \leq 3^{\gamma|\Gamma|},$$

where $|\Gamma|$ denotes the cardinality of the subset Γ. In other words, representations of integers involving finite sub-collections of Λ must be exponentially sparser. For further connections to Riesz products, we also refer the reader to the work of J. Bourgain in [1]. See also Kronecker's Theorem on independent sets in [4, Chapter VI]. This clarifies why methods involving sparse

Fourier spectrum, such as Riesz products and Hadamard lacunary series, cannot readily be used to produce functions appearing in Theorem 2. However, it was recently proved in [8] that methods involving sparse Fourier support can yield other results, of similar flavor.

3. Indispensability of regularity hypothesis

The proof of Theorem 3 will actually be derived from the following stronger statement.

Theorem 11. *Let Φ be a positive increasing function on $(0,\infty)$. There exist increasing positive numbers $(\lambda_n)_n$ with $\lambda_n \uparrow +\infty$, such that*

$$\sum_n \Phi(1/\lambda_n) = +\infty,$$

but if S is a non-zero distribution on $\mathbb{T}$ with $S \in \ell^2(\lambda)$, then $\operatorname{supp}(S) = \mathbb{T}$.

The proof of Theorem 11 rests on a lemma from the work of T. W. Körner, whose surprisingly short and simple proof is attributed to Y. Meyer.

Lemma 12 (Körner–Meyer [5, Lemma 2.1]). *Given $N \geq 1$ and $\gamma, \delta > 0$, there exists $\varepsilon = \varepsilon(N,\gamma,\delta) > 0$, such that the following statement holds: whenever S is a distribution on $\mathbb{T}$ with*

(i) $\sum_{|n|\leq N} \left|\widehat{S}(n)\right|^2 \geq \gamma$;
(ii) $\sup_{|n|>N} \left|\widehat{S}(n)\right| \leq \varepsilon$;

then $\sup_{\zeta\in\mathbb{T}} \operatorname{dist}\big(\zeta, \operatorname{supp}(S)\big) \leq \delta$.

With this lemma at hand, we can give a short proof of our principal observation in this section. The argument below requires a modification of the proof of [5, Theorem 1.2], adapted to our setting.

Proof of Theorem 11. Set $N_1 = \varepsilon_1 = 1$, and suppose that positive integers $N_1 < N_2 < \cdots < N_k$, and positive numbers $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_k > 0$ have already been constructed. Now pick $N_{k+1} > N_k$, such that

$$(N_{k+1} - N_k)\Phi(\varepsilon_k^2 2^{-2k-1}) \geq 1.$$

Invoking the Körner–Meyer Lemma, we can find a number $0 < \varepsilon_{k+1} < \varepsilon_k$ such that the following statement holds: whenever S is a distribution on $\mathbb{T}$ with

(i)$_k$ $\sum_{|n|\leq N_{k+1}} \left|\widehat{S}(n)\right|^2 \geq 2^{-k}$;
(ii)$_k$ $\sup_{|n|>N_{k+1}} \left|\widehat{S}(n)\right| \leq \varepsilon_{k+1}$;

then $\sup_{\zeta\in\mathbb{T}} \operatorname{dist}\big(\zeta, \operatorname{supp}(S)\big) \leq 2^{-k}$. Define the corresponding sequence of positive numbers $(\lambda_n)_n$ as follows: for each $k = 0,1,2,\dots$ set $\lambda_{N_k} = 2^{2k}/\varepsilon_k^2$, and interpolate the intermediate values of $1/\lambda_n$ linearwise,

$$\frac{1}{\lambda_n} := \frac{\varepsilon_k^2}{2^{2k}} \frac{N_{k+1} - n}{N_{k+1} - N_k} + \frac{\varepsilon_{k+1}^2}{2^{2(k+1)}} \frac{n - N_k}{N_{k+1} - N_k}, \quad N_k \leq n < N_{k+1}, \quad k = 0,1,2,\dots$$

Clearly, $(\lambda_n)_n$ is increasing with $\lambda_n \uparrow +\infty$. Furthermore, it follows from the monotonicity of Φ in conjunction with the constructions of $(N_k)_k$ that

$$\sum_{N_k \leq n \leq (N_{k+1}+N_k)/2} \Phi(1/\lambda_n) \geq \frac{(N_{k+1} - N_k)}{2} \Phi(\varepsilon_k^2 2^{-2k-1}) \geq \frac{1}{2}, \quad k = 0,1,2,\dots,$$

hence $\sum_n \Phi(1/\lambda_n) = +\infty$. Now let S be an arbitrary non-zero distribution on $\mathbb{T}$ with

$$\sum_n \left|\widehat{S}(n)\right|^2 \lambda_{|n|} < \infty.$$

We claim that both (i)$_k$ and (ii)$_k$ must hold for all sufficiently large k. Indeed, the former statement follows from $S \in \ell^2$, while the second property follows from the crude estimate:

$$\sup_{|n|>N_{k+1}} \left|\widehat{S}(n)\right|^2 \leq \frac{1}{\lambda_{N_{k+1}}} \sum_{|n|>N_{k+1}} \left|\widehat{S}(n)\right|^2 \lambda_{|n|} \leq \frac{1}{\lambda_{N_{k+1}}} \leq \varepsilon_{k+1}^2,$$

which holds for sufficient large k. Therefore, $\sup_{\zeta\in\mathbb{T}} \operatorname{dist}\big(\zeta, \operatorname{supp}(S)\big) \leq 2^{-k}$ holds for all sufficiently large k, and we conclude that $\operatorname{supp}(S) = \mathbb{T}$. □

Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

Appendix A.

We outline the simple proof of Proposition 1 below.

Proof of Proposition 1. Pick a trigonometric polynomial T_0 which satisfies

(i) $\|T_0\|_{L^\infty} \leq 1$;

(ii) $\|T_0\|_{L^2} \geq 1/2$;

and let $N_0 > 0$ denote its degree. For instance, such polynomials can be found by taking $h(\zeta) = e^{i\operatorname{Re}(\zeta)}$ and passing to appropriate Fejér means. Now choose $n_0 > 2N_0$ and take a subsequence $(n_j)_j$ with the properties that

$$\sum_j \frac{1}{\sqrt{\lambda_{n_j}}} < \infty, \quad n_{j+1} - n_j \geq 2N_0, \quad j = 0,1,2,\dots$$

Set $T_j(\zeta) = \zeta^{n_j} T_0(\zeta)$ for $j = 0,1,2,\dots$, and consider a function f of the form

$$f(\zeta) = \sum_j \frac{1}{\sqrt{\lambda_{n_j}}} T_j(\zeta), \quad \zeta \in \mathbb{T}.$$

Since $\big(1/\sqrt{\lambda_{n_j}}\big)_j$ is summable, the Weierstrass M-test ensures that f is continuous on $\mathbb{T}$. Using the mutually disjoint Fourier support of $(T_j)_j$, we also have

$$\sum_n \left|\widehat{f}(n)\right|^2 \lambda_{|n|} = \sum_j \frac{1}{\lambda_{n_j}} \sum_{n_j \leq |n| \leq n_j + N_0} \left|\widehat{T_j}(n)\right|^2 \lambda_{|n|} \geq \sum_j \sum_n \left|\widehat{T_j}(n)\right|^2 \geq \sum_j \frac{1}{4} = \infty,$$

where we in the penultimate step also utilized the monotonicity of λ_n. □

Now let $w_n \downarrow 0$ and we assume that $\ell^2(w)$ is contained in the Banach space $M(\mathbb{T})$ of complex finite measures on $\mathbb{T}$, equipped with the usual total variation norm. The closed graph theorem then ensures that there exists $C(w) > 0$, such that for any $S \in \ell^2(w)$ one can find $\mu \in M(\mathbb{T})$ with $\widehat{S}(n) = \widehat{\mu}(n)$ for all n, and $\|\mu\|_{M(\mathbb{T})} \leq C(w)\|S\|_{\ell^2(w)}$. Now it is straightforward to verify that $\ell^2(1/w)$ is the dual (and pre-dual) of $\ell^2(w)$, in the classical sequence pairing. Therefore, for any $N > 0$ and any continuous function f in $\mathbb{T}$, we have

$$\sum_{|n|\leq N} \left|\widehat{f}(n)\right|^2 \frac{1}{w_{|n|}} = \sup_{\substack{\|g\|_{\ell^2(w)} \leq 1 \\ \operatorname{supp}(\widehat{S}) \subseteq [-N,N]}} \left|\sum_n \widehat{f}(n)\overline{\widehat{S}(n)}\right| \leq \sup_{\|\mu\|_{M(\mathbb{T})} \leq C(w)} \left|\int_{\mathbb{T}} f \, d\bar{\mu}\right| \leq C(w) \sup_{\zeta\in\mathbb{T}} \left|f(\zeta)\right|.$$

We arrive at a contradiction with Proposition 1 by letting $N \to \infty$. In conclusion, $\ell^2(w)$ contains genuine distributions, whenever $w_n \downarrow 0$.

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