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New eigenfunctions for the negative part of the Connes–Moscovici prolate spectrum

Nouvelles fonctions propres pour la partie négative du spectre prolate de Connes–Moscovici

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Abstract. In 1880, C. Niven, motivated by a problem of conduction of heat, studied the Helmoltz equation on a spheroid. We consider only the prolate spheroid case. There is a separation of variables of the Laplacian in corresponding spheroidal coordinates which is adapted to the problem. Then we get three ordinary differential equations. One is elementary. In the non spherical case, the two others are two copies of the same equation. This equation is called the prolate spheroidal equation. In 1998 Alain Connes discovered a self-adjoint extension $W_{\Lambda,\mathrm{sa}}$ of the prolate operator (of order zero) W_Λ. In 2021 Alain Connes and Henri Moscovici discovered that the restriction of $W_{\Lambda,\mathrm{sa}}$ to the complement of the finite interval $[-\Lambda,\Lambda]$ admits (besides a replica of the classical positive spectrum) negative eigenvalues whose ultraviolet behavior reproduces, for $\Lambda=\sqrt{2}$, that of the squares of the (shifted) zeros of the Riemann zeta function. In this note we use an approach different from the CM approach. It is based on complex analytic functions in place of Sturm–Liouville theory. In particular we introduce a (new) theory of analytic spectra. This allows in particular for an explicit definition of spectral determinants which are entire functions of order $\leq 1/2$ whose zeros are the eigenvalues. An accurate computation of these eigenvalues follows. We also discovered a new equivalent definition, very simple, of the non classical part of the CM spectrum. The corresponding eigenvalues are the naive eigenvalues of the operator on the imaginary axis (that is the eigenfunctions are bounded on this axis). We prove that they are negative, which was conjectured by Connes and Moscovici, and we obtain a very quick and efficient method of computation of these eigenvalues.

Résumé. En 1880, C. Niven, motivé par un problème de conduction de la chaleur, a étudié l'équation de Helmoltz sur un ellipsoïde de révolution. Nous considérons seulement le cas d'un ellipsoïde allongé (prolate). Les coordonnées sphéroïdales correspondantes conduisent à une séparation du laplacien adaptée au problème. On obtient ainsi trois équations différentielles ordinaires. L'une est élémentaire. Dans le cas non sphérique, les deux autres sont deux copies de la même équation. Celle-ci est appelée prolate sphéroïdale. En 1998, Alain Connes a découvert une extension auto-adjointe $W_{\Lambda,\mathrm{sa}}$ de l'opérateur prolate (d'ordre zero) W_Λ. En 2021, Alain Connes et Henri Moscovici ont découvert que la restriction de $W_{\Lambda,\mathrm{sa}}$ au complément de l'intervalle fini $[-\Lambda,\Lambda]$ admet (à côté d'une copie du spectre classique, qui est positif) des valeurs propres négatives dont le comportement ultraviolet reproduit, pour $\Lambda=\sqrt{2}$, celui des carrés des zéros (décalés) de la fonction zeta de Riemann. Dans cette note nous utilisons une approche différente de celle de CM. À la place de la théorie de Sturm–Liouville nous utilisons la théorie des fonctions analytiques. Ceci nous permet en particulier de définir explicitement des déterminants spectraux qui sont des fonctions entières d'ordre

≤ 1/2 dont les zéros sont les valeurs propres. Nous en déduisons un calcul numérique efficace de ces valeurs propres. Nous avons aussi découvert une nouvelle définition, équivalente mais très simple, de la partie non-classique du spectre CM. Les valeurs propres correspondantes sont les valeurs propres naïves de l'opérateur sur l'axe imaginaire (c'est-à-dire que les fonctions propres sont bornées sur cet axe). Nous montrons que ces valeurs propres sont négatives, ce qui avait été conjecturé par Connes et Moscovici, et nous obtenons une méthode très rapide et très efficace de calcul de ces valeurs propres.

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1. Presentation

1.1. *The prolate spheroidal functions*

In 1880, C. Niven, motivated by a problem of conduction of heat, studied the Helmoltz equation $(\Delta + k)\psi = 0$ on a spheroid. A non spherical spheroid can be *prolate* (a rugby ball) or *oblate*. We consider only the prolate case. There is a separation of variables of the Laplacian in spheroidal coordinates which is adapted to the problem. Then we get three ordinary differential equations. One is elementary. In the non spherical case, the two others are two copies of the same equation (the changes are the name of the variable and the domains of the solutions). This equation is called the *prolate spheroidal equation*. The corresponding operator is

$$\mathscr{S}_{\tau,m} := (x^2-1)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + 2x\frac{\mathrm{d}}{\mathrm{d}x} + \left(\tau^2(x^2-1) - \frac{m^2}{x^2-1}\right),$$

where $\tau^2 \in \mathbb{R}_+$ and[1] $m \in \mathbb{N}$; m is the *order*. We will consider only the order zero case, setting $\mathscr{S}_{\tau,0} = \mathscr{S}_\tau$. For our purposes it is better to do an "eigenvalue shift" and we will consider the operator

$$\mathscr{D}_\tau = -\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x} + \tau^2 x^2, \quad \tau \in \mathbb{C}$$

and its spectral version $\mathscr{D}_\tau - \mu = 0$, $\mu \in \mathbb{C}$.

This polynomial operator has three singular points on the Riemann sphere. Two regular singular points at $x = \pm 1$ and an irregular point (of Katz rank one) at infinity. The regular singular points are logarithmic (the two exponents are one) and the exponential exponents at infinity are $\pm i\tau$.

In [8], A. Connes and H. Moscovici use a slightly different version of the prolate spheroidal operator:

$$W_\Lambda = -\frac{\mathrm{d}}{\mathrm{d}x}\left((\Lambda^2 - x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right) + (2\pi\Lambda x)^2.$$

The variable change $x = \Lambda\tilde{x}$ and $\tilde{x} = x$ reduces W_Λ to $\mathscr{D}_\tau$, where $\tau = 2\pi\Lambda^2$. The exponential exponents at infinity of W_Λ are $\pm 2i\pi\Lambda$. The operators $\mathscr{D}_\tau$ and W_Λ are invariant under the symmetry $x \mapsto -x$.

The classical eigenvalues of $\mathscr{D}_\tau$ are, by definition, the $\mu \in \mathbb{C}$ such that there exists a solution f of $\mathscr{D}_\tau - \mu$ which is *bounded* on $[-\Lambda, \Lambda]$. Then f is called a Prolate Spheroidal Wave Function (PSWF); $f \in L^2(\mathbb{R})$ and it extends into an *entire* function. The eigenvalues are real and positive. The number of zeros of the n-th eigenfunction on $[-\Lambda, \Lambda]$ is n.

There are a lot of applications of spheroidal functions. Basic monographs are [14,17,32] and the more recent [20]. Around 1960 a Bell Labs group around David Slepian found a new application in relation with signal theory. They wrote a series of papers on the general theme *"Prolate spheroidal wave functions, Fourier analysis and uncertainty"* [30,31].

[1] This constraint follows from the physical problem. There is a more general family with $m \in \mathbb{C}$ and $\tau \in \mathbb{C}$ [17].

The *Fourier transform* $\mathbb{F}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$, is defined by

$$f(x) \longmapsto \mathbb{F}f = \widetilde{f}(y) = \int_{\mathbb{R}} f(x) e^{-2i\pi xy}\, \mathrm{d}x.$$

We will also use a variant $\mathbb{F}_c$ (Fourier–Laplace), replacing $2i\pi$ by $c \in \mathbb{C}^*$. The Laplace case is $c = 1$.

It is not possible to concentrate simultaneously a function in physical and frequency spaces. A paradox is that it is possible in the "true world"! Claude E. Shannon once posed the following question: *To what extent are functions, which are confined to a finite bandwidth, also concentrated in the time domain ?* An answer is crucial for transmission of signals and an explanation of the above paradox. Shannon question was answered by a group at Bell Laboratories. The central point is related to the eigenvalues of some convolution operator Q with a sine cardinal kernel. The first eigenvalues are *very near of* 1 and afterwards they drop sharply to *very small* positive values. The first eigenfunctions are finite bandwidth and *maximally concentrated* within a finite time interval. We denote:

$$K_\Lambda(x,y) = \operatorname{sinc}\big(2\pi\Lambda(x-y)\big) = \frac{\sin\big(2\pi\Lambda(x-y)\big)}{2\pi\Lambda(x-y)}.$$

Then $f \mapsto \int_{-\Lambda}^{\Lambda} K_\Lambda(x,y) f(x)\,\mathrm{d}x$ defines the sine cardinal convolution operator

$$Q_\Lambda\colon L^2\big([-\Lambda,\Lambda]\big) \longrightarrow L^2\big([-\Lambda,\Lambda]\big).$$

It is self-adjoint and compact. The spectrum is discrete, the eigenvalues $\lambda_n(\Lambda)$ are simple, real positive and:

$$1 > \lambda_1(\Lambda) > \lambda_2(\Lambda) > \cdots > \lambda_n(\Lambda) > \cdots$$

If ψ_n is an eigenfunction, then the eigenvalue $\lambda_n(\Lambda)$ can be interpreted as its *concentration* on the interval $[-\Lambda,\Lambda]$:

$$\lambda_n(\Lambda) = \frac{\|\psi_n\|^2_{[-\Lambda,\Lambda]}}{\|\psi_n\|^2_{\mathbb{R}}}.$$

The *space-limitation operator* $P_\Lambda\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$, is defined by $P_\Lambda f = 1_{[-\Lambda,\lambda]} f$. We set $\widehat{P}_\Lambda = \mathbb{F}P_\Lambda\mathbb{F}^{-1}$. It is the *band-limitation operator*. We have $P_\Lambda \widehat{P}_\Lambda P_\Lambda = 2\Lambda Q_\Lambda$; $P_\Lambda \widehat{P}_\Lambda P_\Lambda$ is called the angle operator. Though the projections P_Λ and $\widehat{P}_\Lambda$ do not commute exactly, even for large Λ, their angle is sufficiently well-behaved (cf. [7]).

For $\Lambda > 0$, we define the *Sonin space* as the subspace of functions $f \in L^2(\mathbb{R})$ such that $P_\Lambda f = \widehat{P}_\Lambda f = 0$.

Except for the first ones, the eigenvalues λ_n of Q_Λ are very small. Then there is a problem to compute numerically the eigenfunctions. Trying to overcome this difficulty, the Bell Labs group discovered[2] a *miracle*: the operator Q_Λ commutes with the prolate operator W_Λ. It is the *lucky accident* (as said Slepian [31]). Then the eigenfunctions of $Q_\Lambda = \frac{1}{2\Lambda} P_\Lambda \widehat{P}_\Lambda P_\Lambda$ are the PSWF, which are easily computed numerically.

1.2. *Prolate and zeta*

In 1998 Alain Connes discovered a self-adjoint extension $W_{\Lambda,\mathrm{sa}}$ of the prolate operator W_Λ. The operator $W_{\Lambda,\mathrm{sa}}$ commutes with P_Λ, $\widehat{P}_\Lambda$ and $\mathbb{F}$. In 2021 Alain Connes and Henri Moscovici discovered that the restriction of this extension of the prolate differential operator to the complement of the finite interval admits (besides a replica of the classical positive spectrum) negative eigenvalues *whose ultraviolet (UV) behavior, for $\Lambda = \sqrt{2}$, reproduces that of the squares of zeros of the Riemann zeta function.* Moreover, they show that, for $\Lambda > 0$, the corresponding eigenfunctions belong to the *Sonin space*.

[2] In fact the commutation appeared before in an article of H. Bateman [1].

In 1999, A. Connes described the "lucky accident" in relation with his spectral interpretation of the critical zeros of the Riemann zeta function [3,8]. Later A. Connes and C. Consani proved the Weil's positivity at the Archimedean place [5], using the compression of the scaling action to the Sonin space. This fits with the fact that the non classical CM eigenfunctions belong to the Sonin space. Prolate spheroidal functions appeared also in [4]. We quote [6]: *"The role of the prolate operator is crucial in both (infrared and ultraviolet) observed agreements with zeros of zeta"*.

1.3. *Our purposes*

Our initial purpose was to study the CM spectrum, extending (partly) the approach of [17] for the classical spectrum. In particular we planned to introduce an equivalent definition of the CM prolate spectrum using a (new) general notion of *analytic spectrum*. We will see that, in the line of [17] for the classical spectrum, this allows for an explicit definition of *spectral determinants* which are *entire functions of order* $\leq 1/2$ whose zeros are the eigenvalues and also for accurate computations of the eigenvalues. Trying, by a complex analytic approach, to get a best understanding of the fact that the non classical CM eigenfunctions belong to the *Sonin space*, which is particularly important in relation with the zeros of zeta, we discovered a new equivalent definition, very simple, of the non classical part of the CM spectrum. The corresponding eigenvalues are the naive eigenvalues[3] of the operator on the imaginary axis. We prove that they are *negative* and obtain a very quick and efficient method of computation.[4]

For a rational *formally symmetric* second order operator $-\frac{\mathrm{d}}{\mathrm{d}x}p\frac{\mathrm{d}}{\mathrm{d}x}+q$, there are (at least...) three notions of spectrum:

(1) naive spectrum (the eigenfunctions are bounded at the extremities);
(2) spectrum defined by a *self-adjoint extension*;
(3) analytic spectrum.

The two first approaches are classical but the notion of analytic spectrum is new (cf. 2.1.4 below). We will describe such spectra for the prolate operator of order zero and compare them. Among these spectra are the classical spectrum, the new spectrum defined by Connes and Moscovici (which contains a replica of the classical spectrum), and a new one that we discovered. More precisely this new one coincides as a set with the negative Connes–Moscovici spectrum but we define it in a completely different way and *the eigenfunctions are different*.[5] As we said above this allows for a proof that the non classical CM eigenvalues are negative and for an efficient computation of these eigenvalues.

2. Analytic spectra of the prolate operator

2.1. *Local solutions of* W_Λ

2.1.1. *Local solutions at the regular singular points*

The local solutions of $\mathscr{D}_\tau-\mu$ at $x=\pm1$ are given in [17, 3.12, Satz 5, p. 222]. Adapting the notations, we get the local solutions of W_Λ at $x=\pm\Lambda$. They involve logarithms.[6]

$$\begin{cases} y_{\mathrm{I}}(x)=\mathfrak{P}_{\mathrm{I}}(\Lambda-x), \quad \mathfrak{P}_{\mathrm{I}}(0)=1,\\ y_{\mathrm{II}}(x)=\mathfrak{P}_{\mathrm{II}}(\Lambda-x)-\frac{1}{2\Lambda}\mathfrak{P}_{\mathrm{I}}(\Lambda-x)\log(\Lambda-x), \end{cases} \tag{1}$$

[3] That is the eigenfunctions are bounded on the imaginary axis.
[4] It is inspired by the "curtain phenomenon" of J. L. Callot [2].
[5] The boundary values of convenient analytic extensions of these functions are the CM eigenfunctions.
[6] We are in the resonant case of the Frobenius method.

where $\mathfrak{P}_{\mathrm{I}}(\xi)$, $\mathfrak{P}_{\mathrm{II}}(\xi)$ are *analytic functions* for $|\xi| < 2\Lambda$. Wronskian: $W(y_{\mathrm{I}}, y_{\mathrm{II}}) = \frac{1}{\Lambda^2 - x^2}$. At $x = -\Lambda$, $x \mapsto -x$ in the above formulas.

Given a direction $d \in S^1$ at $x = \Lambda$ we consider the space $\mathrm{Sol}_{\Lambda,d}$ of solutions on a germ of sector bisected by d. By counterclockwise analytic continuation along a simple loop around Λ, we get a linear automorphism of $\mathrm{Sol}_{\Lambda,d}$, the *monodromy automorphism*. It is *unipotent*, it admits a *unique* line of eigenfunctions: $\mathbb{C}y_{\mathrm{I}}$.

2.1.2. *Local solutions at infinity*

There is a basis

$$\left(\widehat{y}^- = \frac{e^{-2i\pi\Lambda x}}{x}\,\widehat{v}^-, \widehat{y}^+ = \frac{e^{2i\pi\Lambda x}}{x}\,\widehat{v}^+\right)$$

of formal solutions of $W_\Lambda - \mu$ at ∞, where $\widehat{v}^{\mp} = \sum V_n^{\mp} x^{-n}$, with $V_0^- = 1$, $V_0^+ = 1$. The formal power series $\widehat{v}^{\mp}$ are *divergent* and 1*-summable* in every direction, except $\mathbb{R}_- i$ for $\widehat{v}^-$ and $\mathbb{R}_+ i$ for $\widehat{v}^+$. The sum y^- of $\widehat{y}^-$ in the direction $\mathbb{R}_+ i$ is subdominant in the upper half-plane $\Pi^+ = \{\Im x > 0\}$. The sum y^+ of $\widehat{y}^+$ in the direction $\mathbb{R}_- i$ is subdominant in the lower half-plane $\Pi^- = \{\Im x < 0\}$.

If necessary, we will precise the notation: $\widehat{y}^{\pm}_{\tau,\mu}$, $y^{\pm}_{\tau,\mu}$.

2.1.3. *Distinguished lines of solutions*

There are *distinguished lines* of analytic or local sectorial solutions of $W_\Lambda - \mu$ at the 3 singular points and at 0 which is not singular but is a fixed point for the symmetry $x \mapsto -x$ of the operator W_Λ. By definition:

(1) At $x = \pm\Lambda$, there is a unique distinguished line: the set of germs of analytic solutions. It is $\mathbb{C}y_{\mathrm{I}}(x)$, resp. $\mathbb{C}y_1(-x)$.

(2) At infinity, there are two pairs of *formal* distinguished lines. We recall $\widehat{y}^- = \frac{e^{-2i\pi\Lambda x}}{x}\widehat{v}^-$, $\widehat{y}^+ = \frac{e^{2i\pi\Lambda x}}{x}\widehat{v}^+$. Then these pairs are[7]

$$\left(\mathbb{C}\widehat{y}^-, \mathbb{C}\widehat{y}^+\right) \qquad \text{and} \qquad \left(\mathbb{C}(\widehat{y}^- - \widehat{y}^+), \mathbb{C}(\widehat{y}^- + \widehat{y}^+)\right).$$

If a germ of sector V is bisected by a direction d, then a distinguished line of solutions on V is a sum[8] of a formal line in the direction d when this sum exists.[9]

(3) At $x = 0$, the distinguished lines of solutions are respectively defined as the set of germs of even solutions and the set of germs of odd solutions.

2.1.4. *Special solutions and analytic spectra*

By definition a *special solution*[10] of $W_\Lambda - \mu$ is the data of a simple continuous path γ joining two of the three points $\pm\Lambda$, 0 or one of these points to[11] (∞, d) $(d \in S^1)$, or (∞, d_1) to (∞, d_2) and of a solution f connecting along γ two (non trivial) distinguished solutions at the origin and the extremity. If $\mu \in \mathbb{C}$ is such that there exists a special solution, we will say that it is an *eigenvalue* for the analytic spectrum defined by the two points and γ and that f is a corresponding eigenfunction.

[7] The second pair corresponds to odd, resp. even, formal solutions.

[8] In the sense of Borel summability or of 1-summability.

[9] There are two such lines except when $d = \mathbb{R}_\pm i$.

[10] This concept was introduced by M. Klimes, E. Paul and the first author in the more general context of the linear differential equations associated to Painlevé VI and Painlevé V [16,22].

[11] We denote by (∞, d) a point of the divisor of the real blow-up of the Riemann sphere at infinity and γ is a continuous path on this real blow-up, cf. [22].

3. Some results of A. Connes and H. Moscovici

3.1. *Spectrum of the self-adjoint extension of* W_Λ

We recall some results of [8].

In 1998 Alain Connes introduced a self-adjoint extension $W_{\Lambda,\mathrm{sa}}$ of the prolate operator W_Λ. Later, in 2016, Katsnelson described all the self-adjoint extensions of the restriction of W_Λ to $]-\Lambda,\Lambda[$ and proved that there is a unique[12] self-adjoint extension commuting with the Fourier transform $\mathbb{F}$ [15].

We regard W_Λ as an unbounded operator on $L^2(\mathbb{R})$ with core the Schwartz space $\mathscr{S}(\mathbb{R})$. Its closure by the graph norm is $W_{\Lambda,\min}$ and $W_{\Lambda,\max} = W^*_{\Lambda,\min}$, the latter having domain

$$\mathrm{Dom}(W_{\Lambda,\max}) = \left\{\xi \in L^2(\mathbb{R}) \;\middle|\; W_\Lambda \xi \in L^2(\mathbb{R})\right\}.$$

The domain of $W_{\Lambda,\mathrm{sa}}$ consists of the elements $\xi \in \mathrm{Dom}(W_{\Lambda,\max})$ satisfying some boundary conditions at $\pm\Lambda$ and at $\pm\infty$ (conditions [17], [18], [19] of [8]). We recall the condition at $\pm\Lambda$, that is [17]:

$$\lim_{x\to\pm\Lambda} (x^2-\Lambda^2)\partial_x \xi(x) = 0. \tag{$\star$}$$

We use the basis $(y_{\mathrm{I}}, y_{\mathrm{II}})$, cf. (1).

Lemma 1. *Let $\mu \in \mathbb{R}$. For a solution ξ of $W_\Lambda - \mu$ on $\mathbb{R}\setminus\{\pm\Lambda\}$ the following conditions are equivalent:*

(i) *ξ satisfies condition $(\star)$ at Λ;*

(ii) *if, on the right (resp. on the left) of Λ, ξ is induced by the local solution $a y_{\mathrm{I}} + b y_{\mathrm{II}}$, then $b = 0$ (i.e. the local solution is* non-logarithmic*);*

(iii) *ξ is* bounded *"at Λ".*

There is a similar statement at $-\Lambda$.

Lemma 2. *Let $\mu \in \mathbb{R}$ be a* non classical *eigenvalue of $W_{\Lambda,\mathrm{sa}}$ and ϕ a corresponding eigenfunction. Then:*

(i) *ϕ is* identically zero *on* $]-\Lambda,\Lambda[$*;*

(ii) *the dimension of the eigenspace E_μ is* one*;*

(iii) *the Fourier transform $\mathbb{F}\phi$ is* identically zero *on* $]-\Lambda,\Lambda[$*; ϕ belongs to the Sonin space.*

All the end points are LC in the Weyl classification. The endpoints $\pm\Lambda$ are LCNO (non-oscillatory), while the end points $\pm\infty$ are LCO (oscillatory).

We recall some results of [8] (cf. Theorem 1.6, Corollary 1.7 and 6, B) and add some complements.

Theorem 3. *We suppose $\Lambda \in \mathbb{R}$ and we denote $W_{\Lambda,\mathrm{sa}} = W_{\mathrm{sa}}$, $W_{\Lambda,\min} = W_{\min}$.*

(i) *The operator W_{sa} is* self-adjoint *and* commutes *with the Fourier transform $\mathbb{F}$.*

(ii) *W_{sa}* commutes *with the* projections *P_Λ and $\widehat{P}_\Lambda$ and it is the only self-adjoint extension of $W_{\min}$ commuting with P_Λ and $\widehat{P}_\Lambda$.*

(iii) *The spectrum of W_{sa} is* discrete *and* unbounded on two sides*, the* classical *eigenvalues are* double *and the* non classical *eigenvalues are* simple.

(iv) *There are at most finitely many positive non classical eigenvalues.*

A. Connes and H. Moscovici conjectured that all the non classical eigenvalues are negative (cf. [8, 6, B]). We prove that it is the case (cf. Corollary 15).

Corollary 4. *Let ϕ be an eigenfunction of W_{sa}.*

(i) *The eigenfunctions of W_{sa} are* even *or* odd.

[12] It is the restriction of $W_{\Lambda,\mathrm{sa}}$.

(ii) *For some* $\varepsilon > 0$*, the function* ϕ *on* $]\Lambda, \Lambda+\varepsilon[$ *is the restriction of the sum of an analytic series solution of* $W_\lambda - \mu$ *at* $x = \Lambda$ *and the function* ϕ *on* $]\Lambda-\varepsilon, \Lambda[$ *is the restriction of the sum of an analytic series solution at* $x = \Lambda$*. The two series are proportional but they can be different and there is a possible discontinuity at* Λ*.*

(iii) *For some* $\varepsilon > 0$*, the function* ϕ *is bounded on* $]\Lambda-\varepsilon, \Lambda[\cup]\Lambda, \Lambda+\varepsilon[$*.*

(iv) *The leading term of the "asymptotic expansion"*[13] *of* ϕ *at* $+\infty$ *is proportional to* $\sin(2\pi\Lambda x)/x$ *if* ϕ *is even and is proportional to* $\cos(2\pi\Lambda x)/x$ *if* ϕ *is odd.*

(v) *If* ϕ *is a* classical *eigenfunction, then its restriction to* $]-\Lambda, \Lambda[$ *(resp.* $]\Lambda, +\infty[$*) is the restriction of an* entire function *which is also an eigenfunction of* W_{sa}*.*

(vi) *If* ϕ *is a* non classical *eigenfunction, then* ϕ *and* $\mathbb{F}\phi$ *vanish identically on* $]-\Lambda, \Lambda[$*;* ϕ *belongs to the* Sonin space.

3.2. *A relation between the formal solutions at* Λ *and at infinity*

We recall that there exists a formal power series solution of $W_\Lambda - \mu$ at $x = \Lambda$:

$$\widehat{f}_{\mu,\Lambda} = \mathfrak{P}_{\mathrm{I}}(\Lambda - x) = \sum U_n (x-\Lambda)^n, \quad \text{where } U_0 = 1,$$

and a basis $\left(\widehat{y}^- = x^{-1} e^{-2i\pi\Lambda x}\widehat{v}^-, \widehat{y}^+ = x^{-1}e^{2i\pi\Lambda x}\widehat{v}^+\right)$ of formal solutions at ∞, where

$$\widehat{v}^{\mp} = \sum V_n^{\mp} x^{-n}, \quad \text{with } V_0^- = 1,\ V_0^+ = 1.$$

Then we can reformulate [23, Proposition 14]:

Proposition 5. *We have:* $V_n^{\mp} = n!\,(\pm 2i\pi)^{-n} U_n$*.*

The coefficients of the power expansion at $x = \Lambda$ "resurge" in the coefficients of the power series expansions at $x = \infty$.

We have a variant[14] of [8, Lemma 2.1].

Proposition 6. *Let* $\mu \in \mathbb{R}$*,* $\Lambda > 0$*. We denote* $f_{\mu,\Lambda}$ *the unique solution of* $W_\Lambda - \mu$ *on* $]-\Lambda, +\infty[$ *which is convergent at* $x = \Lambda$ *and satisfies* $f_{\mu,\Lambda}(\Lambda) = 1$*. Then:*

(i) *The asymptotic expansion of the unique solution* $\eta_{\mu,\Lambda}$ *of* $W_\Lambda - \mu$ *on* $]\Lambda, +\infty[$ *which, at* ∞*, is asymptotically* $\frac{e^{-2i\pi\Lambda x}}{2i\pi x}$ *is* $\frac{e^{-2i\pi\Lambda x}}{2i\pi x}\widehat{v}$*. This asymptotic expansion is Borel-summable,* $\eta_{\mu,\Lambda}$ *is the restriction to* $]\Lambda, +\infty[$ *of its sum in the direction* $\mathbb{R}_+$ *and it is equal to the Fourier transform of the solution* θ_μ *which is* 0 *on* $]-\infty, \Lambda[$ *and agrees with* $f_{\mu,\Lambda}$ *for* $x \geq \Lambda$*.*

(ii) *Let* $\xi_{\mu,\Lambda}$ *be the unique solution on* $]\Lambda, +\infty[$ *which, at* ∞*, satisfies*

$$\xi_{\mu,\Lambda}(x) = -\frac{\sin(2\pi\Lambda x)}{\pi x} + O(1/x^2).$$

(a) *There exists a unique formal solution such that its sum is* $\xi_{\mu,\Lambda}$*. It is* $\frac{e^{-2i\pi\Lambda x}}{2i\pi x}\widehat{v}^- - \frac{e^{2i\pi\Lambda x}}{2i\pi x}\widehat{v}^+$*.*

(b) *The function* $\xi_{\mu,\Lambda}$ *is the restriction to* $]\Lambda, +\infty[$ *of the Fourier transform of the unique* even *solution* $\varphi_{\mu,\Lambda}$ *which is* 0 *on* $]-\Lambda, \Lambda[$ *and agree with* $f_{\mu,\Lambda}$ *for* $x \geq \Lambda$*.*

There is a variant of part (ii) of Proposition 6, replacing $\sin(2\pi\Lambda x)$ by $\cos(2\pi\Lambda x)$ and supposing $\varphi_{\mu,\Lambda}$ *odd.*

There is a variant of the above proposition, replacing W_Λ by $\mathscr{D}_\tau$. It is only a change of notations. We leave it to the reader.

Proposition 7. *Let* μ *be an even CM-eigenvalue. Let* ψ *be the unique solution which is "asymptotic" to* $-\frac{\sin(2\pi\Lambda x)}{\pi x}$ *at* $+\infty$*, then it is analytic at* $x = \Lambda$ *and:*

---

[13] The terminology "asymptotic expansion" is used abusively.

[14] It allows an arbitrary parameter Λ and the statement is slightly precised.

- *if μ is a* non classical *eigenvalue, then $\psi(\Lambda) = \pm 1$;*
- *if μ is a* classical *eigenvalue, then ψ extends into an entire function and $\psi(\Lambda) = \pm\sqrt{\lambda}$, where (concentration):*

$$0 < \lambda = \frac{\int_{-\Lambda}^{\Lambda} \psi^2 \,\mathrm{d}x}{\int_{-\infty}^{+\infty} \psi^2 \,\mathrm{d}x} < 1.$$

For the first classical eigenvalues, the concentration is very near of 1.

There are three interpretations of λ:

(1) λ is an eigenvalue of the integral convolution operator Q_Λ;
(2) λ is the concentration on $[-\Lambda, \Lambda]$ of a classical eigenfunction;
(3) $\pm\sqrt{\lambda}$ appears in the connection between $+\infty$ and Λ along $]\Lambda, +\infty[$ defined by a classical eigenfunction.

There are similar results for the operator $\mathscr{D}_\tau$, we leave to the reader the corresponding statement.

4. Prolate spectra as analytic spectra and spectral determinants

4.1. *Classical spectrum*

We consider the classical spectrum of[15] $\mathscr{D}_\tau$, mainly following [17].

4.1.1. *Spectrum and Stokes multipliers*

We suppose $\tau > 0$. The *classical spectrum* of $\mathscr{D}_\tau$ is, by definition, the *naive spectrum* on $[-1, 1]$, that is the set of $\mu \in \mathbb{C}$ such that $\mathscr{D}_\tau - \mu$ admits a non trivial *bounded* solution on $]-1, 1[$. (The definition extends to $\tau \in \mathbb{C}$.) This spectrum is real and the eigenvalues are positive or 0.

The Stokes matrices of $D_\tau - \mu$ at infinity are (in a "natural formal basis of solutions")

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}.$$

The trace of the local monodromy around ∞ is $u = 2 + s^2$. It is possible to choose canonically a square root of $u - 2$ using [17] (half-monodromy). We get a base-free definition of s.

For $\tau \neq 0$, a complex number μ is a classical eigenvalue if and only if $s(\mu, \tau) = 0$ [12]. Using [17], we get $s = -2i\pi y_{\mathrm{I}}(0) y'_{\mathrm{I}}(0)$. We extend the function $s(\tau^2, \mu)$ to $\tau = 0$ using this relation. Then the Stokes function $s(\tau^2, \mu)$ is entire on $\mathbb{C}^2$, of order $\leq 1/2$. It is a functional determinant for the classical spectrum. For τ fixed, the zeros of $s(\tau^2, \mu)$ are *simple* (cf. [17, p. 233], 5' proves that $y_{\mathrm{I}}(0)$ and $y'_{\mathrm{I}}(0)$ have only simple zeros, and Hilfssatz 2 that $y_{\mathrm{I}}(0)$ and $y'_{\mathrm{I}}(0)$ don't have common zeros).

We denote by $\mathscr{M}(\mathbb{C})$ the field of meromorphic functions on $\mathbb{C}$. We have various characterisations of the classical prolate spectrum.

Proposition 8. *The complex number μ is a* classical eigenvalue *of $\mathscr{D}_\tau - \mu$ if and only if one of the following conditions is satisfied:*

- *there exists an entire solution;*
- *the local monodromy around ∞ is trivial;*
- *the Stokes phenomenon at infinity is trivial;*
- *$y_{\mathrm{I}}(0) y'_{\mathrm{I}}(0) = 0$;*
- *the global monodromy group of $\mathscr{D}_\tau - \mu$ is triangularisable (equivalently reducible);*
- *the operator $\mathscr{D}_\tau - \mu$ is reducible in the Ore noncommutative ring $\mathscr{M}(\mathbb{C})[\mathrm{d}/\mathrm{d}x]$.*

[15] We choose $\mathscr{D}_\tau$ for our description because we use some results of [17]. It is easy to translate to the case of W_Λ.

4.1.2. *Analytic spectra*

The classical spectrum can be clearly interpreted as:

- the analytic spectrum associated to the two singular points ± 1 joined by $[-1,1]$;
- the union of the two analytic spectra associated to 0 and 1 joined by $[0,1]$ (even and odd cases).

The classical eigenfunctions are entire and the classical spectrum can also be interpreted as an analytic spectrum associated to the data of the two "points" $-\infty$ and $+\infty$ joined by a simple continuous path avoiding ± 1 and connecting two distinguished oscillating solutions (even or odd).

4.2. *CM spectrum*

4.2.1. *An exponential order estimate*

The following result is due to Reinhard Schäfke. (The proof uses in particular a result of [26].)

Proposition 9. *Let* $\Lambda > 0$ *fixed and* $\mu \in \mathbb{C}$. *We consider the solution* y_1 *of* $(W_\lambda - \mu)y$, *with* $y_1(x,\mu) \sim \frac{1}{x} e^{2\pi i \Lambda x}$ *as* $|x| \to +\infty$, *for small* $\arg x$ *(*$\sim$ *means equivalent in the sense that the quotient tends to* 1*). Let* $x_0 > \Lambda$ *fixed. Then* $y_1(x_0,\mu)$ is an entire function *of* μ *and* its exponential growth is at most $1/2$.

We conjecture that $1/2$ is the *exact order*. There are variants, replacing y_1 by the sum of $\widehat{y}^+$ or $\widehat{y}^-$ in a non-singular direction d and x_0 by a point on d such that $|x_0| > \Lambda$. There are also variants replacing W_Λ by $\mathscr{D}_\tau$.

Meixner and Schäfke gave similar results for the simpler cases of an ordinary point or a regular singular point: cf. [17, 1.3, Satz 1 and 2, p. 48–49] (cf. also [26]).

4.2.2. *Spectral determinant of the CM spectrum*

In spectral theory, a *spectral determinant* is an entire function $F(\mu)$ whose zeros are the eigenvalues. In some works spectral determinants are obtained using *analytic continuation* [33]. We use a similar idea, defining spectral determinants by an analytic matching of distinguished solutions.[16] We allow multiplicities for the zeros of our spectral determinants, *a priori* independent of the properties of the eigenvalues.

We work with $D_\tau - \mu$. Let $\psi = -\frac{\sin \tau x}{x} + O(1/x^2)$ *interpreted as a Borel sum of a formal solution.*

We have a basis of solutions[17] of $D_\tau - \mu$ at $x = 1$: $(y_{\rm I}, y_{\rm II})$, $y_{\rm I}$ is analytic, $y_{\rm I}(1) = 1$, $W(y_{\rm I}, y_{\rm II}) = \frac{1}{1-x^2}$ (cf. [17, 3.12, Satz 5, p. 222 and above the similar result for $W_\Lambda - \mu$]).

We have a connection formula for $1 \leftrightarrow +\infty$: $\psi = \alpha y_{\rm I} + \beta y_{\rm II}$, where $\alpha, \beta \in \mathbb{C}$. Then

$$\alpha = W(\psi, y_{\rm II})/W(y_{\rm I}, y_{\rm II}) = (1-x^2)W(\psi, y_{\rm II}),$$
$$\beta = -W(\psi, y_{\rm I})/W(y_{\rm I}, y_{\rm II}) = (x^2-1)W(\psi, y_{\rm I}).$$

For $\tau = \tau_0$ and $x_0 \in\,]1, +\infty[$ fixed, $y_{\rm I}(x_0,\mu)$ and $\psi(x_0,\mu)$ are entire functions of the spectral parameter μ, of order $\le 1/2$. Choosing $x_0 = \sqrt{2}$, we get

$$F(\tau_0, \mu) = W(\psi, y_{\rm I})(\sqrt{2})(\tau_0, \mu) = \beta.$$

The entire functions $\psi(\sqrt{2}, \mu)$ and $y_{\rm I}(\sqrt{2}, \mu)$ are of order $\le 1/2$. This follows from Proposition 9 for ψ and from [17, 1.3, Satz 2, p. 49] for $y_{\rm I}$. Therefore F is a functional determinant for the CM spectrum and it is an entire function of order $\le 1/2$.

[16] A similar method is used by Sibuya in [28, Chapter 6, 29, p. 130].

[17] We use abusively the same notations for the local basis of $D_\tau - \mu$ and $W_\Lambda - \mu$.

We conjecture that the zeros of F are *simple*. (This is supported by some numerical experiments.) If it is so, then we have an infinite product formula for F and this function is independent of the choice of x_0 up to multiplication by a non zero constant.

4.2.3. *Computation of eigenvalues*

Let $\Lambda = \sqrt{2}$, i.e. $\tau = 4\pi$. If μ is a non classical CM eigenvalue, then $\beta = 0$ and $\alpha = \psi(1) = \pm\sqrt{2}\pi$ (cf. Proposition 7). It allows a control for the numerical computation of the eigenvalues by analytical matching. For a classical eigenvalue, we have $\alpha = \psi(1) = \pm\sqrt{2}\sqrt{\lambda}\pi$.

All computations are done with *SageMath*. We compute numerically ψ by truncating the series appearing in the solutions near infinity and we evaluate y_{I} and y_{II} using the tools developped by Marc Mezzarobba for computation of D-finite (or holonomic) functions [19], with a great (certified) precision. We replaced π by a rational number approximating the exact value with 300 digits precision.

We give below two examples. On each line:

- the first value is the approximation given in [8];
- the second one is the value obtained by searching the zeroes of β by dichotomy;
- the third value is the value of β, which must be close to 0;
- the last value is the corresponding value of α, which must be close to $\pm\sqrt{2}\pi$ = ± 4.44288293815837 alternatively.

μ_{-2}	-39	-39.3832165744668224	$-2.09497516339781e-18$	-4.44288293889868
μ_{-148}	-9100	-9104.3331714128495040	$1.88990889191427e-18$	4.44288293815837

5. New eigenfunctions for the negative CM spectrum

5.1. *The naive spectrum of $\mathscr{D}_\tau$ on the imaginary axis*

The classical prolate eigenvalues are the $\mu \in \mathbb{C}$ such that $\mathscr{D}_\tau - \mu$ admits a *bounded* solution on $]-1,1[$. If $\tau \in \mathbb{R}^*$, they are *real positive*.

We consider now, for $\tau > 0$, the naive eigenvalues on the *imaginary axis*, that is the $\mu \in \mathbb{C}$ such that $\mathscr{D}_\tau - \mu$ admits a *bounded* solution on $\mathbb{R}i =]-i\infty, +i\infty[$. We will see later that μ is a naive eigenvalue on $\mathbb{R}i$ if and only if it is a non classical CM eigenvalue (cf. Theorem 14). This implies that the naive spectrum is *infinite*, in particular non void, which does not seem obvious! This implies also that the non classical CM eigenvalues are negative.

The key is the fact, proved in [8], that for the *non classical* CM eigenvalues μ, the eigenfunctions belong to the *Sonin space*.

Lemma 10. *A naive eigenfunction belongs to the Schwartz space $S(\mathbb{R}i)$ and to $L^2(\mathbb{R}i)$. The naive spectrum is the set of $\mu \in \mathbb{C}$ such that $\mathscr{D}_\tau - \mu$ admits a solution in $S(\mathbb{R}i)$ (resp. $L^2(\mathbb{R}i)$).*

Proposition 11.

(i) *If $f, g \in S(\mathbb{R}i)$, then $\langle \mathscr{D}_\tau f, g\rangle = \langle f, \mathscr{D}_\tau g\rangle$;*
(ii) *The naive eigenvalues are* real *and* negative*;*
(iii) *The naive eigenspaces are of* dimension one*;*
(iv) *The naive eigenfunctions are* even *or* odd *and they admit an exponential decay at infinity on $\mathbb{R}i$ (at $\pm i\infty$).*

5.2. *Hardy spaces decomposition*

We have some results in Paley–Wiener style [9,21,25].

We denote by $\Pi^+ = \{\Im x > 0\}$ and $\Pi^- = \{\Im x < 0\}$ the upper and lower half-plane. We consider the *Hardy spaces* $H^2(\Pi^\pm)$. We have linear isometric (non surjective) maps:

$$\mathrm{BL}\colon H^2(\Pi^\pm) \longrightarrow L^2(\mathbb{R}),$$

the *non tangential boundary limits*.

We fix $\tau > 0$. Let $\varphi \in L^2(\mathbb{R})$ and $\psi = \mathbb{F}_{i\tau}\varphi$. We define:

$$\psi^+(x) = \int_0^{+\infty} \varphi(t)e^{-i\tau tx}\,\mathrm{d}t \quad \text{for } \Im x < 0,$$
$$\psi^-(x) = \int_{-\infty}^{0} \varphi(t)e^{-i\tau tx}\,\mathrm{d}t \quad \text{for } \Im x > 0.$$

Then: $\psi^+ \in H^2(\Pi^-) \subset \mathscr{O}(\Pi^-)$ and $\psi^- \in H^2(\Pi^+) \subset \mathscr{O}(\Pi^+)$. We have the Hardy decomposition of ψ:

$$\psi = \mathbb{F}_{i\tau}\varphi = \mathrm{BL}\,\psi^+ + \mathrm{BL}\,\psi^-,$$
$$L^2(\mathbb{R}) = H^2(\Pi^-) \oplus H^2(\Pi^+).$$

If φ (or equivalently ψ) belongs to the Sonin space defined by $]-1,1[$, then we have

$$\mathrm{BL}\,\psi^+(x) + \mathrm{BL}\,\psi^-(x) = 0 \quad \text{for } -1 < x < 1.$$

If μ is a non classical CM eigenvalue and φ a corresponding eigenfuction, then ψ is also a corresponding eigenfunction and ψ^+ (resp. ψ^-) is an *holomorphic solution* of $\mathscr{D}_\tau - \mu$ on the lower (resp. upper) half-plane Π^- (resp. Π^+). The functions φ, ψ belong to the Sonin space.

Therefore, using the *analytic continuations* of the solutions $\psi^\pm$ on the vertical open band $U = \{-1 < \Re x < 1\}$ (abusively denoted similarly), we get $\psi^-(x) + \psi^+(x) = 0$ for $x \in]-1,1[$ and therefore

$$\psi^- = -\psi^+ \quad \text{on } U.$$

The functions $\psi^\pm$ are Borel-sums of *purely exponential* formal solutions at infinity. They are subdominant respectively in the lower and upper vertical direction. Therefore $\psi^+ = -\psi^-$ connects two subdominant solutions by analytic continuation along the imaginary axis. We get a new interpretation of the non classical CM prolate spectrum as an analytic spectrum.

The analytic function ψ^+ is a *subdominant* solution of $\mathscr{D}_\tau - \mu$ on the lower half-plane Π^-. It is (up to a scaling) the sum y^- of $\hat{y}^-$ at $-i\infty$.

The analytic function ψ^- is a *subdominant* solution of $\mathscr{D}_\tau - \mu$ on the upper half-plane Π^+. It is (up to a scaling) the sum y^+ of $\hat{y}^+$ at $+i\infty$.

Proposition 12. *A CM eigenvalue is a naive eigenvalue on the imaginary axis.*

5.3. *The CM eigenfunctions as boundary values of the new eigenfunctions*

A naive eigenvalue on the imaginary axis is a non classical CM eigenvalue and it is possible to recover the (non classical) CM-eigenfunctions on the real axis from the new eigenfunctions on the imaginary axis.

We recall that if $U \subset \mathbb{C}$ is open and if f is holomorphic on U, then f admits a boundary value at $t \in U \cap \mathbb{R}$ if the limit

$$\lim_{\varepsilon \to 0,\, \varepsilon > 0} f(t - i\varepsilon) - f(t + i\varepsilon)$$

exists. By definition, this limit is the boundary value $\mathrm{BV} f(t)$.

Proposition 13. *If μ is a naive eigenvalue on $i\mathbb{R}$, then the naive eigenfunction is the sum of a (even or odd) power series solution at* 0 *of $\mathscr{D}_\tau - \mu$. This solution extends analytically into a holomorphic function f on the cut plane $\mathbb{C} \setminus \big([1,+\infty[\cup]-\infty,-1]\big)$. Then μ is a non classical CM eigenvalue and the boundary value* $\mathrm{BV} f$ *is a CM eigenfunction. If f is even (resp. odd), then* $\mathrm{BV} f$ *is odd (resp. even).*

5.4. *The non classical CM spectrum and the new spectrum*

The non classical CM spectrum coincides with the naive spectrum on the imaginary axis. This follows from 5.2 and 5.3.

Theorem 14. *Let* $\mu < 0$.

(i) *The following conditions are equivalent:*
  (a) μ *is an* even *CM eigenvalue;*
  (b) μ *is an* even *naive eigenvalue on the imaginary axis;*
  (c) $D_\tau - \mu$ *admits a non trivial* odd *power series solution at the origin extending analytically along the imaginary axis into a function with an exponential decay at infinity (at* $\pm i\infty$*).*

(ii) *The following conditions are equivalent:*
  (a) μ *is an* odd *CM eigenvalue;*
  (b) μ *is an* odd *naive eigenvalue on the imaginary axis;*
  (c) $D_\tau - \mu$ *admits a non trivial* even *power series solution at the origin extending analytically along the imaginary axis into a function with an exponential decay at infinity (at* $\pm i\infty$*).*

Corollary 15. *The non classical CM eigenvalues are* negative.

This answers a question of [8]. According to [8], there could exist a finite set of *positive* non classical CM eigenvalues. The above result proves that it is not the case.

By summation of $\widehat{y}^+(\tau,\mu)$ in the direction $i\mathbb{R}_+$, we get an analytic function $y^+(\tau,\mu)(x)$ for $x \in \mathbb{R}i$. Using Proposition 9 and [17, 1.3, Satz 1, p. 48] we get the following result.

Proposition 16. *For a fixed* $\tau > 0$*, the functions* $y^+_{\tau,\mu}(0)$ *and* $(y^+)'_{\tau,\mu}(0)$ *are entire of order* $\leq 1/2$. *Their zeros are respectively the negative even and the negative odd CM eigenvalues.*

This gives spectral determinants for the negative CM spectrum. We conjecture that the zeros are simple. (This is supported by some numerical experiments.)

Remark 17. Our new spectrum is an *analytic spectrum*. An eigenfunction *connects two subdominant solutions in two different Stokes sectors* (along a path passing between the two regular singularities). Fedoryuk and Sibuya considered similar spectra for Schrödinger equations with polynomial potentials [13,28] (cf. also [27, Definition 1]).

5.5. *Numerical computation of eigenvalues*

5.5.1. *The curtain method*

Connes and Moscovici used a quite rough computation of their eigenvalues based on a matching of oscillating functions (it suffices for their purposes in relation with zeta zeros). We described above, in 4.2.3, an efficient method ("analytical matching"), based on analytic continuation *and* Borel summation. However the computations are rather complicated and lengthy, due mainly to problems with numerical resummation.[18] This method can be adapted for the computation of the eigenvalues on the imaginary axis: we connect a Borel sum of $\widehat{v}^-$ (or its derivative) and 0 (cf. Proposition 16). In fact, using our new eigenfunctions we got a *very simple, quick and efficient method* that we will describe.[19]

In 1981, in his thesis in Strasbourg [2], Jean-Louis Callot proposed a description of the Hermite (and similar) spectrum: "opening the curtains". We adapted this idea.

[18] Another inconvenient is the arbitrary choice of the connection point.
[19] With this method, Borel summation is not needed.

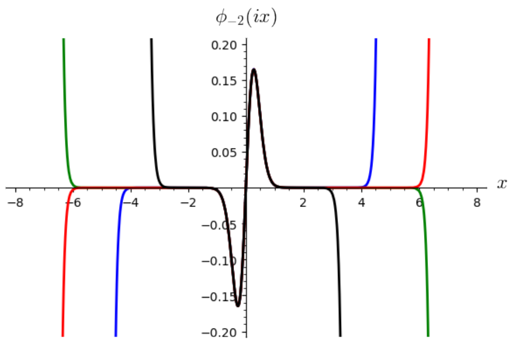
We start from an odd (resp. even) power series solution at $x = 0$ and we search μ such that its sum remains *bounded* on $i\mathbb{R}_+$. We use a dichotomy method. It is similar to the case of the *"chasse au canard"* (duck hunting) studied by the G. Reeb's school.

If $\widetilde{\mu}$ is close to a naive eigenvalue $\mu < 0$, with $\widetilde{\mu} \neq \mu$, then the corresponding eigenfunction "explodes at $\pm i\infty$". We rotate the picture, replacing x by ix and we consider the graphs of the functions on $[0, +\infty[$. Then, we observe an explosion at $\approx x_0$: the graph looks like a vertical half-line going up or down. Before we observe a domain of oscillations on $[0, x_0[$. We move $\widetilde{\mu}$ (upward or downward). If the explosion occurs later (x_0 bigger), then we are nearer μ. If an upper vertical half-line becomes a lower vertical half-line (or conversely), then we have crossed μ.

5.5.2. *Numerical experiments*

We illustrate the curtain method using two examples. In this part $\tau = 4\pi$. The choice between upper or lower explosion on our pictures is "sure" even if not formally certified. Our intervals for the eigenvalues are "sure". Some conjectures on the number of zeros allow to guess the waited sign (up or down) for the explosion.

First even negative eigenvalue μ_{-2}.



Black: $\mu = -39.3832165744668224$

Blue: $\mu = -39.3832165742615394$

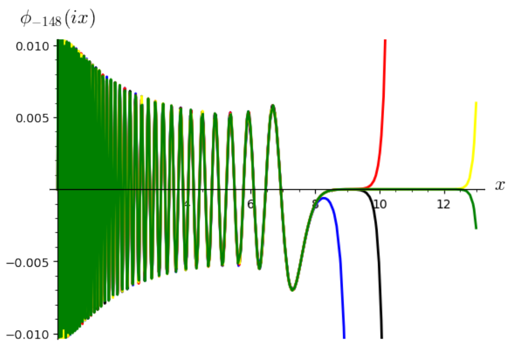
Green: $\mu = -39.38321657426153947615056322$

Red: $\mu = -39.38321657426153947615056317$

We deduce that

$$-39.383216574261539476150563\mathbf{22} < \mu_{-2} < -39.383216574261539476150563\mathbf{17}.$$

Negative eigenvalue of rank 148.



Blue: $\mu = -9104$

Black: $\mu = -9104.3331$

Red: $\mu = -9104.3332$

Yellow: $\mu = -9104.3331714128495040$

Green: $\mu = -9104.3331714128495039$

Then:

$$-9104.3331714128495040 < \mu_{-148} < -9104.3331714128495039.$$

Our best upper and lower bounds are:

$$-9104.33317141284950399\mathbf{94} < \mu_{-148} < -9104.33317141284950399\mathbf{85}.$$

5.6. *Comparison with zeta*

5.6.1. *Non trivial zeros of zeta and prolate negative spectrum*

In his 1859 article [24], Riemann introduced the function

$$\xi(s) = \frac{1}{2} s(1-s)\Gamma(s/2)\pi^{-s/2}\zeta(s).$$

It is an entire function of order one. The *functional equation* of zeta becomes $\xi(s) = \xi(1-s)$. We set $s = \frac{1}{2} + it$, $z = (s-1/2)^2 = -t^2$. Then $\xi\left(\frac{1}{2}+it\right) = \xi\left(\frac{1}{2}-it\right)$ and $\xi\left(\frac{1}{2}+it\right)$ is an analytic function of $z = -t^2$. We set $\Xi(z) = \Xi(-t^2) = \xi(s)$. The function Ξ is an entire function of order $1/2$ of z whose zeros are the opposites of the squares of the imaginary parts of the non trivial zeros of zeta.

According to Connes–Moscovici results and our results, a prolate analog of $\Xi(z)$ is, for $\Lambda > 0$ fixed, the entire function of μ, of order $\leq 1/2$: $y^{+}_{\Lambda,\mu}(0)$ (cf. Proposition 16).

5.6.2. *Trivial zeros of zeta and prolate classical spectrum*

We recall [8, 6, B].

The set of the trivial zeros of zeta is $\{-2n\}_{n\in\mathbb{N}}$.

We set $-2n = \frac{1}{2} + u$. Then $u^2 = (2n+1/2)^2 = 2n(2n+1) + \frac{1}{4}$. We get, up to a translation, the *even Legendre spectrum.*

The trivial zeros correspond by analogy to the zeros of $y_{\Lambda,I}(0)\,y'_{\Lambda,I}(0)$ shifted by $-1/4$.

Asymptotics are similar for each $\Lambda \geq 0$:

$$\mu_{2n} = (2n+1/2)^2 + O(1), \quad n \to +\infty.$$

6. Conclusion

6.1. *What we achieved*

We interpreted the prolate Connes–Moscovici spectra using complex analytic tools. In particular we introduced a notion of analytic spectrum and proved that all the interesting prolate spectra can be interpreted as analytic spectra. This allowed for the explicit definition of some spectral determinants, which are entire functions of order $\leq 1/2$ and for accurate computations of the eigenvalues.

We introduced, in a very elementary way, a new prolate spectrum on the imaginary axis. We proved that this new spectrum coincides with the non classical CM spectrum and that the boundary values of the new eigenfunctions are the non classical CM eigenfunctions, which belong to the Sonin space. This new approach allowed for a proof of a conjecture of Connes–Moscovici: all the non classical CM eigenvalues are negative. This also allowed for a new fast and accurate computation of the negative eigenvalues.

6.2. *Generalizations. Open problems*

(1) The notion of analytic spectrum can be extended to all the linear rational second order equations. In particular to the Heun family and all its confluent form.[20] These spectra have good invariance properties: by Möbius transformations on the independent variable and by s-homotopic transformations and gauge transformations[21] on the dependent variable. We plan to return to this questions in future works, cf. also [10].

[20] The prolate spheroidal equation of arbitrary order $m \in \mathbb{C}$ are a subfamily of the family of confluent Heun equations [10,29].

[21] In particular Darboux transformations.

(2) It seems impossible to extend the Connes–Moscovici definition of the prolate spectrum to the general case of the prolate equations of arbitrary order m. However it is easy to extend our definition on the spectrum on the imaginary axis. The corresponding spectra are analytic but we do not know if they are non void. Some numerical experiments suggest that it is the case and we conjecture that these spectra are infinite. More generally it is possible, up to some technical modifications, to extend our definition of the spectrum on the imaginary axis to the confluent Heun equations.[22]

(3) Bender and Wu studied the quartic oscillator with a parameter and conjectured that there is only one even eigenvalue and only one odd eigenvalue *up to an analytic continuation in the parameter*. This conjecture was proved by Eremenko and Gabrielov [11]. In their monography [17], J. Meixner and F. W. Schäfke did a very complete study of the prolate equations $\mathscr{D}_{\tau,\mu}$ for complex values of τ and μ. In [23], we conjectured that, in analogy with the quartic oscillator case, there is only one even eigenvalue and only one odd eigenvalue up to an analytic continuation in the parameter τ.[23]

Using our new definition of the non classical prolate spectrum, it is possible to define a spectrum for arbitrary (non zero) complex values of τ. However, in this extended case, the directions of the Stokes lines are in general no longer on the imaginary axis and to have a satisfying theory it is necessary to use more complicated paths γ for the definition of the analytic spectra. Such paths are allowed to do several turns around ∞. The "good variable" is not τ but $\log\tau$. The spectral determinants of Proposition 16 can be interpreted as entire analytic functions of $\log\tau$. Then it is possible to study the analytic extension of the eigenvalues as functions of $\log\tau$. We conjecture that there is only one even eigenvalue and only one odd eigenvalue up to an analytic continuation in the parameter $\log\tau$.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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[22] Except for some exceptional cases.

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