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# Analytic and asymptotic properties of solutions to a non-homogeneous functional differential equation

## *Propriétés analytiques et asymptotiques des solutions d'une équation différentielle fonctionnelle non homogène*

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**Abstract.** The paper investigates the analytic properties and asymptotic behaviors of solutions to the non-homogeneous functional differential equation $y'(x) = ay(qx) + by(x) + \frac{1}{1+x}$, where $q$ is a constant satisfying $0 < q < 1$, and $a \neq 0$, $b \neq 0$ are complex numbers, by following methods developed by J. P. Ramis for singular differential equations and $q$-difference equations. First, we study the existence and analytic properties of solutions expressed as series expansions around both zero and infinity. Next, by considering the equation as a perturbation of a differential equation, we derive a solution represented as a sum of integrals containing an infinite number of singularities. Finally, we establish a connection formula between the integral-sum solution and the series expansion at zero, which is crucial for determining the asymptotic behavior of the series solution as $x \to \infty$, given the initial conditions.

**Résumé.** Dans cet article, nous examinons les propriétés analytiques et les comportements asymptotiques des solutions de l'équation différentielle fonctionnelle non homogène $y'(x) = ay(qx) + by(x) + \frac{1}{1+x}$, où $q$ est une constante satisfaisant $0 < q < 1$, et $a \neq 0$, $b \neq 0$ sont des nombres complexes, en suivant les méthodes développées par J. P. Ramis pour les équations différentielles autour des singularités et les équations aux $q$-différences. Nous étudions tout d'abord l'existence et les propriétés analytiques des solutions exprimées sous forme de séries autour de zéro et à l'infini. Ensuite, en considérant l'équation comme une perturbation d'une équation différentielle, nous obtenons une solution représentée par une somme d'intégrales comportant un nombre infini de singularités. Enfin, nous établissons la formule de connexion entre la fonction somme-intégrale et la solution en série à zéro, qui joue un rôle crucial dans la détermination des comportements asymptotiques à l'infini de ladite solution en série, une condition initiale étant donnée.

**Keywords.** Functional differential equation, non-homogeneous term, analytic properties, perturbation of equation, asymptotic expansion.

**Mots-clés.** Équation différentielle fonctionnelle, terme non homogène, propriétés analytiques, perturbation de l'équation, expansion asymptotique.

**2020 Mathematics Subject Classification.** 34K25, 34K09.

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## 1. Introduction

The homogeneous functional differential equation

$$y'(x) = ay(qx) + by(x), \tag{1}$$

where $0 < q < 1$ and $a, b$ are given real numbers, is commonly referred to as the pantograph equation [7,12]. This equation serves as an idealized mathematical model for describing wave motion in the overhead supply lines of an electrified railway system [6]. Researchers have extensively investigated this equation and its extended forms (see [3,13,18] and the literature they cite). For example, Kato and McLeod [8] established the existence and asymptotic form of analytic solutions to equation (1) with respect to the real variable. They found that, given a non-zero initial condition, equation (1) has a unique analytic solution at $x = 0$ when $0 < q < 1$. Lim [9] obtained the asymptotic boundaries of the solutions when the variable $x$ is real. The case where $q > 1$ differs significantly from the previous one; therefore, our focus will be exclusively on the case of $0 < q < 1$ in order to investigate the analytic properties of the solutions.

For complex variables, Zhang [18] conducted a study on the connection formula between power series-type solutions of equation (1). He established an expression for the series at zero as a linear combination of one canonical fundamental solution system at infinity. The main idea behind all the aforementioned studies on asymptotic behaviors is to utilize the connection formula between solutions at zero and infinity.

This paper aims to study the non-homogeneous form related to equation (1), specifically expressed as

$$y'(x) = ay(qx) + by(x) + g(x), \tag{2}$$

where, as previously mentioned, $0 < q < 1$ and $a, b \in \mathbb{C}^*$, by following methods developed by J. P. Ramis *et al.* [14–17] for singular differential equations and $q$-difference equations.

Let $x = -\frac{t}{b}$ with $b \neq 0$ and define $u(t) = y(x) = y\big(-\frac{t}{b}\big)$. Then, the derivative of $u(t)$ with respect to $t$ is given by $u'(t) = -\frac{1}{b}y'\big(-\frac{t}{b}\big)$. Substituting $x = -\frac{t}{b}$ into equation (2), we obtain $u'(t) = \alpha u(qt) - u(t) + f(t)$, where $\alpha = -\frac{a}{b} \neq 0$ and $f(t) = -\frac{1}{b}g\big(-\frac{t}{b}\big)$. Therefore, it suffices to consider equation

$$y'(x) = \alpha y(qx) - y(x) + f(x). \tag{3}$$

Assuming that $f$ is a rational function, we only need to consider $f$ in one of the following three forms:

(1) polynomials;
(2) fractions with a singularity at 0, such as $\frac{1}{x^m}$;
(3) fractions with a singularity at a non-zero constant, such as $\frac{1}{(c+x)^m}$ (where $m > 0$ and $c \neq 0$).

In [4], we studied the asymptotic behaviors of solutions to equation (3) in cases (1) and (2). This paper focuses on case (3), where $f(x) = \frac{1}{(c+x)^m}$ as mentioned before. In this case, the solutions may exhibit an infinite number of singularities, such as $-c, -cq, -cq^2, \ldots$ and $-c/q, -c/q^2, \ldots$ These singularities arise due to the action of the $q$-difference operator. Therefore, we concentrate on the analyticity of these solutions.

Let

$$z(x) = \frac{(-1)^{m-1} y^{(m-1)}(x)}{(m-1)!}.$$

Then, we find that $z(x)$ is an analytic solution of the equation

$$z'(x) = \alpha' z(qx) - z(x) + \frac{1}{(c+x)^m}$$

with $\alpha' = \alpha q^{m-1}$, provided that $y(x)$ is an analytic solution of the equation

$$y'(x) = \alpha y(qx) - y(x) + \frac{1}{c+x},$$

which corresponds to case (3) with $m = 1$. For $c \neq 0$, there are two distinct cases depending on whether $\mathrm{Re}(c) > 0$ or $\mathrm{Re}(c) < 0$. When $\mathrm{Re}(c) > 0$, a fundamental case is when $c = 1$. The present paper primarily focuses on this case. A brief discussion will be given for the case $\mathrm{Re}(c) < 0$ at the end of the paper.

Hence, we shall study the analytic and asymptotic properties of the solutions to the following non-homogeneous functional differential equation:

$$y'(x) = \alpha y(qx) - y(x) + \frac{1}{1+x}. \tag{4}$$

The organization of the paper is as follows. Section 2 presents several essential lemmas and equations necessary for the analysis. In Section 3, we obtain the solutions of (4), which can be expressed as power series around 0 and $\infty$, respectively. We also discuss the analytical continuation properties of these series solutions at 0 and $\infty$. Section 4 explores the equation as a perturbation of a differential equation, allowing us to express the solution as the sum of some integrals. This solution contains an infinite number of logarithmic singularities $\{-1, -q^{-1}, -q^{-2}, \dots\}$. By establishing a link formula between the power series and the integral-sum function, we derive the asymptotic behaviors of the power series solutions with the given initial condition.

## 2. Preliminary results

This section presents some notations and general results on $q$-identities, which will be utilized in the subsequent sections of the paper.

### 2.1. *Some notations and identities*

We denote by $\widetilde{\mathbb{C}}^*$ the Riemann surface associated with the complex logarithm function, and by log the principal branch of the logarithm function defined on $\widetilde{\mathbb{C}}^*$, where $\log(1) = 0$.

For any $\alpha \in \mathbb{C}$ and $n \geq 1$, we define $(\alpha; q)_n$ and $(\alpha)_n$ as follows:

$$(\alpha; q)_n = \prod_{j=0}^{n-1} (1 - \alpha q^j), \qquad (\alpha)_n = \prod_{j=0}^{n-1} (\alpha + j).$$

Let $(\alpha; q)_0 = (\alpha)_0 = 1$. The expressions $(\alpha; q)_n$ and $(\alpha)_n$ tend to $(\alpha; q)_\infty$ and $(\alpha)_\infty$ as $n \to \infty$, respectively.

Define the function $F_0(\alpha; q, x)$ as follows:

$$F_0(\alpha; q, x) = \sum_{n \geq 0} \frac{(-1)^n (\alpha; q)_n}{n!} x^n. \tag{5}$$

It can be shown that $F_0(\alpha; q, x)$ is the unique solution of the pantograph equation $y'(x) = \alpha y(qx) - y(x)$ with initial condition $y(0) = 1$ (see [18]).

According to reference [1, p. 490, Corollary 10.2.2(c)–(d)], we have the following identities:

$$\sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q (-1)^k q^{\frac{k(k-1)}{2}} x^k = (x; q)_N, \qquad \sum_{k=0}^{\infty} \begin{bmatrix} N + k - 1 \\ k \end{bmatrix}_q x^k = \frac{1}{(x; q)_N}, \tag{6}$$

where the $q$-binomial coefficient $\begin{bmatrix} N \\ k \end{bmatrix}_q$ is defined as $\frac{(q;q)_N}{(q;q)_k (q;q)_{N-k}}$.

**Lemma 1.** *Let $k \in \mathbb{N}$ and let $m$ be an integer such that $m \in [0, k]$. The equality*

$$\frac{(\alpha q^{-1}; q^{-1})_m}{(\alpha q^{-1}; q^{-1})_{k+1}} = \sum_{n \geq 0} \frac{(\alpha q^{-(k+1)})^n (q^{k-m+1}; q)_n}{(q; q)_n} \tag{7}$$

*holds for any $|\alpha| < q^{k+1}$.*

**Proof.** We will prove that the equality holds by induction on $k$. If $k=0$, then $m=0$. Equation (7) becomes $\frac{1}{1-\alpha q^{-1}} = \sum_{n\geq 0}(\alpha q^{-1})^n$ for $|\alpha| < q$, which is obviously true. Assume that (7) holds for some $k$. We need to show that it holds for $k+1$. For $k+1$, the left-hand side of (7) becomes $\frac{(\alpha q^{-1};q^{-1})_m}{(\alpha q^{-1};q^{-1})_{k+1}} \cdot \frac{1}{1-\alpha q^{-(k+2)}}$. This can be viewed as the product of two series:

$$\sum_{n\geq 0} \frac{(\alpha q^{-(k+1)})^n (q^{k-m+1};q)_n}{(q;q)_n} \cdot \sum_{n\geq 0} (\alpha q^{-(k+2)})^n.$$

By applying the Cauchy product of series, the above formula is equivalent to

$$\sum_{n\geq 0} (\alpha q^{-(k+2)})^n \sum_{i=0}^{n} \frac{q^i (q^{k-m+1};q)_i}{(q;q)_i}.$$

Using the identity from [5, p. 379, (10.97)] with $b=1$ and $b'=k-m+1$, we have

$$\sum_{i=0}^{n} \frac{q^i (q^{k-m+1};q)_i}{(q;q)_i} = \frac{(q^{k-m+2};q)_n}{(q;q)_n};$$

this equality can also be proved by induction on $n$. Therefore, (7) holds for $k+1$. □

## 2.2. *Index theorems for differential q-difference equation*

Equation (4) is also called a differential $q$-difference equation. We may classify the solutions of (4) by applying index theory for differential $q$-difference equations. Below, we introduce some definitions and results from [11].

**Definition 2.** *Let $\mathbb{C}[[x]]$ be the set of formal power series. Let $s, s' \in \mathbb{R}$ and $A > 0$. For $\widehat{f}(x) = \sum_{n\geq 0} a_n x^n \in \mathbb{C}[[x]]$, if there exists $C > 0$, such that $|a_n| < C|q|^{-\frac{sn(n+1)}{2}} (n!)^{s'} A^n$ for any $n \in \mathbb{N}$, then we say that $\widehat{f}$ is $q$-Gevrey of order $s$ and Gevrey of order $s'$, denoted as $\widehat{f}(x) \in \mathbb{C}[[x]]_{q,s,s',A}$.*

Let $\mathbb{C}[[x]]_{q,s,s'} = \bigcup_{A>0} \mathbb{C}[[x]]_{q,s,s',A}$ and $\mathbb{C}[[x]]_{(q,s,s')} = \bigcap_{A>0} \mathbb{C}[[x]]_{q,s,s',A}$. When $s = s' = 0$, the space $\mathbb{C}[[x]]_{q,0,0}$ reduces to $\mathbb{C}\{x\}$, which is the space of power series with a non-zero radius of convergence.

Let $I, J$ be two positive integers, consider the operator

$$L = \sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k\geq 0} \alpha_{i,j,k}\, x^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i \sigma_q^j$$

acting from $\mathbb{C}[[x]]$ into itself, where $\sigma_q$ is the operator that has a formal series $\sum_{n\geq 0} a_n x^n$ associates the formal series $\sum_{n\geq 0} a_n q^n x^n$. We define

$$p_s^0(L) = \inf_{\alpha_{i,j,k}\neq 0} \big[j + (k-i)s\big], \qquad v_{s,s'}(L) = \inf_{\substack{\alpha_{i,j,k}\neq 0 \\ j+(k-i)s = p_s^0(L)}} \big[(k-i)s' - i\big]$$

and

$$M(s) = \big\{(i,j,k) \mid \alpha_{i,j,k} \neq 0,\ j + (k-i)s = p_s^0(L)\big\},$$
$$N(s,s') = \big\{(i,j,k) \in M(s) \mid (k-i)s' - i = v_{s,s'}(L)\big\}.$$

**Definition 3.** *We say that $s \in \mathbb{R}$ is an exceptional value for $L$ if there are two triples $(i_1, j_1, k_1)$ and $(i_2, j_2, k_2)$ in $M(s)$ such that $k_1 - i_1 \neq k_2 - i_2$. Let $s$ be an exceptional value for $L$, a real number $s'$ is an exceptional value relative to $s$ for the operator $L$ if there exist two triples $(i_1, j_1, k_1)$ and $(i_2, j_2, k_2)$ in $N(s, s')$ such that $k_1 - i_1 \neq k_2 - i_2$.*

**Lemma 4 ([11]).** *Let $L$ be as above. Let $s_1 > s_2 > \cdots > s_l$ be all the possible nonnegative exceptional values for $L$. For each $s_p$, let $s'_{p,1} > s'_{p,2} > \cdots > s'_{p,l_p}$ be all the possible exceptional values relative to $s_p$ (if $s_p = 0$ we limit ourselves to positive or zero $s'$-values). Let $\widehat{f} \in \mathbb{C}[[x]]$ and $g \in \mathbb{C}\{x\}$ such that $L(\widehat{f}) = g$. Then:*

(i) *either* $\widehat{f} \in \mathbb{C}\{x\}$;
(ii) *either there exists a unique real* $s' > 0$ *such that* $\widehat{f} \in \mathbb{C}[[x]]_{q,0,s'}$ *and* $\widehat{f} \notin \mathbb{C}[[x]]_{(q,0,s')}$: *we say that* $\widehat{f}$ *is Gevrey of exact order* $s'$;
(iii) *either there exists a unique real* $s > 0$ *and a unique real* $s'$ *such that* $\widehat{f} \in \mathbb{C}[[x]]_{q,s,s'}$ *and* $\widehat{f} \notin \mathbb{C}[[x]]_{(q,s,s')}$: $\widehat{f}$ *is* $q$*-Gevrey of exact order* $s$ *and Gevrey of exact order* $s'$.

We consider the case where $L = \frac{\mathrm{d}}{\mathrm{d}x} - \alpha\sigma_q + 1$, which is the operator corresponding to equation (4). Based on the lemma referenced above, we can derive results concerning the series solutions of equation (4).

**Corollary 5.** *For any* $f \in \mathbb{C}\{x\}$, *there is an infinite number of solutions* $y \in \mathbb{C}\{x\}$, *such that* $Ly = f$.

**Proof.** For the operator $L$, it is evident that $s = 0$ is an exceptional value, and $s' = -1$ is an exceptional value relative to $s = 0$. Consequently, there are no positive exceptional values for this operator. □

Let $\mathbb{C}[[x^{-1}]]$ be the set of formal series at infinity. Consider the operator $L = \frac{\mathrm{d}}{\mathrm{d}x} - \alpha\sigma_q + 1\colon \mathbb{C}[[x^{-1}]] \mapsto \mathbb{C}[[x^{-1}]]$. Let $t = \frac{1}{x}$. We will transform $L$ into a new operator $\widetilde{L}\colon \mathbb{C}[[t]] \mapsto \mathbb{C}[[t]]$. Write $\widehat{f}(x) = \sum_{n\geq 0} a_n x^{-n} \in \mathbb{C}[[x^{-1}]]$, then $\widehat{g}(t) = \sum_{n\geq 0} a_n t^n \in \mathbb{C}[[t]]$. Additionally,

$$L\big(\widehat{f}(x)\big) = \left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha\sigma_q + 1\right)\big(\widehat{f}(x)\big) = \left(-t^2\frac{\mathrm{d}}{\mathrm{d}t}\sigma_q - \alpha + \sigma_q\right)\sigma_q^{-1}\big(\widehat{g}(t)\big).$$

Define the operator $\widetilde{L} = -t\big(t\frac{\mathrm{d}}{\mathrm{d}t}\big)\sigma_q - \alpha + \sigma_q$. Thus, we establish the relationship

$$L\big(\widehat{f}(x)\big) = \widetilde{L}\circ\sigma_q^{-1}\big(\widehat{g}(t)\big).$$

Consequently, the study of the operator $L$ is equivalent to the study of $\widetilde{L}$.

**Corollary 6.** *For* $L = \frac{\mathrm{d}}{\mathrm{d}x} - \alpha\sigma_q + 1\colon \mathbb{C}[[x^{-1}]] \mapsto \mathbb{C}[[x^{-1}]]$, *if* $f \in \mathbb{C}\{x^{-1}\}$ *and* $y \in \mathbb{C}[[x^{-1}]]$ *satisfy that* $L(y) = f$, *then* $y \in \mathbb{C}\{x^{-1}\}$.

**Proof.** From the analysis above, we only need to consider the operator $\widetilde{L}$. According to Definition 3, it can be determined that $s = -1$ is an exceptional value for $\widetilde{L}$, and $s' = 1$ is an exceptional value relative to $s = -1$. There are no positive exceptional values for the operator $\widetilde{L}$; thus, it fits case (i) in Lemma 4. □

From now on, we assume that $f(x) = \frac{1}{1+x}$, which satisfies conditions in Corollaries 5 and 6.

## 3. Power series solutions at zero and infinity

In this section, we will obtain the solutions of equation (4), which can be expressed as power series at 0, respectively, at $\infty$. Additionally, we will explore the analytic continuation properties of these solutions.

### 3.1. *Power series solution at zero*

In the following, we look for solutions to equation (4) in the form of power series at zero. Using the series expansion $\frac{1}{1+x} = \sum_{n\geq 0}(-x)^n$ ($|x| < 1$) and assuming $y = \sum_{n\geq 0} a_n x^n$, we arrive at the equation

$$\sum_{n\geq 1} n a_n x^{n-1} = \sum_{n\geq 1}(\alpha q^{n-1} - 1)a_{n-1}x^{n-1} + \sum_{n\geq 1}(-x)^{n-1}.$$

By comparing the coefficients of $x^{n-1}$, we get the recurrence relation $a_n = \frac{-(1-\alpha q^{n-1})}{n} a_{n-1} + \frac{(-1)^n}{n}$. Define

$$F_1(\alpha; q, x) = \sum_{n\geq 1}\sum_{k=1}^{n} \frac{(-1)^{n-1}(k-1)!\,(\alpha q^k; q)_{n-k}}{n!} x^n. \tag{8}$$

Let $D(0,1)$ be an open disk centered at 0 and with radius 1. The following result can be established.

**Theorem 7.** *Let $F_0(\alpha;q,x)$ and $F_1(\alpha;q,x)$ be as in* (5) *and* (8).

(i) *Given a constant $c_0\in\mathbb{C}$, the function $F(\alpha;q,c_0,x)=c_0F_0(\alpha;q,x)+F_1(\alpha;q,x)$ is the unique series solution analytic in $D(0,1)$ of equation* (4) *with $y(0)=c_0$.*

(ii) *The series $F_0$ is an entire function, and the radius of convergence of the series $F_1$ is* 1.

(iii) *The function $F_1$ satisfies equation* (4) *with $y(0)=0$, it can be analytically extended to $\mathbb{C}\setminus(-\infty,-1]$, and with the following property: given any integer $n\geq 0$, there exists a function $y_n(x)$ analytic in $|x|<q^{-n-1}$ such that*

$$F_1(\alpha;q,x)=y_n(x)+\sum_{k=0}^{n}A_k(x)\log(1+q^kx),\tag{9}$$

*where*

$$A_k(x)=\alpha^k\sum_{j=0}^{k}\frac{(-1)^jq^{\frac{j(j-1)}{2}}}{(q;q)_j(q;q)_{k-j}}e^{-(1+q^kx)q^{-j}}.$$

**Proof. (i).** From the previous analysis and [18, p. 5, Proposition 1.2], we find that $F$ is the unique solution of (4) that satisfies the initial condition $y(0)=c_0$. To determine the radius of convergence of $F$, we need to prove that (ii) holds.

**(ii).** It is evident that $F_0$ is an entire function, as demonstrated by calculating that its radius of convergence is infinite. Since $(\alpha q^k;q)_{n-k}$ is bounded, the radius of convergence of $F_1$ is at least 1. Suppose that the radius is $R$, then we have $R\geq 1$. Furthermore, $F_1(\alpha;q,qx)$ is analytic for $|x|<\frac{R}{q}$, and the expression $F_1'(\alpha;q,x)-\alpha F_1(\alpha;q,qx)+F_1(\alpha;q,x)$ is analytic for $|x|<R$. If $R>1$, then $y'(x)-\alpha y(qx)+y(x)=\frac{1}{1+x}$ would have no singularities for any $|x|<R$, which contradicts the fact that $\frac{1}{1+x}$ has a singularity at $x=-1$. Therefore, we conclude that $R=1$.

**(iii).** We prove formula (9) by induction on $n$. Let $x\notin(-\infty,-1]$. By Theorem 7 and substituting $F_1(\alpha;q,x)=u(x)e^{-x}$ into the equation, we obtain $u'(x)e^{-x}-\alpha u(qx)e^{-qx}=\frac{1}{1+x}$. From this, we can express $u(x)$ as $\int_0^x\frac{e^t}{1+t}\,\mathrm{d}t+\alpha\int_0^x e^tF_1(\alpha;q,qt)\,\mathrm{d}t$. The function defined by the power series solution $F_1$ satisfies the following equation:

$$F_1(\alpha;q,x)=e^{-x}\int_0^x\frac{e^t}{1+t}\,\mathrm{d}t+\alpha e^{-x}\int_0^x e^tF_1(\alpha;q,qt)\,\mathrm{d}t.\tag{10}$$

Notice that $e^t=e^{-1}\cdot e^{1+t}=e^{-1}\cdot\sum_{m\geq 0}\frac{(1+t)^m}{m!}$. Thus, we have $\int_0^x\frac{e^t}{1+t}\,\mathrm{d}t=e^{-1}\log(1+x)+g_0(x)$, where $g_0$ is an entire function. Hence, formula (9) is true for $n=0$, where $y_0(x)=g_0(x)e^{-x}+\alpha e^{-x}\int_0^x e^tF_1(\alpha;q,qt)\,\mathrm{d}t$ is an analytic function in $|x|<q^{-1}$.

Suppose that assertion (9) is true for some $n$. We substitute it into (10) to obtain:

$$\begin{aligned}F_1(\alpha;q,x)=e^{-(1+x)}\log(1+x)+e^{-x}g_0(x)+\alpha e^{-x}\int_0^x e^ty_n(qt)\,\mathrm{d}t\\+\alpha e^{-x}\int_0^x e^t\sum_{k=0}^{n}A_k(qt)\log(1+q^{k+1}t)\,\mathrm{d}t.\end{aligned}$$

In the following, we will extract terms that contain $\log(1+q^kx)$ from the last integral of the above equation. We have

$$\int_0^x e^t\sum_{k=0}^{n}A_k(qt)\log(1+q^{k+1}t)\,\mathrm{d}t=\sum_{k=1}^{n+1}\alpha^{k-1}\sum_{j=0}^{k-1}\frac{(-1)^jq^{j(j-1)/2}}{(q;q)_j(q;q)_{k-j-1}}\int_0^x e^{-q^{-j}+(1-q^{k-j})t}\log(1+q^kt)\,\mathrm{d}t,$$

where the last integral can be expressed as

$$\frac{e^{-q^{-j}+(1-q^{k-j})x}}{1-q^{k-j}}\log(1+q^k x)+\int_0^x \frac{q^k e^{-q^{-j}+(1-q^{k-j})t}}{(1-q^{k-j})(1+q^k t)}\,\mathrm{d}t.$$

The second integral, $\int_0^x \frac{q^k e^{-q^{-j}+(1-q^{k-j})t}}{(1-q^{k-j})(1+q^k t)}\,\mathrm{d}t$, can be written as $\frac{e^{-q^{-k}}}{1-q^{k-j}}\log(1+q^k x)+g_k(x)$, where $g_k$ is an entire function. Therefore, we can conclude that:

$$\begin{aligned}F_1(\alpha;q,x) &= e^{-(1+x)}\log(1+x)+e^{-x}g_0(x)+\alpha e^{-x}\int_0^x e^t y_n(qt)\,\mathrm{d}t+\sum_{k=1}^{n+1}A_{k-1}(x)g_k(x)\\ &\quad+\sum_{k=1}^{n+1}\alpha^k\sum_{j=0}^{k-1}\frac{(-1)^j q^{\frac{j(j-1)}{2}}\left[e^{-(1+q^k x)q^{-j}}-e^{-(1+q^k x)q^{-k}}\right]}{(q;q)_j(q;q)_{k-j}}\log(1+q^k x).\end{aligned}\tag{11}$$

From the first equation of (6), we find that

$$\sum_{j=0}^{k-1}\frac{(-1)^j q^{\frac{j(j-1)}{2}}}{(q;q)_j(q;q)_{k-j}}=\frac{(-1)^{k-1}q^{\frac{k(k-1)}{2}}}{(q;q)_k}.$$

Substituting this result into (11), we obtain: $F_1(\alpha;q,x)=y_{n+1}(x)+\sum_{k=0}^{n+1}A_k(x)\log(1+q^k x)$, where $y_{n+1}(x)$ is an analytic function on $|x|<q^{-n-2}$. □

**Remark 8.** Notice that $\frac{1}{1+x}=\sum_{n\ge 0}(-x)^n\in\mathbb{C}\{x\}$. According to Corollary 5, the solutions of equation (4) satisfy $y\in\mathbb{C}\{x\}$. Furthermore, from Theorem 7, we have $F(\alpha;q,c_0,x)\in\mathbb{C}\{x\}$, which is consistent with the result stated in Corollary 5.

## 3.2. *Power series solution at infinity*

Next, we seek solutions to equation (4) that can be expressed as power series at infinity. For $|x|>1$, we can rewrite the series as follows: $\frac{1}{1+x}=\frac{1}{x}\cdot\frac{1}{1+\frac{1}{x}}=\sum_{n\ge 0}(-1)^n x^{-n-1}$. Assuming $y=\sum_{n\ge 0}a_n x^{-n-1}$ and substituting this into equation (4), we obtain

$$\sum_{n\ge 1}(-n)a_{n-1}x^{-n-1}=\sum_{n\ge 0}(\alpha q^{-n-1}-1)a_n x^{-n-1}+\sum_{n\ge 0}(-1)^n x^{-n-1}.$$

To derive an expression for all $a_n$ $(n\ge 0)$, we compare the coefficients of $x^{-n-1}$ on both sides of the equation. This comparison yields a recurrence relation for $a_n$.

**Theorem 9.** *Let $\alpha\notin q^{\mathbb{Z}_{>0}}$. Then:*

(i) *the function*

$$G(\alpha;q,x)=\sum_{n\ge 0}\sum_{k=0}^{n}\frac{(-1)^k n!\,(\alpha q^{-1};q^{-1})_k}{k!\,(\alpha q^{-1};q^{-1})_{n+1}}x^{-n-1}\tag{12}$$

*is the unique analytic solution of equation* (4) *in $|x|>q$ vanishing at infinity;*

(ii) *the following relation holds:* $\lim_{x\to\infty}x\cdot G(\alpha;q,x)=\frac{1}{1-\alpha q^{-1}}$*;*

(iii) *the function $G$ can be analytically extended to $\mathbb{C}^*\setminus\{-q,-q^2,-q^3,\dots\}$; in other words, there exist functions $g_n(x)$ analytic in a neighborhood of $x=-q^n$ $(n\ge 1)$, such that*

$$G(\alpha;q,x)=P_n(x)+g_n(x),\tag{13}$$

*where*

$$P_n(x)=(q;q)_{n-1}\left(\frac{q}{\alpha}\right)^n\sum_{k=1}^{n}\frac{(-1)^k(k-1)!\,q^{\frac{k(k-1)}{2}}}{(q;q)_{k-1}(q;q)_{n-k}}\left(\frac{1}{x+q^n}\right)^k.$$

**Proof.** **(i).** Consider

$$G(\alpha;q,x)=\sum_{n\geq 0}\frac{n!}{(\alpha q^{-1};q^{-1})_{n+1}}\sum_{k=0}^{n}u_k x^{-n-1},$$

where $u_k=\frac{(-1)^k(\alpha q^{-1};q^{-1})_k}{k!}$. We find that $\left|\frac{u_k}{u_{k-1}}\right|=\frac{|1-\alpha q^{-k}|}{k}>1$ for sufficiently large $k$ and $n$. Then

$$\left|G(\alpha;q,x)\right|\leq C\sum_{n\geq 0}\frac{n+1}{\left|1-\alpha q^{-n-1}\right|}|x|^{-n-1},$$

for some constant $C$. The above series is convergent for $|x|>q$, then $G$ is analytic on $|x|>q$.

**(ii).** This result can be established by direct computations.

**(iii).** We will prove (13) by induction. The equation $y'(x)=\alpha y(qx)-y(x)+\frac{1}{1+x}$ can be written as $y(x)=\frac{1}{\alpha}\left[y'\left(\frac{x}{q}\right)+y\left(\frac{x}{q}\right)-\frac{q}{x+q}\right]$. This leads to the expression

$$G(\alpha;q,x)=\frac{1}{\alpha}\left[G'\left(\alpha;q,\tfrac{x}{q}\right)+G\left(\alpha;q,\tfrac{x}{q}\right)-\frac{q}{x+q}\right], \tag{14}$$

which is analytic in $|x|>q$. Then $G(\alpha;q,x)=-\frac{q}{\alpha}\cdot\frac{1}{x+q}+g_1(x)$, where $g_1$ is analytic near $x=-q$. Hence, the assertion holds for $n=1$. Suppose (13) holds for some $n\geq 1$, i.e. $G(\alpha;q,x)=P_n(x)+g_n(x)$ as above, where $g_n(x)$ is an analytic function near $x=-q^n$. Substituting (13) into (14), we obtain $G(\alpha;q,x)=\frac{1}{\alpha}\left[P_n'\left(\frac{x}{q}\right)+P_n\left(\frac{x}{q}\right)\right]+g_{n+1}(x)$, where $g_{n+1}(x)=\frac{1}{\alpha}\left[g_n'\left(\frac{x}{q}\right)+g_n\left(\frac{x}{q}\right)-\frac{q}{x+q}\right]$ is an analytic function near $x=-q^{n+1}$. We only need to prove that $\frac{1}{\alpha}\left[P_n'\left(\frac{x}{q}\right)+P_n\left(\frac{x}{q}\right)\right]=P_{n+1}(x)$. From the expression of $P_n(x)$, we have

$$\frac{1}{\alpha}\left[P_n'\left(\frac{x}{q}\right)+P_n\left(\frac{x}{q}\right)\right]=\left(\frac{q}{\alpha}\right)^{n+1}\left\{\sum_{k=1}^{n}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q\frac{(-1)^{k+1}k!\,q^{\frac{k(k+1)}{2}}}{(x+q^{n+1})^{k+1}}\right.$$
$$\left.+\sum_{k=1}^{n}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q\frac{(-1)^{k}(k-1)!\,q^{\frac{k(k-1)}{2}}q^{k-1}}{(x+q^{n+1})^{k}}\right\},$$

where the notation $\begin{bmatrix}n-1\\k-1\end{bmatrix}_q$ is defined in Section 2.1. By replacing $k$ in the last series with $k+1$ and combining the sum of $k$ from 1 to $n-1$, we can express the right-hand side of the above formula as

$$\left(\frac{q}{\alpha}\right)^{n+1}\begin{bmatrix}n\\k\end{bmatrix}_q\sum_{k=1}^{n-1}\frac{(-1)^{k+1}k!\,q^{\frac{k(k+1)}{2}}}{(x+q^{n+1})^{k+1}}+\left(\frac{q}{\alpha}\right)^{n+1}\frac{(-1)^{n+1}n!\,q^{\frac{n(n+1)}{2}}}{(x+q^{n+1})^{n+1}}-\left(\frac{q}{\alpha}\right)^{n+1}\frac{1}{x+q^{n+1}}.$$

Finally, we conclude that $\frac{1}{\alpha}\left[P_n'\left(\frac{x}{q}\right)+P_n\left(\frac{x}{q}\right)\right]=P_{n+1}(x)$, which completes the proof. □

**Remark 10.** By using the uniform convergence theorem, we can see that $F(\alpha;q,c_0,x)$ is analytic for $(\alpha,x)\in\mathbb{C}\times D(0,1)$; $G(\alpha;q,x)$ is analytic for all $(\alpha,x)\in\mathbb{C}\times\left\{|x|>q\right\}$, provided that $\alpha q^{-k}\neq 1$ for any $k\in\mathbb{Z}_{>0}$.

**Remark 11.** The series solution $G(\alpha;q,x)\in\mathbb{C}\{x^{-1}\}$, it is consistent with the result in Corollary 6.

## 4. Asymptotic behaviors of the solution with the initial condition

In this section, we begin by using the equation's perturbation to derive the form of solutions expressed as a sum of integrals and analyze their asymptotic behaviors. Next, we establish the relationship between the power series solution and the integral-sum function. Finally, by applying this established relationship, we determine the asymptotic behaviors of the power series solution under the given initial condition.

Let $\widetilde{\mathbb{C}\setminus\{-c\}}$ be the Riemann surface of $\log(x+c)$, where $c\in\mathbb{C}$. Let $a$ and $b$ be two real numbers such that $a<b$. We denote by $S_n^c(a,b)$ the open sector on $\widetilde{\mathbb{C}\setminus\{-c\}}$ defined as follows:

$$S_n^c(a,b)=\left\{x\in\widetilde{\mathbb{C}\setminus\{-c\}}\;\middle|\;a<\arg(q^n x+c)<b\right\}.$$

Particularly,

$$\begin{aligned} S(a,b) &\triangleq S_0^0(a,b) = \left\{x \in \widetilde{\mathbb{C}^*} \mid a < \arg(x) < b\right\}, \\ S_1(a,b) &\triangleq S_0^1(a,b) = \left\{x \in \widetilde{\mathbb{C}\setminus\{-1\}} \mid a < \arg(x+1) < b\right\}, \\ S_n^1(a,b) &= \left\{x \in \widetilde{\mathbb{C}\setminus\{-1\}} \mid a < \arg(q^n x+1) < b\right\}. \end{aligned} \tag{15}$$

### 4.1. *Perturbation of equations*

Let $\alpha$ be a parameter close to 0. If $\alpha = 0$, then equation (4) reduces to

$$y'(x) = -y(x) + \frac{1}{1+x}, \tag{16}$$

which has the same form as the Euler equation $y'(x) = -y(x) + \frac{1}{x}$. Let $S(a,b)$ be as in (15). Notice that the Euler equation has a formal series solution $\sum_{n\geq 0} n!\, x^{-n-1}$, whose Borel-sum, denoted by $E(x)$, is an analytic function defined on $S\left(-\frac{\pi}{2}, \frac{5\pi}{2}\right)$. The function $E(x)$ consists of the analytic continuation of all functions of the form $\int_0^{\infty e^{id}} \frac{e^{-xt}}{1-t}\, \mathrm{d}t$ with integral direction $d \in (-\pi,\pi)$. Similarly, equation (16) has a divergent series solution $\sum_{n\geq 0} n!\,(1+x)^{-n-1}$. This divergent series is Borel-summable (see [10, p. 175, Definition 1.3.1.2]). Let $S_1(a,b)$ be as in (15). Using a method similar to the Borel summation for the Euler series, equation (16) has a Borel-sum solution $E(x+1)$, which is defined on $S_1\left(-\frac{\pi}{2}, \frac{5\pi}{2}\right)$.

If we view equation (4) as a perturbation of equation (16) and expand the solution of (4) as a series $\sum_{n\geq 0} y_n(x)\alpha^n$, then the recursive relation for the coefficients is given by

$$y'_{n+1}(x) + y_{n+1}(x) = y_n(qx), \quad n \geq 0. \tag{17}$$

**Lemma 12.** *Let $S_n^1(a,b)$ be as in* (15). *There is a family of functions $\left\{I_n(x)\right\}_{n\geq 0}$ defined on $S_n^1\left(-\frac{\pi}{2}, \frac{5\pi}{2}\right)$ that satisfies the system of equations given in* (17).

**Proof.** Let $S_1(a,b)$ be as in (15). Choose $y_0(x) = \int_0^{\infty e^{-i\pi}} \frac{e^{-(1+x)t}}{1-t}\, \mathrm{d}t$ as a solution of (16) which is defined on $S_1\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Let $y_{n+1} = e^{-x}z$. Then, we have $z = \int_{\ell_x} e^{t_n} y_n(qt_n)\, \mathrm{d}t_n$, where $\ell_x = \left\{t + i\cdot \mathrm{Im}(x) \mid t \in \left(-\infty, \mathrm{Re}(x)\right]\right\}$. Therefore,

$$y_{n+1} = e^{-x}z = \int_{\ell_x} e^{-x+t_n} y_n(qt_n)\, \mathrm{d}t_n.$$

For $n = 0$:

$$y_1(x) = \int_{\ell_x}\int_0^{\infty e^{-i\pi}} e^{-x+t_0}\cdot \frac{e^{-(qt_0+1)t}}{1-t}\, \mathrm{d}t\, \mathrm{d}t_0 = \int_0^{\infty e^{-i\pi}} \frac{e^{-(qx+1)t}}{(1-t)(1-qt)}\, \mathrm{d}t.$$

For $n = 1$:

$$y_2(x) = \int_{\ell_x}\int_0^{\infty e^{-i\pi}} e^{-x+t_1}\cdot \frac{e^{-(q^2t_1+1)t}}{(1-t)(1-qt)}\, \mathrm{d}t\, \mathrm{d}t_1 = \int_0^{\infty e^{-i\pi}} \frac{e^{-(q^2x+1)t}}{(1-t)(1-qt)(1-q^2t)}\, \mathrm{d}t.$$

By induction, we establish that for $n \geq 0$, the functions $y_n$ are given by:

$$y_n(x) = \int_0^{\infty e^{-i\pi}} \frac{e^{-(q^nx+1)t}}{(1-t)(1-qt)\cdots(1-q^nt)}\, \mathrm{d}t.$$

Let $S_n^1(a,b)$ be as in (15). Each function $y_n$ is well-defined on the domain $S_n^1\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Replacing the integral path $[0, \infty e^{-i\pi})$ by $[0, \infty e^{id})$, where $d \in (-2\pi, 0)$, we obtain

$$I_n^{[d]}(x) = \int_0^{\infty e^{id}} \frac{e^{-(q^nx+1)t}}{(t;q)_{n+1}}\, \mathrm{d}t.$$

Let

$$V_n^d = S_n^1\left(-d - \frac{\pi}{2}, -d + \frac{\pi}{2}\right); \tag{18}$$

we will prove the analyticity of the function $I_n^{[d]}$ in any relatively-compact subset of $V_n^d$.

For real number $q$, we have $\left|(t;q)_{n+1}\right| = |1-t||1-qt|\cdots|1-q^n t| \geq \left[\sin(d)\right]^n$. By writing $t = re^{id}$, one can get that, in any relatively-compact subset of $V_n^d$,

$$\left|\frac{e^{-(q^n x+1)t}}{(t;q)_{n+1}}\right| \leq \frac{e^{\mathrm{Re}\{-(q^n x+1)t\}}}{\left[\sin(d)\right]^n} = \frac{e^{-[\mathrm{Re}(q^n x+1)\cos d - \mathrm{Im}(q^n x+1)\sin d]r}}{\left[\sin(d)\right]^n} \leq \frac{e^{-\epsilon_n r}}{\left[\sin(d)\right]^n}$$

with $\epsilon_n = \min_{x\in K}\left\{\mathrm{Re}(q^n x+1)\cos(d) - \mathrm{Im}(q^n x+1)\sin(d)\right\} > 0$. Since $\epsilon_n \to \cos(d)$ as $n \to \infty$, we have $\epsilon_n \geq \epsilon' > 0$. Then

$$\left|I_n^{[d]}(x)\right| \leq \int_0^{+\infty} \frac{e^{-\epsilon' r}}{\left[\sin(d)\right]^n}\,\mathrm{d}r = \frac{1}{\left[\sin(d)\right]^n \epsilon'}.$$

Then $I_n^{[d]}(x)$ is analytic on $V_n^d$ for every fixed $d \in (-2\pi, 0)$. By analytic continuation, gluing all functions $I_n^{[d]}(x)$ yields a function defined on the domain $S_n^1\left(-\frac{\pi}{2}, \frac{5\pi}{2}\right)$, denoted by $I_n(x)$. □

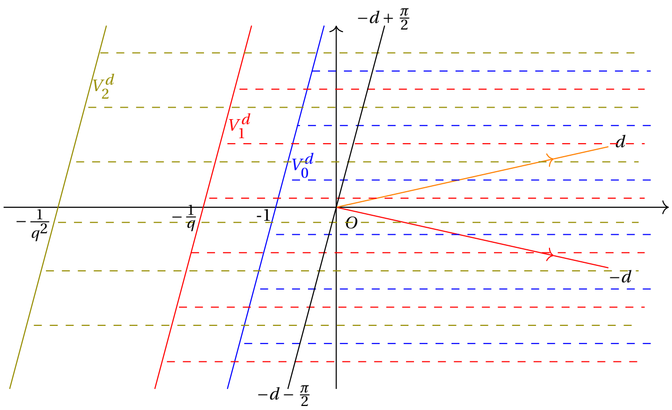


**Figure 1.** Analytic domain of $I_n^{[d]}(x)$

**Theorem 13.** *Let* $|\alpha| < 1$ *and let* $S_1(-\pi,\pi)$ *be as in* (15)*. There are two analytic functions* $T_1$ *and* $T_2$ *on* $S_1(-\pi,\pi)$*, such that:*

(i) *they are both solutions of equation* (4)*;*

(ii) $T_2(\alpha;q,x) - T_1(\alpha;q,x) = (\eta_2 - \eta_1)F_0(\alpha;q,x)$*, where* $F_0$ *is given in* (5)*, and where* $\eta_1$ *and* $\eta_2$ *are two constants defined as follows:*

$$\eta_1 = \sum_{k\geq 0}\alpha^k \int_0^{\infty e^{id_1}} \frac{e^{-t}}{(t;q)_{k+1}}\,\mathrm{d}t, \qquad \eta_2 = \sum_{k\geq 0}\alpha^k \int_0^{\infty e^{id_2}} \frac{e^{-t}}{(t;q)_{k+1}}\,\mathrm{d}t, \tag{19}$$

*with* $d_1 \in \left(0,\frac{\pi}{2}\right)$ *and* $d_2 \in \left(-\frac{\pi}{2},0\right)$*;*

(iii) *for* $x \in S_1\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$*,*

$$T_2(\alpha;q,x) - T_1(\alpha;q,x) = 2\pi i \sum_{n\geq 0}\alpha^n \sum_{j=0}^{n} \frac{(-1)^j q^{\frac{j(j-1)}{2}}}{(q;q)_j (q;q)_{n-j}} e^{-(q^n x+1)q^{-j}}. \tag{20}$$

**Proof.** **(i).** From the analysis at the beginning of Section 4.1 and Lemma 12, we only need to consider the analyticity of the series $\sum_{n\geq 0}\alpha^n I_n^{[d]}(x)$. Let $V_n^d$ be defined as in (18). If $d \in \left(-\frac{\pi}{2},0\right) \cup \left(0,\frac{\pi}{2}\right)$, it follows from Figure 1 that $V_0^d \subset V_1^d \subset \cdots \subset V_n^d \subset \cdots$ and $\bigcap_{n\geq 0} V_n^d = V_0^d$. However, if $d \in \left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, then $\bigcap_{n\geq 0} V_n^d$ will be empty, and we cannot prove the analyticity of $\sum_{n\geq 0}\alpha^n I_n^{[d]}(x)$ for all $n$. Therefore, for sufficiently small $\epsilon > 0$, we choose $d_1 = \frac{\pi}{2} - \epsilon \in \left(0,\frac{\pi}{2}\right)$ and $d_2 = -\frac{\pi}{2} + \epsilon \in \left(-\frac{\pi}{2},0\right)$. Let $S_1(a,b)$ and $S_n^1(a,b)$ be as in (15). Define $T_1(\alpha;q,x)$ and $T_2(\alpha;q,x)$ as two functions that are analytic for $x \in S_1(-\pi+\epsilon,\epsilon)$ and $x \in S_1(-\epsilon,\pi-\epsilon)$, respectively. These functions can be viewed as the analytic continuations of the series $\sum_{n\geq 0}|\alpha|^n I_n^{[d_1]}$ and $\sum_{n\geq 0}|\alpha|^n I_n^{[d_2]}$, respectively.

From the previous definition, we first examine the analyticity of $T_1$ on the sector $S_1(-\pi+\epsilon,\epsilon)$. It is sufficient to demonstrate the analyticity of the series $\sum_{n\geq 0}|\alpha|^n I_n^{[d_1]}$ (a similar result applies to the function $T_2$). For $d_1 \in \left(0,\frac{\pi}{2}\right)$, we have the inclusion sequence $V_0^{d_1} \subset V_1^{d_1} \subset \cdots \subset V_n^{d_1} \subset \cdots$. From the proof of Lemma 12, we obtain the inequality

$$\left|\sum_{n\geq 0} \alpha^n I_n^{[d_1]}(x)\right| \leq \sum_{n\geq 0}\left(\frac{|\alpha|}{\cos(\epsilon)}\right)^n \frac{1}{\epsilon'}.$$

For $|\alpha|<1$, there exists a sufficiently small $\epsilon>0$, such that $|\alpha|<\cos(\epsilon)$. Under this condition, the aforementioned series is normally convergent in $K$. Therefore, the function $T_1$ is defined and analytic on $S_1(-\pi+\epsilon,\epsilon)$. Similarly, the function $T_2$ is defined and analytic on $S_1(-\epsilon,\pi-\epsilon)$.

**(ii).** Notice that $T_1$ and $T_2$ are well-defined at $x=0$. Let $\eta_1$ and $\eta_2$ be as in (19). According to Theorem 7, we obtain that $T_1$ and $T_2$ can be analytically extended to $\mathbb{C}\setminus(-\infty,-1]$, such that $T_1(\alpha;q,0)=\eta_1$ and $T_2(\alpha;q,0)=\eta_2$, respectively.

**(ii).** Let $L$ be a smooth, anti-clockwise curve whose interior contains $\{q^{-j}\}$ for $0\leq j\leq n$. Since

$$T_2(\alpha;q,x)-T_1(\alpha;q,x)=\sum_{n\geq 0}\alpha^n\int_L \frac{e^{-(q^n x+1)t}}{(t;q)_j(1-tq^j)(tq^{j+1};q)_{n-j}}\,\mathrm{d}t,$$

(20) can be proved by the residue theorem. □

**Remark 14.** In the following, we will focus exclusively on the function $T_1$, as the results obtained will also apply to the function $T_2$.

## 4.2. *Asymptotic behaviors of the integral-sum functions*

Before analyzing the asymptotic behaviors of $T_1$ and $T_2$, we first introduce the following lemma, which provides the asymptotic expansion of the function $I_n$ for every fixed $n$. For the definition of Gevrey asymptotic expansion, see [2, p. 70].

**Lemma 15.** *Let $S_1(a,b)$ be as in* (15). *Consider the family of analytic functions $\{I_n(x)\}_{n\geq 0}$ mentioned in Section 4.1. Then for every relatively-compact subsector $S \Subset S_1(-\pi,\pi)$, there exists $C_S>0$, such that the following Gevrey asymptotic expansion*

$$I_n(x)=\sum_{m=0}^{M}\frac{m!\,(q^{m+1};q)_n}{(q;q)_n}\cdot\frac{1}{(q^n x+1)^{m+1}}+r_{n,M}(x) \tag{21}$$

*holds for any $n, M\in\mathbb{N}$ and any $x\in S$, where*

$$\left|r_{n,M}(x)\right| \leq \frac{(-q^{M+1};q)_n}{(C_S)^{M+2}(q;q)_n}\cdot(M+1)!\cdot\frac{1}{|q^n x+1|^{M+2}}.$$

**Proof.** We will follow steps similar to those in the proof of [2, p. 79, Theorem 22]. Let $S$ be any relatively-compact subsector of $S_1(-\pi,\pi)$. Then there exists $0<\epsilon<\pi$, such that $S\Subset S_1(-\pi+\epsilon,\pi-\epsilon)$. Define $d_1=\frac{\pi}{2}-\frac{\epsilon}{2}$, $d_2=-\frac{\pi}{2}+\frac{\epsilon}{2}$. We then set

$$\begin{aligned}D_1&=\left\{x\in\widetilde{\mathbb{C}\setminus\{-1\}}\;\middle|\;\arg(x+1)\in\left[-d_1-\tfrac{\pi}{2}+\tfrac{\epsilon}{2},-d_1+\tfrac{\pi}{2}-\tfrac{\epsilon}{2}\right]\right\}\\&=\left\{x\in\widetilde{\mathbb{C}\setminus\{-1\}}\;\middle|\;\arg(x+1)\in[-\pi+\epsilon,0]\right\}\end{aligned}$$

and

$$\begin{aligned}D_2&=\left\{x\in\widetilde{\mathbb{C}\setminus\{-1\}}\;\middle|\;\arg(x+1)\in\left[-d_2-\tfrac{\pi}{2}+\tfrac{\epsilon}{2},-d_2+\tfrac{\pi}{2}-\tfrac{\epsilon}{2}\right]\right\}\\&=\left\{x\in\widetilde{\mathbb{C}\setminus\{-1\}}\;\middle|\;\arg(x+1)\in[0,\pi-\epsilon]\right\}.\end{aligned}$$

Since $S_1(-\pi+\epsilon,\pi-\epsilon)\subset D_1\cup D_2$, we have $S\subset D_1\cup D_2$. Let $X=q^n x+1$. We only need to prove that $I_n^{[d_1]}(x)=\int_0^{\infty e^{id_1}}\frac{e^{-Xt}}{(t;q)_{n+1}}\,\mathrm{d}t$ has the same asymptotic expansion as in (21). The result for other directions follows the same process. From the second equation of (6), one can obtain that

$$\frac{1}{(t;q)_{n+1}}=\sum_{m=0}^{M}\frac{(q^{m+1};q)_n}{(q;q)_n}t^m+\widehat{r}_{n,M}(t). \tag{22}$$

From the first equation of (6), we derive:

$$\sum_{m=0}^{M}\frac{(q^{m+1};q)_n}{(q;q)_n}t^m=\sum_{m=0}^{M}\sum_{k=0}^{n}\frac{(-1)^k q^{\frac{k(k+1)}{2}}}{(q;q)_k(q;q)_{n-k}}(tq^k)^m=\sum_{k=0}^{n}\frac{(-1)^k q^{\frac{k(k+1)}{2}}}{(q;q)_k(q;q)_{n-k}}\cdot\frac{1-(tq^k)^{M+1}}{1-tq^k}.$$

Substituting it into (22) and using the identity $\frac{1}{(t;q)_{n+1}}=\sum_{k=0}^{n}\frac{(-1)^k q^{\frac{k(k+1)}{2}}}{(q;q)_k(q;q)_{n-k}}\cdot\frac{1}{1-tq^k}$, the function $\widehat{r}_{n,M}(t)$ can be simplified as

$$\widehat{r}_{n,M}(t)=\frac{1}{(t;q)_{n+1}}-\sum_{k=0}^{n}\frac{(-1)^k q^{\frac{k(k+1)}{2}}}{(q;q)_k(q;q)_{n-k}}\cdot\frac{1-(tq^k)^{M+1}}{1-tq^k}=\sum_{k=0}^{n}\frac{(-1)^k q^{\frac{k(k+1)}{2}}}{(q;q)_k(q;q)_{n-k}}\cdot\frac{(tq^k)^{M+1}}{1-tq^k}.$$

Then, for $d_1=\frac{\pi}{2}-\frac{\epsilon}{2}$, we have $\mathrm{Re}(t)>0$, and $\left|\widehat{r}_{n,M}(t)\right|\le\frac{(-q^{M+1};q)_n}{(q;q)_n}|t|^{M+1}$.

From $\int_0^{\infty e^{id_1}}t^m e^{-Xt}\,\mathrm{d}t=\frac{m!}{X^{m+1}}$, we obtain

$$\int_0^{\infty e^{id_1}}\sum_{m=0}^{M}\frac{(q^{m+1};q)_n}{(q;q)_n}t^m e^{-Xt}\,\mathrm{d}t=\sum_{m=0}^{M}\frac{m!\,(q^{m+1};q)_n}{X^{m+1}(q;q)_n}.$$

Define $r_{n,M}(x)=\int_0^{\infty e^{id_1}}\widehat{r}_{n,M}(t)e^{Xt}\,\mathrm{d}t$. By changing variables, we have

$$r_{n,M}(x)=\int_0^{\infty e^{i[d_1+\arg(X)]}}\frac{\widehat{r}_{n,M}\left(\frac{t}{X}\right)}{X}e^t\,\mathrm{d}t.$$

By setting $t=se^{i[d_1+\arg(X)]}$ (where $s\in\mathbb{R}$) and using the estimation of $\widehat{r}_{n,M}(t)$, we have

$$\left|r_{n,M}(x)\right|\le\frac{(-q^{M+1};q)_n}{(q;q)_n}\cdot\frac{1}{|X|^{M+2}}\int_0^{+\infty}s^{M+1}e^{\cos(d_1+\arg(X))s}\,\mathrm{d}s.$$

For any $x\in D_1$, the above integral satisfies

$$\int_0^{+\infty}s^{M+1}e^{\cos(d_1+\arg(X))s}\,\mathrm{d}s\le\int_0^{+\infty}s^{M+1}e^{-\kappa s}\,\mathrm{d}s=\frac{(M+1)!}{\kappa^{M+2}},$$

where $\kappa$, dependent on $D_1$, is a postive constant equal to $\inf\{-\cos(d_1+\arg(X))\}=\sin\frac{\epsilon}{2}$. The direction $d_2$ follows the same process. Therefore, for every $S\Subset S_1(-\pi,\pi)$, there is a positive constant $C_S=\sin\frac{\epsilon}{2}$ (dependent on $S$), such that the function $I_n(x)$ has the Gevrey asymptotic expansion shown in (21), and $\left|r_{n,M}(x)\right|\le\frac{(M+1)!(-q^{M+1};q)_n}{(C_S)^{M+2}(q;q)_n}\cdot\frac{1}{|X|^{M+2}}$ for any $x\in S$. □

**Lemma 16.** *For any relatively-compact subsector $S$ of $\mathbb{C}\setminus[0,-\infty)$, there is a positive constant $C_S$, such that for any couple of integers $M$ and $m\in[0,M]$, the function $\frac{t^{m+1}}{(1+t)^{m+1}}$ has the following expansion:*

$$\frac{t^{m+1}}{(1+t)^{m+1}}=\sum_{k=m}^{M}\frac{(-1)^{k+m}k!}{m!\,(k-m)!}\cdot t^{k+1}+\nu_{m,M}(t), \tag{23}$$

*where*

$$\left|\nu_{m,M}(t)\right|\le\frac{(1+C_S)^{M+1}}{(C_S)^{m+1}}\cdot|t|^{M+2}$$

*for any $t\in S$.*

**Proof.** Let $S$ be any relatively-compact subsector of $\mathbb{C}\setminus[0,-\infty)$. There exists a sufficiently small $\epsilon>0$, such that $S \Subset \{t\in\mathbb{C}^* \mid \arg(t)\in(-\pi+\epsilon,\pi-\epsilon)\}$. Assume that $t\in S$. Let $\rho=|t|\sin\frac{\epsilon}{2}$ and $L_\rho=\{z\in\mathbb{C}\mid z=t+\rho e^{i\theta},\ \theta\in[0,2\pi]\}$ rotating counterclockwise centered at $t$.

For any fixed $n$ and $m\in[0,M]$, we have

$$\frac{t^{m+1}}{(1+t)^{m+1}}=\frac{(-1)^m t^{m+1}}{m!}\cdot\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left(\frac{1}{1+t}\right). \tag{24}$$

For $t\in S$, we can express $\frac{1}{1+t}=\sum_{k=0}^{M}(-t)^k+\frac{(-t)^{M+1}}{1+t}$. Substituting this expression into (24), we obtain

$$\frac{t^{m+1}}{(1+t)^{m+1}}=\sum_{k=m}^{M}\frac{(-1)^{k+m}k!}{m!(k-m)!}t^{k+1}+\nu_{m,M}(t),$$

where

$$\nu_{m,M}(t)=\frac{(-1)^m t^{m+1}}{m!}\cdot\frac{\mathrm{d}^m}{\mathrm{d}t^m}\Big(\frac{(-t)^{M+1}}{1+t}\Big)=\frac{(-1)^m t^{m+1}}{2\pi i}\int_{L_\rho}\frac{(-z)^{M+1}}{1+z}\cdot\frac{1}{(z-t)^{m+1}}\,\mathrm{d}z.$$

For $z\in L_\rho\subset\{z\in\mathbb{C}\mid \arg(z)\in(-\pi+\frac{\epsilon}{2},\pi-\frac{\epsilon}{2})\}$, we have $|1+z|\ge\sin\frac{\epsilon}{2}$ and $|z|\le|t|\big[1+\sin\frac{\epsilon}{2}\big]$. Then

$$\big|\nu_{m,M}(t)\big|\le\frac{|t|^{m+1}}{2\pi\sin\frac{\epsilon}{2}}\int_0^{2\pi}\frac{|t|^{M+1}\big(1+\sin\frac{\epsilon}{2}\big)^{M+1}}{\big(|t|\sin\frac{\epsilon}{2}\big)^m}\mathrm{d}\theta=\frac{\big(1+\sin\frac{\epsilon}{2}\big)^{M+1}}{\big(\sin\frac{\epsilon}{2}\big)^{m+1}}|t|^{M+2}.$$

This completes the proof. □

Then we obtain the asymptotic behavior of $T_1$ (or $T_2$).

**Theorem 17.** *Let $|\alpha|<q^{M+2}$ and let $S(-\pi,\pi)$ be as in* (15). *For every relatively-compact subsector $S\Subset S(-\pi,\pi)$, there exists a constant $D_{S,M}>0$, so that for every non-negative integer $M$ and $x\in S$,*

$$T_1(\alpha;q,x)=\sum_{k=0}^{M}\sum_{m=0}^{k}\frac{(-1)^k k!(\alpha q^{-1};q^{-1})_m}{m!(\alpha q^{-1};q^{-1})_{k+1}}\cdot\frac{1}{x^{k+1}}+R_{S,M}(x), \tag{25}$$

*where* $\big|R_{S,M}(x)\big|\le D_{S,M}\cdot\dfrac{1}{|x|^{M+2}}$.

**Proof.** From (15), we have $S(-\pi,\pi)\subset S_1(-\pi,\pi)$. Letting $t=\frac{1}{q^n x}$ in (23), we obtain that for any relatively-compact subsector $S$ of $S(-\pi,\pi)$, there is a constant $C_S=\sin\frac{\epsilon}{2}>0$, such that

$$\frac{1}{(q^n x+1)^{m+1}}=\sum_{k=m}^{M}\frac{(-1)^{k+m}k!\,q^{-n(k+1)}}{m!(k-m)!}\cdot\frac{1}{x^{k+1}}+\nu_{n,m,M}(x), \tag{26}$$

where

$$\big|\nu_{n,m,M}(x)\big|\le\frac{(1+C_S)^{M+1}q^{-n(M+2)}}{(C_S)^{m+1}}\cdot\frac{1}{|x|^{M+2}} \tag{27}$$

for any $n$ and any $x\in S$.

For any fixed $n$ and any $x\in S$, we can estimate the remainder $r_{n,M}(x)$ in Lemma 15 by replacing $t$ with $\frac{1}{q^n x}$ in (23) as follows:

$$\big|r_{n,M}(x)\big|\le\frac{\big(1+\frac{1}{C_S}\big)^{M+1}(M+1)!(-q^{M+1};q)_n q^{-n(M+2)}}{(C_S)^{M+3}(q;q)_n}\cdot\frac{1}{|x|^{M+2}}. \tag{28}$$

Substituting (26) and (28) into (21), we have

$$T_1(\alpha;q,x)=\sum_{n\ge0}\alpha^n I_n(x)=\sum_{n\ge0}\sum_{m=0}^{M}\sum_{k=m}^{M}\frac{(-1)^{k+m}(\alpha q^{-(k+1)})^n(q^{m+1};q)_n k!}{(q;q)_n(k-m)!}\cdot\frac{1}{x^{k+1}}+R_M(x), \tag{29}$$

where

$$R_M(x)=\sum_{n\ge0}\alpha^n r_{n,M}(x)+\sum_{n\ge0}\alpha^n\sum_{m=0}^{M}\nu_{n,m,M}(x). \tag{30}$$

Since (27) shows

$$\left|x^{M+2}\sum_{n\geq 0}\alpha^n\sum_{m=0}^{M}v_{n,m,M}(x)\right|\leq\sum_{n\geq 0}\sum_{m=0}^{M}\frac{(1+C_S)^{M+1}\left|\alpha q^{-(M+2)}\right|^n}{(C_S)^{m+1}},$$

the last series of (30) converges as $|\alpha|<q^{M+2}$. Together with (28), we have

$$\left|x^{M+2}R_M(x)\right|\leq\sum_{n\geq 0}\frac{\left(1+\frac{1}{C_S}\right)^{M+1}(M+1)!(-q^{M+1};q)_n\left|\alpha q^{-(M+2)}\right|^n}{(C_S)^{M+3}(q;q)_n}+\sum_{n\geq 0}\sum_{m=0}^{M}\frac{(1+C_S)^{M+1}\left|\alpha q^{-(M+2)}\right|^n}{(C_S)^{m+1}}$$
$$\triangleq D_{S,M}$$

for any $x\in S$. We replace $m$ with $k-m$ in (29) and simplify the coefficients of $\frac{1}{x^{k+1}}$ (for $m\in\mathbb{N}$), the proof is then completed by using (7). □

**Remark 18.** The functions $T_1$ and $T_2$ exhibit asymptotic behavior as described in formula (25), where the first $M$ terms match those of the function $G$. However, formula (25) is valid only when both the parameter $\alpha$ and the coefficient $D_{S,M}$ depend on $M$. Consequently, formula (25) does not represent a Gevrey asymptotic expansion.

### 4.3. *Asymptotic behaviors of the power series solutions*

To analyze the asymptotic behavior of the power series solution with the initial conditions, we first examine the relationship between the power series $F$ and the integral-sum functions $T_1$ and $T_2$. Let $\eta_1$ and $\eta_2$ be defined as in (19). We have the initial conditions $T_1(\alpha;q,0)=\eta_1$ and $T_2(\alpha;q,0)=\eta_2$. According to Theorem 7, the function $F(\alpha;q,c_0,x)$ satisfies equation (4) with $y(0)=c_0$. We obtain the following relationship:

$$F(\alpha;q,c_0,x)=T_1(\alpha;q,x)+(c_0-\eta_1)F_0(x), \tag{31}$$

for any $x\in S(-\pi,\pi)$. Similarly, we have: $F(\alpha;q,c_0,x)=T_2(\alpha;q,x)+(c_0-\eta_2)F_0(x)$.

Let $\mu\in\mathbb{C}$ be a fixed number such that $\alpha=q^{\mu}$. Due to the periodic nature of the exponential function in the complex plane, there are infinity numbers of $\mu_l=\mu+i\kappa l$ ($l\in\mathbb{Z}$ and $\kappa=-\frac{2\pi}{\ln q}$), such that $q^{\mu_l}=\alpha$. In the following, we introduce a $q$-periodic function to represent the relationship between $F$ and $T_1$ (or $T_2$).

**Lemma 19.** *For every fixed n, the function*

$$g_n(\mu;q,x)=\sum_{l\in\mathbb{Z}}\Gamma(n+\mu+i\kappa l)x^{-i\kappa l},\quad n\geq 0,$$

*is a solution of equation: $y(qx)=y(x)$. Furthermore, $g_n$ is called $q$-periodic function and is bounded as $x\to\infty$ in any direction* $\arg(x)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

**Proof.** Let $x_0$ be a parameter in $\left[e^{id},\frac{1}{q}e^{id}\right]$ with $d\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. For every solution of $y(qx)=y(x)$, we have $y(x_0)=y\left(\frac{1}{q}x_0\right)=\ldots$ By using the continuity and letting

$$C=\max_{x\in\left[e^{id},\frac{1}{q}e^{id}\right]}\left|g_n(\mu;q,x)\right|,$$

we obtain that $\left|g_n(\mu;q,x)\right|\leq C$ for $x=te^{id}$ with $t\geq 1$. □

**Theorem 20.** *Let $\alpha\notin q^{\mathbb{Z}_{\leq 0}}$. The relation*

$$F(\alpha;q,c_0,x)=T_1(\alpha,q,x)+C_\alpha x^{-\mu}\sum_{n\geq 0}\frac{(-1)^n q^{n(n+1)/2}g_n(\mu;q,x)}{(q;q)_n}x^{-n} \tag{32}$$

*holds for $x\in S\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, where*

$$C_\alpha=\frac{\kappa(c_0-\eta_1)(\alpha;q)_\infty}{2\pi(q;q)_\infty}.$$

**Proof.** One can complete the proof by using (31) and the connection formula in [18, p. 6, (1.7)]:

$$F_0(\alpha;q,x)=\frac{\kappa(\alpha;q)_\infty}{2\pi(q;q)_\infty}\sum_{n\ge0}\frac{(-1)^n q^{n(n+1)/2}g_n(\mu;q,x)}{(q;q)_n}x^{-\mu-n}$$

for $\arg(x)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. □

**Remark 21.** Equation (32) still holds after replacing $T_1$ with $T_2$, where $C_\alpha=\frac{\kappa(c_0-\eta_2)(\alpha;q)_\infty}{2\pi(q;q)_\infty}$.

From Theorems 17 and 20, we get the asymptotic behavior of $F(\alpha;q,c_0,x)$ as follows.

**Corollary 22.** *Assume that* $|\alpha|<q^2$. *Then:*

(i) $F(\alpha;q,c_0,x)=\frac{q}{q-\alpha}\cdot x^{-1}+o(x^{-1})$ *as* $x\to\infty$ *in any relatively-compact subsector of* $S\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$;

(ii) $F(\alpha;q,c_0,x)=(\alpha;q)_\infty e^{-x}+o(e^{-x})$ *as* $x\to\infty$ *in any relatively-compact subsector of* $S\left(\frac{\pi}{2},\pi\right)\cup S\left(\pi,\frac{3\pi}{2}\right)$.

**Proof.** **(i).** For $x$ in the right-half plane $\mathbb{C}^+$. From $q^\mu=\alpha$, we have

$$|x^{-\mu}|=\left|e^{-\mathrm{Re}(\mu)\log|x|+\mathrm{Im}(\mu)\arg x+i[\mathrm{Re}(\mu)\arg x-\mathrm{Im}(\mu)\log|x|]}\right|=e^{\mathrm{Im}(\mu)\arg x}|x|^{-\mathrm{Re}(\mu)}.$$

(1) If $c_0\ne\eta_1,\eta_2$, then we need to compare the values of $-\mathrm{Re}(\mu)$ and $-1$. Since $|\alpha|<q^2$, we have $-\mathrm{Re}(\mu)=-\ln|\alpha|/\ln q<-1$. From (32) and Theorem 17 ($M=0$), we find that the function $F$ exhibits the same asymptotic behavior as $T_1$ and $T_2$ for $\arg(x)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Therefore,

$$F(\alpha;q,c_0,x)=x^{-1}\left[\frac{1}{1-\alpha q^{-1}}+O(x^{-\mathrm{Re}(\mu)+1})\right]$$

as $x\to\infty$ in any relatively-compact subsector of $\mathbb{C}^+$.

(2) If $c_0=\eta_1$ or $c_0=\eta_2$, then $C_\alpha=0$. The function $F$ still has the same asymptotic behavior as $T_1$ and $T_2$.

**(ii).** For $x\in\mathbb{C}^-$, according to [18, p. 8, Theorem 2.1], we obtain that $F_0(\alpha;q,x)=(\alpha;q)_\infty e^{-x}+o(e^{-x})$ as $x\to\infty$. One can obtain the result from the analytic region of $T_1$ (or $T_2$). □

For equation $y'(x)=\alpha y(qx)-y(x)+\frac{1}{c+x}$ with $\mathrm{Re}(c)<0$, by considering $-y(x)$ instead of $y(x)$, we only need to study equation

$$y'(x)=\alpha y(qx)-y(x)+\frac{1}{\widehat{c}-x},\tag{33}$$

where $\widehat{c}=-c$ and thus $\mathrm{Re}(\widehat{c})>0$. Similar to the analysis in Sections 3.1 and 3.2, the series solutions at 0 and $\infty$ of equation (33) have a similar form as the solutions $F$ and $G$ in Theorems 7 and 9, just by letting

$$F_1(\alpha;q,x)=\sum_{n\ge1}\sum_{k=1}^{n}\frac{(-1)^{n-1}(k-1)!(\alpha q^k;q)_{n-k}}{\widehat{c}^k n!}x^n$$

and

$$G(\alpha;q,x)=\sum_{n\ge0}\sum_{k=0}^{n}\frac{(-\widehat{c})^k n!(\alpha q^{-1};q^{-1})_k}{k!(\alpha q^{-1};q^{-1})_{n+1}}x^{-n-1},$$

respectively. Then, we can obtain an analytic solution of (33) for $\arg(x-\widehat{c})\in(0,2\pi)$, which is an integral-sum function and similar to the form of $T(\alpha;q,x)$. Finally, we derive the asymptotic behavior of the series solution of equation (33) with the given initial condition (denoted as $\widehat{F}$). Assume that $|\alpha|<q^2$. Then, for $x\in\mathbb{C}^+$, $\widehat{F}(\alpha;q,c_0,x)=\frac{q}{\alpha-q}\cdot x^{-1}+o(x^{-1})$ as $x\to\infty$ in any relatively-compact subsector of $S\left(-\frac{\pi}{2},0\right)\cup S\left(0,\frac{\pi}{2}\right)$; for $x\in\mathbb{C}^-$, $\widehat{F}(\alpha;q,c_0,x)=(\alpha;q)_\infty e^{-x}+o(e^{-x})$ as $x\to\infty$ in any relatively-compact subsector of $S\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$.

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