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Regularity of semi-valuation rings and homotopy invariance of algebraic K-theory

Régularité d'anneaux de semi-valuation et invariance par homotopie pour la K-théorie algébrique

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Abstract. We show that the algebraic K-theory of semi-valuation rings with stably coherent regular semi-fraction ring satisfies homotopy invariance. Moreover, we show that these rings are regular if their valuation is non-trivial. Thus they yield examples of regular rings which are not homotopy invariant for algebraic K-theory. On the other hand, they are not necessarily coherent, so that they provide a class of possibly non-coherent examples for homotopy invariance of algebraic K-theory. As an application, we show that Temkin's relative Riemann–Zariski spaces also satisfy homotopy invariance for K-theory under some finiteness assumption.

Résumé. Nous montrons que la K-théorie algébrique des anneaux de semi-valuation avec un anneau de semi-fractions régulier et stablement cohérent satisfait à l'invariance par homotopie. De plus, nous montrons que ces anneaux sont réguliers si leur valuation est non-triviale. Ainsi, ils donnent des exemples d'anneaux réguliers qui ne sont pas invariants par homotopie pour la K-théorie algébrique. D'autre part, ils ne sont pas nécessairement cohérents, de sorte qu'ils fournissent une classe d'exemples éventuellement non cohérents pour l'invariance d'homotopie de la K-théorie algébrique. Comme application, nous montrons que les espaces de Riemann–Zariski relatifs de Temkin satisfont également l'invariance d'homotopie pour la K-théorie sous certaines hypothèses de finitude.

Keywords. K-theory, regularity, homotopy invariance, semi-valuation rings, Riemann–Zariski spaces.

Mots-clés. K-théorie, regularité, invariance par homotopie, anneaux de semi-valuation, espaces de Riemann–Zariski.

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1. Introduction

Algebraic K-theory of rings is not $\mathbf{A}^1$-invariant in general, but homotopy invariance is known for some classes of rings, for instance:

(1) stably coherent regular rings (e.g. noetherian regular, valuation, or Prüfer rings);
(2) perfect $\mathbf{F}_p$-algebras;
(3) certain rings arising from C^*-algebras;
(4) certain rings of continuous functions.

Here we use a very general notion of regularity generalising the usual one in the noetherian context, see Definition 7. In this note, we expand this list by certain semi-valuation rings (Definition 1) which can be non-coherent (Lemma 16 and Lemma 18).

Theorem (Corollary 23). *Let $(A^+, \mathfrak{p})$ be a semi-valuation ring whose semi-fraction ring $A = A^+_{\mathfrak{p}}$ is stably coherent and regular. Then the canonical maps*

(1) $\mathrm{K}_{\geq 0}(A^+) \longrightarrow \mathrm{K}(A^+)$,
(2) $\mathrm{K}(A^+) \longrightarrow \mathrm{K}\big(A^+[t_1,\ldots,t_k]\big)$, *and*
(3) $\mathrm{K}(A^+) \longrightarrow \mathrm{KH}(A^+)$

are equivalences. Here, $\mathrm{K}(A^+)$ *denotes the non-connective algebraic K-theory spectrum of A^+ and* $\mathrm{KH}(A^+)$ *its homotopy invariant K-theory spectrum.*

The relationship between regularity of rings and homotopy invariance of their algebraic K-theory has been studied since the origins of K-theory. The case of noetherian regular rings has been proved by Quillen [27] and the same proof works more generally for stably coherent regular rings, see Swan [30]. Another proof for stably coherent regular rings was given by Waldhausen [34, Theorem 3 & Theorem 4]. For valuation rings there are also proofs by Kelly–Morrow [19, Theorem 3.2] and Kerz–Strunk–Tamme [22, Lemma 4.3]. Banerjee–Sadhu showed that Prüfer domains are stably coherent and regular [3]. Recently, Antieau–Mathew–Morrow showed homotopy invariance for perfect $\mathbf{F}_p$-algebras [1, Proposition 5.1]. For C^*-algebras, Higson has shown the statement for stable ones [17, Section 6] and Cortiñas–Thom for rings $A \otimes_{\mathbf{C}} I$ with an H-unital C^*-algebra A and a sub-harmonic ideal I satisfying $I = [I, I]$ [8, Theorem 8.2.5] as well as for rings $S \otimes_{\mathbf{C}} C$ with a smooth $\mathbf{C}$-algebra S and a commutative C^*-algebra C [9, Theorem 1.5]. Recently, Aoki showed the case of continuous functions on compact Hausdorff spaces with values in local division rings [2]. Of course, this list of references is not exhaustive.

So far, it has been an open problem whether the K-theory of a non-coherent regular ring satisfies homotopy invariance, e.g. consider [6, p. 8198]. Showing that semi-valuation rings with non-trivial valuation are regular (Lemma 12) we add to this line of research the following.

Remark (Remark 24). Certain semi-valuation rings give examples of non-coherent regular rings whose algebraic K-theory does not satisfy homotopy invariance.

By private communication, the author also learned of another such example found by Luca Passolunghi.

Semi-valuation rings have been introduced by Temkin [32] and they occur as the stalks of his relative Riemann–Zariski spaces $\mathrm{RZ}_Y(X)$ which are locally ringed spaces associated with any separated morphism $f\colon Y \to X$ between quasi-compact and quasi-separated (qcqs) schemes. Furthermore, semi-valuation rings are important since they are the stalks of discretely ringed adic spaces, see Remark 5. Exploiting that algebraic K-theory commutes with the formation of stalks, we deduce the following statement for the K-theory sheaves K^{RZ} and $\mathrm{KH}^{\mathrm{RZ}}$ on this space, see Definition 31 for a precise definition.

Theorem (Proposition 34, Corollary 35). *Let $f\colon Y \to X$ be a separated morphism between qcqs schemes. Assume that Y is of finite dimension, that all its stalks are stably coherent regular rings,* *and that the morphism $f\colon Y \to X$ admits a compactification (e.g. f is of finite type). Then the canonical maps*

(1) $\mathrm{K}^{\mathrm{RZ}}(-) \longrightarrow \mathrm{K}^{\mathrm{RZ}}\big((-)[t_1,\ldots,t_k]\big)$ *and*
(2) $\mathrm{K}^{\mathrm{RZ}}(-) \longrightarrow \mathrm{KH}^{\mathrm{RZ}}(-)$

are equivalences of spectrum-valued sheaves on the topological space $\mathrm{RZ}_Y(X)$.

In the special setting that the morphism $Y \to X$ is the immersion of a regular dense open subscheme Y into a divisorial noetherian scheme X and assuming $k = 1$, this specialises to the author's previous result for the K-theory of admissible Zariski–Riemann spaces $\langle X\rangle_Y$ [11], see Corollary 35 and Remark 36.

Notation. Discrete categories are denoted by upright letters whereas genuine ∞-categories are denoted by bold letters. For a ring R we denote by $\mathrm{K}(R)$ its non-connective algebraic K-theory spectrum à la Blumberg–Gepner–Tabuada [5, Section 9.1] which is an object of the ∞-category **Sp** of spectra [25, p. 1.4.3.1]; its associated object in the homotopy category of spectra is equivalent to the K-theory spectrum of Thomason–Trobaugh [33, Definition 3.1]. In particular, the connective part $\mathrm{K}_{\geq 0}(R)$ is equivalent to Quillen's K-theory [27], combine [33, Proposition 3.10] with [5, Section 7.2].

2. Semi-valuation rings

For the convenience of the reader we recollect the definition and some basic facts about semi-valuation rings. All this is due to Temkin [32]. Let us repeat some relevant terminology: a *valuation* on a ring R is a map $|-|\colon R \to \Gamma \cup \{0\}$ for a totally ordered multiplicative abelian group Γ such that $|1| = 1$, $|xy| = |x|\cdot|y|$, and $|x+y| \leq \max\{|x|,|y|\}$ for all $x, y \in R$ with the convention that $0\cdot\gamma = 0$ and $0 < \gamma$ for all $\gamma \in \Gamma$. It follows that $|0| = 0$ and that the support $\mathrm{supp}\big(|-|\big) := |-|^{-1}(0)$ is a prime ideal. Two valuations $|-|_1$ and $|-|_2$ on R are called *equivalent* if for all $x, y \in R$ the conditions $|x|_1 \leq |y|_1$ and $|x|_2 \leq |y|_2$ are equivalent.

Definition 1 (Temkin [32, Section 2.1]). *A* semi-valuation ring *is a pair $\big(A^+, |-|\big)$ consisting of a ring A^+ and a valuation $|-|\colon A^+ \to \Gamma \cup \{0\}$ such that*

(1) *every zero divisor of A^+ lies in the kernel of $|-|$, and*
(2) *for all $x, y \in A^+$ with $|x| \leq |y| \neq 0$ one has $y \,\big|\, x$.*

If A^+ is a semi-valuation ring and $\mathfrak{p} := \mathrm{supp}\big(|-|\big)$, then the local ring $A^+_{\mathfrak{p}}$ is called its semi-fraction ring *and the ring $A^+/\mathfrak{p}$ is a valuation ring. We call $\mathfrak{p}$ the* valuative ideal *of A^+.*

We have the following characterisation of semi-valuation rings.

Lemma 2. *For a semi-valuation ring A^+ with valuative ideal $\mathfrak{p}$, semi-fraction ring $A := A^+_{\mathfrak{p}}$, valuation ring $V := A^+/\mathfrak{p}$, and residue field $k := A^+_{\mathfrak{p}}/\mathfrak{p}$ the induced square*

$$\begin{array}{ccc} A^+ & \longrightarrow & A \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ V & \longrightarrow & k \end{array} \tag{$\square$}$$

is a Milnor square (i.e. a bicartesian square of rings where two parallel arrows are surjective). In particular, A^+ is a local ring with maximal ideal $\mathfrak{m}^+ := \pi^{-1}(\mathfrak{m}_V)$.

Consequently, the following data are equivalent:

(1) *a semi-valuation ring $\big(A^+, |-|\big)$ up to equivalence of the valuation $|-|$;*
(2) *a local ring $(A^+, \mathfrak{m}^+)$ and a prime ideal $\mathfrak{p} \subset A^+$ such that $A^+/\mathfrak{p}$ is a valuation ring;*
(3) *a local ring $(A, \mathfrak{p})$ and a valuation ring $V \subseteq A/\mathfrak{p}$.*

Proof. The implications (1) ⇒ (2) ⇔ (3) are clear. Assuming (2), we obtain a valuation $|-|\colon A^+ \xrightarrow{\pi} V \to V^\times/k^\times \cup \{0\}$ since V is assumed to be a valuation ring. We will check the conditions (1) and (2) of Definition 1. As V is a domain, we get (1). Now let $x, y \in A^+$ with $|x| \leq |y| \neq 0$, hence $y \notin \mathfrak{p}$. Since the morphism $A \to k$ reflects units, we get that $y \in A^\times$ and that for every $p \in \mathfrak{p}$ the ratio $\frac{p}{y} \in A$ lies in A^+. By standard properties of valuation rings we find an element $z \in A^+$ such that $\pi(x) = \pi(y) \cdot \pi(z)$ in V, hence $x = yz + p$ for a suitable element $p \in \mathfrak{p}$. It follows that $x = yz + p = y\big(z + \frac{p}{y}\big)$ in A^+ as desired. □

Notation 3. In the sequel, we refer to a semi-valuation ring as a pair $(A^+, \mathfrak{p})$ as in Lemma 2, even though the valuation is only defined up to equivalence. We always assume the square (□) to be implicitly defined.

Lemma 4. *Let $(A^+, \mathfrak{p})$ be a semi-valuation ring with a non-trivial valuation (equivalently, its valuation ring $V = A^+/\mathfrak{p}$ is not a field). Then* $\operatorname{colim}_{x \in \mathfrak{m}^+ \setminus \mathfrak{p}} (x)$ *is a* filtered *colimit and its canonical morphism to* $\mathfrak{m}^+$ *is an isomorphism of ideals of* A^+.

Proof. Since the valuation is assumed to be non-trivial, the set $\mathfrak{m}^+ \setminus \mathfrak{p}$ is non-empty. According to condition (2) in Definition 1, the elements of I are totally ordered with respect to divisibility, hence the colimit is filtered. This implies the claim since every module is the filtered colimit of its finitely generated submodules. □

Remark 5. Semi-valuation rings are precisely the rings A^+ occuring in *local Huber pairs* (A, A^+) as defined by Hübner–Schmidt [18, pp. 407f.] which are the local rings for discretely ringed adic spaces [18, 10.9(i)].

3. Coherence and regularity of semi-valuation rings

In this section, we recall the notions of coherence and regularity for rings. For an elaborate treatment we refer to the book of Glaz [15]. Afterwards, we will examine these notions for semi-valuation rings.

Definition 6. *Let A be a ring. An A-module M is said to be...*

(1) coherent *if every finitely generated submodule is finitely presented. The ring A is said to be coherent if it is a coherent A-module.*
(2) finitely n-presented *for $n \in \mathbf{N}$ if there exists an exact sequence*
$$F_n \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$
with finitely generated free A-modules $F_0, \ldots, F_n$.
(3) pseudo-coherent *if there exists an exact sequence*
$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$
with finitely generated projective A-modules $(P_i)_{i \in \mathbf{N}}$.

It follows immediately that a ring is noetherian if and only if every finitely generated module is coherent and that a ring is coherent if and only if every finitely presented module is coherent.

For regularity, we have several notions appearing in the literature which are slightly different, see Remark 9 below.

Definition 7. *Let A be a ring and let $n \in \mathbf{N}$. We say that A is...*

(1) regular *if every pseudo-coherent module has finite projective dimension;*
(2) n-regular *if every finitely n-presented A-module has finite projective dimension;*
(3) uniformly regular *if the projective dimensions of all pseudo-coherent modules are uniformly bounded;*

(4) uniformly n-regular *if the projective dimensions of all finitely n-presented modules are uniformly bounded;*
(5) Glaz-regular *if every finitely generated ideal of A has finite projective dimension.*

Proposition 8. *For any ring, we have the following implications:*

(1) *for all $n \geq 0$: (uniformly) n-regular $\Rightarrow$ (uniformly) $(n+1)$-regular;*
(2) *for all $n \geq 0$: uniformly n-regular $\Rightarrow$ n-regular.*

For local rings and any $n \geq 0$, we have the implication:

(3) *n-regular $\Rightarrow$ regular.*

For a coherent ring A, the following are equivalent:

(4) *A is regular;*
(5) *A is n-regular for some $n \geq 1$;*
(6) *A is n-regular for all $n \geq 1$;*
(7) *A is Glaz-regular.*

Proof. The implications of (1) and (2) hold by design. The implication in (3) follows since projective modules over local rings are free. The implication (6) ⇒ (5) is trivial. If A is coherent, every finitely presented A-module is pseudo-coherent and vice versa, hence (4) ⇔ (5). The equivalence with (7) goes by induction on the number of generators [15, Theorem 6.2.1]. □

Remark 9. The notion "1-regular" is called "regular" by Gersten [14, Definition 1.3]. The notion "Glaz-regular" is called "regular" by Glaz [15, Chapter 6, Section 2]. For a coherent ring, these notions agree with Waldhausen's notion "regular coherent" [34, p. 138]. Antieau–Mathew–Morrow [1, Section 2] call a ring "weakly regular" if it has finite flat dimension (for them a regular ring is noetherian). For coherent rings, the notion "weakly regular" is equivalent to the notion "uniformly 1-regular" and it is *stronger* than the notions used by Gersten and Glaz. For arbitrary rings, the notion "regular" from Definition 7 seems to be most meaningful (at least to the author).

Regularity of semi-valuation rings

Lemma 10. *Let $(R,\mathfrak{m})$ be a local ring and M an R-module.*

(1) *If M is finitely presented and $\mathrm{Tor}_1^R(M,R/\mathfrak{m})=0$, then M is free.*
(2) *If M is finitely $(n+1)$-presented and $\mathrm{Tor}_{n+1}^R(M,R/\mathfrak{m})=0$ for some $n \geq 0$, then M has projective dimension $\leq n$.*

Proof. We follow the standard proof [29, Chapter IV, Theorem 8] where the noetherian hypothesis is stated, but not needed.

(1). Let $x_1,\ldots,x_n \in M$ such that their images form a $R/\mathfrak{m}$-basis of $M/\mathfrak{m}M$ so that the morphism $\varphi\colon R^n \to M$, $e_i \mapsto x_i$, induces an isomorphism $\overline{\varphi}\colon (R/\mathfrak{m})^n \to M/\mathfrak{m}M$. By Nakayama's lemma we get an exact sequence $0 \to N \to R^n \xrightarrow{\varphi} M \to 0$, inducing an exact sequence

$$0 = \mathrm{Tor}_1^R(M,R/\mathfrak{m}) \longrightarrow N/\mathfrak{m}N \longrightarrow (R/\mathfrak{m})^n \xrightarrow{\cong} M/\mathfrak{m}M \longrightarrow 0$$

so that $N/\mathfrak{m}N = 0$. If M is finitely presented, then N is finitely generated, hence $N=0$ by Nakayama's lemma.

(2). Let

$$F_{n+1} \xrightarrow{f_{n+1}} F_n \xrightarrow{f_n} \cdots \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \longrightarrow 0$$

be an exact sequence where $F_{n+1},\ldots,F_0$ are finitely generated free. Setting $K_i := \ker(f_i)$ we get that

$$\mathrm{Tor}_1^R(K_{n-1},R/\mathfrak{m}) = \mathrm{Tor}_2^R(K_{n-2},R/\mathfrak{m}) = \ldots = \mathrm{Tor}_n^R(K_0,R/\mathfrak{m}) = \mathrm{Tor}_{n+1}^R(M,R/\mathfrak{m}) = 0$$

by assumption. Since M is finitely $(n+1)$-presented, K_{n-1} is finitely presented, hence free by (1), so that M has projective dimension $\leq n$. □

Example 11. Let $A := \mathbf{Q}_p[\![X,Y]\!]$ and let A^+ be defined to be the pullback in the Milnor square

$$\begin{array}{ccc} A^+ & \longrightarrow & A \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_{0,0}} \\ \mathbf{Z}_p & \longrightarrow & \mathbf{Q}_p. \end{array}$$

Then A^+ has maximal ideal $\mathfrak{m}^+ = \pi^{-1}(p\mathbf{Z}_p) = (p,X,Y) = (p)$ as $X = p\cdot\frac{X}{p}, Y = p\cdot\frac{Y}{p} \in (p)$. Thus $\mathrm{Tor}_2^{A^+}(M, A^+/\mathfrak{m}^+) = 0$ for any A^+-module M, so that every finitely 2-presented A^+-module has projective dimension ≤ 1 by Lemma 10. Thus A^+ is 2-regular.

We can generalise this example to any semi-valuation ring with non-trivial valuation.

Lemma 12. *Let $(A^+,\mathfrak{p})$ be a semi-valuation with non-trivial valuation. Then every finitely 2-presented A^+-module has projective dimension ≤ 1. In particular, the ring A^+ is 2-regular.*

Proof. By Lemma 4, we can write $\mathfrak{m} = \mathrm{colim}_{x\in I}(x)$ as a filtered colimit with $I = \mathfrak{m}^+ \setminus \mathfrak{p} \neq \emptyset$. According to condition (1) in Definition 1, the sequence $0 \to A^+ \xrightarrow{\cdot x} A^+ \to A^+/(x) \to 0$ is exact for every $x \in I$. Since tensor products are cocontinuous in each variable and since filtered colimits are exact, we get that

$$\mathrm{Tor}_2^{A^+}(M, A^+/\mathfrak{m}^+) = \operatorname*{colim}_{x\in I} \mathrm{Tor}_2^{A^+}\big(M, A/(x)\big) = 0$$

for any A^+-module M. By Lemma 10, every finitely 2-presented A^+-module has projective dimension ≤ 1 as desired. □

Regularity in Milnor squares

More generally than in the situation of semi-valuation rings, one can wonder under which circumstances pullback rings in Milnor squares are regular. Let

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A' & \xrightarrow{f'} & B' \end{array} \tag{M}$$

be a Milnor square (i.e. a cartesian square of rings such that p and q are surjective). It follows that the square (M) is also a pushout square, i.e. $B' \cong B \otimes_A A'$.

Example 13. Let k be a field of characteristic $\neq 2$ and consider the node $A = k[X,Y]/(Y^2 - X^3 - X^2)$. It fits into a Milnor square (M) with $B = k[T]$, f being the normalisation (i.e. $X \mapsto T^2 - 1$ and $Y \mapsto T(T^2-1)$), and $A' = A/(X,Y)$ being the origin. Then A', B, and B' are regular noetherian rings, but A is not regular.

Question 14. *Given a Milnor square* (M) *such that the morphism $A \xrightarrow{f} B$ is of finite Tor-dimension and assuming that the rings A', B, and B' all are regular. Is then the ring A regular?*

A helpful result for answering the question might be the following statement.

Lemma 15. *Let*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

be a pullback square of $\mathbf{E}_1$-ring spectra. Then an A-module M is perfect if and only if the base changes $M \otimes^{\mathrm{L}}_A B$ and $M \otimes^{\mathrm{L}}_A A'$ both are perfect.

Proof. If an A-module is perfect, this also holds true for any base change. For the other implication we use that there exists a commutative square

$$\begin{array}{ccc} \mathbf{Perf}(A) & \hookrightarrow & \mathbf{Perf}(B) \overrightarrow{\times}_{\mathbf{Perf}(B')} \mathbf{Perf}(A') \\ \downarrow & & \downarrow \\ \mathbf{RMod}(A) & \hookrightarrow & \mathbf{RMod}(B) \overrightarrow{\times}_{\mathbf{RMod}(B')} \mathbf{RMod}(A') \end{array}$$

where $\mathbf{RMod}(-)$ denotes the presentable, stable ∞-category of (derived) right modules, $\mathbf{Perf}(-)$ its full subcategory of perfect modules (which are precisely the compact objects), and "$\overrightarrow{\times}$" the lax pullback [31, Definition 5]. The horizontal functors (which are induced by the base change functors) are fully faithful [23, p. 1.7] and the left vertical functor is the inclusion of compact objects by definition. The right vertical functor identifies also with the inclusion of compact objects [31, Proposition 13]. Now the claim follows since base change, and hence the horizontal functors preserve filtered colimits [31, Lemma 8(iii)]. □

Non-coherence of semi-valuation rings

Lemma 16. *Let $(A^+, \mathfrak{p})$ be a semi-valuation ring such that $A = A^+_{\mathfrak{p}}$ is not finitely generated as a module over A^+ and such that there exists a regular sequence X, Y in $\mathfrak{p}$. Then the ring A^+ is not coherent.*

Proof. Consider the homomorphism $\varphi\colon A^+ \times A^+ \to A^+$, $(f, g) \mapsto fX - gY$. We see immediately that $A\cdot(Y, X) \subseteq \ker(\varphi)$. On the other hand, for $(f, g) \in \ker(\varphi)$ we get $gY = 0$ in $A^+/(X)$ by regularity so that $g \in X \cdot A^+$. By symmetry, $f \in Y \cdot A^+$. Setting $g = g'X$ and $f = f'Y$ (for suitable elements f' and g' in A^+) we get $0 = (f' - g')XY$, hence $f' = g'$ so that $(f, g) = f'(Y, X) \in A \cdot (Y, X)$. □

Example 17. A concrete instance for the setting of Lemma 16 is given by the semi-valuation ring of Example 11, i.e. $A = \mathbf{Q}_p[\![X, Y]\!]$ and $A^+ = \{f \in A \mid f(0,0) \in \mathbf{Z}_p\}$ (and X and Y as themselves).

Lemma 18. *Let $(A^+, \mathfrak{p})$ be a semi-valuation ring with non-trivial valuation. If $A = A^+_{\mathfrak{p}}$ is coherent and not regular, then the ring A^+ is not coherent.*

Proof. We know that the ring A^+ is 2-regular by Lemma 12. If it was also coherent, then it would be Glaz-regular by Proposition 8, hence A would be Glaz-regular [15, p. 6.2.3]. □

Example 19. An instance of the setting of Lemma 18 comes from starting with any coherent local ring $(A, \mathfrak{p})$ that is not regular and a non-trivial valuation on its residue field $k = A/\mathfrak{p}$, e.g. $A = \mathbf{Q}_p[X, Y]_{(X,Y)}/(Y^2 - X^3)$ with the p-adic valuation on $k = \mathbf{Q}_p$, so that $V = \mathbf{Z}_p$.

Remark 20. Assuming that the ideal $\mathfrak{p} \subset A^+$ is a *flat* A^+-module, then A^+ is a coherent ring provided that A is a coherent ring [15, Theorem 5.1.3].

4. K-theory and G-theory of semi-valuation rings

Theorem 21. *For a semi-valuation ring* $(A^+,\mathfrak{p})$ *with semi-fraction ring* $A = A^+_{\mathfrak{p}}$*, valuation ring* $V = A^+/\mathfrak{p}$*, and field of fractions* $k = A^+_{\mathfrak{p}}/\mathfrak{p}$ *the induced square*

$$\begin{array}{ccc} \mathrm{K}\big(A^+[t_1,\dots,t_r]\big) & \longrightarrow & \mathrm{K}\big(A[t_1,\dots,t_r]\big) \\ \downarrow & & \downarrow \\ \mathrm{K}\big(V[t_1,\dots,t_r]\big) & \longrightarrow & \mathrm{K}\big(k[t_1,\dots,t_r]\big) \end{array}$$

of non-connective algebraic K-theory spectra is cartesian for every $r \in \mathbf{N}$.

Proof. Let $\mathbf{F}$ be the prime field of k so that the valuation ring $V \cap \mathbf{F}$ has rank ≤ 1. Let I be the set of all subextensions $\ell \subset k$ that have finite transcendence degree over $\mathbf{F}$. Note that $k \cong \operatorname{colim}_{\ell \in I} \ell$ within the category of rings. Now let $\ell \in I$. By the "dimension inequality" [12, Theorem 3.4.3, Corollary 3.4.4], the valuation ring $V_\ell := V \cap \ell$ has finite rank. Hence V is a filtered colimit of valuation rings of finite rank. Set A_ℓ to be the preimage of ℓ in A which is a local ring with residue field ℓ. Hence the preimage A^+_ℓ of V in A_ℓ is a semi-valuation ring and the square (□) is the filtered colimit of its restrictions along $\ell \hookrightarrow k$ for all $\ell \in I$.

Since K-theory commutes with filtered colimits of rings we may assume that V has finite rank, hence it is a microbial valuation ring (i.e. has a prime ideal of height 1). In this case, there exists an element $s \in A^+$ such that $V[\bar{s}^{-1}] = K$ where $\bar{s} = \pi(s)$ for the projection $\pi\colon A^+ \to V$. Then one checks easily that $s \in A^\times$, $A^+[s^{-1}] = A$, and $\mathfrak{p} \subseteq s^n \cdot A^+$ for all $n \in \mathbf{N}$. Hence the map $\pi\colon A^+ \to V$ is an analytic isomorphism along $S = \{s^n \mid n \in \mathbf{N}\}$ and the same holds true for the induced maps $A^+[t_1,\dots,t_r] \to V[t_1,\dots,t_r]$ for all $r \in \mathbf{N}$. By Weibel's analytic isomorphism theorem [35, Theorem 1.3] the square of Theorem 21 is cartesian. Note that *every* Milnor square induces a cartesian square on non-positive K-theory $\mathrm{K}_{\leq 0}$ [4, Chapter XII, Theorem (8.3), p. 677]. □

Corollary 22. *Let* $(A^+,\mathfrak{p})$ *be a semi-valuation ring. Then for* $n < 0$ *the canonical morphism* $\mathrm{K}_n(A^+) \to \mathrm{K}_n(A)$ *is an isomorphism. If* $A = A^+_{\mathfrak{p}}$ *is noetherian of finite dimension* d*, then*

(1) $\mathrm{K}_n(A^+) = 0$ *for* $n < -\dim(A)$*,*
(2) $\mathrm{K}_n(A^+) \xrightarrow{\simeq} \mathrm{K}_n\big(A^+[t_1,\dots,t_k]\big)$ *for* $n \leq -d$ *and* $k \geq 0$*, and*
(3) $\mathrm{K}_{-d}(A^+) \cong \mathrm{H}^d_{\mathrm{cdh}}(\mathrm{Spec}(A),\mathbf{Z})$.

Proof. This follows from Theorem 21, the vanishing of negative K-theory of polynomial rings over valuation rings, and the corresponding statements for the ring A which hold due to a result of Kerz–Strunk–Tamme [20, Theorem B & Theorem D]. □

Corollary 23. *Let* $(A^+,\mathfrak{p})$ *be a semi-valuation ring whose semi-fraction ring* $A = A^+_{\mathfrak{p}}$ *is stably coherent and regular. Then the canonical maps*

(1) $\mathrm{K}_{\geq 0}(A^+) \longrightarrow \mathrm{K}(A^+)$,
(2) $\mathrm{K}(A^+) \longrightarrow \mathrm{K}\big(A^+[t_1,\dots,t_k]\big)$, *and*
(3) $\mathrm{K}(A^+) \longrightarrow \mathrm{KH}(A^+)$

are equivalences.

Proof. The claimed equivalences (1)–(3) follow formally from Theorem 21 and by the known corresponding statements for the K-theory of stably coherent regular rings [30] and valuation rings [34], see also Kelly–Morrow [19, Theorem 3.3]. □

Remark 24 (Regularity does not imply K-regularity). We say that a ring A is *K-regular* if for every $k \geq 1$ the canonical map $\mathrm{K}(A) \to \mathrm{K}\big(A[t_1,\dots,t_k]\big)$ is an equivalence of spectra. From Theorem 21 we deduce that a semi-valuation ring A^+ is K-regular if and only if its semi-fraction ring A is K-regular. Given a coherent local ring A which is *not* K-regular together with a non-trivial valuation on its residue field (e.g. the ring A in Example 19), then the associated semi-valuation ring A^+ is a non-coherent (Lemma 18) and regular (Lemma 12) ring which is *not* K-regular.

Remark 25 (G-theory). Let A be a ring. Denote by $\mathrm{PCoh}(A)$ the full subcategory of $\mathrm{Mod}(A)$ spanned by pseudo-coherent modules (Definition 6); it is an exact subcategory [36, p. II.7.1.4]. The *G-theory* of A is defined as

$$\mathrm{G}(A) := \mathrm{K}\big(\mathrm{PCoh}(A)\big)$$

where K denotes Schlichting's *non-connective* K-theory for exact categories [28]; cf. Thomason–Trobaugh [33, p. 3.11.1] and Weibel's *K-book* [36, p. V.2.7.4]; note that this can also be realised as the K-theory of a stable ∞-category, see [16, Section 8]. If a ring A is regular, then the canonical map $\mathrm{K}(A) \to \mathrm{G}(A)$ is an equivalence, since—by definition of regularity—the inclusion $\mathrm{Vec}(A) \hookrightarrow \mathrm{PCoh}(A)$ satisfies the conditions of the resolution theorem [36, p. V.3.1].

5. K-theory of relative Riemann–Zariski spaces

In this section we generalise the results from the author's previous article on the K-theory of admissible Zariski–Riemann spaces [11] to the setting of Temkin's relative Riemann–Zariski spaces [32];[1] the former are defined for the inclusion of an open subscheme whereas the latter are defined for an arbitrary separated morphism. The statements are reduced to the stalks of these spaces which are semi-valuation rings so that we can use the results from Section 4.

Notation. In this section let $f\colon Y \to X$ be a separated morphism between quasi-compact and quasi-separated schemes.

Definition 26 (Temkin [32, Section 2.1]). *A* **Y**-modification *of X is a factorisation $Y \xrightarrow{g_i} X_i \xrightarrow{f_i} X$ of f into a schematically dominant morphism g_i and a proper morphism f_i. We denote by $\mathrm{Mdf}_Y(X)$ the category of Y-modifications of X together with compatible morphisms. The* relative Riemann–Zariski space $\mathrm{RZ}_Y(X)$ *is the limit $\lim_i X_i$ within the category of locally ringed spaces, indexed by the cofiltered category $\mathrm{Mdf}_Y(X)$.*

Lemma 27. *Let $(A^+,\mathfrak{p})$ be a semi-valuation ring with semi-fraction ring $A = A^+_{\mathfrak{p}}$. Then the canonical projection*

$$\mathrm{RZ}_{\mathrm{Spec}(A)}\big(\mathrm{Spec}(A^+)\big) \longrightarrow \mathrm{Spec}(A^+)$$

is an isomorphism.

Proof. We have a bicartesian square

$$\begin{array}{ccc} \mathrm{Spec}(A^+_{\mathfrak{p}}/\mathfrak{p}) & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A^+/\mathfrak{p}) & \longrightarrow & \mathrm{Spec}(A^+). \end{array}$$

Every $\mathrm{Spec}(A)$-modification $X \to \mathrm{Spec}(A^+)$ yields a lift $\mathrm{Spec}(A^+/\mathfrak{p}) \to X$ by the valuative criterion of properness. Hence we get a section $\mathrm{Spec}(A^+) \to X$ so that $\mathrm{id}_{\mathrm{Spec}(A^+)}$ is cofinal in $\mathrm{Mdf}_{\mathrm{Spec}(A)}\big(\mathrm{Spec}(A^+)\big)$. □

Corollary 28. *Let A and A^+ be as in Lemma 16. Then the relative Riemann–Zariski space $\mathrm{RZ}_{\mathrm{Spec}(A)}\big(\mathrm{Spec}(A^+)\big)$ is not cohesive (i.e. its structure sheaf is not coherent over itself).*

Proof. This follows from Lemma 16 together with Lemma 27. □

[1] The terms "Zariski" and "Riemann" appear in different orders in the literature, cf. [32] vs. [21], and the author tries to be coherent with these sources.

The following corollary answers the question whether admissible Zariski–Riemann spaces are cohesive, see [11, Proposition 3.10] and the preceding paragraph in loc. cit.

Corollary 29. *Let X be a quasi-compact and quasi-separated scheme, $U \subseteq X$ a quasi-compact open subscheme, $\langle X\rangle_U$ the associated admissible Zariski–Riemann space, and $i\colon \widetilde{Z} \hookrightarrow \langle X\rangle_U$ be the inclusion of the closed complement with reduced structure. There exists an example of this situation such that the locally ringed space $\langle X\rangle_U$ is not cohesive.*

Proof. Consider the rings $A := \mathbf{Q}_p[[X,Y]]$ and $A^+ = \{f \in A \mid f(0,0) \in \mathbf{Z}_p\}$ from Example 11. Since $A = A^+[p^{-1}]$, the induced morphism $\operatorname{Spec}(A) \to \operatorname{Spec}(A^+)$ is an open immersion, so that $\langle A^+\rangle_A \cong \mathrm{RZ}_{\operatorname{Spec}(A)}(\operatorname{Spec}(A^+))$ is not cohesive. □

Lemma 30. *For any point $x \in \mathrm{RZ}_Y(X)$, the stalk $\mathcal{O}_{\mathrm{RZ}_Y(X),x}$ is a semi-valuation ring.*

Proof. This follows from [32, Prop 2.2.1] since the morphism $f\colon Y \to X$ is assumed to be separated which is equivalent to being decomposable by [32, Theorem 1.1.3]. □

Definition 31. *For an open subset V of $\mathrm{RZ}_Y(X)$ we define the set $\operatorname{Model}(V)$ whose elements are open subsets V' of some $X' \in \operatorname{Mdf}_Y(X)$ such that $p_{X'}^{-1}(V') = V$. Defining $V' \le V''$ if $V'' = q^{-1}(V')$ for a morphism $q\colon X'' \to X'$ in $\operatorname{Mdf}_Y(X)$ we get a partial order on $\operatorname{Model}(V)$. Since $\operatorname{Mdf}_Y(X)$ is cofiltered, the sets $\operatorname{Model}(V)$ are filtered posets. If V is quasi-compact, then $\operatorname{Model}(V)$ is non-empty [13, Chapter 0, 2.2.9]. In particular, $\operatorname{Mdf}_Y(X) = \operatorname{Model}(\mathrm{RZ}_Y(X))$. We define the* K-theory on the Riemann–Zariski space *to be the presheaf*

$$\mathrm{K}^{\mathrm{RZ}}\colon \operatorname{Open}^{\mathrm{qc}}(\mathrm{RZ}_Y(X))^{\mathrm{op}} \longrightarrow \mathbf{Sp}, \quad V \mapsto \operatorname*{colim}_{V' \in \operatorname{Model}(V)} \mathrm{K}(V'),$$

on the poset $\operatorname{Open}^{\mathrm{qc}}(\mathrm{RZ}_Y(X))$ of quasi-compact open subsets of $\mathrm{RZ}_Y(X)$. Analogously, we define the presheaf $\mathrm{KH}^{\mathrm{RZ}}(-)$.

Proposition 32. *The presheaves $\mathrm{K}^{\mathrm{RZ}}(-)$ and $\mathrm{KH}^{\mathrm{RZ}}(-)$ are sheaves of spectra. In particular, we get induced sheaves on the topological space $\mathrm{RZ}_Y(X)$.*

Proof. The topology on the category $\operatorname{Open}^{\mathrm{qc}}(\mathrm{RZ}_Y(X))$ equals the topology induced by the cd-structure of its cartesian squares. Hence it suffices to show that for any open subset V of $\mathrm{RZ}_Y(X)$ which is covered by two open subsets $V_1, V_2 \subset V$ with intersection $V_3 := V_1 \cap V_2$ the induced square

$$\begin{array}{ccc} \mathrm{K}^{\mathrm{RZ}}(V) & \longrightarrow & \mathrm{K}^{\mathrm{RZ}}(V_1) \\ \downarrow & & \downarrow \\ \mathrm{K}^{\mathrm{RZ}}(V_2) & \longrightarrow & \mathrm{K}^{\mathrm{RZ}}(V_3) \end{array} \qquad (\spadesuit)$$

is cartesian in $\mathbf{Sp}$. For $V_1' \in \operatorname{Model}(V_1)$ and $V_2' \in \operatorname{Model}(V_2)$ we may assume that they live on a common Y-modification of X. Then $V_1' \cup V_2' \in \operatorname{Model}(V)$ and $V_1' \cap V_2' \in \operatorname{Model}(V_3)$. By cofinality, the square ($\spadesuit$) is equivalent to a colimit of cartesian squares since K-theory is a Zariski-sheaf on the category of qcqs schemes. The statement for $\mathrm{KH}^{\mathrm{ZR}}(-)$ has the same proof. Since the quasi-compact open subsets of $\mathrm{RZ}_Y(X)$ form a basis of the topology, the sheaves extend from $\operatorname{Open}^{\mathrm{qc}}(\mathrm{RZ}_Y(X))$ to $\operatorname{Open}(\mathrm{RZ}_Y(X))$. □

Lemma 33. *For $x \in \mathrm{RZ}_Y(X)$ the stalk of $\mathrm{K}_x^{\mathrm{RZ}}$ is equivalent to $\mathrm{K}(\mathcal{O}_{\mathrm{RZ}_Y(X),x})$.*

Proof. Since K-theory commutes with colimits, we can compute

$$\begin{aligned}
\mathrm{K}_x^{\mathrm{RZ}} &\simeq \operatorname*{colim}_{x\in V} \mathrm{K}^{\mathrm{RZ}}(V) \\
&\simeq \operatorname*{colim}_{x\in V} \operatorname*{colim}_{V'\in \mathrm{Model}(V)} \mathrm{K}(V') \\
&\simeq \operatorname*{colim}_{X'\in \mathrm{Mdf}_Y(X)} \operatorname*{colim}_{p_{X'}(x)\in V'} \mathrm{K}\big(\mathscr{O}_{X'}(V')\big) \\
&\simeq \operatorname*{colim}_{X'\in \mathrm{Mdf}_Y(X)} \mathrm{K}(\mathscr{O}_{X',p_{X'}(x)}) \\
&\simeq \mathrm{K}(\mathscr{O}_{\mathrm{RZ}_Y(X),x})
\end{aligned}$$

where the step from the second to the third line is due to cofinality of the indexing categories. □

Proposition 34. *Assume that all stalks of Y are stably coherent regular rings and that* $\mathrm{Mdf}_Y(X)$ *admits a cofinal subcategory $\mathscr{M}_d$ such that* $\dim(X') \le d$ *for all $X' \in \mathscr{M}_d$ for some $d \in \mathbf{N}$. Then the canonical maps*

1. $\mathrm{K}^{\mathrm{RZ}}(-) \longrightarrow \mathrm{K}^{\mathrm{RZ}}\big((-)[t_1,\dots,t_k]\big)$ *and*
2. $\mathrm{K}^{\mathrm{RZ}}(-) \longrightarrow \mathrm{KH}^{\mathrm{ZR}}(-)$

are equivalences of spectrum-valued sheaves on $\mathrm{RZ}_Y(X)$.

Proof. For every $X' \in \mathscr{M}_d$, the sheaf topos $\mathbf{Sh}(X')$ of space-valued sheaves on X' has homotopy dimension $\le d$ by a result of Clausen–Mathew [7, p. 3.12]. This implies that $\mathbf{Sh}\big(\mathrm{RZ}_Y(X)\big)$ has homotopy dimension $\le d$ [7, p. 3.11]. Analogously, every quasi-compact open subset V of RZ_Y is a cofiltered limit of schemes of finite dimension so that $\mathbf{Sh}(V)$ has homotopy dimension $\le d$. Thus $\mathbf{Sh}\big(\mathrm{RZ}_Y(X)\big)$ is locally of homotopy dimension $\le d$, hence Postnikov complete [24, p. 7.2.1.10]. Since the ∞-category $\mathbf{Sh}_{\mathbf{Sp}}\big(\mathrm{RZ}_Y(X)\big)$ is equivalent to the category $\mathbf{Sh}_{\mathbf{Sp}}\big(\mathbf{Sh}(\mathrm{RZ}_Y(X))\big)$ [26, p. 1.3.1.7], it is left-complete [26, pp. 1.3.3.10, 1.3.3.11]. Thus we can check equivalences of sheaves of spectra on $\mathrm{RZ}_Y(X)$ on stalks (this is folklore, see [10, p. A.1.32]).

Hence we can check the statements (1) and (2) on the stalks $\mathscr{O}_{\mathrm{RZ}_Y(X),x}$ which are semi-valuation rings (Lemma 30) so that the claim follows from Lemma 33 and Corollary 23. □

Corollary 35. *Assume that Y is of finite dimension, that all its stalks are stably coherent regular rings, and that the morphism $f\colon Y \to X$ admits a compactification (e.g. f is of finite type). Then the properties* (1) *and* (2) *of Proposition 34 hold true.*

Proof. This follows since $\dim(X') \le \dim(Y)$ for every $X' \in \mathrm{Mdf}_Y(X)$ by the assumptions. □

Remark 36. If the morphism f is the inclusion of a schematically dense open subscheme Y of X, the space $\mathrm{RZ}_Y(X)$ is isomorphic to the admissible Zariski–Riemann space $\langle X\rangle_Y$ [11, p. 2.7]. In case that the morphism $f\colon Y \to X$ admits a compactification $Y \hookrightarrow \overline{X} \to X$ (e.g. if f is of finite type), then the canonical morphism $\langle \overline{X}\rangle_Y \cong \mathrm{RZ}_Y(\overline{X}) \to \mathrm{RZ}_Y(X)$ is an isomorphism. If moreover $\overline{X}$ is noetherian, then Proposition 34 with $k = 1$ already follows from previous work of the author [11, Corollary 4.18].

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