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The independent isolation number of a tree

L'indice d'isolement indépendant d'un arbre

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Abstract. Let G be a simple graph. An isolating set of G is a set I of vertices such that removing I and its neighborhood leaves no edge. In addition, the set I is said to be an independent isolating set of G if I is an isolating set of G and $G[I]$ has no edge, where $G[I]$ represents the subgraph of G induced by I. The independent isolation number of G, denoted by $\iota^i(G)$, is the minimum cardinality of among all independent isolating sets of G. In this paper, we prove that for every tree T except a star,
$$\frac{n(T)-|L(T)|-|S(T)|+3}{4} \le \iota^i(T) \le \frac{n(T)-|L(T)|+2|S(T)|}{4},$$
where $n(T)$, $L(T)$ and $S(T)$ represent the order, the sets of leaves and support vertices, respectively. Finally, we characterize the extremal trees attaining the bounds.

Résumé. Soit G un graphe simple. Un ensemble isolant de G est un ensemble I de sommets tel que la suppression de I et de son voisinage laisse le graphe sans arêtes. De plus, l'ensemble I est dit un ensemble isolant indépendant de G si I est un ensemble isolant de G et si $G[I]$ ne contient pas d'arêtes, où $G[I]$ représente le sous-graphe de G induit par I. Le nombre d'isolement indépendant de G, noté $\iota^i(G)$, est la cardinalité minimum parmi tous les ensembles isolants indépendants de G. Dans cet article, nous prouvons que pour chaque arbre T sauf une étoile,
$$\frac{n(T)-|L(T)|-|S(T)|+3}{4} \le \iota^i(T) \le \frac{n(T)-|L(T)|+2|S(T)|}{4},$$
où $n(T)$, $L(T)$ et $S(T)$ représentent respectivement l'ordre de T, l'ensemble des feuilles et l'ensemble des sommets de support. Enfin, nous caractérisons les arbres extrémaux atteignant les bornes.

Keywords. Tree, independent isolating set, independent isolation number.

Mots-clés. Arbre, ensemble isolant indépendant, numéro d'isolation indépendant.

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1. Introduction

We consider simple graphs only. For a graph $G = \big(V(G), E(G)\big)$, we denote its order $|V(G)|$ simply by $n(G)$ throughout the paper. For a subset $S \subseteq V(G)$, the open neighborhood of S is the set $N(S) = \big\{u \in V(G)\setminus S : uv \in E(G), v \in S\big\}$, while the closed neighborhood of S is the set $N[S] = N(S) \cup S$. For any vertex $v \in V(G)$, $N(v) = \big\{u \in V(G) : uv \in E(G)\big\}$ and $N[v] = N(v) \cup \{v\}$ represent the open neighborhood and the closed neighborhood of v, respectively. The degree of v in G, denoted by $d_G(v)$, is the cardinality of the open neighborhood of v, that is, $d_G(v) = |N(v)|$.

Given a set $A \subseteq V(G)$, $G - A$ denotes the graph obtained from G by removing all the vertices in A and all the edges incident with a vertex in A, and $G[A]$ denotes the subgraph of G induced by A. The minimum degree $\delta(G)$ of G is the minimum among all vertex degrees of G. Likewise, the maximum degree $\Delta(G)$ of G is the maximum among all vertex degrees of G. The diameter of a graph G is the maximum distance between two vertices of G, denoted by $\mathrm{diam}(G)$. We denote the set of leaves adjacent to a vertex v by L_v. A graph G is bipartite if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y. Next, we define some sets that will be used in the sequel.

- $L(G) = \{v \in V(G) : d_G(v) = 1\}$ is the set of leaves of G.
- $S(G) = \{v \in V(G) : |N(v) \cap L(G)| \geq 1\}$ is the set of support vertices of G.
- $S_s(G) = \{v \in S(G) : |N(v) \cap L(G)| \geq 2\}$ is the set of strong support vertices of G.
- $L_s(G) = \{v \in L(G) : |N(v) \cap S_s(G)| = 1\}$ is the set of strong leaves of G.
- $L_w(G) = L(G) \setminus L_s(G)$ is the set of weak leaves of G.
- $SS(G) = \{v \in V(G) \setminus (L(G) \cup S(G)) : |N(v) \cap S(G)| \geq 1\}$ is the set of semi-support vertices of G.
- $S^*(G) = \{v \in S(G) : d_G(v) = |L_v| + 1\}$.
- $SS^*(G) = \{v \in SS(G) : |N(v) \cap S^*(G)| \geq 1\}$.
- $SS_2(G) = \{v \in SS^*(G) : d_G(u) = 2 \text{ for some vertex } u \in N(v) \cap S^*(G)\}$.
- $R(G) = \{v \in V(G) \setminus (L(G) \cup S(G) \cup SS(G)) : |N(v) \cap SS(G)| \geq 1\}$.
- $R_2(G) = \{v \in R(G) : d_G(u) = 2 \text{ for some vertex } u \in N(v) \cap SS_2(G)\}$.

Let T be a tree. An inner vertex of T is a vertex that is not a leaf. We denote by $K_{1,m}$ a star of order $m+1$. A tree is a double star if it contains exactly two vertices that are not leaves. A double star with, respectively, a and b leaves attached at every support vertex is denoted by $DS_{a,b}$. The path on n vertices is written as P_n. For any vertex v in a rooted tree T, the maximal subtree at v is the subtree of T induced by v and all descendants of v, denoted by T_v. Moreover, given a tree T, by attaching a path P to a vertex $v \in V(T)$ we mean adding the path P and joining v by an edge to a leaf of P.

Let $\mathscr{F}$ be a family of graphs. Caro and Hansberg [6] defined an $\mathscr{F}$-isolating set of G as a set $I \subseteq V(G)$ such that $G - N[I]$ contains no member of $\mathscr{F}$ as a subgraph. The minimum cardinality of an $\mathscr{F}$-isolating set of G is denoted $\iota(G,\mathscr{F})$ and called $\mathscr{F}$-isolation number of G. If $\mathscr{F} = \{K_1\}$, then an $\mathscr{F}$-isolating set coincides with the definition of a dominating set and $\iota(G, K_1) = \gamma(G)$ is the domination number of G. Moreover, a dominating set of G is said to be an independent dominating set of G if no two vertices in I are adjacent. The independent domination number of G, denoted by $i(G)$, is the minimum cardinality among all independent dominating sets of G. If $\mathscr{F} = \{K_2\}$, then terms $\mathscr{F}$-isolating set and $\mathscr{F}$-isolation number, and the notation $\iota(G,\mathscr{F})$, are abbreviated to isolating set, isolation number, and $\iota(G)$, respectively. In [1] it was observed that an isolating set coincides with the concept of vertex-edge dominating set, as introduced earlier by Lewis et al. [15].

In 2004, Ma and Chen [16] proved that $i(G) \leq \frac{n(G)}{2}$ for any connected bipartite graph G. Favaron [10] has proved $\gamma(T) \leq i(T) \leq \frac{n(T)+|L(T)|}{3}$ for every nontrivial tree T, and characterized the extremal graphs. In 2023, Cabrera-Martínez, Peiró and Rueda-Vázquez [5] proved that $\gamma(T) \leq \frac{n(T)+|S(T)|}{3}$ for any tree T of order $n(T) \geq 3$, which improved the result of Favaron. Recently, Cabrera-Martínez [3] gave an alternative proof for the bound $\gamma(T) \leq \frac{n(T)+|S(T)|}{3}$ and improved this upper bound. Krishnakumari, Venkatakrishnan and Krzywkowski in [12] established the following bounds on the isolation number of a tree. For every tree T of order $n(T) \geq 3$, we have $\frac{n(T)-|L(T)|-|S(T)|+3}{4} \leq \iota(T) \leq \frac{n(T)}{3}$. They also characterized the extremal trees. More parameters concerning domination and independent domination in trees have been extensively studied, see [2,4,7,9,13,17–19].

In this paper we instead consider a restriction on the isolating set I itself. Specifically, an independent isolating set is one where no two vertices in I are adjacent. Further, we define the independent isolation number $\iota^i(G)$ of a graph G as the minimum size of such a set. The parameter was originally introduced by Lewis [14] as independent vertex-edge domination number, and further studied in [8,11,20]. Finally, we establish the following bounds on the independent isolation number of any tree T except a star,

$$\frac{n(T)-|L(T)|-|S(T)|+3}{4} \le \iota^i(T) \le \frac{n(T)-|L(T)|+2|S(T)|}{4},$$

and we also characterize the trees attaining the bounds.

2. Upper bound

We provide an upper bound on the independent isolation number of a connected bipartite graph.

Theorem 1. *If G is a connected bipartite graph except a star, then $\iota^i(G) \le \frac{n(G)-|L(G)|}{2}$.*

Proof. Let L be the set of leaves of G and $G' = G - L$. Suppose that I is a minimum independent dominating set of G'. Then I is an independent isolating set of G. Since G' is a connected bipartite graph and $i(G') \le \frac{n(G')}{2}$, we have

$$\iota^i(G) \le i(G') \le \frac{n(G')}{2} \le \frac{n(G)-|L(G)|}{2},$$

completing the proof. □

The following corollary is immediate from Theorem 1.

Corollary 2. *If T is a tree except a star, then $\iota^i(T) \le \frac{n(T)-|L(T)|}{2}$.*

Next we show a new upper bound on the independent isolation number of a tree. To characterize the trees attaining the bound given in Theorem 5, we introduce a new family $\mathscr{T}$. Let r be an integer. We define the family $\mathscr{T}$ of all trees T that can be obtained from a sequence of trees $T_0, T_1, \ldots, T_r = T$, with $r \ge 0$ and $T_0 = P_6$. If $r \ge 1$, then for each $i \in \{1, \ldots, r\}$, the tree T_i can be obtained from $T' = T_{i-1}$ by one of the following three operations defined below.

Operation O_1: Attach a path P_1 to a vertex $v \in S(T')$.
Operation O_2: Attach a path P_3 to the vertex $v \in SS_2(T) \cup R_2(T)$.
Operation O_3: Attach a path P_4 to the vertex $v \in L_w(T')$.

Lemma 3. *Let T be a tree of order $n \ge 3$.*

(i) *If T is obtained from any tree T' by attaching a path P_1 to a vertex $v \in S(T')$, then $\iota^i(T) = \iota^i(T')$.*
(ii) *If T is obtained from any tree T' by attaching a path P_3 to a vertex $v \in SS_2(T') \cup R_2(T')$, then $\iota^i(T) = \iota^i(T') + 1$.*
(iii) *If T is obtained from any tree T' by attaching a path P_4 to a vertex $v \in L(T')$, then $\iota^i(T) = \iota^i(T') + 1$.*

Proof. (i). The proof of (i) is clear.

(ii). Suppose that T is obtained from T' by adding the path $P_3 = v_1v_2v_3$ and joining v_3 to v, with $v \in SS_2(T') \cup R_2(T')$. Notice that any independent isolating set of T' can be extended to an independent isolating set of T by adding the vertex v_2. Hence, $\iota^i(T) \le \iota^i(T') + 1$.

Let I be a minimum independent isolating set of T such that $\left|I \cap \{v_1, v_2, v_3\}\right|$ is minimum. Since I is minimum and independent, $1 \le \left|I \cap \{v_1, v_2, v_3\}\right| \le 2$. If $\left|I \cap \{v_1, v_2, v_3\}\right| = 2$, then $v_1, v_3 \in I$ and $I \setminus \{v_1\}$ is an independent isolating set of T, which is a contradiction to I is minimum. So

$\left|I \cap \{v_1, v_2, v_3\}\right| = 1$. If $v_i \in I$ for $i \in \{1,2\}$, $I \setminus \{v_i\}$ is an independent isolating set of T'. Next we consider $v_3 \in I$, which implies $v \notin I$. We discuss $v \in SS_2(T')$ and $v \in R_2(T')$, respectively.

If $v \in SS_2(T')$, then there exist $u \in N(v) \cap S(T')$ and $w \in N(u) \cap L(T')$ such that $d_{T'}(u) = 2$. Since I is minimum and independent, $u \in I$ or $w \in I$. If $u \in I$, then $I \setminus \{v_3\}$ is an independent isolating set of T'. If $w \in I$, then $I \setminus \{v_3, w\} \cup \{u\}$ is an independent isolating set of T'.

If $v \in R_2(T')$, then there exist the vertex $u \in SS_2(T')$, $w \in S(T')$ and $x \in L(T')$ such that $uv, wu, xw \in E(T')$ and $d_T(u) = d_T(w) = 2$. Due to $v \notin I$, we obtain that $\left|\{u, w, x\} \cap I\right| = 1$. Next the process of the proof is similar to the above discussion.

Based on the above cases, we obtain $\iota^i(T') \le \iota(T) - 1$, and further, $\iota(T) = \iota(T') + 1$, as desired.

(iii). Assume that T is obtained from T' by adding the path $P_4 = v_1v_2v_3v_4$ and joining v_4 to v, with $v \in L(T')$. Notice that any independent isolating set of T' can be extended to an independent isolating set of T by adding the vertex v_3. Hence, $\iota^i(T) \le \iota^i(T') + 1$.

Let I be a minimum independent isolating set of T such that $\left|I \cap \{v_1, v_2, v_3, v_4\}\right|$ is minimum. Since I is an independent set, we deduce that $1 \le \left|I \cap \{v_1, v_2, v_3, v_4\}\right| \le 2$. If $\left|I \cap \{v_1, v_2, v_3, v_4\}\right| = 2$ and $v_i, v_j \in I$ with $i, j \in \{1,2,3,4\}$, then $I \setminus \{v_i, v_j\} \cup \{v_3\}$ is an independent isolating set of T', a contradiction. Thus $\left|I \cap \{v_1, v_2, v_3, v_4\}\right| = 1$. In this case, if $v_4 \in I$, then I isn't a minimum independent isolating set of T, a contradiction. If $v_i \in I$ for $i \in \{1,2,3\}$, then $I \setminus \{v_i\}$ is an independent isolating set of T'. So $\iota^i(T') \le \iota^i(T) - 1$. Therefore, $\iota^i(T) = \iota^i(T') + 1$, as desired. □

Lemma 4. *Let T be a tree except a star. If $T \in \mathscr{T}$, then $\iota^i(T) = \frac{n(T)-|L(T)|+2|S(T)|}{4}$.*

Proof. We proceed by induction on the order of a tree $T \in \mathscr{T}$. Since $T \ne K_{1,m}$ with $m \ge 0$, $n(T) \ge 4$. If $T \in \mathscr{T}$ and $4 \le n(T) \le 6$, then $T = P_6$ and $\iota^i(T) = 2 = \frac{n(T)-|L(T)|+2|S(T)|}{4}$, as required. Suppose that $n(T) > 6$ and that every tree T^* in $\mathscr{T}$, with $6 \le n(T^*) < n(T)$, satisfies that $\iota^i(T^*) = \frac{n(T^*)-|L(T^*)|+2|S(T^*)|}{4}$. Since $T \in \mathscr{T}$, it is clear that T can be obtained from a sequence of trees $T_0, T_1, \ldots, T_r = T$, with $T_0 = P_6$ and $r \ge 1$. Let $T' = T_{r-1}$, which implies $T' \in \mathscr{T}$, and by the induction hypothesis, it follows that $\iota^i(T') = \frac{n(T')-|L(T')|+2|S(T')|}{4}$. We consider the following three cases, depending on which operation is used to obtain the tree T from T'.

Case 1: T is obtained from T' by operation O_1. Let T be obtained from T' by adding the path $P_1 = u$ and the edge uv where $v \in S(T')$. We have $n(T') = n(T) - 1$, $|S(T')| = |S(T)|$ and $|L(T')| = |L(T)| - 1$. So by Lemma 3(i), we have the following equality chain:

$$\begin{aligned}\iota^i(T) = \iota^i(T') &= \frac{n(T') - |L(T')| + 2|S(T')|}{4} \\ &= \frac{n(T) - 1 - \big(|L(T)| - 1\big) + 2|S(T)|}{4} \\ &= \frac{n(T) - |L(T)| + 2|S(T)|}{4}.\end{aligned}$$

Case 2: T is obtained from T' by operation O_2. Let T be obtained from T' by adding the path $P_3 = v_1v_2v_3$ and the edge vv_3 where $v \in SS_2(T') \cup R_2(T')$. We have $n(T') = n(T) - 3$, $|L(T')| = |L(T)| - 1$ and $|S(T')| = |S(T)| - 1$. Thus, by Lemma 3(ii), we have the following equality chain:

$$\begin{aligned}\iota^i(T) = \iota^i(T') + 1 &= \frac{n(T') - |L(T')| + 2|S(T')|}{4} + 1 \\ &= \frac{n(T) - 3 - \big(|L(T)| - 1\big) + 2\big(|S(T)| - 1\big)}{4} + 1 \\ &= \frac{n(T) - |L(T)| + 2|S(T)|}{4}.\end{aligned}$$

Case 3: T is obtained from T' by operation O_3. Let T be obtained from T' by adding the path $P_4 = v_1v_2v_3v_4$ and the edge vv_4 with $v \in L_w(T')$. Then we have $n(T') = n(T) - 4$, $|S(T')| = |S(T)|$ and $|L(T')| = |L(T)|$. Thus, by Lemma 3(iii), we have the following equality chain:

$$\begin{aligned}\iota^i(T) = \iota^i(T') + 1 &= \frac{n(T') - |L(T')| + 2|S(T')|}{4} + 1\\ &= \frac{n(T) - 4 - |L(T)| + 2|S(T)|}{4} + 1\\ &= \frac{n(T) - |L(T)| + 2|S(T)|}{4}.\end{aligned}$$

As a consequence, we conclude that $\iota^i(T) = \frac{n(T)-|L(T)|+2|S(T)|}{4}$, which completes the proof. □

Theorem 5. *If T is a tree except a star, then $\iota^i(T) \le \frac{n(T)-|L(T)|+2|S(T)|}{4}$ with equality if and only if $T \in \mathscr{T}$.*

Proof. By Lemma 4, we know that if $T \in \mathscr{T}$, then $\iota^i(T) = \frac{n(T)-|L(T)|+2|S(T)|}{4}$. Now we prove $\iota^i(T) \le \frac{n(T)-|L(T)|+2|S(T)|}{4}$ by induction on $n(T)$, and if $\iota^i(T) = \frac{n(T)-|L(T)|+2|S(T)|}{4}$, then $T \in \mathscr{T}$. Since $T \ne K_{1,m}$ with $m \ge 0$, we know $n(T) \ge 4$ and $\mathrm{diam}(T) \ge 3$. If $4 \le n(T) \le 6$, then it is easy to obtain that $\iota^i(T) \le \frac{n(T)-|L(T)|+2|S(T)|}{4}$, with equality holding only if $T = P_6$. Let T be a tree of order $n(T) \ge 7$. Suppose that the tree T' of order $n(T') \le n(T)-1$ satisfies $\iota^i(T') \le \frac{n(T')-|L(T')|+2|S(T')|}{4}$ and the tree T' with $\iota^i(T') = \frac{n(T')-|L(T')|+2|S(T')|}{4}$ is in $\mathscr{T}$. Take a diametrical path $v_1v_2\cdots v_{d+1}$ in T and root T at v_{d+1}. Clearly, $v_1, v_{d+1} \in L(T)$. If $3 \le \mathrm{diam}(T) \le 4$, then $\{v_3\}$ is a minimum independent isolating set of T and $\iota^i(T) = 1 < \frac{n(T)-|L(T)|+2|S(T)|}{4}$. Next we suppose that $\mathrm{diam}(T) \ge 5$.

If there exists a vertex $v \in S(T)$ such that $|L_v| \ge 2$, then there exists a leaf $w \in N(v)$. We define $T' = T - \{w\}$. Since $\mathrm{diam}(T) \ge 5$, we have $\mathrm{diam}(T') > 3$. So $n(T') = n(T) - 1$, $|L(T')| = |L(T)| - 1$ and $|S(T')| = |S(T)|$. Hence, by the induction hypothesis and Lemma 3(i), we have

$$\begin{aligned}\iota^i(T) = \iota^i(T') &\le \frac{n(T') - |L(T')| + 2|S(T')|}{4}\\ &= \frac{n(T) - 1 - \big(|L(T)| - 1\big) + 2|S(T)|}{4}\\ &= \frac{n(T) - |L(T)| + 2|S(T)|}{4}.\end{aligned}$$

The equality requires that $\iota^i(T') = \frac{n(T')-|L(T')|+2|S(T')|}{4}$. By the induction hypothesis, $T' \in \mathscr{T}$. Due to $|S(T')| = |S(T)|$, we obtain that $v \in S(T')$. So T can be obtained from T' by operation O_1 and consequently, $T \in \mathscr{T}$.

Next we consider that $|L_v| = 1$ for any vertex $v \in S(T)$. Then we have $|L(T)| = |S(T)|$. Since $v_2, v_d \in S(T)$, we deduce that $d_T(v_2) = d_T(v_d) = 2$.

Case 1: $d_T(v_4) = 2$. Let $T' = T - T_{v_4}$. We have $n(T') \le n(T) - 4$, $|S(T')| \le |L(T')| \le |L(T)|$ and $|S(T')| \le |S(T)|$. Suppose that I' is a minimum independent isolating set of T'. Since $\mathrm{diam}(T) \ge 5$, we obtain that $\mathrm{diam}(T') \ge 1$. If $1 \le \mathrm{diam}(T') \le 2$, then $\{v_3, v_5\}$ is a minimum independent isolating set of T and $\iota^i(T) = 2 < \frac{n(T)-|L(T)|+2|S(T)|}{4}$ because $n(T) \ge 7$ and $|S(T)| = |L(T)| \ge 2$. So we consider that $\mathrm{diam}(T') \geq 3$. It is easy to see that $I' \cup \{v_3\}$ is an independent isolating set of T. Thus, by the induction hypothesis, we have

$$\begin{aligned}\iota^i(T) \leq \iota^i(T') + 1 &\leq \frac{n(T') - |L(T')| + 2|S(T')|}{4} + 1\\ &\leq \frac{n(T') - |S(T')| + 2|S(T')|}{4} + 1\\ &\leq \frac{n(T) - 4 + |S(T)|}{4} + 1\\ &= \frac{n(T) - |L(T)| + 2|S(T)|}{4}.\end{aligned}$$

The equality requires $\iota^i(T) = \iota^i(T') + 1$, $n(T') = n(T) - 4$, $|L(T')| = |L(T)|$, $|S(T')| = |S(T)|$ and $\iota^i(T') = \frac{n(T')-|L(T')|+2|S(T')|}{4}$. By the induction hypothesis, $T' \in \mathscr{T}$. Since $n(T') = n(T) - 4$ and $|L(T')| = |L(T)|$, we can obtain $d_T(v_3) = d_T(v_5) = 2$. Due to $|S(T')| = |S(T)|$, we deduce that $v_6 \in S(T')$ and $v_6 \notin S(T)$. So $v_5 \in L_w(T')$ and T can be obtained from T' by operation O_3. Therefore, $T \in \mathscr{T}$.

Case 2: $d_T(v_4) \geq 3$.

Subcase 2.1: $d_T(v_3) \geq 3$. Observe that $v_3 \in S(T) \cup SS(T)$ by the choice of the diametrical path.

Through the above assumption that $|L_v| = 1$ for every $v \in S(T)$, if $v_3 \in S(T)$, then we have $|L_{v_3}| = 1$. So there exists only one leaf $u \in N(v_3)$. Let $T' = T - \{u\}$ and I' be a minimum independent isolating set of T'. Clearly, $\mathrm{diam}(T') > 3$. We have $n(T') = n(T)-1$, $|L(T')| = |L(T)|-1$, $|S(T')| = |S(T)|-1$. If $v_3 \in N[I']$, then I' is also an independent isolating set of T; otherwise, $v_1 \in I'$ and $I' \setminus \{v_1\} \cup \{v_2\}$ is an independent isolating set of T. By the induction hypothesis, we have

$$\begin{aligned}\iota^i(T) \leq \iota^i(T') &\leq \frac{n(T') - |L(T')| + 2|S(T')|}{4}\\ &= \frac{n(T) - 1 - \big(|L(T) - 1|\big) + 2\big(|S(T)| - 1\big)}{4}\\ &= \frac{n(T) - |L(T)| + 2|S(T)| - 2}{4}\\ &< \frac{n(T) - |L(T)| + 2|S(T)|}{4}.\end{aligned}$$

Now we assume that $v_3 \in SS(T)$. Then $|L_u| = 1$ for every $u \in N(v_3) \cap S(T)$. Thus $d_T(u) = 2$. Since $d_T(v_4) \geq 3$, observe that $v_4 \in S(T) \cup SS(T) \cup R(T)$.

First we consider $v_4 \in S(T)$. There exists only one leaf $w \in N(v_4)$. Let $T' = T - \{w\}$. We have $n(T') = n(T)-1$, $|L(T')| = |L(T)|-1$, $|S(T')| = |S(T)|-1$. Suppose that I' is a minimum independent isolating set of T'. If $v_4 \in N[I']$, then I' is also an independent isolating set of T. Now considering $v_4 \notin N[I']$. Denote $X = \big(N(v_3) \setminus \{v_4\}\big) \cap I'$. So $|X| \geq 1$ and $(I' \setminus X) \cup \{v_3\}$ is an independent isolating set of T. Thus, by the induction hypothesis, we have

$$\begin{aligned}\iota^i(T) \leq \iota^i(T') &\leq \frac{n(T') - |L(T')| + 2|S(T')|}{4}\\ &= \frac{n(T) - 1 - \big(|L(T) - 1|\big) + 2\big(|S(T)| - 1\big)}{4}\\ &< \frac{n(T) - |L(T)| + 2|S(T)|}{4}.\end{aligned}$$

Next we consider $v_4 \in SS(T)$. There exists $w \in S(T)$ such that $|L_w| = 1$, where $w \in N(v_4) \setminus \{v_3, v_5\}$. If $d_T(w) \geq 3$, then there exists only one leaf $u \in N(w)$ and let $T' = T - \{u\}$. By the similar demonstration as the above discussion, we obtain that $\iota^i(T) \leq \iota^i(T') < \frac{n(T)-|L(T)|+2|S(T)|}{4}$. Now we consider that $d_T(w) = 2$. Let $T' = T - T_w - T_{v_2'}$ with $v_2' \in N(v_3) \setminus \{v_2, v_4\}$ and $d_T(v_2') = 2$.

Clearly, $\mathrm{diam}(T') > 3$. Observe that $n(T') = n(T) - 4$, $|L(T')| = |L(T)| - 2$ and $|S(T')| = |S(T)| - 2$. If $v_3 \in I'$, then $I' \cup \{w\}$ is an independent isolating set of T. If $v_4 \in I'$, then $I' \cup \{v_2'\}$ is an independent isolating set of T. If $v_3, v_4 \notin I'$, then there exist $v_1 \in I'$ or $v_2 \in I'$. So we know that $\big(I' \setminus \big(X \cup \{v_1\}\big)\big) \cup \{v_3, w\}$ is an independent isolating set of T, with $X = \big(N(v_3) \setminus \{v_4\}\big) \cap I'$. So by the induction hypothesis, we have

$$\begin{aligned}\iota^i(T) \le \iota^i(T') + 1 &\le \frac{n(T') - |L(T')| + 2|S(T')|}{4} + 1 \\ &= \frac{n(T) - 4 - \big(|L(T)| - 2\big) + 2\big(|S(T)| - 2\big)}{4} + 1 \\ &< \frac{n(T) - |L(T)| + 2|S(T)|}{4}.\end{aligned}$$

Finally we consider $v_4 \in R(T)$. Then $w \in SS(T)$ for every $w \in N(v_4) \setminus \{v_3, v_5\}$. Suppose that $N(v_4) \setminus \{v_5\} = \{w_1, w_2, \ldots, w_l\}$ and $w_1 = v_3$. We define $T' = T - T_{v_4}$. Observe that $n(T') \le n(T) - (3l+3)$, $|L(T')| \le |L(T)| - l$ and $|S(T')| \le |S(T)| - l$. Clearly, $\mathrm{diam}(T') \ge 2$. If $\mathrm{diam}(T') = 2$, then T is a tree with $\mathrm{diam}(T) = 6$, $n(T) \ge 3l+6$ and $|L(T)| = |S(T)| \ge l+2$ because $d_T(v_3) \ge 3$. We can observe that $\{v_5\} \cup \big(\bigcup_{i=1}^{l} \{w_i\}\big)$ is an independent isolating set of T and $\iota^i(T) \le l+1 < \frac{n(T)-|L(T)|+2|S(T)|}{4}$. Now we consider $\mathrm{diam}(T') \ge 3$. It is easy to see that $I' \cup \big(\bigcup_{i=1}^{l} \{w_i\}\big)$ is an independent isolating set of T. So by the induction hypothesis and the fact that $|S(T')| \le |L(T')|$, we have

$$\begin{aligned}\iota^i(T) \le \iota^i(T') + l &\le \frac{n(T') - |L(T')| + 2|S(T')|}{4} + l \\ &\le \frac{n(T') - |S(T')| + 2|S(T')|}{4} + l \\ &\le \frac{n(T) - (3l+3) + \big(|S(T)| - l\big)}{4} + l \\ &< \frac{n(T) - |L(T)| + 2|S(T)|}{4}.\end{aligned}$$

Subcase 2.2: $d_T(v_3) = 2$. Denote $T' = T - T_{v_3}$. Then $n(T') = n(T) - 3$, $|L(T')| = |L(T)| - 1$ and $|S(T')| = |S(T)| - 1$. Obviously, $\mathrm{diam}(T') \ge 3$. Suppose that I' is a minimum independent isolating set of T'. It's easy to see that $I' \cup \{v_2\}$ is an independent isolating set of T. Hence, by the induction hypothesis, we have

$$\begin{aligned}\iota^i(T) \le \iota^i(T') + 1 &\le \frac{n(T') - |L(T')| + 2|S(T')|}{4} + 1 \\ &= \frac{n(T) - 3 - \big(|L(T)| - 1\big) + 2\big(|S(T)| - 1\big)}{4} + 1 \\ &= \frac{n(T) - |L(T)| + 2|S(T)|}{4}.\end{aligned}$$

The equality holds if $\iota^i(T) = \iota^i(T') + 1$ and $\iota^i(T') = \frac{n(T')-|L(T')|+2|S(T')|}{4}$. So $T' \in \mathscr{T}$ by the induction hypothesis. Since $d_T(v_4) \ge 3$, we have $v_4 \notin L(T')$. Observe that $v_4 \in S(T') \cup SS(T') \cup R(T')$.

If $v_4 \in S(T')$, then there exists only one leaf $u \in N(v_4)$. Let $T'' = T - \{u\}$. We obtain that $n(T'') = n(T) - 1$, $|L(T'')| = |L(T)| - 1$ and $|S(T'')| = |S(T)| - 1$. Assume that I'' is a minimum independent isolating set of T''. If $v_4 \in N[I'']$, then I'' is also an independent isolating set of T. Otherwise, $v_2 \in I''$ and $\big(I'' \setminus \{v_2\}\big) \cup \{v_3\}$ is an independent isolating set of T. So by the induction hypothesis, we have $\iota^i(T) \le \iota^i(T'') < \frac{n(T)-|L(T)|+2|S(T)|}{4}$, a contradiction. Therefore, $v_4 \notin S(T')$.

If $v_4 \in SS(T')$, then there exists a vertex $u \in N(v_4) \cap S(T')$ such that $|L_u| = 1$. If $d'_T(u) \ge 3$, then there exists only one leaf $w \in N(u)$. Let $T'' = T - \{w\}$. Clearly, $\mathrm{diam}(T'') > 3$. By the similar demonstration as Subcase 2.1, we get $\iota^i(T) \le \iota^i(T'') < \frac{n(T)-|L(T)|+2|S(T)|}{4}$, a contradiction. Thus

$d'_T(u) = 2$ and further, we have $v_4 \in SS_2(T')$. As a consequence, T can be obtained from T' by operation O_2 and $T \in \mathscr{T}$.

If $v_4 \in R(T')$, then $u \in SS(T')$ for all $u \in N(v_4)\backslash\{v_3, v_5\}$. If there exists $u \in N(v_4) \cap SS(T')$ such that $d_{T'}(u) \geq 3$, then let $T'' = T - T_{v_4}$. Next by the similar demonstration process as Subcase 2.1, we can obtain that $\iota^i(T) < \frac{n(T)-|L(T)|+2|S(T)|}{4}$, a contradiction. So $d_{T'}(u) = 2$ for every $u \in N(v_4) \cap SS(T')$ and further, $v_4 \in R_2(T')$. Therefore, T can be obtained from T' by operation O_2 and $T \in \mathscr{T}$, which completes the proof. □

Through a proof process similar to the above discussion, we can obtain the following theorem regarding a new upper bound of the isolation number of a tree.

Theorem 6. *If T is a tree except a star, then $\iota(T) \leq \frac{n(T)-|L(T)|+2|S(T)|}{4}$ with equality if and only if $T \in \mathscr{T}$.*

Corollary 7. *If T is a tree of order $n(T) \geq 3$, then $\iota(T) \leq \iota^i(T) \leq \frac{n(T)+|S(T)|}{4}$.*

Theorem 8. *Let T be a tree on $n(T) \geq 6$ vertices in which all non-leaves have degree at least 3, $\iota^i(T) \leq \frac{n(T)-2}{4}$ and this is sharp.*

Proof. Let V' be the set of inner vertices and L the set of leaves of T. Since all inner vertices of T have degree at least 3, we obtain the following result:

$$\sum_{v\in V} d_T(v) = \sum_{v\in L} d_T(v) + \sum_{v\in V'} d_T(v) = 2|E(T)| \geq |L| + 3|V'|.$$

According to the relation of vertices to the edges in the tree T, we have

$$2(n(T)-1) = 2|E(T)| \geq 3|V'| + |L| = 3|V'| + n(T) - |V'| = n(T) + 2|V'|.$$

This implies that $|V'| \leq \frac{n(T)-2}{2}$. Let D be a minimum independent dominating set of the tree $T - L$. When $|V(T-L)| = |V'| = 1$, T is a star $K_{1,n-1}$ and $\iota^i(T) = 1 \leq \frac{n(T)-2}{4}$. If $|V'| \geq 2$, then D is an independent isolating set of T. Since T is also a bipartite graph, we have $\iota^i(T) \leq |D| \leq \frac{|V'|}{2} \leq \frac{n(T)-2}{4}$.

For the sharpness, construct a tree T as follows. For any integer $t \geq 2$, take the paths $P = v_0 v_1 \ldots v_t$ and $T_1, T_2, \ldots, T_{t-1} = P_3$. Fix $u_i \in V(T_i)$ with $u_i \in S(T_i)$ for each $i \in \{1, 2, \ldots, t-1\}$. Set $T = P \cup \bigcup_{i=1}^{t-1}(T_i + u_i v_i)$. Observe that $\{u_1, u_2, \ldots, u_{t-1}\}$ is a minimum independent isolating set of T. Therefore, $\iota^i(T) = t - 1 = \frac{n(T)-2}{4}$. □

3. Lower bound

As previously exposed, Krishnakumari et al. [12] proved $\iota(T) \geq \frac{n(T)-|L(T)|-|S(T)|+3}{4}$. In addition, they characterized the family $\mathscr{D}$ of trees which the equality above holds. They defined the family $\mathscr{D}$ of all trees T that can be obtained from a sequence of trees $T_0, T_1, \ldots, T_r = T$, with $r \geq 0$ and $T_0 = P_5$. If $r \geq 1$, then for each $i \in \{1, 2, \ldots, r\}$, the tree T_i can be obtained from $T' = T_{i-1}$ by one of the following operations defined below.

Operation O_1: Attach a path P_1 to a vertex $v \in S(T')$.
Operation O_2: Attach a path P_2 to a vertex $v \in SS^*(T')$.
Operation O_3: Attach a path P_4 to a vertex $v \in L_w(T')$.

Theorem 9 ([12]). *If T is a nontrivial tree, then $\iota(T) \geq \frac{n(T)-|L(T)|-|S(T)|+3}{4}$ with equality if and only if $T \in \mathscr{D}$.*

Lemma 10. *If T is obtained from any tree $T' \in \mathscr{D}$ by attaching a path P_2 to a vertex $v \in SS^*(T')$, then $\iota^i(T) = \iota^i(T')$.*

Proof. Assume that T is obtained from any tree $T' \in \mathscr{D}$ by attaching a path $P_2 = u_1u_2$ to a vertex $v \in SS^*(T')$. Let u_2 be joined to v. First we show that $\iota^i(T) \le \iota^i(T')$. Let I' be a minimum independent isolating set of T'. If $v \in I'$, then I' is also an independent isolating set of T and $\iota^i(T) \le \iota^i(T')$. Now we consider $v \notin I'$. If $v \notin N[I']$, then there exist $u \in N(v) \cap S(T')$ and $w \in N(u) \cap L(T')$ such that $w \in I'$ because $v \in SS^*(T')$. So we can observe that $I' \setminus \{w\} \cup \{v\}$ is an independent isolating set of T and $\iota^i(T) \le \iota^i(T')$.

Next we consider that $v \in N(I')$. Since $T' \in \mathscr{D}$, we have $\mathrm{diam}(T') \ge 4$. If $\mathrm{diam}(T') = 4$, then we can observe that $\iota^i(T) = \iota^i(T') = 1$. Next we suppose that $\mathrm{diam}(T') \ge 5$. Since $v \in SS^*(T')$, there exist $u \in N(v) \cap S(T')$ and $w \in N(u) \cap L(T')$ such that u or $w \in I'$. Let $M = \{x \mid x \in N(v) \cap I'\}$, $M_1 = M \cap S(T')$ and $M_2 = M \setminus M_1$. According to the construction of $\mathscr{D}$, we know that if there exist $x \in M_2$ and $x' \in N(x) \setminus \{v\}$ in T', then $d_{T'}(x) = d_{T'}(x') = 2$. In addition, we define $X_1 = \{x \in M_2 : N(x) \setminus \{v\} \nsubseteq N(I' \setminus \{x\})\}$ and $X_2 = \{x \in M_2 : N(x) \setminus \{v\} \subseteq N(I' \setminus \{x\})\}$. Clearly, $M_2 = X_1 \cup X_2$. Notice that $(I' \setminus (M \cup \{w\})) \cup N(X_1)$ is an independent isolating set of T. Since $|N(X_1)| = |X_1| + 1$, we can deduce that $\iota^i(T) \le \iota^i(T')$.

Now we prove that $\iota^i(T') \le \iota^i(T)$. Let I be a minimum independent isolating set of T. If $u_1 \in I$, then $I \setminus \{u_1\}$ is an independent isolating set of T' and $\iota^i(T') \le \iota^i(T) - 1$. Thus we have $\iota^i(T') + 1 \le \iota^i(T) \le \iota^i(T')$ by the above discussion, a contradiction. So $u_1 \notin I$. Since $v \in SS^*(T)$, there exists a vertex $u \in N(v) \cap S(T)$. Now we suppose that $u_2 \in I$. If $L_u \cap I \neq \emptyset$, then $(I \setminus L_u \cup \{u_2\}) \cup \{u\}$ is an independent isolating set of T'. Otherwise, we have $u \in I$ and $I \setminus \{u_2\}$ is an independent isolating set of T'. Consequently, $\iota^i(T') \le \iota^i(T) - 1$, a contradiction. Thus $u_2 \notin I$. We can obtain that $v \in I$ and I is also an independent isolating set of T'. Therefore, we get $\iota^i(T') \le \iota^i(T)$.

As a consequence, $\iota^i(T) = \iota^i(T')$, which completes the proof. □

Lemma 11. *For every nontrivial tree T, if $T \in \mathscr{D}$, then $\iota^i(T) = \frac{n(T)-|L(T)|-|S(T)|+3}{4}$.*

Proof. We proceed by induction on the order of a tree $T \in \mathscr{D}$. If $T \in \mathscr{D}$ and $n(T) \le 5$, then $T = P_5$ and $\iota^i(T) = 1 = \frac{n(T)-|L(T)|-|S(T)|+3}{4}$, as required. Suppose that $n(T) \ge 6$ and that every tree T^* in $\mathscr{D}$, with $5 \le n(T^*) < n(T)$, satisfies that $\iota^i(T^*) = \frac{n(T^*)-|L(T^*)|-|S(T^*)|+3}{4}$. Since $T \in \mathscr{D}$, it is clear that T can be obtained from a sequence of trees $T_0, T_1, \ldots, T_r = T$, with $r \ge 1$ and $T_0 = P_5$. Let $T' = T_{r-1}$, and by the induction hypothesis, it follows that $\iota^i(T') = \frac{n(T')-|L(T')|-|S(T')|+3}{4}$. Next we consider the following three cases, depending on which operation is used to obtain the tree T from T'.

Case 1: T is obtained from T' by operation O_1. Let T be obtained from T' by adding the path $P_1 = u$ and the edge uv where $v \in S(T')$. So we have $n(T') = n(T) - 1$, $|L(T')| = |L(T)| - 1$ and $|S(T')| = |S(T)|$. By Lemma 3(i), we obtain the following result:

$$\begin{aligned}\iota^i(T) = \iota^i(T') &= \frac{n(T') - |L(T')| - |S(T')| + 3}{4} \\ &= \frac{n(T) - 1 - (|L(T)| - 1) - |S(T)| + 3}{4} \\ &= \frac{n(T) - |L(T)| - |S(T)| + 3}{4}.\end{aligned}$$

Case 2: T is obtained from T' by operation O_2. Let T be obtained from T' by adding the path $P_2 = u_1u_2$ and the edge u_2v where $v \in SS^*(T')$. Thus, we have $n(T') = n(T) - 2$, $|L(T')| = |L(T)| - 1$ and $|S(T')| = |S(T)| - 1$. By Lemma 10, we obtain the following result:

$$\begin{aligned}\iota^i(T) = \iota^i(T') &= \frac{n(T') - |L(T')| - |S(T')| + 3}{4} \\ &= \frac{n(T) - 2 - (|L(T)| - 1) - (|S(T)| - 1) + 3}{4} \\ &= \frac{n(T) - |L(T)| - |S(T)| + 3}{4}.\end{aligned}$$

Case 3: T is obtained from T' by operation O_3. Let T be obtained from T' by adding the path $P_4 = u_1u_2u_3u_4$ and the edge u_4v where $v \in L_w(T')$. Then we have $n(T') = n(T)-4$, $|L(T')| = |L(T)|$ and $|S(T')| = |S(T)|$. By Lemma 3(ii), we have the following result:

$$\begin{aligned}\iota^i(T) = \iota^i(T')+1 &= \frac{n(T')-|L(T')|-|S(T')|+3}{4}+1\\ &= \frac{n(T)-4-|L(T)|-|S(T)|+3}{4}+1\\ &= \frac{n(T)-|L(T)|-|S(T)|+3}{4}.\end{aligned}$$

This completes the proof. □

Theorem 12. *For any nontrivial tree T, $\iota^i(T) = \frac{n(T)-|L(T)|-|S(T)|+3}{4}$ if and only if $T \in \mathscr{D}$.*

Proof. By Lemma 11, we know that if $T \in \mathscr{D}$, then $\iota^i(T) = \frac{n(T)-|L(T)|-|S(T)|+3}{4}$. Now we are ready to prove that if $\iota^i(T) = \frac{n(T)-|L(T)|-|S(T)|+3}{4}$, then $T \in \mathscr{D}$. Suppose, to the contrary, that $T \notin \mathscr{D}$, we obtain that $\iota(T) > \frac{n(T)-|L(T)|-|S(T)|+3}{4}$ by Theorem 9. Since $\iota^i(T) \geq \iota(T)$ for every tree T, we can deduce that $\iota^i(T) > \frac{n(T)-|L(T)|-|S(T)|+3}{4}$, a contradiction. Therefore, $T \in \mathscr{D}$, which completes the proof. □

Since $\iota^i(T) \geq \iota(T)$, it follows that the value $\frac{n(T)-|L(T)|-|S(T)|+3}{4}$ represents a lower bound for $\iota^i(T)$. As an immediate consequence of Lemma 11 and Theorem 12, we have the following result.

Theorem 13. *If T is a nontrivial tree, then $\iota^i(T) \geq \frac{n(T)-|L(T)|-|S(T)|+3}{4}$ with equality if and only if $T \in \mathscr{D}$.*

4. Conclusion

It is clear that $\iota(T) \leq \iota^i(T)$ for every tree T. One naturally asks whether it is true that $\iota(T) = \iota^i(T)$ for every tree T. The answer is negative. For example, $\iota(S_{p,q}) = 2 < \iota^i(S_{p,q})$, where $S_{p,q}$ is defined as a tree obtained by connecting p copies of P_2 and q copies of P_2 to the two vertices u and v of an edge uv for $p, q \geq 2$, respectively.

As an immediate consequence of the results in Sections 2 and 3, we conclude the following.

Corollary 14. *If $T \in \mathscr{T} \cup \mathscr{D}$, then $\iota(T) = \iota^i(T)$.*

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

References

[1] G. Boyer and W. Goddard, “Disjoint isolating sets and graphs with maximum isolation number”, *Discrete Appl. Math.* **356** (2024), pp. 110–116.

[2] A. Cabrera-Martínez, “A new lower bound for the independent domination number of a tree”, *RAIRO, Oper. Res.* **57** (2023), no. 4, pp. 1951–1956.

[3] A. Cabrera-Martínez, “An improved upper bound on the domination number of a tree”, *Discrete Appl. Math.* **343** (2024), pp. 44–48.

[4] A. Cabrera-Martínez and A. Conchado Peiró, “On the {2}-domination number of graphs”, *AIMS Math.* **7** (2022), no. 6, pp. 10731–10743.

[5] A. Cabrera-Martínez, A. C. Peiró and J. M. Rueda-Vázquez, "Further results on the total Italian domination number of trees", *AIMS Math.* **8** (2023), no. 5, pp. 10654–10664.

[6] Y. Caro and A. Hansberg, "Partial domination—the isolation number of a graph", *Filomat* **31** (2017), no. 12, pp. 3925–3944.

[7] M. Chellali and N. Meddah, "Trees with equal 2-domination and 2-independence numbers", *Discuss. Math., Graph Theory* **32** (2012), no. 2, pp. 263–270.

[8] X.-G. Chen, K. Yin and T. Gao, "A note on independent vertex-edge domination in graphs", *Discrete Optim.* **25** (2017), pp. 1–5.

[9] N. Dehgardi, S. M. Sheikholeslami, M. Valinavaz, H. Aram and L. Volkmann, "Domination number, independent domination number and 2-independence number in trees", *Discuss. Math., Graph Theory* **41** (2021), no. 1, pp. 39–49.

[10] O. Favaron, "A bound on the independent domination number of a tree", *Vishwa Int. J. Graph Theory* **1** (1992), no. 1, pp. 19–27.

[11] O. Favaron and P. Kaemawichanurat, "Inequalities between the K_k-isolation number and the independent K_k-isolation number of a graph", *Discrete Appl. Math.* **289** (2021), pp. 93–97.

[12] B. Krishnakumari, Y. B. Venkatakrishnan and M. Krzywkowski, "Bounds on the vertex-edge domination number of a tree", *C. R. Math.* **352** (2014), no. 5, pp. 363–366.

[13] M. Lemańska, "Lower bound on the domination number of a tree", *Discuss. Math., Graph Theory* **24** (2004), no. 2, pp. 165–169.

[14] J. R. Lewis, *Vertex-edge and edge-vertex domination in graphs*, PhD thesis, Clemson University (USA), 2007.

[15] J. R. Lewis, S. T. Hedetniemi, T. W. Haynes and G. H. Fricke, "Vertex-edge domination", *Util. Math.* **81** (2010), pp. 193–213.

[16] D.-X. Ma and X.-G. Chen, "A note on connected bipartite graphs having independent domination number half their order", *Appl. Math. Lett.* **17** (2004), no. 8, pp. 959–962.

[17] J. D. McFall and R. Nowakowski, "Strong independence in graphs", *Congr. Numerantium* **29** (1980), pp. 639–656.

[18] G. Zhang and B. Wu, "Domination, independent domination and k-independence in trees", *Taiwanese J. Math.* **26** (2022), no. 2, pp. 221–231.

[19] G. Zhang and B. Wu, "2-independent domination in trees", *J. Oper. Res. Soc. China* **12** (2024), no. 2, pp. 485–494.

[20] S. Zhang, T. Jia and M. Li, "Partial domination of network modelling", *AIMS Math.* **8** (2023), no. 10, pp. 24225–24232.