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Nonvariational double phase problems with variable exponents depending on the gradient of the solution with convection term

Problèmes à double phase non variationnels avec exposants variables dépendant du gradient de la solution avec terme de convection

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Abstract. In this work, we study nonvariational double phase problems driven by a novel operator with variable exponents depending on the solutions and its gradient, along with a convection term. Using the Galerkin method, we prove the existence of solutions and apply a truncation technique to demonstrate multiplicity. Additionally, we address the uniqueness of solutions. The paper presents self-contained techniques to prove the multiplicity result, which may also be useful for addressing other problems with a convection term.

Résumé. Dans ce travail, nous étudions des problèmes à double phase non variationnels gouvernés par un nouvel opérateur à exposants variables dépendant de la solution et de son gradient, auxquels s'ajoute un terme de convection. En utilisant la méthode de Galerkin, nous établissons l'existence de solutions et appliquons une technique de troncature pour démontrer la multiplicité. Nous abordons également la question de l'unicité des solutions. L'article présente des techniques autonomes permettant d'établir des résultats de multiplicité, lesquelles peuvent également être utiles pour traiter d'autres problèmes comportant un terme de convection.

Keywords. Double phase operators, multiplicity results, gradient term, variable exponents, nonvariational methods.

Mots-clés. Opérateurs à double phase, résultats de multiplicité, terme en gradient, exposants variables, méthodes non variationnelles.

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1. Introduction

Since the early 2000s, variable exponent problems have been widely studied for their applications in fields such as image processing, where they are particularly effective in image denoising. This approach allows the exponents to depend on the solution and its gradient, as in problems driven by operators like the $p[\nabla u]$-Laplacian [1], $p(u)$-Laplacian [8], and $p\big(x,[\nabla u]\big)$-Laplacian in [15].

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In this paper, we study the following variable exponents double phase problem:

$$-\operatorname{div}\big(|\nabla u|^{p(x,|\nabla u|)-2}\nabla u+\mu(x)|\nabla u|^{q(x,|\nabla u|)-2}\nabla u\big)+V(x)\big(|u|^{p(x,|u|)-2}u+\mu(x)|u|^{q(x,|u|)-2}u\big) = f(x,u,\nabla u), \quad x\in\mathbb{R}^d, \quad (\mathscr{P})$$

where $0\le\mu(\cdot)\in C^{0,1}(\mathbb{R}^d)$, $p,q\colon\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ and $f\colon\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ are three Carathéodory functions satisfying certain assumptions, while $V\colon\mathbb{R}^d\to\mathbb{R}$ is a measurable function. Since the nonlinear term f depends on ∇u, the problem exhibits a nonvariational structure, making the existence of multiple solutions more challenging and complex. Moreover, few studies address this topic (see [17]). This paper aims to bridge this gap by introducing a novel approach that combines the Galerkin method with a truncation technique.

The problem $(\mathscr{P})$ is governed by a novel operator that falls within the class of so-called double phase operators. These operators originate from the study of functionals of the form

$$v\longmapsto\int_{\mathbb{R}^d}|\nabla v|^p+\mu(x)|\nabla v|^q\,\mathrm{d}x,\quad 1<p<q<d, \tag{1}$$

as introduced by Zhikov [18] to model strongly anisotropic materials. In the past decade, several authors have studied functionals of the form (1), particularly focusing on the regularity properties of local minimizers. We refer to the works [5,9,11] for detailed discussions. Double phase differential operators and the corresponding energy functionals have attracted significant attention in various physical applications, such as transonic flows (see Bahrouni–Rădulescu–Repovš [4]), quantum physics (see Benci–D'Avenia–Fortunato–Pisani [6]), reaction-diffusion systems (see Cherfils–Il'yasov [7]), and non-Newtonian fluids (see Liu–Dai [16]).

The study of the double phase operator and its related functional spaces was notably advanced in 2022 by Crespo-Blanco et al. in [10]. Their work delves into a quasilinear elliptic equation, expressed through the following double phase operator with variable exponents:

$$\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u+\mu(x)|\nabla u|^{q(x)-2}\nabla u\big), \tag{2}$$

where $p,q\in C(\overline{\Omega})$, with $1<p(x)<d$ and $p(x)<q(x)$ for all $x\in\Omega$, while $0\le\mu(\cdot)\in L^\infty(\Omega)$.

In this work, we focus on a new type of double phase operator with variable exponents given by

$$\operatorname{div}\big(|\nabla u|^{p(x,|\nabla u|)-2}\nabla u+\mu(x)|\nabla u|^{q(x,|\nabla u|)-2}\nabla u\big)+V(x)\big(|u|^{p(x,|u|)-2}u+\mu(x)|u|^{q(x,|u|)-2}u\big), \tag{3}$$

which has recently been studied in [3]. The key novelty of this operator, compared to previously studied double phase operators, lies in the dependence of the exponents on the gradient of the solution, introducing new and significant challenges. To the best of our knowledge, the works in [2,3] were the first to explore such problems, where the variable exponent was treated as a special case within a broader framework, further highlighting the originality of this study.

The operator (3) is well-defined in the Musielak–Orlicz Sobolev space $W^{1,\mathscr{H}}_V(\mathbb{R}^d)$ (see Section 2), where $\mathscr{H}\colon\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ is defined by $\mathscr{H}(x,t):=\int_0^{|t|}h(x,\eta)\,\mathrm{d}\eta$, with

$$h(x,t)=\begin{cases}a\big(x,|t|\big)t, & \text{for } t\neq 0,\\ 0, & \text{for } t=0,\end{cases}\qquad\text{and}\qquad a\big(x,|t|\big)=|t|^{p(x,|t|)-2}+\mu(x)|t|^{q(x,|t|)-2}. \tag{4}$$

Here $p,q\colon\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ are two Carathéodory functions that satisfy the following assumptions:

(H) (i) the functions p and q are bounded, that is,

$$2\le p^-:=\operatorname*{ess\,inf}_{(x,t)\in\mathbb{R}^d\times\mathbb{R}_+}p(x,t)\le p^+:=\operatorname*{ess\,sup}_{(x,t)\in\mathbb{R}^d\times\mathbb{R}_+}p(x,t)<d,$$

$$2\le q^-:=\operatorname*{ess\,inf}_{(x,t)\in\mathbb{R}^d\times\mathbb{R}_+}q(x,t)\le q^+:=\operatorname*{ess\,sup}_{(x,t)\in\mathbb{R}^d\times\mathbb{R}_+}q(x,t)<+\infty,$$

and

$$p(x,t)<q(x,t)<p^-_*:=\frac{dp^-}{d-p^-}\quad\text{for a.a. } x\in\mathbb{R}^d \text{ and for all } t\ge 0;$$

(ii) the functions $t \mapsto p(x,t)$ and $t \mapsto q(x,t)$ exhibit a constant behavior equal $p(x,t) := p(x,1)$ and $q(x,t) := q(x,1)$, respectively, for all $t \in [0,1]$ and a nondecreasing behavior for $t \geq 1$;

(iii) there exist $c_p, c_q > 0$ such that

$$\left|p(x,t)-p(y,t)\right| \leq c_p|x-y| \qquad \text{and} \qquad \left|q(x,t)-q(y,t)\right| \leq c_q|x-y|,$$

for all $t \geq 0$ and for all $x, y \in \mathbb{R}^d$;

(iv) $\dfrac{q^+}{p^-} < 1 + \dfrac{1}{d}$.

Since problem $(\mathscr{P})$ is defined in the whole space $\mathbb{R}^d$, to address the issue of lack of compactness, we apply a recent embedding result [3, Theorem 1.4]. For this purpose, we assume, in addition to condition (H), the following hypothesis related to the potential V:

(V) (i) $V \in C(\mathbb{R}^d, \mathbb{R})$ and there exists $V_0 > 0$ such that $V(x) \geq V_0$ for any $x \in \mathbb{R}^d$;
(ii) the set $\{x \in \mathbb{R}^d : V(x) < L\}$ has finite Lebesgue measure for each $L > 0$.

Another distinctive feature of our problem $(\mathscr{P})$ is the presence of the gradient term ∇u in the nonlinearity f, commonly referred to in the literature as the "convection term." This presence renders the problem $(\mathscr{P})$ nonvariational. Elliptic problems with convection terms, particularly those concerning the existence of multiple solutions, present significant challenges due to their nonvariational nature and their importance in modeling various phenomena. For a discussion on the physical motivations behind these problems, see [12,13]. To tackle these challenges, we develop specialized tools designed for this specific context. Then, the primary aim of this paper is to bridge this gap by employing a combination of Galerkin and truncation methods (see Theorem 9). Additionally, we address the uniqueness of solutions, see Theorem 6.

For the nonlinearity f, we suppose the following hypotheses.

(H_f) Let $f\colon \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory function such that $f(\cdot,0,\cdot) < 0$ and the following hold:

(i) there exist a generalized N-function $\mathscr{B}(x,t) = \displaystyle\int_0^t b(x,s)\,\mathrm{d}s$, $\beta \in L^{\widetilde{\mathscr{B}}}(\mathbb{R}^d, \mathbb{R}_+^*)$ and $\kappa_1, \kappa_2 > 0$ such that

$$\left|f(x,t,\xi)\right| \leq \beta(x) + \kappa_1 b\big(x,|t|\big) + \kappa_2|\xi|^{p(x,|\xi|)-1},$$

for a.a. $x \in \mathbb{R}^d$, for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^d$, and

$$\begin{cases} q^+ < b^- := \inf\limits_{t>0} \dfrac{b(x,t)\,t}{\mathscr{B}(x,t)} \leq b^+ := \sup\limits_{t>0} \dfrac{b(x,t)\,t}{\mathscr{B}(x,t)} < p_*^-, \text{ for a.a. } x \in \mathbb{R}^d \text{ and all } t \geq 0, \\ C_1 \leq \mathscr{B}(x,1) \leq C_2, \text{ for a.a. } x \in \mathbb{R}^d, \text{ with } C_1, C_2 > 0, \\ \limsup\limits_{|t|\to 0} \dfrac{\mathscr{B}(x,t)}{\mathscr{H}(x,t)} < \infty, \text{ uniformly in } x \in \mathbb{R}^d, \text{ and } \mathscr{B} \ll \mathscr{H}_*, \end{cases} \tag{5}$$

where $\ll$ and $\widetilde{\mathscr{B}}$ are defined in Definition 2, while $\mathscr{H}_*$ is introduced in Definition 3;

(ii) there exist $\eta_- < 0$ and $\eta_+ > 0$ such that

$$f(x,\eta_+,\xi) < 0 < f(x,\eta_-,\xi), \quad \text{for a.a. } x \in \mathbb{R}^d \text{ and all } \xi \in \mathbb{R}^d.$$

Remark 1. The following function satisfies the condition (H_f):

$$f(x,t,\xi) = \vartheta(t)\Big(\beta(x) + \kappa_1 b\big(x,|t|\big) + \kappa_2|\xi|^{p(x,|\xi|)-1}\Big),$$

with $\vartheta \in C(\mathbb{R}^d)$, bounded, $\vartheta(0), \vartheta(1) < 0$ and $\vartheta(-1) > 0$.

2. Preliminaries and notations

In this section, we work under assumptions (H) and (V), and we recall the main properties of the function spaces associated with problem ($\mathscr{P}$); see [2,3]. To this end, let $M(\mathbb{R}^d)$ denote the space of all measurable functions $u\colon \mathbb{R}^d \to \mathbb{R}$. Before proceeding further, we present the following basic definitions concerning the relation between two generalized N-functions, as well as the Sobolev conjugate of the N-function $\mathscr{H}$.

Definition 2. *Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be two generalized N-functions.*

(1) *We say that $\mathscr{B}_1$ increases essentially slower than $\mathscr{B}_2$ near infinity and we write $\mathscr{B}_1 \ll \mathscr{B}_2$, if for any $k>0$*
$$\lim_{t\to\infty}\frac{\mathscr{B}_1(x,kt)}{\mathscr{B}_2(x,t)}=0,\quad \textit{uniformly in } x\in\mathbb{R}^d.$$
(2) *The function $\widetilde{\mathscr{B}_1}\colon \mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ defined by*
$$\widetilde{\mathscr{B}_1}(x,t):=\sup_{\tau\geq 0}\big(t\tau-\mathscr{B}_1(x,\tau)\big),\quad \textit{for all } x\in\mathbb{R}^d \textit{ and all } t\geq 0, \tag{6}$$
is called the complementary function of $\mathscr{B}_1$.

Definition 3. *The Sobolev conjugate of $\mathscr{H}$ is defined as the generalized N-function $\mathscr{H}_*$ obeying*
$$\mathscr{H}_*(x,t):=\widehat{\mathscr{H}}\big(x,\mathscr{N}^{-1}(x,t)\big),\quad \textit{for } x\in\mathbb{R}^d \textit{ and } t\geq 0,$$
where $\mathscr{N}\colon \mathbb{R}^d\times[0,+\infty)\to[0,+\infty)$ is the function given by
$$\mathscr{N}(x,t):=\left(\int_0^t\left(\frac{\tau}{\widehat{\mathscr{H}}(x,\tau)}\right)^{\frac{1}{d-1}}\mathrm{d}\tau\right)^{\frac{1}{d'}},\quad \textit{for } x\in\mathbb{R}^d \textit{ and } t\geq 0,$$
with
$$\widehat{\mathscr{H}}(x,t):=\begin{cases}2\max\Big\{\mathscr{H}\big(x,\mathscr{H}^{-1}(x,1)\,t\big),2t-1\Big\}-1 & \textit{if } t\geq 1,\\ \limsup\limits_{|x|\to+\infty}\max\Big\{\mathscr{H}\big(x,\mathscr{H}^{-1}(x,1)\,t\big),2t-1\Big\} & \textit{if } 0\leq t<1,\end{cases}$$
for all $x\in\mathbb{R}^d$ and $d'=\frac{d}{d-1}$.

Invoking [2, Propositions 3.1 & 3.2], we know that $\mathscr{H}$ is a generalized N-function that satisfies the Δ_2-condition, namely:
$$p^-\leq\frac{h(x,t)\,t}{\mathscr{H}(x,t)}\leq q^+\quad\text{for all } t>0 \text{ and a.a. } x\in\mathbb{R}^d. \tag{7}$$
The corresponding modular to $\mathscr{H}$ is given by
$$\rho_{\mathscr{H}}(u)=\int_{\mathbb{R}^d}\mathscr{H}\big(x,|u|\big)\,\mathrm{d}x$$
and the associated Musielak–Orlicz space $L^{\mathscr{H}}(\mathbb{R}^d)$ is then defined by
$$L^{\mathscr{H}}(\mathbb{R}^d)=\big\{u\in M(\mathbb{R}^d)\,:\,\rho_{\mathscr{H}}(u)<+\infty\big\}$$
endowed with the Luxemburg norm
$$\|u\|_{\mathscr{H}}=\inf\Big\{\tau>0\,:\,\rho_{\mathscr{H}}\Big(\frac{u}{\tau}\Big)\leq 1\Big\}.$$
Similarly, the Musielak–Orlicz Sobolev space $W^{1,\mathscr{H}}(\mathbb{R}^d)$ is defined by
$$W^{1,\mathscr{H}}(\mathbb{R}^d)=\big\{u\in L^{\mathscr{H}}(\mathbb{R}^d)\,:\,|\nabla u|\in L^{\mathscr{H}}(\mathbb{R}^d)\big\}$$
equipped with the norm $\|u\|_{1,\mathscr{H}}=\|u\|_{\mathscr{H}}+\|\nabla u\|_{\mathscr{H}}$, where $\|\nabla u\|_{\mathscr{H}}=\big\||\nabla u|\big\|_{\mathscr{H}}$. Next, we define the weighted Musielak–Orlicz Sobolev space $W_V^{1,\mathscr{H}}(\mathbb{R}^d)$ by
$$W_V^{1,\mathscr{H}}(\mathbb{R}^d):=\left\{u\in W^{1,\mathscr{H}}(\mathbb{R}^d)\,:\,\rho_{\mathscr{H}_V}(u)=\int_{\mathbb{R}^d}V(x)\mathscr{H}\big(x,|u|\big)<+\infty\right\}.$$

We know that $L^{\mathscr{H}}(\mathbb{R}^d)$, $W^{1,\mathscr{H}}(\mathbb{R}^d)$ and $W_V^{1,\mathscr{H}}(\mathbb{R}^d)$ are reflexive Banach spaces and we can equip $W_V^{1,\mathscr{H}}(\mathbb{R}^d)$ with the norm $\|u\| := \|\nabla u\|_{\mathscr{H}} + \|u\|_{\mathscr{H}_V}$ where $\|u\|_{\mathscr{H}_V} = \inf\{\lambda > 0 : \rho_{\mathscr{H}_V}\big(\frac{|u|}{\lambda}\big) \le 1\}$, see [2].

Moreover, the modular $\rho(u) := \rho_{\mathscr{H}}\big(|\nabla u|\big) + \rho_{\mathscr{H}_V}\big(|u|\big)$ satisfies

$$\min\big\{\|u\|^{p^-}, \|u\|^{q^+}\big\} \le \rho(u) \le \max\big\{\|u\|^{p^-}, \|u\|^{q^+}\big\} \quad \text{for all } u \in W_V^{1,\mathscr{H}}(\mathbb{R}^d), \tag{8}$$

and the embedding

$$W_V^{1,\mathscr{H}}(\mathbb{R}^d) \hookrightarrow L^{\mathscr{R}}(\mathbb{R}^d) \tag{9}$$

is compact, where $\mathscr{R}$ is a generalized N-function satisfying (5) (see [3, Theorem 1.3]).

Let $A\colon W_V^{1,\mathscr{H}}(\mathbb{R}^d) \to \big(W_V^{1,\mathscr{H}}(\mathbb{R}^d)\big)^*$ be the nonlinear map defined by

$$\begin{aligned}\langle A(u), v\rangle := \int_{\mathbb{R}^d} &\big(|\nabla u|^{p(x,|\nabla u|)-2}\nabla u + \mu(x)|\nabla u|^{q(x,|\nabla u|)-2}\nabla u\big)\cdot\nabla v\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^d} V(x)\big(|u|^{p(x,|u|)-2}u + \mu(x)|u|^{q(x,|u|)-2}u\big)v\,\mathrm{d}x\end{aligned} \tag{10}$$

for all $u, v \in W_V^{1,\mathscr{H}}(\mathbb{R}^d)$, where $\langle\cdot,\cdot\rangle$ is the duality pairing between $W_V^{1,\mathscr{H}}(\mathbb{R}^d)$ and its dual space $\big(W_V^{1,\mathscr{H}}(\mathbb{R}^d)\big)^*$. The operator $A\colon W_V^{1,\mathscr{H}}(\mathbb{R}^d) \to \big(W_V^{1,\mathscr{H}}(\mathbb{R}^d)\big)^*$ has the following properties, see [3, Theorem 4.3].

Proposition 4. *Assume that hypotheses* (H) *and* (V) *hold. Then, the operator A defined in* (10) *is bounded, continuous, and a homeomorphism. Furthermore, A is strictly monotone and of type (S_+); that is, if $u_n \rightharpoonup u$ in $W_V^{1,\mathscr{H}}(\mathbb{R}^d)$ and*

$$\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \le 0,$$

then $u_n \to u$ in $W_V^{1,\mathscr{H}}(\mathbb{R}^d)$.

3. Existence and multiplicity results

In this section, we introduce and establish our main results concerning the existence and multiplicity of solutions.

Definition 5. *We say that $u \in W_V^{1,\mathscr{H}}(\mathbb{R}^d)$ is a weak solution of problem $(\mathscr{P})$ if*

$$\begin{aligned}\int_{\mathbb{R}^d} &\big(|\nabla u|^{p(x,|\nabla u|)-2}\nabla u + \mu(x)|\nabla u|^{q(x,|\nabla u|)-2}\nabla u\big)\cdot\nabla v\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^d} V(x)\big(|u|^{p(x,|u|)-2} + \mu(x)|u|^{q(x,|u|)-2}\big)uv\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} f(x,u,\nabla u)v\,\mathrm{d}x, \quad \forall\, v \in W_V^{1,\mathscr{H}}(\mathbb{R}^d).\end{aligned} \tag{11}$$

The following proposition follows immediately from Proposition 4.

Theorem 6. *Assume that hypotheses* (H) *and* (V) *hold. Suppose that*

$$f(x,t,\xi) = \beta(x), \quad \textit{for } (x,t,\xi) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d,$$

where β and $\mathscr{B}$ are defined in (H_f)(i). *Then, problem $(\mathscr{P})$ admits a unique nontrivial weak solution.*

Proof. Since $\mathscr{B}$ satisfies assumptions (5), it follows from (9) that $W_V^{1,\mathscr{H}}(\mathbb{R}^d) \hookrightarrow L^{\mathscr{B}}(\mathbb{R}^d)$, hence $\beta \in L^{\widetilde{\mathscr{B}}}(\mathbb{R}^d) = \big(L^{\mathscr{B}}(\mathbb{R}^d)\big)^* \hookrightarrow \big(W_V^{1,\mathscr{H}}(\mathbb{R}^d)\big)^*$. Consequently, the uniqueness of the weak solution is guaranteed by the bijectivity of the operator A (see Proposition 4). □

We present the following technical lemma, which will be useful in the subsequent analysis.

Lemma 7. *Assume that hypotheses* (H) *and* (V) *are satisfied. Then, the following inequality holds:*

$$\int_{\mathbb{R}^d} |\nabla u|^{p(x,|\nabla u|)-1}|u|\,dx \leq T_0\rho(u), \quad \text{where } T_0 = (p^+-1)\left(1+V_0^{\frac{-p^+}{p^-}}\right). \tag{12}$$

Proof. Define the function $h_1 : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ by $h_1(x,t) := |t|^{p(x,|t|)-2}t$ if $t \neq 0$ and equal to 0 if $t = 0$. From assumption (H), we can show that $\mathscr{H}_1(x,t) := \int_0^{|t|} h_1(x,s)\,\mathrm{d}s$ is a generalized N-function that satisfies the Δ_2-condition. To simplify the proof, we consider two cases.

Case 1. If $V_0 \geq 1$, applying Young's inequality and [2, Lemma 2.3], we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} h_1\big(x,|\nabla u|\big)|u|\,\mathrm{d}x &\leq \int_{\mathbb{R}^d}\Big(\widetilde{\mathscr{H}_1}\Big(x,h_1\big(x,|\nabla u|\big)\Big)+\mathscr{H}_1\big(x,|u|\big)\Big)\,\mathrm{d}x\\
&\leq \int_{\mathbb{R}^d}\big((p^+-1)\mathscr{H}_1\big(x,|\nabla u|\big)+V_0\mathscr{H}_1\big(x,|u|\big)\big)\,\mathrm{d}x\\
&\leq (p^+-1)\int_{\mathbb{R}^d}\Big(\mathscr{H}\big(x,|\nabla u|\big)+V(x)\mathscr{H}\big(x,|u|\big)\Big)\,\mathrm{d}x\\
&= (p^+-1)\rho(u).
\end{aligned} \tag{13}$$

Case 2. If $0 < V_0 < 1$, by Young's inequality and [2, Lemma 2.3 & Proposition 2.7], one has

$$\begin{aligned}
\int_{\mathbb{R}^d} h_1\big(x,|\nabla u|\big)|u|\,\mathrm{d}x &= \int_{\mathbb{R}^d}\frac{1}{V_0^{\frac{1}{p^-}}}h_1\big(x,|\nabla u|\big)V_0^{\frac{1}{p^-}}|u|\,\mathrm{d}x\\
&\leq \int_{\mathbb{R}^d}\left(\widetilde{\mathscr{H}_1}\left(x,\frac{1}{V_0^{\frac{1}{p^-}}}h_1\big(x,|\nabla u|\big)\right)+\mathscr{H}_1\big(x,V_0^{\frac{1}{p^-}}|u|\big)\right)\mathrm{d}x\\
&\leq \int_{\mathbb{R}^d}\frac{1}{V_0^{\frac{p^+}{p^-}}}\Big(\widetilde{\mathscr{H}_1}\Big(x,h_1\big(x,|\nabla u|\big)\Big)+V_0\mathscr{H}_1\big(x,|u|\big)\Big)\,\mathrm{d}x\\
&\leq \int_{\mathbb{R}^d}\left(\frac{p^+-1}{V_0^{\frac{p^+}{p^-}}}\mathscr{H}_1\big(x,|\nabla u|\big)+V_0\mathscr{H}_1\big(x,|u|\big)\right)\mathrm{d}x\\
&\leq \frac{p^+-1}{V_0^{\frac{p^+}{p^-}}}\int_{\mathbb{R}^d}\Big(\mathscr{H}_1\big(x,|\nabla u|\big)+V(x)\mathscr{H}_1\big(x,|u|\big)\Big)\,\mathrm{d}x\\
&\leq \frac{p^+-1}{V_0^{\frac{p^+}{p^-}}}\rho(u).
\end{aligned} \tag{14}$$

Hence, the inequality (12) holds from (13) and (14). □

We are now ready to establish the existence result.

Theorem 8. *Let hypotheses* (H), (H_f)(i), *and* (V) *hold. Furthermore, suppose that*

$$p^- - \kappa_2 T_0 > 0,$$

where κ_2 is defined in (H_f) and T_0 is given in Lemma 7. Then, there exists $\theta > 0$ such that if $\|\beta\|_{\widetilde{\mathscr{B}}} \leq \theta$, the problem $(\mathscr{P})$ admits at least one nontrivial weak solution.

Proof. The proof is divided into two steps.

Step 1. We prove the existence of an approximate sequence of solutions. The proof relies on the Galerkin approximative method. Recalling that $W_V^{1,\mathscr{H}}(\mathbb{R}^d)$ is a separable Banach space, then we can find a Galerkin basis $\{\phi_n\}_{n\in\mathbb{N}}$ of $W_V^{1,\mathscr{H}}(\mathbb{R}^d)$. For each positive integer $m \geq 1$, let $V_m = \{\phi_1, \phi_2, \dots, \phi_m\}$ be the m-dimensional subspace of $W_V^{1,\mathscr{H}}(\mathbb{R}^d)$. Notice that, for $\xi = (\xi_1, \xi_2, \dots, \xi_m) \in \mathbb{R}^m$, the real number $|\xi|_m = \left\|\sum_{j=1}^m \xi_j \phi_j\right\|$ defines a norm in $\mathbb{R}^m$. Hence, we can identify the normed space $\left(V_m, \|\cdot\|\right)$ and $\left(\mathbb{R}^m, |\cdot|_m\right)$ by the isometric isomorphism $u = \sum_{j=1}^m \xi_j \phi_j \leftrightarrow \xi = (\xi_1, \dots, \xi_m)$. We define the function $\Psi_m \colon \mathbb{R}^m \to \mathbb{R}^m$ using that final isometric linear transformation by

$$\Psi_m(\xi) = \Psi_m(\xi_1, \dots, \xi_m) = \left(\Psi_m^{(1)}(\xi), \Psi_m^{(2)}(\xi), \dots, \Psi_m^{(m)}(\xi)\right),$$

where

$$\begin{aligned}\Psi_m^{(j)}(\xi) = \int_{\mathbb{R}^d} &|\nabla u|^{p(x,|\nabla u|)-2} \nabla u \nabla \phi_j + \mu(x) |\nabla u|^{q(x,|\nabla u|)-2} \nabla u \nabla \phi_j \, \mathrm{d}x \\ &+ \int_{\mathbb{R}^d} V(x)\left(|u|^{p(x,|u|)-2} u \phi_j + \mu(x) |u|^{q(x,|u|)-2} u \phi_j\right) \mathrm{d}x \\ &- \int_{\mathbb{R}^d} f(x, u, \nabla u)\, \phi_j \, \mathrm{d}x, \quad \forall\, 1 \leq j \leq m,\end{aligned}$$

with $u = \sum_{j=1}^m \xi_j \phi_j$.

If $\|u\| \leq R := \min\left\{1, \left(\frac{p^- - \kappa_2 T_0}{2(\kappa_1 b^+ + 1)\overline{\gamma_{\mathscr{B}}}}\right)^{\frac{1}{b^- - q^+}}\right\}$, we infer, from (H_f)(i), (7), (8), Young's inequality and Lemma 7, that

$$\begin{aligned}\left\langle \Psi_m(\xi), \xi \right\rangle &= \int_{\mathbb{R}^d} \left(|\nabla u|^{p(x,|\nabla u|)} + \mu(x) |\nabla u|^{q(x,|\nabla u|)} + V(x)\left(|u|^{p(x,|u|)} + \mu(x)|u|^{q(x,|u|)}\right)\right) \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^d} f(x, u, \nabla u) u \, \mathrm{d}x \\ &\geq p^- \rho(u) - \int_{\mathbb{R}^d} \kappa_2 |\nabla u|^{p(x,|\nabla u|)-1} |u| \, \mathrm{d}x - \kappa_1 \int_{\mathbb{R}^d} b\left(x, |u|\right) |u| \, \mathrm{d}x - \int_{\mathbb{R}^d} \beta(x) |u| \, \mathrm{d}x \\ &\geq (p^- - \kappa_2 T_0) \rho(u) - \kappa_1 b^+ \int_{\mathbb{R}^d} \mathscr{B}\left(x, |u|\right) \mathrm{d}x - \int_{\mathbb{R}^d} \mathscr{B}\left(x, |u|\right) \mathrm{d}x - \int_{\mathbb{R}^d} \widetilde{\mathscr{B}}\left(x, \beta(x)\right) \mathrm{d}x \\ &\geq (p^- - \kappa_2 T_0) \|u\|^{q^+} - (\kappa_1 b^+ + 1) \max\{\|u\|_{\mathscr{B}}^{b^-}, \|u\|_{\mathscr{B}}^{b^+}\} - \int_{\mathbb{R}^d} \widetilde{\mathscr{B}}\left(x, \beta(x)\right) \mathrm{d}x \\ &\geq (p^- - \kappa_2 T_0) \|u\|^{q^+} - (\kappa_1 b^+ + 1) \overline{\gamma_{\mathscr{B}}} \|u\|^{b^-} - \int_{\mathbb{R}^d} \widetilde{\mathscr{B}}\left(x, \beta(x)\right) \mathrm{d}x \\ &\geq \frac{(p^- - \kappa_2 T_0)}{2} \|u\|^{q^+} - \int_{\mathbb{R}^d} \widetilde{\mathscr{B}}\left(x, \beta(x)\right) \mathrm{d}x,\end{aligned}$$

where $\overline{\gamma_{\mathscr{B}}} = \max\{\gamma_{\mathscr{B}}^{b^+}, \gamma_{\mathscr{B}}^{b^+}\}$ with $\gamma_{\mathscr{B}}$ denotes the Sobolev embedding constant of $W_V^{1,\mathscr{H}}(\mathbb{R}^d)$ into $L^{\mathscr{B}}(\mathbb{R}^d)$. Hence, if $\|\beta\|_{\widetilde{\mathscr{B}}} \leq \theta = \min\left\{\left(\frac{(p^- - \kappa_2 T_0)}{2} R^{q^+}\right)^{\frac{b^- - 1}{b^-}}, \left(\frac{(p^- - \kappa_2 T_0)}{2} R^{q^+}\right)^{\frac{b^+ - 1}{b^+}}\right\}$, we have $\left\langle \Psi_m(\xi), \xi \right\rangle > 0$ when $|\xi|_m = \|u\| = R$. Therefore, in view of the Brouwer fixed point theorem (see [14, Theorem 5.2.5]), there exists $\widetilde{\xi} = (\widetilde{\xi}_1, \dots, \widetilde{\xi}_m) \in \mathbb{R}^m$ such that $|\widetilde{\xi}|_m \leq R$ and $\Psi_m(\widetilde{\xi}) = (0, \dots, 0) \in \mathbb{R}^m$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^d} &\left(|\nabla u_m|^{p(x,|\nabla u_m|)-2} \nabla u_m \nabla \phi_j + \mu(x) |\nabla u_m|^{q(x,|\nabla u_m|)-2} \nabla u_m \nabla \phi_j\right) \mathrm{d}x \\ &+ \int_{\mathbb{R}^d} V(x)\left(|u_m|^{p(x,|u_m|)-2} u_m \phi_j + \mu(x) |u_m|^{q(x,|u_m|)-2} u_m \phi_j\right) \mathrm{d}x \\ &= \int_{\mathbb{R}^d} f(x, u_m, \nabla u_m) \phi_j \, \mathrm{d}x, \quad \forall\, 1 \leq j \leq m,\end{aligned}$$

where $u_m = \sum_{j=1}^m \tilde{\xi}_j \phi_j$. It yields

$$\begin{aligned}\int_{\mathbb{R}^d} \big(|\nabla u_m|^{p(x,|\nabla u_m|)-2} \nabla u_m \nabla v + \mu(x) |\nabla u_m|^{q(x,|\nabla u_m|)-2} \nabla u_m \nabla v\big)\,\mathrm{d}x \\ + \int_{\mathbb{R}^d} V(x)\big(|u_m|^{p(x,|u_m|)-2} u_m v + \mu(x)|u_m|^{q(x,|u_m|)-2} u_m v\big)\,\mathrm{d}x \\ = \int_{\mathbb{R}^d} f(x,u_m,\nabla u_m) v\,\mathrm{d}x, \quad \forall\, v \in V_m. \quad (15)\end{aligned}$$

Step 2. Passage to the limit as $m \to +\infty$. Given that R is independent of m and considering the fact that $|\tilde{\xi}|_m = \|u_m\| < R$, we conclude that there exists a function $u \in W_V^{1,\mathscr{H}}(\mathbb{R}^d)$ such that $u_m \rightharpoonup u$ weakly in $W_V^{1,\mathscr{H}}(\mathbb{R}^d)$, and $u_m(x) \to u(x)$ a.e. in $\mathbb{R}^d$. First, from (H_f)(i), using Hölder's inequality, [2, Lemma 2.3] and the embedding (9), one has

$$\left|\int_{\mathbb{R}^d} f(x,u,\nabla u) v\,\mathrm{d}x\right| \le \bar{\gamma}\big((b^+-1)\|u\|_{\mathscr{B}} + (p^+-1)\|\nabla u\|_{\mathscr{H}_1} + \|\beta\|_{\tilde{\mathscr{B}}}\big)\|v\|, \quad (16)$$

for all $u, v \in W_V^{1,\mathscr{H}}(\mathbb{R}^d)$, where $\bar{\gamma} = \max\{\gamma_{\mathscr{B}}, \gamma_{\mathscr{H}_1}\}$. The inequality (16) shows that the Nemytskii operator $N_f\colon W_V^{1,\mathscr{H}}(\mathbb{R}^d) \to \big(W_V^{1,\mathscr{H}}(\mathbb{R}^d)\big)^*$ corresponding to the Carathéodory function $f\colon \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, namely $N_f(u) := f\big(\cdot, u(\cdot), \nabla u(\cdot)\big)$, $\forall\, u \in W_V^{1,\mathscr{H}}(\mathbb{R}^d)$, is well-defined, bounded and continuous. Through the assumption (H_f)(i), in conjunction with Hölder's inequality and [2, Lemma 2.3], we arrive at

$$\left|\int_{\mathbb{R}^d} f(x,u_m,\nabla u_m)(u_m-u)\,\mathrm{d}x\right| \le (b^+-1)\Big(\big(\|u_m\|_{\mathscr{B}} + \|\beta\|_{\tilde{\mathscr{B}}}\big)\|u_m-u\|_{\mathscr{B}} + \|\nabla u_m\|_{\mathscr{H}_1}\|u_m-u\|_{\mathscr{H}_1}\Big),$$

for all $m \ge 1$. It follows, due to the boundedness of $\{u_m\}_{m\ge1}$ in $W_V^{1,\mathscr{H}}(\mathbb{R}^d)$, that

$$\left|\int_{\mathbb{R}^d} f(x,u_m,\nabla u_m)(u_m-u)\,\mathrm{d}x\right| \le M\big(\|u_m-u\|_{\mathscr{B}} + \|u_m-u\|_{\mathscr{H}_1}\big), \quad \forall\, m \ge 1,$$

for some constant $M > 0$. Hence, by (9), it follows that

$$\lim_{m\to+\infty}\left|\int_{\mathbb{R}^d} f(x,u_m,\nabla u_m)(u_m-u)\,\mathrm{d}x\right| = 0. \quad (17)$$

Based on (15) and (17), we conclude that

$$\begin{aligned}\int_{\mathbb{R}^d} \Big(a(x,|\nabla u_m|)\nabla u_m - a\big(x,|\nabla u|\big)\nabla u\Big)\nabla(u_m-u)\,\mathrm{d}x \\ + \int_{\mathbb{R}^d} V(x)\Big(a\big(x,|u_m|\big)u_m - a\big(x,|u|\big)u\Big)(u_m-u)\,\mathrm{d}x \\ = \int_{\mathbb{R}^d} f(x,u_m,\nabla u_m)(u_m-u)\,\mathrm{d}x \\ = o_m(1).\end{aligned}$$

Therefore, invoking Simon's inequality [3, Lemma 2.5], we get

$$\begin{aligned} o_m(1) &\ge 4\int_{\mathbb{R}^d}\left[\mathscr{H}\left(x, \frac{\big|\nabla(u_m-u)\big|}{2}\right) + V(x)\mathscr{H}\left(x, \frac{|u_m-u|}{2}\right)\right]\mathrm{d}x \\ &\ge \min\left\{\left\|\frac{u_m-u}{2}\right\|^{p^-}, \left\|\frac{u_m-u}{2}\right\|^{q^+}\right\} \\ &\ge 0.\end{aligned}$$

Hence, $u_m \to u$ in $W_V^{1,\mathscr{H}}(\mathbb{R}^d)$ as $m \to +\infty$. So we conclude that u is a weak solution of the problem $(\mathscr{P})$ and from (H_f)(i), we have that $u \not\equiv 0$. □

Consequently, we are able to present our multiplicity result.

Theorem 9. *Let the conditions of Theorem 8 and* (H_f)(ii) *hold. Then, if* $\|\beta\|_{\widetilde{\mathscr{B}}} \leq \theta$ (θ *is defined in Theorem* (8))*, there exist at least one nonpositive weak solution and at least one nonnegative weak solution to the problem* ($\mathscr{P}$).

Proof. Let us introduce the Carathéodory functions $f_{\pm} : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ by

$$f_+(x,t,\xi) := \begin{cases} f(x,t^+,\xi) & \text{if } t \leq \eta_+, \\ f(x,\eta_+,\xi) & \text{if } t > \eta_+, \end{cases} \qquad \text{and} \qquad f_-(x,t,\xi) := \begin{cases} f(x,t^-,\xi) & \text{if } t \geq \eta_-, \\ f(x,\eta_-,\xi) & \text{if } t < \eta_-, \end{cases} \tag{18}$$

where η_+ and η_- are defined in (H_f)(ii). It is clear that f_+ and f_- satisfy the assumption (H_f)(i). Then, invoking Theorem 8, there exists at least one weak solution, $u_0 \in W_V^{1,\mathscr{H}}(\mathbb{R}^d) \setminus \{0\}$, such that

$$\begin{aligned} -\operatorname{div}\Big(\big(|\nabla u_0|^{p(x,|\nabla u_0|)-2} + \mu(x)|\nabla u_0|^{q(x,|\nabla u_0|)-2}\big)\nabla u_0\Big) + V(x)h(x,u_0)& \\ = f_+(x,u_0,\nabla u_0), \quad x \in \mathbb{R}^d.& \end{aligned} \tag{19}$$

Then, we act with $v = u_0^- = \max\{0,-u_0\}$ in (11) with f_+ in place of f, one has

$$\begin{aligned} &\int_{\mathbb{R}^d} \big(|\nabla u_0|^{p(x,|\nabla u_0|)-2}\nabla u_0 + \mu(x)|\nabla u_0|^{q(x,|\nabla u_0|)-2}\nabla u_0\big)\cdot \nabla u_0^- \,\mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^d} V(x)\big(|u_0|^{p(x,|u_0|)-2} + \mu(x)|u_0|^{q(x,|u_0|)-2}\big)u_0 u_0^- \,\mathrm{d}x \\ &\qquad\qquad = \int_{\mathbb{R}^d} f_+(x,u_0,\nabla u_0)u_0^- \,\mathrm{d}x, \end{aligned}$$

which gives, by (7) and the truncation (18), that

$$\rho(u_0^-) \leq \int_{\mathbb{R}^d} \Big(\big(|\nabla u_0|^{p(x,|\nabla u_0|)} + \mu(x)|\nabla u_0|^{q(x,|\nabla u_0|)}\big) + V(x)\big(|u_0|^{p(x,|u_0|)} + \mu(x)|u_0|^{q(x,|u_0|)}\big)\Big)\,\mathrm{d}x \leq 0.$$

Hence, $u_0^- = 0$, and so $u_0 \geq 0$. It remains to show the boundedness of u_0. We act with $w = (u_0-\eta^+)^+ = \max\{u_0-\eta^+,0\}$ in (11) with f_+ in place of f and using (H_f)(ii), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^d} \big(|\nabla u_0|^{p(x,|\nabla u_0|)-2}\nabla u_0 + \mu(x)|\nabla u_0|^{q(x,|\nabla u_0|)-2}\nabla u_0\big)\cdot \nabla (u_0-\eta^+)^+ \,\mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^d} V(x)\big(|u_0|^{p(x,|u_0|)-2} + \mu(x)|u_0|^{q(x,|u_0|)-2}\big)u_0 (u_0-\eta^+)^+ \,\mathrm{d}x \\ &\qquad\qquad = \int_{\mathbb{R}^d} f_+(x,u_0,\nabla u_0)(u_0-\eta^+)^+ \,\mathrm{d}x \\ &\qquad\qquad = \int_{\mathbb{R}^d} f_+(x,\eta^+,\nabla u_0)(u_0-\eta^+)^+ \,\mathrm{d}x \\ &\qquad\qquad \leq 0. \end{aligned} \tag{20}$$

Then, from (20) and Simon's inequality [3, Lemma 2.5], we infer that

$$\begin{aligned} 0 &\geq \int_{\{u_0>\eta^+\}} \Big(h\big(x,|\nabla u_0|\big) - h\big(x,|\nabla \eta^+|\big)\Big)\cdot \nabla(u_0-\eta^+)\,\mathrm{d}x + \int_{\{u_0>\eta^+\}} \big(h(x,u_0)-h(x,\eta^+)\big)(u_0-\eta^+)\,\mathrm{d}x \\ &\geq 4\int_{\{u_0>\eta^+\}} \left[\mathscr{H}\left(x,\frac{|\nabla(u_0-\eta^+)|}{2}\right) + V(x)\mathscr{H}\left(x,\frac{|u_0-\eta^+|}{2}\right)\right]\mathrm{d}x \\ &= 4\int_{\mathbb{R}^d} \left[\mathscr{H}\left(x,\frac{|\nabla(u_0-\eta^+)^+|}{2}\right) + V(x)\mathscr{H}\left(x,\frac{|(u_0-\eta^+)^+|}{2}\right)\right]\mathrm{d}x. \end{aligned}$$

This implies that $(u_0-\eta^+)^+ \equiv 0$, meaning $u_0 \in [0,\eta^+]$. By applying the truncation (18), we conclude that u_0 is a nonnegative bounded weak solution to problem ($\mathscr{P}$).

Similarly, let v_0 be a nontrivial weak solution of problem (19), with f_- in place of f_+. Proceeding as before, we find that $v_0 \in [\eta^-,0]$. This completes the proof of Theorem 9. □

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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