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A dynamical system approach to the Chandrasekhar–Hamilton–Jacobi equation

Une approche systèmes dynamiques de l'équation de Chandrasekhar–Hamilton–Jacobi

Marie-Françoise Bidaut-Véron [a] and Laurent Véron [a]

[a] Institut Denis Poisson, CNRS UMR 7013, Université de Tours, 37200 Tours, France
E-mails: veronmf@univ-tours.fr, veronl@univ-tours.fr

To Haïm Brezis, outstanding mathematician, great mind and sincere friend

Abstract. We study the local properties of positive solutions of the equation $-\Delta u = e^u - M|\nabla u|^q$ in a punctured domain $\Omega \setminus \{0\}$ of $\mathbb{R}^N$ in the range of parameters $q > 1$ and $M > 0$. We prove a series of a priori estimates near a singular point. In the case of radial solutions we use various techniques inherited from the dynamical systems theory to obtain the precise behaviour of singular solutions. We prove also the existence of singular solutions with these precise behaviours.

Résumé. Nous étudions les propriétés locales des solutions de l'équation $-\Delta u = e^u - M|\nabla u|^q$ dans un domaine épointé $\Omega \setminus \{0\}$ de $\mathbb{R}^N$ avec des paramètres $q > 1$ et $M > 0$. Nous donnons une série d'estimations a priori près d'un point singulier. Dans le cas de solutions radiales, nous obtenons le comportement précis des solutions avec des méthodes issues de la théorie des systèmes dynamiques. Nous démontrons aussi l'existence de solutions singulières avec les comportements singuliers obtenus.

Keywords. Elliptic equations, limit sets, saddle points, stable manifolds, energy functions.

Mots-clés. Équations elliptiques, ensembles limites, points selles, variété stable, fonctions d'énergie.

2020 Mathematics Subject Classification. 35J62, 35B08, 37C25, 37C75.

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1. Introduction

The aim of this paper is to study properties of *radial* solutions of

$$\mathscr{L}_{m,p,q} u := -\Delta u + M|\nabla u|^q - e^u = 0 \quad \text{in } \Omega, \tag{1}$$

where M is a real positive number, $q > 1$ and Ω is either a punctured domain if we are interested in isolated singularities, or an exterior domain if we study the asymptotic behaviour of solutions.

Equation (1) belongs to the family of equations of diffusion-reaction-absorption type

$$-\Delta u \pm f\big(|\nabla u|\big) = g(u), \tag{2}$$

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where f is a power function and g a power or an exponential function.

For these equations, there are several deep problems:

- obtention of a priori estimates near a singularity or at infinity for equations in an exterior domain;
- description of singular solutions or behaviour at infinity of solutions;
- existence of global regular or singular solutions.

Equations with absorption correspond to the cases where $g(u) = -e^u$ or $g(u) = -u^p$. The equation

$$-\Delta u + M|\nabla u|^q + u^p = 0,$$

has been studied in [20] when $M > 0$ and in [4] when $M < 0$. The equation

$$-\Delta u + M|\nabla u|^q + e^u = 0,$$

with $M < 0$ is considered in [18], particularly in the most relevant dimension $N = 2$ since in higher dimension the isolated singularities are removable.

Next we focus on the equation where $g(u)$ acts as a source term. Equation with a power u^p with $p > 0$,

$$-\Delta u + M|\nabla u|^q - u^p = 0, \tag{3}$$

has been first considered with $M = 1$ in [13,14] in the radial case where two critical values appear: $p = \frac{N+2}{N-2}$ and $q = \frac{2p}{p+1}$. This study was developed in [23,24], putting into light the question of existence of global solutions, still not completely solved. The general problem with $M \in \mathbb{R}$ has been studied in the non-radial case in [21] when $p < \frac{N+2}{N-2}$ and $q < \frac{2p}{p+1}$, then in [2] when $q > \frac{2p}{p+1}$. The critical case $q = \frac{2p}{p+1}$ with $p > 1$ has been exhaustively solved in the radial case in [3].

In the present article we assume that $g(u)$ is an exponential source term and we consider equation (1) with $M > 0$. To study the properties of the solutions of (1) we associate three underlying equations:

- the *Emden–Chandrasekhar equation*,
$$-\Delta u - e^u = 0; \tag{4}$$
- the *viscous Hamilton–Jacobi equation*,
$$-\Delta u + M|\nabla u|^q = 0; \tag{5}$$
- and the *eikonal equation*,
$$M|\nabla u|^q - e^u = 0. \tag{6}$$

Each of these three equations has been already much studied.

The first exhaustive study of radial solutions of the Emden–Chandrasekhar equation is due to Chandrasekhar [12]. Non-radial solutions which are rather simple to study in the 2-dimensional case are described in [22]. In the 3-dimensional case the level of difficulties is not at all comparable and this is due to the fact that the reaction term is much larger than the classical Sobolev exponent $p = \frac{N+2}{N-2}$ which turns out to be equal to 5 since $N = 3$. It is proved in [5] that no uniform estimate near a singularity can hold, but if a solution u of (4) in $B_1 \setminus \{0\}$ (resp. B_1^c) satisfies $|x|^2 e^u \in L^\infty(B_1)$ (resp. $|x|^2 e^u \in L^\infty(B_1^c)$) then its behaviour near $x = 0$ (resp. when $|x| \to \infty$) can be completely described. Besides the upper estimate another severe problem of convergence occurs and is overcome by the introduction of deep results from complex geometry.

The viscous Hamilton–Jacobi equation is simpler to study. In the radial case it reduces to an explicitly integrable equation

$$-u_{rr} - \frac{N-1}{r} u_r + M|u_r|^q = 0.$$

This equation admits two critical exponents $q = q_c = \frac{N}{N-1}$ and $q = 2$. By using Bernstein method, it is proved in [17] that if $q > 1$ any not necessarily radial solution u of (5) in $B_1 \setminus \{0\}$ satisfies

$$\left|\nabla u(x)\right| \leq c|x|^{-\frac{1}{q-1}} \quad \text{in } B_{\frac{1}{2}} \setminus \{0\},$$

and that its behaviour near zero is similar to the one of the radial (and explicit) solutions, in particular if $q \geq q_c$ any nonnegative solution remains bounded, while if $1 < q < q_c$ and if it is singular either it satisfies $u(x) \sim c|x|^{2-N}$ for some $c \neq 0$, or $u(x) \sim \xi_M |x|^{\frac{q-2}{q-1}}$, where ξ_M is a specific positive constant expressed in (35).

The eikonal equation is also explicitly integrable in the radial case and it always reduces to an equation of the form

$$|\nabla v| = 1$$

with $v = qM^{\frac{1}{q}} e^{-\frac{u}{q}}$ in general. The uniqueness of viscosity solutions of the eikonal equation in $\mathbb{R}^N \setminus \{0\}$ is proved in [11].

In [7] we present a general study of non-necessarily radial solutions of (1) and emphasis on their singular solutions if they are defined in $B_1 \setminus \{0\}$, and their asymptotic behaviour if they defined in B_1^c. This study involves all the tools used in [5], often under a more refined manner, and $q = 2$ appears as a fundamental critical value. In particular if $1 < q < 2$ we have to impose the a priori bound $|x|^2 e^u \leq C$ in $B_1 \setminus \{0\}$ in order to obtain the description of singular solutions, while if $q > 2$ no a priori bound on a solution is needed.

In the present article we consider functions satisfying

$$-u_{rr} - \frac{N-1}{r} u_r + M|u_r|^q - e^u = 0, \tag{7}$$

with $q > 1$ and $M > 0$, either in $(0, 1]$ or in $[1, \infty)$. This assumption of radiality has the advantage of authorising the use of finite dimensional dynamical systems theory, a theory which allows us to go much deeper in the local study of the solutions. For $q \neq 2$ there is no invariance for (7) by a scaling transformation (the case $q = 2$ which reduces to a Lane–Emden type equation will not be considered). Consequently it is not possible to reduce this equation to a second order autonomous equation of any type as it is possible for (4), (5) and (6), and also for (3) when $q = \frac{2p}{p+1}$. Instead, we introduce various systems of order 3 to perform our analysis of singular and asymptotic behaviours: we give a precise description of singular solutions (or their asymptotic behaviour), and we prove the existence of solutions with all the possible behaviours that we have put into light. These existence results were out of reach for non-necessarily radial solutions studied in [7].

An important observation, dealing with the globality of the radial solutions is the following statement proved in Proposition 6 and Theorem 32.

Theorem A. *Let $q > 1$.*

(1) *If $N \geq 2$, the maximal interval of definition of a radial solution $r \mapsto u(r)$ of (7) defined in a right neighborhood of $r = 0$ and nondecreasing there is $(0, \infty)$. Furthermore $u_r < 0$ and u tends to $-\infty$ at ∞.*
(2) *If $1 < q < 2$ and $N \geq 2$, the maximal interval of definition of a solution $r \mapsto u(r)$ of (7) is $(0, \infty)$. If $1 < q < 2$ and $N = 1$ any maximal solution of (7) is defined on $\mathbb{R}$ and symmetric with respect to some value a.*

We obtain also a full description of the isolated singularities of solutions of (7).

Theorem B. *Let $1 < q < 2$ and $N \geq 3$. Then:*

(1) *either $\lim_{r\to 0}\left(u(r) - 2\ln\frac{1}{r}\right) = \ln 2(N-2)$;*
(2) *or there exists $u_0 \in \mathbb{R}$ such that $\lim_{r\to 0} u(r) = u_0$ and $\lim_{r\to 0} u_r(r) = 0$ (such a u is called a regular solution);*

(3) *or* $\lim_{r\to 0} u(r) = -\infty$ *and:*
- *if* $1 < q < \frac{N}{N-1}$, *then* $\lim_{r\to 0} r^{N-2} u(r) = \gamma < 0$;
- *if* $q = \frac{N}{N-1}$, *then* $\lim_{r\to 0} r^{N-2} |\ln r|^{N-1} u(r) = -\frac{1}{N-2}\left(\frac{N-1}{M}\right)^{N-1}$;
- *if* $\frac{N}{N-1} < q < 2$, *then* $\lim_{r\to 0} r^{\frac{2-q}{q-1}} u(r) = -\xi_M := -\frac{q-1}{2-q}\left(\frac{(N-1)q-N}{M(q-1)}\right)^{\frac{1}{q-1}}$.

This result shows that the singular behaviour of solutions is governed either by the Emden–Chandrasekhar equation or by the viscous Hamilton–Jacobi equation. Existence of solutions with the above behaviour is obtained in Theorems 16 and 17 by fixed point arguments or by the study near the equilibrium of the associated dynamical system.

When $q > 2$ the situation changes completely and we prove that the singularities of radial solutions are either governed by the eikonal equation or by the Hamilton–Jacobi equation.

Theorem C. *Let* $q > 2$ *and* $N \geq 2$. *Then:*

(1) *either* $\lim_{r\to 0} r^q e^{u(r)} = Mq^q$ *and* $\lim_{r\to 0} r u_r(r) = -q$;

(2) *or there exists* $u_0 \in \mathbb{R}$ *such that* $\lim_{r\to 0} u(r) = u_0$ *and* $\lim_{r\to 0} u_r(r) = 0$, *in that case* u *is a regular solution;*

(3) *or there exists* $u_0 \in \mathbb{R}$ *such that* $\lim_{r\to 0} u(r) = u_0$ *and there holds* $u(r) = u_0 + C_{M,N} r^{\frac{q-2}{q-1}} \big(1 + o(1)\big)$ *where* $C_{M,N}$ *is explicited in Theorem E. In such a case the singularity is only on the derivative.*

We also obtain the existence of singular solutions with the prescribed behaviour given above. Concerning solutions of eikonal type we prove existence in Theorems 23 and 24 by a very delicate method based upon inverse function arguments.

Theorem D. *Let* $q > 2$.

(1) *If* $N = 1$, *there exists one and only one solution* u^* *of* (7) *on* $(0,\infty)$ *such that*

$$\lim_{r\to 0} r^q e^{u^*(r)} = Mq^q \qquad \text{and} \qquad \lim_{r\to 0} r u^*_r(r) = -q. \tag{8}$$

Furthermore the function u^* *is the increasing limit when* $n \to \infty$ *of the regular solutions* u_n *(i.e.* $u_n(0) = n$ *and* $u_{n\,r} = 0$*).*

(2) *If* $N \geq 2$, *there exists at least one solution* u^* *of* (7) *on* $(0,\infty)$ *satisfying* (8).

Concerning solutions of Hamilton–Jacobi type, we prove the following result.

Theorem E. *Let* $q > 2$ *and* $u_0 \in \mathbb{R}$ *arbitrary.*

(1) *If* $N = 1$, *there exists at least one solution of* (7) *on* $(0,\infty)$ *satisfying*

$$u(r) = u_0 - \frac{q-1}{q-2}\left(\frac{1}{M(q-1)}\right)^{\frac{1}{q-1}} r^{\frac{q-2}{q-1}} \big(1 + o(1)\big) \quad \text{as } r \to 0.$$

Furthermore the function u *is decreasing on* $(0,\infty)$.

(2) *If* $N \geq 2$, *there exists at least one solution of* (7) *on* $(0,\infty)$ *satisfying*

$$u(r) = u_0 + \frac{q-1}{q-2}\left(\frac{(N-1)q-N}{M(q-1)}\right)^{\frac{1}{q-1}} r^{\frac{q-2}{q-1}} \big(1 + o(1)\big) \quad \text{as } r \to 0.$$

Furthermore the function u *is increasing on* $(0,\infty)$.

This result is proved in Theorems 27 and 28 by methods coming from the analysis of the stable and unstable manifolds associated to the stationary points of the relevant dynamical system.

The description of the asymptotic behaviour of radial solutions of (1) in an exterior domain exchanges the ranges $1 < q < 2$ and $q > 2$. Note also that only one type of behaviour is possible. The following statements are proved in Theorems 30 and 31.

Theorem F.

(1) *Let* $1 < q < 2$ *and* $N \geq 3$. *If* u *is a radial solution of* (1) *in* $\overline{B}_{r_0^c}$ *it satisfies* $\lim_{r\to\infty} r^q e^{u(r)} = Mq^q$ *and* $\lim_{r\to\infty} r u_r(r) = -q$.

(2) *Let* $q > 2$ *and* $N \geq 3$. *If* u *is a radial solution of* (1) *in* $\overline{B}_{r_0^c}$ *it satisfies* $\lim_{r\to\infty} r^2 e^{u(r)} = 2(N-2)$.

We end the article with an appendix which shows that when $q > 2$, all the radial solutions u of (1) which satisfy $\lim_{r\to 0} r^q e^{u(r)} = Mq^q$ and $\lim_{r\to 0} r u_r(r) = -q$, which means that they are of eikonal type, have the same expansion at any order near 0. This is a clue that uniqueness of solutions of (1) with such behaviour could hold in any dimension as it does hold when $N = 1$.

We leave as an open problem the study of

$$-\Delta u + M|\nabla u|^q = V(x)e^u \tag{9}$$

for many types of potential $V(x)$. Deep results in the case $N = 2$, $M = 0$ have been obtained by Brezis and Merle [9].

2. Estimates

2.1. *Estimates of radial supersolutions*

Theorem 1. *Let* $N \geq 1$ *and* $q > 1$.

(1) *If* $u \in C\big(\overline{B_{r_0}} \setminus \{0\}\big)$ *is a radial supersolution of* (1) *in* $B_{r_0} \setminus \{0\}$ *it satisfies*

$$e^{u(r)} \leq Cr^{-\max\{2,q\}} \quad \text{in } B_{\frac{r_0}{2}} \setminus \{0\}, \tag{10}$$

where $C = C(N,M,q,u) > 0$ *in the general case and* $C = C(N,M,q) > 0$ *if* $r \mapsto u(r)$ *is nonincreasing.*

(2) *If* $u \in C(\overline{B^c_{r_0}})$ *is a supersolution of* (1) *in* $B^c_{r_0}$ *it satisfies*

$$e^{u(r)} \leq Cr^{-\min\{2,q\}} \quad \text{in } B^c_{2r_0}, \tag{11}$$

where $C = C(N,M,q,u) > 0$ *is as in case* (1).

Proof. If u is a radial supersolution it is clear that it has at most one local maximum. Indeed at each local extremal point $\tilde{r}$ there holds $-u_{rr}(\tilde{r}) \geq e^{u(\tilde{r})} > 0$. Hence u_r keeps a constant sign near 0. We set $w = e^u$, then $w \geq 0$ satisfies

$$-\Delta w + \frac{|\nabla w|^2}{w} + M\frac{|\nabla w|^q}{w^{q-1}} - w^2 = -w_{rr} - \frac{N-1}{r}w_r + \frac{w_r^2}{w} + M\frac{|w_r|^q}{w^{q-1}} - w^2 \geq 0. \tag{12}$$

(1). We first assume that the function w is nondecreasing on $(0, r_1]$ for some $r_1 \in (0, r_0]$, the estimate (10) holds with C depending on u.

Next we assume that m is nonincreasing on $(0, r_1]$, then it stays decreasing on the whole interval $(0, r_0]$. For $\epsilon \in \big(0, \frac{1}{2}\big)$ let ϕ_ϵ be a $C^\infty(\mathbb{R}_+)$ nonnegative function vanishing on $[0, 1-\epsilon] \cup [1+\epsilon, \infty)$, with value 1 on $\big[1-\frac{\epsilon}{2}, 1+\frac{\epsilon}{2}\big]$, such that $\epsilon|\phi_{\epsilon\, r}|$ and $\epsilon^2|\phi_{\epsilon\, rr}|$ are bounded on $\big[1-\epsilon, 1-\frac{\epsilon}{2}\big] \cup \big[1+\frac{\epsilon}{2}, 1+\epsilon\big]$. If $0 < R < \frac{r_0}{2}$ we define v by

$$v(x) = w(x) - w(R)\phi_\epsilon\left(\frac{|x|}{R}\right)$$

Clearly $v(R)=0$. There exists $\tilde r_{R,\epsilon}\in\big[(1-\epsilon)R,(1+\epsilon)R\big]$ where v achieves a nonpositive minimum, hence $w(\tilde r_{R,\epsilon})\le w(R)$, $v_r(\tilde r_{R,\epsilon})=0$ and $\Delta v(\tilde r_{R,\epsilon})\ge 0$. There holds $w_r(\tilde r_{R,\epsilon})=w(R)\phi_{\epsilon\, r}\big(\frac{r_{R,\epsilon}}{R}\big)$ and $\Delta w(\tilde r_{R,\epsilon})\ge w(R)\Delta\phi_\epsilon\big(\frac{|x_{R,\epsilon}|}{R}\big)$. Therefore,

$$\begin{aligned}
w^2(\tilde r_{R,\epsilon}) &\le -\Delta w(\tilde r_{R,\epsilon})+\frac{\big|\nabla w(\tilde r_{R,\epsilon})\big|^2}{w(\tilde r_{R,\epsilon})}+M\frac{\big|\nabla w(\tilde r_{R,\epsilon})\big|^q}{w^{q-1}(\tilde r_{R,\epsilon})}\\
&\le -w(R)\Delta\phi_\epsilon\Big(\frac{r_{R,\epsilon}}{R}\Big)+w^2(R)\frac{\big|\nabla\phi_\epsilon\big(\frac{r_{R,\epsilon}}{R}\big)\big|^2}{w(\tilde r_{R,\epsilon})}+w^q(R)\frac{\big|\nabla\phi_\epsilon\big(\frac{r_{R,\epsilon}}{R}\big)\big|^q}{w^{q-1}(\tilde r_{R,\epsilon})}\\
&\le C\Big(\frac{w(R)}{\epsilon^2R^2}+\frac{w^2(R)}{\epsilon^2R^2w(\tilde r_{R,\epsilon})}+\frac{w^q(R)}{\epsilon^qR^qw^{q-1}(\tilde r_{R,\epsilon})}\Big),
\end{aligned}\tag{13}$$

where $C=C(N,q,M)>0$.

Now, if $q>2$, we multiply by $w^{q-1}(\tilde r_{R,\epsilon})$ and obtain the estimate

$$w^{q+1}(\tilde r_{R,\epsilon})\le C\Big(\frac{w(R)}{\epsilon^2R^2}w^{q-1}(R)+\frac{w^2(R)}{\epsilon^2R^2}w^{q-2}(R)+\frac{w^q(R)}{\epsilon^qR^q}\Big)\le C'\frac{w^q(R)}{\epsilon^qR^q}.$$

Because $w^{q+1}(\tilde r_{R,\epsilon})\ge\min_{(1-\epsilon)R\le r\le(1+\epsilon)R}w^{q+1}(r)\ge w^{q+1}\big((1+\epsilon)R\big)$ we obtain

$$w^{q+1}\big((1+\epsilon)R\big)\le C\frac{w^{\frac{q}{q+1}}(R)}{\epsilon^{\frac{q}{q+1}}R^{\frac{q}{q+1}}}.$$

We apply the bootstrap method of [6, Lemma 2.1] with $\Phi(\rho)=\rho^{-\frac{q}{q+1}}$ and $d=-h=\frac{q}{q+1}$ and we conclude that

$$w(R)\le C\big(\Phi(R)\big)^{\frac{1}{1-d}}=CR^{-q}.$$

If $1<q<2$ we have from (13),

$$\begin{aligned}
w^3(\tilde r_{R,\epsilon}) &\le C\Big(\frac{w(R)}{\epsilon^2R^2}w(\tilde r_{R,\epsilon})+\frac{w^2(R)}{\epsilon^2R^2}+\frac{w^q(R)}{\epsilon^qR^qw^{2-q}}(\tilde r_{R,\epsilon})\Big)\\
&\le C\Big(\frac{w(R)}{\epsilon^2R^2}w(R)+\frac{w^2(R)}{\epsilon^2R^2}+\frac{w^q(R)}{\epsilon^qR^qw^{2-q}(r)}\Big)\\
&\le C'\Big(\frac{w^2(R)}{\epsilon^2R^2}+\frac{w^2(R)}{\epsilon^qR^2}\Big)\\
&\le C''\frac{w^2(R)}{\epsilon^2R^2}.
\end{aligned}$$

Then,

$$w\big((1+\epsilon)R\big)\le c\frac{w^{\frac23}(R)}{\epsilon^{\frac23}R^{\frac23}}.$$

Now, applying [6, Lemma 2.1] with $\Phi(\rho)=\rho^{-\frac23}$, $d=-h=\frac23$, we obtain

$$w(R)\le C\big(\Phi(R)\big)^{\frac{1}{1-d}}=CR^{-2}.$$

Note that in the two cases, the upper estimate is independent of u.

(2). Assume that u satisfies (10) in $B^c_{r_0}$. Here also the function w is monotone on $[r_1,\infty)$ for some $r_1\ge r_0$. Let $0<\epsilon\le\frac12$ and $R>r_1+1$. If w is nondecreasing on $\big[R(1-\epsilon),R(1+\epsilon)\big]$, then $w(R)\ge w(\tilde r_{R,\epsilon})\ge w\big(R(1-\epsilon)\big)$ and therefore

$$w\big(R(1-\epsilon)\big)\le w(\tilde r_{R,\epsilon})\le C\Big(\frac{w^q(R)}{\epsilon^2R^2}+\frac{w^q(R)}{\epsilon^qR^q}\Big)^{\frac{1}{q+1}}\le\begin{cases}2C\dfrac{w^{\frac{q}{q+1}}}{\epsilon^{\frac{2}{q+1}}R^{\frac{2}{q+1}}} & \text{if } q>2,\\[2mm] 2C\dfrac{w^{\frac{q}{q+1}}}{\epsilon^{\frac{q}{q+1}}R^{\frac{q}{q+1}}} & \text{if } 1<q<2.\end{cases}$$

From the bootstrap argument,

$$w(R) \leq \begin{cases} C_1 R^{-2} & \text{if } q > 2, \\ 2CR^{-q} & \text{if } 1 < q < 2, \end{cases}$$

which contradicts the fact that w is nondecreasing. Hence w is nonincreasing and we obtain the estimate

$$w\big(R(1+\epsilon)\big) \leq \begin{cases} 2C \dfrac{w^{\frac{q}{q+1}}}{\epsilon^{\frac{2}{q+1}} R^{\frac{2}{q+1}}} & \text{if } q > 2, \\ 2C \dfrac{w^{\frac{q}{q+1}}}{\epsilon^{\frac{q}{q+1}} R^{\frac{q}{q+1}}} & \text{if } 1 < q < 2. \end{cases}$$

Therefore we obtain (10) as before. □

Remark 2. Note that in the non-radial case, such a kind of estimates of supersolutions are extended in [7] to similar estimates of the spherical minimum of u.

2.2. *General gradient estimates*

Here we obtain a general estimate of the gradient of a solution, non-necessarily radial, in terms of the function itself. It is based upon a combination of Bernstein and Keller–Osserman methods.

Theorem 3. *Let $N \geq 1$ and $q > 1$. If $u \in C\big(\overline{B_{r_0}} \setminus \{0\}\big)$ is a solution of* (1) *in $B_{r_0} \setminus \{0\}$. Then for any $\rho \in \big(0, \frac{r_0}{2}\big]$ and any $x \in B_\rho \setminus \{0\}$*

$$\big|\nabla u(x)\big| \leq c_1 |x|^{-\frac{1}{q-1}} + c_2 \max_{B_\rho(x)} e^{\frac{u}{q}} + c_3 \max_{B_\rho(x)} e^{\frac{u}{2(q-1)}}, \tag{14}$$

where $c_j = c_j(N,q,M) > 0$, for $j = 1,2,3$.

Proof. The technique is standard and we recall it for the sake of completeness. We set $z = |\nabla u|^2$, then by Schwarz inequality,

$$-\frac{1}{2}\Delta z + \frac{(\Delta u)^2}{N} + \langle \nabla \Delta u, \nabla u\rangle \leq 0.$$

Then, for $\epsilon > 0$ small enough,

$$-\frac{1}{2}\Delta z + \frac{(Mz^{\frac{q}{2}} - e^u)^2}{N} \leq e^u z + \frac{Mq}{2} z^{\frac{q}{2}-2}\big|\langle \nabla z, \nabla u\rangle\big| \leq e^u z + \epsilon z^q + C_{\epsilon,q,M}\frac{|\nabla z|^2}{z}.$$

Using again Hölder and Young inequalities, we obtain

$$\begin{aligned} -\frac{1}{2}\Delta z + \frac{M^2 z^q}{N} &\leq 2Me^u z^{\frac{q}{2}} + e^u z + \epsilon z^q + C_{\epsilon,q,M}\frac{|\nabla z|^2}{z} \\ &\leq \epsilon_1 z^q + C_{\epsilon_1} e^{2u} + \epsilon_2 z^q + C_{\epsilon_2} e^{q'u} + \epsilon z^q + C_{\epsilon,q,M}\frac{|\nabla z|^2}{z}. \end{aligned}$$

With the choice ϵ, ϵ_1 and ϵ_2 small enough and the last inequality turns out into

$$-\frac{1}{2}\Delta z + \frac{M^2 z^q}{2N} \leq C_1 e^{2u} + C_2 e^{q'u} + C_{q,N,M}\frac{|\nabla z|^2}{z}. \tag{15}$$

Then we use a variant of the Osserman–Keller inequality proved in [6, Lemma 3.1] and we obtain (14). □

The next result which holds only if $q > 2$ is a universal a priori estimate of u and ∇u solution of (1). The proof is delicate and can be found in [7].

Theorem 4. *Let $N \geq 2$ and $q > 2$. If $u \in C\big(\overline{B_{r_0}} \setminus \{0\}\big)$ is a solution of* (1) *in $B_{r_0} \setminus \{0\}$. Then there exists $C > 0$ depending on N, q, M, u such that*

$$e^{u(x)} \leq \frac{C}{|x|^q} \quad \text{and} \quad \big|\nabla u(x)\big| \leq \frac{C}{|x|} \quad \text{for all } x \in B_{\frac{r_0}{2}} \setminus \{0\}. \tag{16}$$

3. The dynamical system approach

A radial solution of (1) satisfies

$$\mathscr{L}^{\mathrm{rad}}_{m,p,q}u := -u_{rr} - \frac{N-1}{r}u_r + M|u_r|^q - e^u = 0. \tag{17}$$

In all the sequel we consider essentially radially symmetric solutions u of (1), that means functions satisfying (17). Furthermore, solutions are at least C^3 on their maximal interval of existence.

3.1. *Monotonicity and global properties*

Lemma 5. *Let $q > 1$ and u be a radial solution of* (17) *on* $(0, r_0)$*. Then u has at most one local maximum and u_r keeps a constant sign near $r = 0$ and the following dichotomy holds:*

(1) *either $u_r(r) < 0$ on $(0, r_0)$ and $u(r) \to \infty$ when $r \to 0$;*
(2) *or $u_r(r) < 0$ on $(0, r_0)$ and $u(r) \to u_0$ when $r \to 0$;*
(3) *or $u_r(r) > 0$ near 0 and $u(r) \to u_0$ when $r \to 0$;*
(4) *or $u_r(r) > 0$ near 0 and $u(r) \to -\infty$ when $r \to 0$.*

Proof. The proof follows easily from the monotonicity of u. □

Proposition 6. *Let $N \geq 2$ and $q > 1$. If u is a solution of* (17) *defined in the maximal interval* $(0, R)$ *and decreasing near* 0*, then $R = \infty$ and $u_r < 0$ on* $(0,\infty)$*. Furthermore $u(r) \to -\infty$ and $u_r(r) \to 0$ when $r \to \infty$. As a consequence, if u is positive near* 0*, it has a unique zero on* $(0,\infty)$*.*

Proof. By Lemma 5, $u_r < 0$ on $(0, R)$. Set

$$\mathscr{H}(r) = e^u + \frac{u_r^2}{2}. \tag{18}$$

Then

$$\mathscr{H}_r(r) = -\frac{N-1}{r}u_r^2 + M|u_r|^q u_r = -\frac{N-1}{r}u_r^2 - M|u_r|^{q+1}.$$

Hence $\mathscr{H}$ is decreasing on $(0, R)$. As a consequence for any $\tilde{r} \in (0, R)$, u_r^2 is bounded on $[\tilde{r}, R)$. If $R < \infty$, by integration $u(r)$ is also bounded on $[\tilde{r}, R)$ which is impossible since R is maximal. Hence $R = \infty$. Since $e^{u(r)}$ is decreasing and positive, there exists $\ell \geq 0$ such that $e^{u(r)} \to \ell$ when $r \to \infty$. Moreover, since it is positive and decreasing $\mathscr{H}(r)$ admits a limit $\lambda \geq 0$ when $r \to \infty$ and therefore $u_r(r)$ shares this property. This implies that $\mathscr{H}_r$ is integrable and therefore $u_r(r) \to 0$ when $r \to \infty$. Since $e^{u(r)} \to \ell \geq 0$, and $u_r(r) \to 0$ we obtain that $-u_{rr}(r) \to -\ell$. If $\ell > 0$ we would obtain that $u(r) \to \ln \ell$, which is not compatible. Hence $\ell = 0$ and $u(r) \to -\infty$ when $r \to \infty$. □

Remark 7. More generally, if u is a solution defined on a maximal interval I_u containing $r_0 > 0$ and if $u_r(r_0) \leq 0$, then $u_r(r) < 0$ for $r \in I_u \cap (r_0,\infty) = (r_0,\infty)$, $u(r) \to -\infty$ and $u_r(r) \to 0$ when $r \to \infty$. On the contrary, if $u_r(r_0) > 0$, then either u admits a unique maximum at $r_1 > r_0$ and therefore $u(r) \to -\infty$ and $u_r(r) \to 0$ when $r \to \infty$, or u is increasing on $I_u \cap (r_0,\infty)$. In such a case two situations could occur: either $I_u \cap [r_0,\infty) = [r_0, r_1)$ and $\lim_{r\to r_1} u(r) = \infty$ (u is a large solution), or $I_u \cap [r_0,\infty) = [r_0,\infty)$ and $\lim_{r\to\infty} u(r) = \infty$.

3.2. *Associated differential systems*

To the equation (17) we associate several systems autonomous or not.

3.2.1. *Non-autonomous systems of order 2*

Lemma 8. *Let x, X and Φ be defined by*

$$x(t) = r^2 e^{u(r)}, \quad X(t) = r^q e^{u(r)}, \quad \Phi(t) = -r u_r(r) \quad \textit{with } t = \ln r. \tag{19}$$

Then u is a solution of (17) *if and only if*

$$\begin{cases} x_t = x(2-\Phi) \\ \Phi_t = (2-N)\Phi - Me^{(2-q)t}|\Phi|^q + x, \end{cases} \tag{20}$$

where $x > 0$. This system is also equivalent to

$$\begin{cases} X_t = X(q-\Phi) \\ \Phi_t = (2-N)\Phi - e^{(2-q)t}\big(-M|\Phi|^q + X\big). \end{cases} \tag{21}$$

Proof. We set

$$u(r) = U(t) \quad \text{with } t = \ln r. \tag{22}$$

Then

$$-U_{tt} - (N-2)U_t - e^{2t}e^U + Me^{(2-q)t}|U_t|^q = 0, \tag{23}$$

and $\Phi = -U_t = -r\frac{v_r}{v}$ with $v = e^u$. The proof follows. □

Remark 9. For the Emden equation (4) the system in (x,Φ) is

$$\begin{cases} x_t = x(2-\Phi) \\ \Phi_t = (2-N)\Phi + x. \end{cases}$$

The equilibrium are $(0,0)$ and $\big(2(N-2),2\big)$. If $N > 2$, the characteristic values of the linearisation at $(0,0)$ are $\lambda_1 = 2-N$ with eigenvector $(0,1)$ and $\lambda_2 = 2$ with eigenvector $(N,1)$. Hence $(0,0)$ is a saddle point. The stable trajectory at $(0,0)$ is located on $x = 0$ and actually it is $\big(x(t),\Phi(t)\big) \equiv (0, ce^{(2-N)t})$. It is not admissible. The unstable trajectory at $(0,0)$ satisfies $\big(x(t),\Phi(t)\big) = (Ne^{2t}, e^{2t})\big(c + o(1)\big)$ $(c > 0)$ when $t \to -\infty$, which corresponds to a solution u satisfying $\big(u(0), u_r(0)\big) = (\ln cN, 0)$. Replacing $\ln cN$ by u_0, we obtain all the regular with $u(0) = u_0$.

3.2.2. *First autonomous systems of order 3*

It is well-known that a non-autonomous system of order 2 can be transformed into an autonomous system of order 3. Actually several transformations are possible.

Lemma 10. *Let u be a solution of* (17). *Set*

$$x = r^2 e^u, \quad X = r^q e^u, \quad \Phi = -r u_r, \quad V = r|u_r|^{q-1} \quad \textit{with } t = \ln r. \tag{24}$$

Then there holds in variable (x,Φ,V)

$$\begin{cases} x_t = x(2-\Phi) \\ \Phi_t = (2-N)\Phi - M|\Phi|V + x \\ V_t = V\Big(N - (N-1)q - (q-1)\big(MV\,\text{sign}(\Phi) - \frac{x}{\Phi}\big)\Big), \end{cases} \tag{25}$$

and also with (X,Φ,V)

$$\begin{cases} X_t = X(q-\Phi) \\ \Phi_t = (2-N)\Phi - V|\Phi|\big(M - \frac{X}{|\Phi|^q}\big) \\ V_t = V\Big(N - (N-1)q - (q-1)\big(M - \frac{X}{|\Phi|^q}\big)\,\text{sign}(\Phi)V\Big). \end{cases} \tag{26}$$

Proof. By a direct computation

$$r^{2-q} = e^{(2-q)t} = |\Phi|^{q-1} V,$$

$$\Phi_t = (2-N)\Phi - M|\Phi|V + x = (2-N)\Phi + |\Phi|^{1-q} V\big(-M|\Phi|^q + X\big)$$

and, whatever is the sign of Φ,

$$\begin{aligned}\frac{V_t}{V} = (2-q) + (q-1)\frac{\Phi_t}{\Phi} &= 2-q+(q-1)\left(2-N+\frac{-M|\Phi|V+x}{\Phi}\right)\\ &= q-N(q-1)+(q-1)\frac{-M|\Phi|V+x}{\Phi}\\ &= q-N(q-1)+(q-1)\left(-MV\operatorname{sign}(\Phi)+\frac{x}{\Phi}\right)\\ &= q-N(q-1)+(q-1)\left(-M+\frac{X}{|\Phi|^q}\right)V\operatorname{sign}(\Phi),\end{aligned}$$

which leads to (24) and (26). □

We also introduce another system of order 3 in the variables (x,Φ,Θ), where $\Theta(t) = e^{(2-q)t}$:

$$\begin{cases} x_t = x(2-\Phi)\\ \Phi_t = x + (2-N)\Phi - M|\Phi|^q\Theta\\ \Theta_t = (2-q)\Theta. \end{cases} \tag{27}$$

This system will be interesting when $t \to -\infty$ (i.e. singularity in x) if $1 < q < 2$ and when $t \to \infty$ if $q > 2$ since in these two cases $\Theta(t) \to 0$ when $t \to -\infty$ or $t \to \infty$ according $r \to 0$ or $r \to \infty$. It admits two equilibria, $(0,0,0)$ and $\big(2(N-2),2,0\big)$.

3.2.3. *Reduction to an autonomous quadratic system of order 3*

Using a suitable change of variable, the equation is transformed into a remarkable quadratic system of Lotka–Volterra type as the next lemma shows it.

Lemma 11. *Let u be a solution of* (17). *At any point where $u_r(r) \neq 0$ define*

$$Z = -\frac{re^u}{u_r}, \quad V = r|u_r|^{q-1}, \quad \Phi = -ru_r \quad \textit{with } t = \ln r. \tag{28}$$

Then there holds

$$\begin{cases} Z_t = Z(N-\Phi+sMV-Z)\\ V_t = V\big(N-(N-1)q+(q-1)(Z-sMV)\big)\\ \Phi_t = \Phi\big(2-N+Z-sMV\big) \end{cases} \tag{29}$$

where $s = \operatorname{sign}(\Phi)$.

Proof. We start from (25) and define $Z = \frac{x}{V} = -\frac{re^u}{u_r}$. Then

$$\frac{Z_t}{Z} = \frac{x_t}{x} - \frac{\Phi_t}{\Phi} = 2-\Phi+N-2+M\frac{|\Phi|}{\Phi} - Z,$$

which implies (29). □

Remark 12. For $q \neq 2$ the system (28) admits four equilibria

$$O = (0,0,0), \quad Q_0 = (N-2,0,2), \quad N_0 = (N,0,0), \quad P_0 = \left(0, s\tfrac{(N-1)q-N)}{q-1}, 0\right).$$

The linearised system at Q_0 is

$$\begin{cases} Z_t = (N-2)(N-2-Z+MV-\Phi)\\ V_t = (2-q)V\\ \Phi_t = (2-N+Z-MV). \end{cases} \tag{30}$$

The characteristic polynomial is

$$P(\lambda) = (2 - q - \lambda)\big(\lambda^2 + (N-2)\lambda + 2(N-2)\big).$$

One root is $2 - q$, the two other roots have negative real part. Hence if $1 < q < 2$ there exist a 2-dimensional manifold of trajectories converging to Q_0 when $t \to \infty$ and an unstable trajectory issued from Q_0. This unstable trajectory corresponds to a solution u of (1) satisfying $\lim_{r\to 0} r^2 e^{u(r)} = 2(N-2)$. If $q > 2$, Q_0 is a sink which attracts all local trajectories.

The linearised system at N_0 is

$$\begin{cases} Z_t = N\big(-\Phi - MV - Z + N\big) \\ V_t = qV \\ \Phi_t = 2\Phi. \end{cases} \tag{31}$$

The characteristic polynomial is

$$P(\lambda) = -(q-\lambda)(2-\lambda)(N+\lambda).$$

It can be checked that the trajectories converging to N_0 at $-\infty$ correspond to the regular trajectories.

Finally the trajectories converging to P_0 at $-\infty$ with $q > 2$ correspond to the solutions of Hamilton–Jacobi type as it is shown in the proof of Theorem 28.

4. Isolated singularities when $1 < q < 2$

4.1. *Singular behaviour*

In this section we use a perturbation argument due to [19, Proposition 4.1] that we recall below.

Proposition 13. *Let $h\colon \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function. Assume that there exists locally continuous function $h^*: \mathbb{R}^N \to \mathbb{R}^N$ such that for all compact sets $C \subset \mathbb{R}^N$ and all $\epsilon > 0$ there exists $T = T(\epsilon, C) > 0$ such that*

$$\operatorname*{ess\,sup}_{x\in C} \sup_{t\geq T} \big|h(\tau,x) - h^*(x)\big| \leq \epsilon.$$

If $x(t)$ is a bounded solution of $x_\tau = h(\tau,x)$ on $\mathbb{R}_+$ such that $x(0) = x_0$, then the omega-limit set of the positive trajectory of x is a non-empty connected compact set of $\mathbb{R}^N$ which is invariant under the flow of the equation $x_\tau = h^*(x)$.

Mutatis mutandis a similar result holds if $\mathbb{R}_+$ is replaced by $\mathbb{R}_-$ and omega-limit set by alpha-limit set.

Our first result is a complete description of the behaviour of any solution near 0.

Theorem 14. *Let $1 < q < 2$ and $N \geq 3$. If u is any radial solution of* (1) *in $B_{r_0} \setminus \{0\}$ there holds:*

(1) *either*

$$\lim_{r\to 0} r^2 e^{u(r)} = 2(N-2), \quad \textit{that is} \quad \lim_{r\to 0}\left(u(r) - 2\ln\frac{1}{r}\right) = \ln 2(N-2); \tag{32}$$

(2) *or u is a regular solution, i.e. there exists $u_0 \in \mathbb{R}$ such that $\lim_{r\to 0} u(r) = u_0$ and $\lim_{r\to 0} u_r(r) = 0$;*

(3) *or* $\lim_{r\to 0} u(r) = -\infty$ *and*

$$\lim_{r\to 0} r^{N-2} u(r) = \gamma < 0 \quad \textit{when } 1 < q < \frac{N}{N-1}, \tag{33}$$

$$\lim_{r\to 0} r^{N-2} |\ln r|^{N-1} u(r) = -\frac{1}{N-2}\left(\frac{N-1}{M}\right)^{N-1} \quad \textit{when } q = \frac{N}{N-1}, \tag{34}$$

$$\lim_{r\to 0} r^{\frac{2-q}{q-1}} u(r) = -\xi_M := -\frac{q-1}{2-q}\left(\frac{(N-1)q-N}{M(q-1)}\right)^{\frac{1}{q-1}} \quad \textit{when } \frac{N}{N-1} < q < 2. \tag{35}$$

Proof. By Theorem 1 we have that $r^2 e^{u(r)} \le C = C(N,p,q,M) > 0$ when $0 < r \le \frac{r_0}{2}$. We consider the system (20) in $(x,\Phi) = (r^2 e^{u(\cdot)}, -r u_r)$ in the variable $t = \ln r$, that is

$$\begin{cases} x_t = x(2-\Phi) \\ \Phi_t = x + (2-N)\Phi - Me^{(2-q)t}|\Phi|^q. \end{cases} \tag{36}$$

Step 1. We assume that u is decreasing near $r = 0$. Then $\Phi > 0$ and $x(r) \le C$ for $0 < r \le \frac{r_0}{2}$.

If Φ is unbounded on $(-\infty, \widetilde{T}]$ with $\widetilde{T} = \ln \widetilde{r}$, we encounter two possibilities.

- Either Φ is monotone as $t \to -\infty$ and $\Phi(t) \to \infty$. Then for any $A > 0$ we have that $x_t(t) \le -Ax(t)$ for $t \le t_A \le \widetilde{T}$, which implies that $t \mapsto e^{At}x(t)$ is decreasing on $(-\infty, t_A]$, thus $x(t) \ge e^{A(t_A - t)}x(t_A)$ on this interval, an inequality which contradicts the boundedness of $x(t)$.
- Or Φ is not monotone and thus there exists a sequence $\{t_n\}$ tending to $-\infty$ such as $\Phi(t_n)$ is a local maximum of Φ, and the sequence $\{\Phi(t_n)\}$ tends to ∞ when $n \to \infty$. From (36) we have that $x(t_n) = (N-2)\Phi(t_n) + Me^{(2-q)t_n}\left|\Phi(t_n)\right|^q \to \infty$. This contradicts the boundedness of x.

Hence Φ is bounded. Therefore there exists $\kappa > 0$ such that $\sup\{\left|x(t)\right|, \left|\Phi(t)\right|\} \le \kappa$ for $t \le \ln \widetilde{T}$. Since $q < 2$ this system is an exponential perturbation as $t \to -\infty$ of the system

$$\begin{cases} x_t = x(2-\Phi) \\ \Phi_t = x + (2-N)\Phi, \end{cases} \tag{37}$$

in the sense of Proposition 13, which is the system associated to the Chandrasekhar–Emden equation $-\Delta u = e^u$. This system admits the equilibria $\big(2(N-2), 2\big)$ and $(0,0)$ which both are hyperbolic. The point $\big(2(N-2), 2\big)$ is a sink while $(0,0)$ is a saddle point. Therefore $\big(2(N-2), 2\big)$ is repulsive when $t \to -\infty$ and the only solution in its neighbourhood is the constant solution $\big(2(N-2), 2\big)$ which corresponds to the explicit solution of the Chandrasekhar–Emden equation, $u(r) = \ln \frac{2(N-2)}{r^2}$. To the stable trajectory of (37) converging to $(0,0)$ corresponds the regular solution of the same equation.

By Proposition 13 any bounded solution of (36) admits a limit set at $-\infty$ which is invariant under the flow of (37), actually, the only possibilities are $\big(2(N-2), 2\big)$ and $(0,0)$. Then either $\big(x(t), \Phi(t)\big) \to \big(2(N-2), 2\big)$ or $\big(x(t), \Phi(t)\big) \to (0,0)$ when $t \to -\infty$.

In the first case $u(r)$ satisfies (32). In the second case we introduce the system (27) with $\Theta(t) = e^{(2-q)t}$ and we are in the situation where $\big(x(t, \Phi(t), \Theta(t)\big) \to (0,0,0)$ when $t \to -\infty$. We recall the system (27) in (x, Φ, Θ) with $\Theta(t) = e^{(2-q)t}$,

$$\begin{cases} x_t = x(2-\Phi) \\ \Phi_t = x + (2-N)\Phi - M|\Phi|^q\Theta \\ \Theta_t = (2-q)\Theta. \end{cases} \tag{38}$$

The associated linearised system at $(0,0,0)$ is

$$\begin{cases} x_t = 2x \\ \Phi_t = x + (2-N)\Phi \\ \Theta_t = (2-q)\Theta. \end{cases} \tag{39}$$

The eigenvalues are $\lambda_1 = 2 > 0$, $\lambda_2 = 2-N < 0$ and $\lambda_3 = 2-q > 0$. Hence there exists a 2-dimensional unstable manifold $\mathscr{M}_s$ of trajectories issued of $(0,0,0)$ as $t \to -\infty$; $\mathscr{M}_s$ is relative to the eigenvalues λ_1 and λ_3 with corresponding eigenvectors $\omega_1 = (N,1,0)$ and $\omega_3 = (0,0,1)$ which generate the tangent 2-plane at this point. In the manifold $\mathscr{M}$ there exists one trajectory $\mathscr{T}_f$, called the fast trajectory, associated to λ_1 and admitting the vector ω_1 for tangent vector at this point. However this trajectory is located in the plane $\Theta = 0$ since the point $(0,0)$ of the restriction of (38) to this plane is a saddle point, hence this trajectory is also the unstable trajectory of $(0,0)$. Therefore this trajectory is not admissible. Hence our trajectory is associated to λ_3 with tangent vector ω_3 at $(0,0,0)$. Along this trajectory there holds $x(t) = o(e^{(2-q)t}) = o\big(\Theta(t)\big)$ and $\Phi(t) = o(e^{(2-q)t})$. Then $r\big|u_r(r)\big| = o(r^{2-q})$ when $r \to 0$, and thus $u(r)$ has a finite limit u_0 at $r = 0$, which implies $x(t) = e^{u_0}e^{2t}\big(1+o(1)\big)$ as $t \to -\infty$. It follows from the equation (1) and elliptic equation regularity that u can be extended as a C^2 solution, thus $u_r(0) = 0$ and u is a regular solution.

Step 2. We assume that u is increasing near $r = 0$. Then either $u(r)$ has a finite limit u_0 when $r \to 0$ or $u(r) \to -\infty$. Moreover u_r is monotone near 0: indeed at any point $\tilde{r}$ where $u_{rr}(\tilde{r}) = 0$ the following identity

$$u_{rrr}(\tilde{r}) = \left(\frac{N-1}{\tilde{r}} - e^{u(\tilde{r})}\right)u_r(\tilde{r}) > 0.$$

Then:

- either $u_r(r) \to 0$ but in that case $u(r) \to u_0$ at 0 and u is concave near 0, hence it is decreasing, contradiction;
- or u_r has a positive limit c_0 at 0, then again $u(r) \to u_0$ but from the equation u_{rr} is not integrable at 0, contradiction;
- or u_r tends to ∞. In that case $e^{u(r)} = o\big(u_r^q(r)\big)$. In that case equation (17) can be written under the form

$$-u_{rr} - \frac{N-1}{r}u_r + \widetilde{M}(r)u_r^q = 0 \tag{40}$$

where $\widetilde{M}(r) = M - (u_r(r)^{-q})e^{u(r)} \to M$ when $r \to 0$. Equation (40) is explicitly integrable and we have

$$r^{-(N-1)(q-1)}X(r) = r_0^{-(N-1)(q-1)}X(r_0) + (q-1)\int_r^{r_0} s^{-(N-1)(q-1)}\widetilde{M}(s)\,\mathrm{d}s. \tag{41}$$

where $X(r) = u_r^{1-q}$. By performing a direct integration of (41) we obtain (3). □

4.2. *Existence of singular solutions of Emden–Chandrasekhar type*

Theorem 15. *Let* $1 < q < 2$ *and* $N \geq 3$. *Then there exists a unique radial solution* u_ω *of* (17) *defined on* $(0,\infty)$ *such that*

$$\lim_{r\to 0} r^2 e^{u(r)} = 2(N-2) \qquad \textit{and} \qquad \lim_{r\to 0} r\,u_r(r) = -2. \tag{42}$$

Proof. We still consider the systems (36) and (38) relative to (x,Φ,Θ). Here we prove the existence of a unique trajectory $\mathcal{T}$ of (38) converging to the point $P_0 = \big(2(N-2),2,0\big)$ as $t \to -\infty$. The linearised system at P_0 with $x = 2(N-2)+\overline{x}$ and $\Phi = 2+\overline{\Phi}$ is

$$\begin{cases} \overline{x}_t = -2(N-2)\overline{\Phi} \\ \overline{\Phi}_t = \overline{x} + (2-N)\overline{\Phi} - M2^q\overline{\Theta} \\ \Theta_t = (2-q)\Theta. \end{cases} \tag{43}$$

Set

$$A = \begin{pmatrix} 0 & -2(N-2) & 0 \\ 1 & 2-N & -2^q M \\ 0 & 0 & 2-q \end{pmatrix}.$$

The characteristic polynomial associated is

$$\det(A-\lambda I) := P(\lambda) = (2-q-\lambda)\big(\lambda^2 + (N-2)\lambda + 2(N-2)\big). \tag{44}$$

The corresponding eigenvalues are $\lambda_1 = 2-q > 0$ and λ_2, λ_3 which are real and negative if $N \geq 10$ and complex with negative real part if $3 \leq N \leq 9$. Hence we have the standard decomposition

$$\mathbb{R}^3 = \ker(A-\lambda_1 I) \oplus H$$

where H is either $\ker(A-\lambda_2 I) \oplus \ker(A-\lambda_3 I)$ if $\lambda_2 \neq \lambda_3$ are real, or $\ker(A-\lambda_2 I)^2$ if $\lambda_2 = \lambda_3$ (necessarily real), or $\mathcal{R}e\big(\ker(A-\lambda_2 I) \oplus \ker(A-\lambda_3 I)\big)$ if $\lambda_2 \neq \lambda_3$ are not real but conjugate. Then $\ker(A-\lambda_1 I) = \mathrm{span}\{\omega_1\}$ where ω_1 has for components

$$\omega_1 = \big(2(N-2), q-2, 2^q M f(q)\big) \quad \text{where} \quad f(q) = q^2 - (N+2)q + 4(N-1).$$

We prove that $f(q) \neq 0$. This is clear if $N < 10$. If $N \geq 10$ it admits two positive roots $q_1 < q_2$, with $2 < q_2$. Since $f(2) > 0$, then $2 < q_1$. Therefore $f(q) > 0$. Then there exists a unique trajectory $\mathcal{T}^*$ associated to (x^*,Φ^*,Θ^*) such that $\lim_{t\to-\infty}\big(x^*(t),\Phi^*(t),\Theta^*(t)\big) = \big(2(N-2),2,0\big)$ with tangent vector at the trajectory at P_0 is colinear to ω_1 and such that $\Theta^*(t) > 0$ as $t \to \infty$. To this trajectory is associated a solution u_ω of (17) satisfies (42). It is decreasing; by Proposition 6 it is defined on $(0,\infty)$ and it satisfies $\lim_{r\to\infty} u(r) = -\infty$ and $\lim_{r\to\infty} u_r(r) = 0$. Any other solution corresponding to the same trajectory is just a time shift of $\big(x^*(t),\Phi^*(t),\Theta^*(t)\big)$ and it corresponds to the same function u_ω. □

4.3. *Existence of singular solutions of Hamilton–Jacobi type*

Next we show the existence of solutions satisfying (35).

Theorem 16. *Let* $\frac{N}{N-1} < q < 2$. *Then there exists infinitely many radial solutions u of* (17) *defined on* $(0,\infty)$ *such that*

$$\lim_{r\to 0} r^\beta u(r) = -\xi_M, \tag{45}$$

where $\beta = \frac{2-q}{q-1}$.

Proof. We set $U = -u$, then

$$-\Delta U - M|\nabla U|^q + e^{-U} = 0,$$

and we put

$$U(r) = r^{-\beta}\xi(t), \quad U_r(r) = -r^{-\frac{1}{q-1}}\eta(t) = -r^{-\beta-1}\eta(t) \quad \text{with } t = \ln r. \tag{46}$$

We are led to the system

$$\begin{cases} \xi_t = \beta\xi - \eta \\ \eta_t = -\kappa\eta + M\eta^q - e^{\frac{qt}{q-1}}e^{-e^{-\beta t}\xi(t)}, \end{cases} \tag{47}$$

with $\kappa = \frac{(N-1)q-N}{q-1} > 0$. The couple (ξ,η) is a solution if $\big(\xi(t),\eta(t)\big) \to (\xi_M, \beta\xi_M)$ when $t \to -\infty$. Set $\bar\xi = \xi - \xi_M$ and $\bar\eta = \eta - \beta\xi_M$. Then

$$\begin{cases} \bar\xi_t = \beta\bar\xi - \bar\eta \\ \bar\eta_t = \kappa(q-1)\bar\eta + F(\bar\eta) - e^{\frac{qt}{q-1}} e^{-e^{-\beta t}(\bar\xi+\xi_M)}, \end{cases} \tag{48}$$

where $F(\bar\eta) = O\big(|\bar\eta|^2\big)$. Let $0 < \theta < 4\kappa(2-q)$, then there exists $\mu := \mu(\theta,\kappa) > 0$ such that

$$\theta\beta x^2 - \theta xy + \kappa(q-1)y^2 \geq \mu(\theta x^2 + y^2).$$

Hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\theta\bar\xi^2 + \bar\eta^2) \geq \mu(\theta\bar\xi^2 + \bar\eta^2) + \bar\eta\big(F(\bar\eta) - e^{\frac{qt}{q-1}} e^{-e^{-\beta t}(\bar\xi+\xi_M)}\big) \tag{49}$$

There exists (x_0, y_0) such that:

(1) $|x_0| \leq \frac{1}{4}\xi_M$;

(2) $|y_0| \leq \min\{\frac{\beta}{4}\xi_M, c\} \;\Rightarrow\; \big|F(y_0)\big| \leq \frac{\mu}{2}|y_0|^3 \leq \frac{\mu}{2}|y_0|^2$;

and if we choose $(\bar\xi_0,\bar\eta_0)$ such that $|\bar\xi_0| \leq |x_0|$ and $|\bar\eta_0| \leq |y_0|$. We denote by $\big(\bar\xi(t),\bar\eta(t)\big)_{t\leq t_0}$ the solution of (48) with initial data $\big(\bar\xi(t_0),\bar\eta(t_0)\big) = (\bar\xi_0,\bar\eta_0)$. As long as $\big|\bar\xi(t)\big| \leq |x_0|$ and $\big|\bar\eta(t)\big| \leq |y_0|$ we have

$$|\eta| e^{\frac{qt}{q-1}} e^{-e^{-\beta t}(\bar\xi+\xi_M)} \leq \frac{\mu}{4}\eta^2 + \frac{4}{\mu} e^{\frac{2qt}{q-1}} e^{-2e^{-\beta t}\frac{\xi_M}{2}}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\theta\bar\xi^2 + \bar\eta^2) \geq \frac{\mu}{2}(\theta\bar\xi^2 + \bar\eta^2) - \frac{c(t)}{2},$$

where $C(t)$ is a positive function which satisfies

$$\lim_{t\to-\infty} e^{-at}c(t) = 0 \quad \text{for all } a > 0.$$

This implies

$$\theta\bar\xi(t)^2 + \bar\eta(t)^2 \leq e^{\mu(t-t_0)}(\theta\bar\xi_0^2 + \bar\eta_0^2) + e^{\mu t}\int_t^{t_0} e^{-\mu s}c(s)\,\mathrm{d}s \quad \text{for } t \leq t_0. \tag{50}$$

As long as $\big|\bar\xi(t)\big| \leq |x_0|$ and $\big|\bar\eta(t)\big| \leq |y_0|$, the above inequality holds. If we take $\theta\bar\xi_0^2 + \eta_0^2 \leq \min\{\theta x_0^2, y_0^2\}$, inequality (50) holds for all $t \leq t_0$. Hence $\big(\bar\xi(t),\bar\eta(t)\big) \to (0,0)$ when $t \to -\infty$, which implies

$$\lim_{t\to-\infty}\big(\xi(t),\eta(t)\big) = (\xi_M, \beta\xi_M). \tag{51}$$

If we set $r_0 = e^{t_0}$ and $u(r) = -r^{-\beta}\big(\xi_M + \bar\xi(\ln r)\big)$, then u is a solution of (17) in $(0, r_0]$ which satisfies (45). Such a solution can be extended to $(0,\infty)$ by Proposition 6 and the choice of its data at $r = r_0$ has for unique restriction $u(r_0)$ and $u_r(r_0)$ corresponding to $(\bar\xi_0,\bar\eta_0)$. □

Next we show the existence of solutions of (17) satisfying (33) by a fixed point method.

Theorem 17. *Let $N \geq 3$ and $1 < q < \frac{N}{N-1}$. Then there exist $\rho_0 > 0$ and $k_0 > 0$ such that for $0 < \rho \leq \rho_0$ and $-k_0 < \gamma \leq 0$ there exists a radial function u_γ satisfying*

$$\begin{aligned} -\Delta u_\gamma + M|\nabla u_\gamma|^q - e^{u_\gamma} &= c_N\gamma\delta_0 && \text{in } \mathscr{D}'(B_\rho), \\ u_\gamma &= 0 && \text{on } \partial B_\rho. \end{aligned} \tag{52}$$

Furthermore

$$\lim_{r\to 0} r^{N-2}u_\gamma(r) = \gamma. \tag{53}$$

Proof. We look for a radial solution u satisfying

$$\lim_{r\to 0} r^{N-2}u(r)=\gamma \qquad \text{and} \qquad \lim_{r\to 0} r^{N-1}u(r)=(1-N)\gamma. \tag{54}$$

If such a solution exists e^u and $|\nabla u|^q$ are integrable and (52) holds. The function $U=-u$ has to satisfy (note that $-\gamma>0$)

$$\begin{aligned}
&-U_{rr}-\frac{N-1}{r}U_r-M|\nabla U|^q+e^{-U}=0 \quad \text{in } (0,\rho),\\
&\lim_{r\to 0} r^{N-2}U(r)=-\gamma,\\
&\lim_{r\to 0} r^{N-1}U_r(r)=(N-1)\gamma,\\
&U(\rho)=0.
\end{aligned} \tag{55}$$

Hence the function $V(r)=U_r(r)$ satisfies

$$V(r)=(N-1)\gamma r^{1-N}-r^{1-N}\int_0^r\left(M|V|^q-e^{-U}\right)s^{N-1}\,\mathrm{d}s. \tag{56}$$

We combine this with

$$U(r)=-\int_r^\rho V(s)\,\mathrm{d}s, \tag{57}$$

and define the operator $(U,V)\mapsto K(U,V)=\big(K_1(U,V),K_2(U,V)\big)$ with

$$\begin{aligned}
&K_1(U,V)(r)=-\int_r^\rho V(s)\,\mathrm{d}s,\\
&K_2(U,V)(r)=(N-1)\gamma r^{1-N}-r^{1-N}\int_0^r\left(M|V|^q-e^{-|U|}\right)s^{N-1}\,\mathrm{d}s.
\end{aligned} \tag{58}$$

We define K on the subspace $\mathscr{K}$ of $C\big((0,\rho]\big)\times C\big((0,\rho]\big)$ of functions $W=(U,V)$ which satisfy

$$\|W\|_{\mathscr{K}}=\big\|(U,V)\big\|_{\mathscr{K}}=\max\left\{\sigma\sup_{0<r<\rho} r^{N-2}\big|U(r)\big|,\ \sup_{0<r<\rho} r^{N-1}\big|V(r)\big|\right\}:=\max\{\sigma N_1(U),N_2(V)\}<\infty,$$

where $0<\sigma<1$.

Step 1: Lipschitz estimate. We have

$$\begin{aligned}
\big\|K(U_1,V_1)-K(U_2,V_2)\big\|_{\mathscr{K}}=\max\Bigg\{&\sigma\sup_{0<r<\rho} r^{N-2}\left|\int_r^\rho (V_1-V_2)\,\mathrm{d}s\right|,\\
&\sup_{0<r<\rho}\left|\int_0^r\left(M\big(|V_1|^q-|V_2|^q\big)-\big(e^{-|U_1|}-e^{-|U_2|}\big)\right)s^{N-1}\,\mathrm{d}s\right|\Bigg\}\\
=\max\{I_1,I_2\}&.
\end{aligned}$$

Since for $r<s<\rho$

$$\big|(V_1-V_2)(s)\big|\le s^{1-N}\sup_{0<r<\rho} r^{N-1}\big|(V_1-V_2)(r)\big|,$$

we have that

$$I_1\le\frac{\sigma}{N-2}\sup_{0<r<\rho} r^{N-1}\big|(V_1-V_2)(r)\big|=\frac{\sigma}{N-2}N_2(V_1-V_2).$$

Concerning I_2, we have

$$\left|e^{-|U_1|}-e^{-|U_2|}\right|\le\big||U_1|-|U_2|\big|\le|U_1-U_2|,$$

hence

$$\sup_{0<r<\rho}\left|\int_0^r \left(e^{-|U_1|}-e^{-|U_2|}\right)s^{N-1}\,\mathrm{d}s\right| \le \left(\sup_{0<r<\rho} r^{N-2}\left|U_1(r)-U_2(r)\right|\right)\int_0^\rho s\,\mathrm{d}s$$
$$\le \frac{\rho^2}{2}\sup_{0<r<\rho} r^{N-2}\left|U_1(r)-U_2(r)\right|$$
$$\le \frac{\rho^2}{2}N_1(U_1-U_2),$$

and

$$\left||V_1|^q-|V_2|^q\right|(s) \le q\sup\left\{\left|V_1(s)\right|^{q-1},\left|V_2(s)\right|^{q-1}\right\}|V_1-V_2|(s)$$
$$\le q\sup_{0<r<\rho}\max\left\{\left|r^{N-1}V_1(r)\right|^{q-1},\left|r^{N-1}V_2(r)\right|^{q-1}\right\}s^{(q-1)(1-N)}|V_1-V_2|(s),$$

and

$$\sup_{0<r<\rho}\left|\int_0^r m\left(|V_1|^q-|V_2|^q\right)s^{N-1}\,\mathrm{d}s\right|$$
$$\le \sup_{0<r<\rho}\frac{mq\max\left\{\left|r^{N-1}V_1(r)\right|^{q-1},\left|r^{N-1}V_2(r)\right|^{q-1}\right\}}{N-q(N-1)}\sup_{0<r<\rho} r^{N-1}\left|V_1(r)-V_2(r)\right|$$
$$\le \sup_{0<r<\rho}\frac{mq\max\left\{\left|r^{N-1}V_1(r)\right|^{q-1},\left|r^{N-1}V_2(r)\right|^{q-1}\right\}}{N-q(N-1)}N_2(V_1-V_2).$$

Finally,

$$\left\|K(U_1,V_1)-K(U_2,V_2)\right\|_{\mathcal{K}} \le \max\left\{\frac{\sigma}{N-2}N_2(V_1-V_2),\right.$$
$$\left.\frac{\rho^2}{2}N_1(U_1-U_2)+\frac{Mq\max\{N_2^{q-1}(V_1),N_2^{q-1}(V_2)\}}{N-q(N-1)}N_2(V_1-V_2)\right\}. \quad (59)$$

Step 2: Bounds on the mapping K. We still have to estimate the terms $N_2^{q-1}(V_j)$ and for such a task we have to find a ball $B^{\mathcal{K}}$ in $C\big((0,\rho]\big)\times C\big((0,\rho]\big)$ endowed with the norm $\|\cdot\|_{\mathcal{K}}$ which is invariant under K. If $(U,V)\in B^{\mathcal{K}}=B^{\mathcal{K}}_R\big((0,0)\big)$ we have by assumption

$$\left|U(r)\right| \le \frac{R}{\sigma}r^{2-N} \quad \text{and} \quad \left|V(r)\right| \le Rr^{1-N} \quad \text{for all } 0<r\le\rho. \quad (60)$$

Then

$$N_1\big(K_1(U,V)\big) \le \sup_{0<r\le\rho} r^{N-2}\int_r^\rho Rs^{1-N}\,\mathrm{d}s \le \frac{R}{N-2}. \quad (61)$$

Since by (60)

$$\int_0^r\left|M|V|^q-e^{-|U|}\right|s^{N-1}\,\mathrm{d}s \le \frac{r^N}{N}+\frac{MR^q r^{N-q(N-1)}}{N-q(N-1)}$$

we obtain

$$N_2\big(K_2(U,V)\big) \le \frac{\rho^2}{2}N_1(U)+(N-2)|\gamma|+\frac{\rho^N}{N}+\frac{MR^q\rho^{N-q(N-1)}}{N-q(N-1)}, \quad (62)$$

and finally

$$\left\|K(U,V)\right\|_{\mathcal{K}} \le \max\left\{\frac{\sigma R}{N-2},\frac{\rho^2 R}{2\sigma}+(N-2)|\gamma|+\frac{\rho^N}{N}+\frac{MR^q\rho^{N-q(N-1)}}{N-q(N-1)}\right\} \quad (63)$$

We fix $\sigma=\frac{3}{4}$. Therefore for any $R>0$ there exist $0<\rho_0<1$ and $k_0>0$ such that for $0\le|\gamma|\le k_0$ and $0<\rho\le\rho_0$ there holds

$$\left\|(U,V)\right\|_{\mathcal{K}} \le R \quad \Longrightarrow \quad \left\|K(U,V)\right\|_{\mathcal{K}} \le R. \quad (64)$$

Step 3: End of the proof. With the choice of σ, we have $\frac{\sigma}{N-2}=\frac{3}{4(N-2)}\leq\frac{3}{4}$. Furthermore, if $\left\|(U,V)\right\|_{\mathcal{K}}\leq R$ we have also $\sup_{0<r\leq\rho}r^{N-1}\left|V(r)\right|\leq R$, hence $N_2^{q-1}(V)\leq R^{q-1}$. Combining this fact with estimate (59) we obtain that if (U_1,V_1) and (U_2,V_2) are in $B_R^{\mathcal{K}}$ there holds if

$$\left\|K(U_1,V_1)-K(U_2,V_2)\right\|_{\mathcal{K}}\leq\max\left\{\tfrac{3}{4}N_2(V_1-V_2),\tfrac{\rho^2}{2}N_1(U_1-U_2)+\tfrac{MqR^{q-1}}{N-q(N-1)}N_2(V_1-V_2)\right\}\tag{65}$$

Up to reducing the value of R, always with $R\leq 1$, we can assume that $\frac{mqR^{q-1}}{N-q(N-1)}\leq\frac{1}{4}$. A straightforward verification shows that

$$\max\left\{\tfrac{3}{4}N_2(V_1-V_2),\tfrac{\rho^2}{2}N_1(U_1-U_2)+\tfrac{1}{4}N_2(V_1-V_2)\right\}\leq\max\left\{\tfrac{3}{4},\rho^2\right\}\|U-V\|_{\mathcal{K}}.\tag{66}$$

Since $\rho<1$, the mapping K admits a fixed point (U,V). Hence U satisfies

$$\begin{aligned}&-U_{rr}-\frac{N-1}{r}U_r-M|\nabla U|^q+e^{-|U|}=0\quad\text{in }(0,\rho),\\&\lim_{r\to0}r^{N-2}U(r)=-\gamma,\end{aligned}$$

and it vanishes on $r=\rho$. Since $-\gamma>0$ and $\lim_{r\to0}r^{N-2}U(r)=-\gamma$ there exists $\rho_1\in(0,\rho]$ such that $U(r)>0$ for $0<r\leq\rho_1$. Hence $e^{-|U|}=e^{-U}$. Thus $u=-U$ satisfies (52) in B_{ρ_1}. By Proposition 6 u can be extended as a solution of

$$-\Delta u+M|\nabla u|^q-e^u=c_N\gamma\delta_0\quad\text{in }\mathcal{D}'(\mathbb{R}^N).\tag{67}$$

Note that the existence of solutions to (67) is natural as soon as it is proved that $|\nabla u|^q$ is integrable by using the classical result of [8]. □

5. Isolated singularities when $q>2$

We first denote by U_{eik} the solution of the eikonal equation in $\mathbb{R}^N\setminus\{0\}$

$$M|u_r|^q-e^u=0.\tag{68}$$

Its expression is

$$U_{\text{eik}}(r)=\ln\frac{Mq^q}{r^q}=-q\ln r+\Lambda_{N,q,M}\quad\text{with }\Lambda_{N,q,M}=\ln Mq^q.\tag{69}$$

Notice that if $N=2$, U_{eik} is a solution of (17) in $\mathbb{R}^2\setminus\{0\}$. If $N\geq2$ it is a supersolution.

Our first key result is the description of singularities at $r=0$ when $q>2$.

5.1. *Singular behaviour*

Theorem 18. *Let $N\geq1$ and $q>2$. If $u\in C^2\left(B_{r_0}\setminus\{0\}\right)$ is a radial solution of* (1) *in $B_{r_0}\setminus\{0\}$, the following dichotomy holds:*

(1) *either*

$$\lim_{r\to0}r^qe^{u(r)}=Mq^q\qquad\text{and}\qquad\lim_{r\to0}ru_r(r)=-q;\tag{70}$$

(2) *or there exists $u_0\in\mathbb{R}$ such that*

$$u(r)=u_0+c_{M,N,q}r^{\frac{q-2}{q-1}}\left(1+o(1)\right)\quad\text{as }r\to0,\tag{71}$$

where $c_{M,N,q}=\frac{q-1}{q-2}\left(\frac{(N-1)q-N}{M(q-1)}\right)^{\frac{1}{q-1}}$ if $N>1$ and $c_{M,1,q}=-\frac{q-1}{q-2}\left(\frac{1}{M(q-1)}\right)^{\frac{1}{q-1}}$;

(3) *or u is regular at* 0 *in the sense that there exists $u_0\in\mathbb{R}$ such that*

$$\lim_{r\to0}u(r)=u_0\qquad\text{and}\qquad\lim_{r\to0}u_r(r)=0;\tag{72}$$

(4) *or $N=1$ there exists $u_0\in\mathbb{R}$ such that*

$$\lim_{r\to0}u(r)=u_0\in\mathbb{R}\qquad\text{and}\qquad\lim_{r\to0}u_r(r)=b\in\mathbb{R}_*.\tag{73}$$

For proving this result we need two intermediate lemmas.

Lemma 19. *Assume $N \geq 1$ and $q > 2$. If u is a solution of* (17) *in* $(0,1)$*, then either for any $C \in (0,1)$ there exists $r_C > 0$ such that*

$$r^q e^{u(r)} \geq CMq^q \quad \textit{for all } r \geq r_C, \tag{74}$$

or u satisfies (2), (3) *or* (4) *in the previous statement.*

Proof. Step 1: The case of decreasing solutions. Assume $u_r < 0$ near 0. Then $u_r < 0$ in $(0,1)$ by Lemma 5. Following an idea of Serrin and Zou [23], for any $C \in (0,1)$ we define the function F_C by

$$F_C(r) = e^u - CM|u_r|^q. \tag{75}$$

(i). We first prove that given $C \in (0,1)$, F_C has a constant sign near 0. There holds

$$F_C'(r) = -e^u|u_r| + CMq|u_r|^{q-1}\left(M|u_r|^q + \frac{N-1}{r}|u_r| - e^u\right).$$

At a point r^* such that $F_C(r^*) = 0$ we have

$$\begin{aligned} F_C'(r^*) &= -CM|u_r|^{q+1} + CMq|u_r|^{q-1}\left(M(1-C)|u_r|^q + \frac{N-1}{r^*}|u_r|\right) \\ &= -CM|u_r|^{q+1} + CMq|u_r|^{q}\left(M(1-C)|u_r|^{q-1} + \frac{N-1}{r^*}\right) \\ &= CM|u_r|^{q}\left(qM(1-C)|u_r|^{q-1} + q\frac{N-1}{r^*} - |u_r|\right). \end{aligned}$$

As a consequence, if furthermore $F_C'(r^*) < 0$, we obtain that for $r = r^*$

$$qM(1-C)|u_r|^{q-1} + q\frac{N-1}{r^*} < |u_r|. \tag{76}$$

Since $q > 2$, this inequality implies that $u_r(r^*)$ is bounded and more precisely $\left|u_r(r^*)\right| \leq A_{C,M,q} = \left(qM(1-C)\right)^{\frac{1}{2-q}}$ which in turn implies $e^{u(r^*)} = CM\left|u_r(r^*)\right|^q \leq B_{C,M,q} = CM\left(qM(1-C)\right)^{\frac{q}{2-q}}$. This implies also that $q\frac{N-1}{r^*}$ is bounded. Therefore, for $N > 1$ this cannot happen and F_C has a constant sign near 0.

Next, if $N = 1$, u_r is monotone near 0 by Lemma 5. If the function F_C is oscillating near 0, there exist two sequences $\{r_n\}$ and $\{\tilde{r}_n\}$ tending to 0 such that $F_C(r_n) = F_C(\tilde{r}_n) = 0$ with $F_C'(r_n) < 0$ and $F_C'(\tilde{r}_n) > 0$. This implies $\left|u_r(r_n)\right| < A_{C,M,q}$ and $\left|u_r(\tilde{r}_n)\right| > A_{C,M,q}$, which contradicts the monotonicity of u_r. As a consequence, for $N = 1$ also, F_C keeps a constant sign near 0.

(ii). Suppose that for any $C \in (0,1)$ there exists $r_C \in (0,1)$ such that $F_C(r) > 0$ for $0 < r < r_C$. Then $e^{\frac{u}{q}} + (CM)^{\frac{1}{q}}u_r \geq 0$ on this interval which implies that $r \mapsto (CM)^{\frac{1}{q}}r - qe^{-\frac{u(r)}{q}}$ in increasing on $(0, r_C)$. Since $u_r < 0$, either $u(r) \to \infty$ when $r \to 0$, in which case $(CM)^{\frac{1}{q}}r - qe^{-\frac{u(r)}{q}} \geq 0$ and therefore $e^{u(r)} \geq CMq^q r^{-q}$ which is (74). Or $u(r) \to u_0$ and the assumption $F_C > 0$ implies that u_r is bounded on $(0, r_C]$. By standard regularity theory, u_{rr} is also bounded. This implies that $u_r(r)$ has a limit when $r \to 0$, and this limit is necessarily zero from the equation if $N > 1$. Therefore in that case u satisfies (73) or (72).

(iii). Suppose now that for some $C \in (0,1)$ there exists $r_C \in (0,1)$ such that $F_C(r) < 0$ for $0 < r < r_C$. Then

$$-u_{rr} - \frac{N-1}{r}u_r + M(1-C)|u_r|^q \leq 0.$$

Set $\mu = M(1-C)$ and $W = -r^{N-1}u_r = r^{N-1}|u_r|$, then

$$\mu r^{-(N-1)(q-1)} + W^{-q}W_r \leq 0.$$

By integration, it implies that the function

$$r \mapsto W^{1-q}(r) + \frac{\mu(q-1)}{(N-1)q-N} r^{N-(N-1)q}$$

is nondecreasing. When $N > 1$ we have that $(N-1)q - N > 0$ since $q > 2$ and we get a contradiction. When $N = 1$, $W = -u_r$ and $r \mapsto \left|u_r(r)\right|^{1-q} - \mu(q-1)r$ is nondecreasing. Hence it admits a limit $\ell \geq 0$ when $r \to 0$. If $\ell = 0$ then $\left|u_r(r)\right| \to \infty$. Since $\left|u_r(r)\right|^{1-q} - \mu(q-1)r \to 0$, we have that $\left|u_r(r)\right|^{1-q} \geq \mu(q-1)r$. Hence $\left|u_r(r)\right| \leq \left(\mu(q-1)r\right)^{-\frac{1}{q-1}}$. By integration we obtain that $u(r)$ has a limit u_0. Since $-u_{rr} + M|u_r|^q\left(1+o(1)\right) = 0$ we deduce by integration that (71) holds. If $\ell > 0$ then $\left|u_r(r)\right| \to \frac{\ell}{q-1}$ and (73) holds.

Step 2: The case of increasing solutions. Since u is bounded from above, it follows by Theorem 3 that $\left|u_r(r)\right| \leq cr^{-\frac{1}{q-1}}$ near 0. Since $q > 2$ it follows that $u(r)$ admits a limit u_0 when $r \to 0$. The function $v = u - u_0$ satisfies

$$-v_{rr} - \frac{N-1}{r} v_r + M|v_r|^q = e^{u_0}(e^v - 1) := \phi v$$

where $\phi = e^{u_0}\frac{e^v-1}{v} \to e^{u_0}$ as $r \to 0$. Set $W = r^{N-1}v_r$, then

$$W_r = r^{N-1}\left(Mr^{-(N-1)q}W^q - \phi v\right).$$

The function u_r is monotone near 0. Indeed, at a point $\tilde{r}$ where $u_{rr}(\tilde{r}) = 0$ there holds

$$u_{rrr}(\tilde{r}) = \left(\frac{N-1}{\tilde{r}^2} - e^{u(\tilde{r})}\right)u_r(\tilde{r});$$

as $u(r) \to u_0$ when $r \to 0$, $\frac{N-1}{r^2} - e^u$ is positive near 0 if $N \geq 2$, negative if $N = 1$ and in any case this expression keeps a constant sign.

Either $u_r \to \infty$ when $r \to 0$ or u_r has a finite limit. In the first case we have that $\phi v = o(v_r)$ when $r \to 0$. Thus we can write the equation satisfied by v under the form

$$-r^{1-N}(r^{N-1}v_r)_r + M^*(r)|v_r|^q = 0$$

where $M^*(r) = M\left(1+o(1)\right)$. This equation is just a perturbation of $-r^{1-N}(r^{N-1}v_r)_r + M|v_r|^q = 0$ and it is explicitly integrable. If $N > 1$ it yields $v_r(r) = \Theta_{M,N}r^{-\frac{1}{q-1}}\left(1+o(1)\right)$ where $\Theta_{M,N} = \left(\frac{(N-1)q-N}{M(q-1)}\right)^{\frac{1}{q-1}}$. We obtain (71). If $N = 1$ we obtain a contradiction. In the case where u_r has a finite limit, the function u satisfies (72) or (73). □

Next we give the precise behaviour of solutions of (17) such that $r^q e^u$ is positively bounded from below. The result is obtained thanks to an energy function adapted from Leighton's method [1].

Lemma 20. *Let $N \geq 1$ and $q > 2$. If u is a solution of* (17) *such that $r^q e^u \geq C_1$ for some $C_1 > 0$ in a neighbourhood of* 0*, then* (70) *holds.*

Proof. Since $u(r) \geq -q\ln r + c$ and u_r has constant sign near 0, it is negative. We set

$$X(t) = r^q e^{u(r)}, \quad Y(t) = -r^{q+1}e^{u(r)}u_r(r) \quad \text{with } t = \ln r. \tag{77}$$

Then (X, Y) satisfies

$$\begin{aligned} X_t &= qX - Y := f(X,Y), \\ Y_t &= (q-N+2)Y - \frac{Y^2}{X} + e^{(2-q)t}\left(X^2 - M\frac{Y^q}{X^{q-1}}\right) = g(X,Y,t). \end{aligned} \tag{78}$$

This system is a variant of system (20) where the unknown are $X(t)$ and $\Phi(t) = -ru_r(r)$. Hence $Y(t) = X(t)\Phi(t)$. Note that the use of (X,Y) is the most natural transformation for the equation associated to (12). We recall that Leighton's method relative to an autonomous system

$$\mathscr{X}_t = \tilde{f}(\mathscr{X},\mathscr{Y}),$$
$$\mathscr{Y}_t = \tilde{g}(\mathscr{X},\mathscr{Y}),$$

which has the property that the relation $\tilde{f}(\mathscr{X},\mathscr{Y}) = 0$ is equivalent to $\mathscr{Y} = \tilde{h}(\mathscr{X})$ consists in analysing the variations of the function $t \mapsto \mathscr{F}\big(\mathscr{X}(t),\mathscr{Y}(t)\big)$ defined by

$$\mathscr{F}(\mathscr{X},\mathscr{Y}) = \int_{h(\mathscr{X})}^{\mathscr{Y}} \tilde{f}(\mathscr{X},s)\,\mathrm{d}s - \int_{h(\mathscr{X})}^{\mathscr{Y}} \tilde{g}\big(s,h(s)\big)\,\mathrm{d}s. \tag{79}$$

We adapt this methods to the non-autonomous system (78) in considering first

$$\begin{aligned} F(t) &= \int_{qX(t)}^{Y(t)} f(X(t),s)\,\mathrm{d}s - \int_{qX(t)}^{Y(t)} g(s,qs,t)\,\mathrm{d}s \\ &= (N-2)q\frac{X(t)}{2} - \frac{X_t^2(t)}{2} + e^{(2-q)t}\left(Mq^q\frac{X^2(t)}{2} - \frac{X^3(t)}{3}\right). \end{aligned}$$

In this expression the term $e^{(2-q)t}$ tends to ∞ when $t \to -\infty$. Therefore we replace F by

$$G(t) := e^{(q-2)t}F(t) = Mq^q\frac{X^2(t)}{2} - \frac{X^3(t)}{3} + e^{(q-2)t}\left((N-2)q\frac{X(t)}{2} - \frac{X_t^2(t)}{2}\right). \tag{80}$$

Then we obtain

$$\begin{aligned} G_t &= (Mq^q - X)XX_t + (q-2)e^{(q-2)t}\left((N-2)q\frac{X}{2} - \frac{X_t^2}{2}\right) + e^{(q-2)t}X_t\left(\frac{(N-2)q}{2} - X_{tt}\right) \\ &= (Mq^q - X)XX_t + (q-2)e^{(q-2)t}\left((N-2)q\frac{X}{2} - \frac{X_t^2}{2}\right) \\ &\quad + e^{(q-2)t}X_t\left(\frac{(N-2)q}{2} - (N-2)Y - \frac{X_t^2}{X} - e^{(2-q)t}\left(M\frac{Y^q}{X^{q-1}} - X^2\right)\right) \\ &= \left(Mq^qX - M\frac{Y^q}{X^{q-1}}\right)X_t + e^{(q-2)t}\Psi(t), \end{aligned}$$

where

$$\begin{aligned} \Psi(t) &= (q-2)\left((N-2)q\frac{X}{2} - \frac{X_t^2}{2}\right) + X_t\left(\frac{(N-2)q}{2} - (N-2)Y - \frac{X_t^2}{X}\right) \\ &= (q-2)\left((N-2)q\frac{X}{2} - \frac{X_t^2}{2}\right) + \frac{(N-2)q}{2}X_t - \frac{X_t^3}{X} + (N-2)(X_t - qX)X_t. \end{aligned}$$

Replacing X_t by its value we obtain the following expression for $G_t(t)$

$$G_t(t) = MX^{1-q}(qX - Y)(q^qX^q - Y^q) + e^{(q-2)t}\Psi(t). \tag{81}$$

Using the assumption and the bound from Theorem 1, we have in a neighbourhood of 0

$$C_1 \le r^q e^{u(r)} \le C_2.$$

This implies that $X(t)$ is bounded from above and from below. By the proof of Lemma 19 we have for any $C < 1$, $CM|u_r|^q \le e^{u(r)}$ near 0, hence $|u_r| \le \frac{C_3}{r}$ near 0. Since $X(t)$ is bounded when $t \to -\infty$ and $Y(t) = \phi(t)X(t) = -ru_r(r)X(t)$, we have also that $\Phi(t)$ and $Y(t)$ are bounded at $-\infty$. Hence $X_t(t) = qX(t) - Y(t)$ shares the same property. Furthermore $\frac{X_t^3}{X(t)} = X^2(t)\big(q - \Phi(t)\big)^2$ is bounded, a fact which implies that the function $\Psi(t)$ is bounded. Noticing that $(qX - Y)\big((qX)^q - Y^q\big) \ge 0$, we obtain

$$G_t(t) = MX^{1-q}(qX - Y)\big((qX)^q - Y^q\big) + e^{(q-2)t}\Psi \ge -Ce^{(q-2)t} \quad \text{for } t \le 0.$$

This implies that the function $t \mapsto G(t) + \frac{C}{q-2}e^{(q-2)t}$ is increasing. Hence either it tends to some finite ℓ or it tends to $-\infty$ when $t \to -\infty$. Since $X(t)$ is bounded from above and below and $Y(t)$ is bounded, we deduce that ℓ is finite. By the definition of $G(t)$ we have that $Mq^q\frac{X^2(t)}{2} - \frac{X^3(t)}{3} \to \ell$ when $t \to -\infty$, hence $X(t)$ converges to some λ such that $Mq^q\frac{\lambda^2}{2} - \frac{\lambda^3}{3} = \ell$, and $\ell \neq 0$ since $X(t)$ is bounded from below. Concerning $Y(t)$, either it has a limit Λ and $X_t(t) \to q\lambda - \Lambda$. But the only possible limit for $X_t(t)$ at $-\infty$ is zero, thus $\Lambda = q\lambda$. Or $Y(t)$ is not monotone, thus it oscillates at infinity and for a sequence $\{t_n\}$ of extremal points of Y tending to $-\infty$, we have that $(q-N+2)Y(t_n) - \frac{Y^2(t_n)}{X(t_n)} = O(e^{(q-2)t_n}) \to 0$ from the equation. Since $X(t_n) \to \lambda$ and $e^{(q-2)t_n} \to 0$ it follows that $Y(t_n)$ admits a limit when $t_n \to \infty$, which implies in turn that $Y^q(t_n) \to \frac{\lambda^{q+1}}{M}$. Hence, even if it oscillates, $Y(t)$ admits a limit that we denote $\widetilde{\Lambda}$ at $-\infty$, and $\widetilde{\Lambda} = \left(\frac{\lambda^{q+1}}{M}\right)^{\frac{1}{q}}$. Therefore $\lim_{r\to 0} r^q e^{u(r)} = Mq^q$. Since $\Phi(t) = \frac{Y(t)}{X(t)}$, then $\lim_{t\to-\infty}\Phi(t) = q = -\lim_{r\to 0} r u_r(r)$. □

In the next lemma we give precise estimates of the behaviour of such solutions by a method inspired from [4].

Lemma 21. *Let $N > 1$ and $q > 2$. If u is a solution of* (17) *satisfying* (70), *then*

$$u_{rr}(r) = \frac{q}{r^2}\big(1 + o(1)\big) \quad \text{as } r \to 0. \tag{82}$$

Furthermore

$$u(r) = \ln\frac{Mq^q}{r^q} - \frac{N-2}{Mq^q(q-1)} r^{q-2}\big(1+o(1)\big) \quad \text{as } r \to 0, \tag{83}$$

and

$$u_r(r) = -\frac{q}{r}\left(1 - \frac{(N-2)(q-2)}{Mq^q(q-1)} r^{q-2}\big(1+o(1)\big)\right) \quad \text{as } r \to 0. \tag{84}$$

Proof. In the proof we use the system (21) in $X(t) = r^q e^{u(r)}$ and $\Phi(t) = r u_r(r)$ with $t = \ln r$. We recall it below

$$\begin{cases} X_t = X(q-\Phi) \\ \Phi_t = (2-N)\Phi + e^{(2-q)t}\big(X - M|\Phi|^q\big). \end{cases}$$

Then $\Phi > 0$ since $u_r < 0$. Note that (Mq^q, q) is an equilibrium of (21) only if $N = 2$.

Step 1. We claim that

$$\lim_{t\to-\infty} e^{(2-q)t}\big(X(t) - M\Phi^q(t)\big) = (N-2)q. \tag{85}$$

Set

$$\Psi(t) = e^{(2-q)t}\big(X(t) - M\Phi^q(t)\big) = \Phi_t(t) + (N-2)\Phi(t).$$

Then

$$\begin{aligned} \Psi_t &= \Phi_{tt} + (N-2)\Phi_t \\ &= (2-q)e^{(2-q)t}(X - M\Phi^q) + e^{(2-q)t}(X(q-\Phi) - Mq\Phi^{q-1}\Phi_t) \\ &= (2-q)\Psi + e^{(2-q)t}X(q-\Phi) - Mqe^{(2-q)t}\Phi^{q-1}\big((2-N)\Phi + \Psi\big) \\ &= \big(2-q-Mqe^{(2-q)t}\Phi^{q-1}\big)\Psi + e^{(2-q)t}\big(X(q-\Phi) + Mq(N-2)\Phi^q\big) \\ &= e^{(2-q)t}\Big(X(q-\Phi) + Mq(N-2)\Phi^q + \Psi\big((2-q)e^{(q-2)t} - Mq\Phi^{q-1}\big)\Big). \end{aligned}$$

If Ψ is not monotone near infinity, at each $t^* < 0$ where $\Psi_t(t^*) = 0$ there holds

$$\big((q-2)e^{(q-2)t^*} + Mq\Phi^{q-1}(t^*)\big)\Psi(t^*) = X(t^*)\big(q - \Phi(t^*)\big) + Mq(N-2)\Phi^q(t^*).$$

Equivalently

$$\left(\frac{q-2}{Mq\Phi^{q-1}(t^*)}e^{(q-2)t^*} + 1\right)\Psi(t^*) = \frac{X(t^*)\big(q-\Phi(t^*)\big)}{Mq\Phi^{q-1}(t^*)} + (N-2)\Phi(t^*).$$

By assumption $\Phi(t) \to q$ and $X(t) \to Mq^q$ as $t \to -\infty$. If $t^* = t_n \to -\infty$, we have that $\Psi(t_n) \to (N-2)q$, then $\Psi(t) \to (N-2)q$.

If Ψ is monotone near infinity, then it admits a limit $L \in [-\infty, \infty]$. Since $\Phi_t(t) \to L-(N-2)q$ and $\Phi(t) \to q$ when $t \to -\infty$, then necessarily L is finite, $\Phi_t(t) \to 0$ and finally $L = (N-2)q$ as in the previous case. It follows that

$$\lim_{t\to-\infty} \Psi(t) = (N-2)q = \lim_{t\to-\infty} e^{(2-q)t}\big(X(t) - M\Phi(t)\big),$$

which can be written

$$X(t) - M\Phi(t) = (N-2)qe^{(q-2)t}\big(1+o(1)\big) \quad \text{as } t \to -\infty,$$

which is the claim.

Step 2: End of the proof. Relation (85) can be expressed by

$$r^2\Big(e^{u(r)} - M\big|u_r(r)\big|^q\Big) = (N-2)q\big(1+o(1)\big) \quad \text{as } r \to 0. \tag{86}$$

By (17) we obtain (82) since

$$u_{rr} = \frac{N-1}{r}|u_r| - e^u + M|u_r|^q = \frac{(N-1)q}{r^2}\big(1+o(1)\big) - \frac{(N-2)q}{r^2}\big(1+o(1)\big) = \frac{q}{r^2}\big(1+o(1)\big).$$

Since $M\big|u_r(r)\big|^q = e^{u(r)}\big(1+o(1)\big)$ when $r \to 0$, we define W by $u_r(r) = -\frac{e^{\frac{u(r)}{q}}}{M^{\frac{1}{q}}}\big(1+W(r)\big)$. Then $W(r) \to 0$ and we have when $r \to 0$,

$$r^2\Big(e^{u(r)} - M\big|u_r(r)\big|^q\Big) = r^2e^{u(r)}\Big(1-\big(1+W(r)\big)^q\Big) = -qr^2e^{u(r)}W(r)\big(1+o(1)\big).$$

This implies

$$W(r) = -\frac{(N-2)e^{-u(r)}}{r^2}\big(1+o(1)\big) = -\frac{(N-2)r^{q-2}}{Mq^q}\big(1+o(1)\big).$$

Therefore

$$-u_r(r) = \frac{e^{\frac{u(r)}{q}}}{M^{\frac{1}{q}}}\left(1 - \frac{(N-2)r^{q-2}}{Mq^q}\big(1+o(1)\big)\right), \tag{87}$$

which can be written as

$$M^{\frac{1}{q}}e^{-\frac{u(r)}{q}}u_r(r) + 1 - \frac{(N-2)r^{q-2}}{Mq^q}\big(1+o(1)\big) = 0.$$

By integration,

$$-qM^{\frac{1}{q}}e^{-\frac{u(r)}{q}} + r\left(1 - \frac{(N-2)r^{q-2}}{M(q-1)q^q}\big(1+o(1)\big)\right) = 0.$$

This yields the expansion of $e^{u(r)}$,

$$e^{u(r)} = \frac{Mq^q}{r^q}\left(1 + \frac{(N-2)r^{q-2}}{M(q-1)q^{q-1}}\big(1+o(1)\big)\right), \tag{88}$$

from which follows

$$u(r) = \ln\frac{Mq^q}{r^q} + \frac{(N-2)r^{q-2}}{Mq^q(q-1)}\big(1+o(1)\big) \quad \text{as } r \to 0. \tag{89}$$

Combining (87) and (88) we obtain

$$-u_r(r) = \frac{q}{r}\left(1 - \frac{(N-2)(q-2)r^{q-2}}{Mq^q(q-1)}\big(1+o(1)\big)\right) \quad \text{as } r \to 0. \tag{90}$$

This ends the proof. □

Remark 22. By (88) we obtain the

$$r^qe^{u(r)} > Mq^q \quad (\text{resp. } r^qe^{u(r)} < Mq^q) \quad \text{if } N \geq 2 \quad (\text{resp. } N = 1). \tag{91}$$

5.2. *Existence of singular solutions of eikonal type*

In this section we prove existence results for eikonal type solutions. The proof is difficult and in dimension $N \geq 2$ it is based upon the construction for $N = 1$.

5.2.1. *The case $N = 1$*

Theorem 23. *Let $N = 1$ and $q > 2$. Then there exists one and only one solution u^* of* (17) *in* $(0,\infty)$ *such that* (70) *holds. Furthermore $u^* = \lim_{n\to\infty} u_n$ where u_n is the regular solution of* (17) *in* $(0,\infty)$ *such that $u_{n\,r}(0) = 0$ and $u_n(0) = n$.*

Proof of the uniqueness. We already know that such a solution u is decreasing and satisfies $\lim_{r\to\infty} u(r) = -\infty$. Hence $r \mapsto u(r)$ is a decreasing diffeomorphism from $(0,\infty)$ onto $(-\infty,\infty)$; we consider r as a function of u and set

$$z(u) = u_r^2(r) = u_r^2\big(r(u)\big).$$

Then z is defined on $\mathbb{R}$ and there holds

$$\frac{\mathrm{d}z}{\mathrm{d}u} = 2u_r u_{rr}\frac{\mathrm{d}r}{\mathrm{d}u} = 2u_{rr},$$

hence

$$\frac{\mathrm{d}z}{\mathrm{d}u} = 2Mz^{\frac{q}{2}} - 2e^u. \tag{92}$$

The associated system in (r,z) as functions of u is

$$\begin{cases} z_u = 2Mz^{\frac{q}{2}} - 2e^u \\ r_u = -\frac{1}{\sqrt{z}}. \end{cases} \tag{93}$$

By (70) we have that

$$z(u) = \frac{e^{\frac{2u}{q}}}{M^{\frac{2}{q}}}\big(1+o(1)\big) \quad \text{and} \quad r(u) = M^{\frac{1}{q}} q e^{-\frac{u}{q}}\big(1+o(1)\big) \quad \text{as } u\to\infty. \tag{94}$$

The point is that there exists a *unique* solution of (92) satisfying (94). To see that, let z_1 and z_2 be two such solutions, then by subtracting the two corresponding equations in z_j $(j = 1,2)$, the term $-2e^u$ disappears and we obtain

$$\frac{\mathrm{d}(z_1 - z_2)}{\mathrm{d}u} = 2M\big(z_1^{\frac{q}{2}} - z_2^{\frac{q}{2}}\big) = Mq\xi^{\frac{q-2}{2}}(z_1 - z_2),$$

where $\xi := \xi(u) = \theta z_1(u) + (1-\theta)z_2(u)$ for some $\theta \in [0,1]$. Because of (70) we have that

$$Mq\xi^{\frac{q-2}{2}} = qM^{\frac{2}{q}} e^{(1-\frac{2}{q})u}\big(1+o(1)\big) \quad \text{as } u\to\infty.$$

Hence the function

$$u \mapsto H(u,v) := e^{-Mq\int_v^u \xi(s)^{\frac{q-2}{2}}\,\mathrm{d}s}\big(z_1(u) - z_2(u)\big)$$

is constant for any v. Since by l'Hospital's rule

$$-Mq\int_v^u \xi(s)^{\frac{q-2}{2}}\,\mathrm{d}s = -\frac{Mq^2}{q-2}e^{(1-\frac{2}{q})u}\big(1+o(1)\big) \quad \text{as } u\to\infty,$$

it follows from (94) that $H(u,v) \to 0$ as $u \to \infty$, hence $H(v,v) = 0$ which implies $z_1(v) = z_2(v)$. Therefore if u_1 and u_2 are solutions, the uniqueness of z_1 and z_2 implies $u_{1\,r}(r) = u_{2\,r}(r) < 0$ for any $r > 0$ and $u_1(r) = u_2(r) + c$. The fact that u_1 and u_2 satisfy (17) implies $c = 0$.

Proof of the existence. Equation (17) reduces to

$$-u_{rr}+M|u_r|^q-e^u=0, \tag{95}$$

equivalently to the *autonomous* system of order 2,

$$\begin{cases} u_r=-v \\ v_r=e^u-M|v|^q. \end{cases} \tag{96}$$

Let $a_1\geq 0$ and denote by u_a any regular solution of (95) such that $u(0)=a>a_1$ and $u_r(0)=0$. Since u_a is monotone it is decreasing and necessarily it tends to $-\infty$ at infinity; from Proposition 6 there exists a unique ρ_1 such that $u_a(\rho_1)=0$.

Step 1. *We claim that there exists* $K=K_{a_1,M,q}>0$ *such that*

$$\left|u_{ar}(\rho_1)\right|\leq K. \tag{97}$$

We consider the function F_C defined in (75) for $C=\frac{1}{2}$ and $F_{\frac{1}{2}}(r)=e^{u_a(r)}-\frac{1}{2}M\left|u_{ar}(r)\right|^q$. Then $F_{\frac{1}{2}}(0)=e^a$ and

$$F_{\frac{1}{2}}(\rho_1)=e^{a_1}-\tfrac{1}{2}M\left|u_{ar}(\rho_1)\right|^q.$$

If $F_{\frac{1}{2}}(\rho_1)\geq 0$ we obtain

$$\left|u_{ar}(\rho_1)\right|\leq\left(\frac{2e^{a_1}}{M}\right)^{\frac{1}{q}}.$$

If $F_{\frac{1}{2}}(\rho_1)<0$ there exists $r_0\in(0,\rho_1)$ such that $F_{\frac{1}{2}}(r_0)=0$ and $F'_{\frac{1}{2}}(r_0)<0$ and by (76), we deduce $\left|u_{ar}(r_0)\right|^q>0$, then $2^{-1}qM\left|u_{ar}(r_0)\right|^{q-1}\leq\left|u_{ar}(r_0)\right|$, hence

$$\left|u_{ar}(r_0)\right|\leq(2^{-1}qM)^{\frac{1}{q-2}}:=\widetilde{K}_{a_1,M,q},$$

and

$$-u_{arr}(r_0)=e^{u_a(r_0)}-M\left|u_{ar}(r_0)\right|^q=-2^{-1}M\left|u_{ar}(r_0)\right|^q<0,$$

Since $u_{arr}(0)=-e^a<0$, there exists $r_2\in(0,r_0)$ such that $u_{arr}(r_2)=0$, and this point is unique since $-u_{arrr}(r_2)=e^{u_a(r_2)}u_{ar}(r_2)<0$. Then $u_{arr}>0$ on (r_2,ρ_1). This implies that u_{ar} is increasing on (r_2,ρ_1) and thus $u_{ar}(r_2)<u_{ar}(\rho_1)$. Consequently,

$$\left|u_{ar}(\rho_1)\right|\leq\left|u_{ar}(r_0)\right|\leq\widetilde{K}_{a_1,M,q}.$$

Estimate (97) follows with $K=\max\left\{\left(\frac{2e^{a_1}}{M}\right)^{\frac{1}{q}},\widetilde{K}_{a_1,M,q}\right\}$.

Step 2. *We claim that there exist singular solutions.* We denote by $\mathscr{H}$ the vector field

$$\mathscr{H}(u,v)=\left(-v,e^u-M|v|^q\right):=\left(\mathscr{H}_1(u,v),\mathscr{H}_2(u,v)\right)$$

in the phase plane $Q:=\left\{(u,v):v>0\right\}$. We denote by $\mathscr{C}$ the curve $\left\{(u,v)\in Q:Mv^q=e^u\right\}$ and the two regions

$$\mathscr{R}:=\left\{(u,v)\in Q:Mv^q>e^u\right\}\qquad\text{and}\qquad\mathscr{S}:=\left\{(u,v)\in Q:Mv^q<e^u\right\}.$$

On $\mathscr{C}$ the vector field is horizontal and entering in $\mathscr{R}$. Thus this region is positively invariant. On the axis $v=0$ the vector field is vertical and inward to Q.

A regular trajectory $\mathscr{T}_a:=\left\{(u_a,-u_{ar})\right\}_{r>0}$ starts from $(a,0)$ with a vertical slope. Since there exists a unique $r_2>0$ such that $u_{arr}(r_2)=0$, $\mathscr{T}_a$ intersects $\mathscr{C}$ in Q and remains in $\mathscr{R}$ which is indeed positively invariant. Conversely, any trajectory issued from a point $(\bar{u},\bar{v})\in\mathscr{C}$ for $r=\bar{r}$ is included in $\mathscr{S}$ for $r<\bar{r}$ as long as it remains in Q. Since $\mathscr{H}$ admits no equilibrium in $\mathscr{S}$, this backward trajectory cannot remain bounded for $r<\bar{r}$. If this backward trajectory does not intersect the axis $v=0$, then $u(r)\to\infty$ when $r<\bar{r}$ decreases to the infimum of the maximal interval of existence and $v(r)$ admits a finite limit. From (96) this implies that $v_r(r)\to\infty$, contradiction. Then the trajectory intersect $v=0$ at some $\tilde{r}<\bar{r}$. This implies that $\bar{u}(r-\tilde{r})$ is the regular trajectory issued from $\bar{u}(\tilde{r})$.

We first take $a_1=0$. Hence any regular trajectory cuts the axis $u=0$ at a point c such that $0<c\leq K_{0,M,q}$. Therefore, any trajectory through a point $(0,c)$ with $c>K_{0,M,q}$ is not a regular one. We denote by Reg the set of $c>0$ such that the trajectory through $(0,c)$ is regular and put

$$c^*=\sup\{c>0 : c\in \mathrm{Reg}\}. \tag{98}$$

By the implicit function theorem, the set of Reg is open and it is an interval since two trajectories cannot intersect. Therefore c^* is not in this set. Let $\mathcal{T}_*$ be the backward trajectory through $(0,c^*)$. It does not intersect $\mathcal{C}$, thus it is located in the region $\mathcal{R}$ where $Mv^q>e^u$. This means $u_re^{-\frac{u}{q}}+M^{-\frac{1}{q}}<0$, thus the function $r\mapsto M^{-\frac{1}{q}}r-qe^{-\frac{u(r)}{q}}$ is nonincreasing, consequently, for such a trajectory, the variable r is bounded from below. Therefore any solution u with trajectory $\mathcal{T}^*$ is defined on a maximal interval (R^*,∞) with $R^*>-\infty$. Now the function $r\mapsto u^*:=u(r-R^*)$ which a solution is defined on the maximal interval $(0,\infty)$, it is nonincreasing, thus it is necessarily singular and the trajectory $\mathcal{T}_a$ of any regular solution lies below $\mathcal{T}^*$ in the half-plane Q. By Theorem 18 we have two possibilities.

(i) Either there exists $a_1>0$ such that $u^*(r)\to a_1$ and $u^*_r(r)=-\big(\frac{1}{M(q-1)}\big)^{\frac{1}{q-1}}r^{-\frac{1}{q-1}}\big(1+o(1)\big)$ when $r\to 0$. In such a case, for any $a>a_1$, the regular solution u_a which satisfies $u_a(\rho_1)=a_1$ for some $\rho_1>0$ satisfies also $\left|u_{a\,r}(\rho_1)\right|\leq K_{a_1,M,q}$ by (97). This means that in the phase plane, on the trajectory $\mathcal{T}_a$, there holds $v_a\leq K_{a_1,M,q}$ at the point $(a_1,v_a)=\big(u_a(\rho_1),-u_{r\,a}(\rho_1)\big)$. Consider now the trajectory passing through $(a_1,1+K_{a_1,M,q})$, it is below $\mathcal{T}^*$ and above $\mathcal{T}_a$ for all $a>a_1$. Therefore it intersects the axis $u=0$ at some $\tilde{c}<c^*$ but which is also larger or equal to $\sup\{c : c\in\mathrm{Reg}\}$ which is c^*. This is a contradiction.

(ii) Or $\lim_{r\to 0}u^*(r)=\infty$. By Theorem 18 this implies that (70) holds.

Step 3: Convergence of the regular solutions. We consider the regular solutions u_n, this means $u_n(0)=n$ and $u_{n\,r}(0)=0$. Let $b\leq 0$, since u_n is decreasing and tends to $-\infty$ when $r\to\infty$ there exists $\rho_{n,b}>0$ such that $u_n(r)>b$ on $[0,\rho_{n,b})$ and $u_n(\rho_{n,b})=0$. Similarly $u^*(r)>b$ on $[0,\rho_{*,b})$. We can write

$$\rho_{n,b}=\int_b^n\frac{\mathrm{d}u}{v_n(u)}=\int_b^n\frac{\mathrm{d}u}{\chi_{(0,n)}(u)v_n(u)}\qquad\text{and}\qquad \rho_{*,b}=\int_b^\infty\frac{\mathrm{d}u}{v^*(u)}.$$

Because two different trajectories cannot intersect, $v_n(u)$ is an increasing function of n. Hence, by the monotone function theorem

$$\rho_{*,b}=\inf_n\rho_{n,b}=\lim_{n\to\infty}\rho_{n,b}.$$

This implies that all the regular solutions are greater than b on $[0,\rho_{*,b})$. By Theorem 1(1) we have for any $R>0$

$$e^{u_n(r)}\leq Cr^{-q}\quad\text{for all } r\in(0,R]$$

where $C>0$ depends on R but not on n since u_n is decreasing on $[0,\infty)$, therefore there also holds

$$0\leq e^{u_n-b}\leq Ce^{-b}r^{-q}\quad\text{on } [0,\rho_{*,b}].$$

From (14) in Theorem 3 we have

$$\left|u_{n\,r}(r)\right|\leq C(r^{-\frac{1}{q-1}}+r^{-1}+1)\quad\text{for all } r\in\big(0,\tfrac{R}{2}\big],$$

where $C>0$ is independent of n and thus

$$\left|u_{n\,rr}(r)\right|\leq C(r^{-q}+r^{-\frac{q-2}{q-1}}+r^{-2}+1)\quad\text{on } [0,\rho_{*,b}].$$

There exists a subsequence $\{n_j\}$ and a solution $U_b\geq b$ of (95) such that $\{u_{n_j}\}$ converges to U_b in the $C^1_{\mathrm{loc}}\big((0,\rho_{*,b})\big)$-topology. The corresponding trajectory $\mathcal{T}_{U_b}$ is below $\mathcal{T}_*$, thus $\mathcal{T}_{U_b}$ crosses the axis $u=0$ at a point which is necessarily c^* from (98), hence $\mathcal{T}_{U_b}=\mathcal{T}_*$. Because of uniqueness, see Step 1, the whole sequence $\{u_{n_j}\}$ converges to u^*. □

5.2.2. *The case* $N\geq 2$

Theorem 24. *Let* $N>1$ *and* $q>2$. *Then there exists at least one solution* u^* *of* (17) *in* $(0,\infty)$ *such that* (70) *holds.*

Proof. Since the proof uses the result in dimension 1, we will denote by $u^{(N)}$ or $u^{(1)}$ the solutions in N-dim or in 1-dim. Thus $u_a^{(N)}$ and $u_a^{(1)}$ denote the regular solutions respectively in $\mathbb{R}^N$ and in $\mathbb{R}$ with initial data a. For $b\leq 0$ we denote by $[0,\rho_{a,b}^{(N)})$ the maximal interval where $u_a^{(N)}>b$.

Step 1. We claim that $\rho_{a,b}^{(N)}>\rho_{a,b}^{(1)}$ and $u_a^{(N)}>u_a^{(1)}$ on $(0,\rho_{a,b}^{(1)})$ for $N>1$. Since $r\mapsto u^{(j)}(r)$ $(j=1,N)$ is decreasing, we set $z^{(j)}(u)=(u_r^{(j)})^2\big((u^{(j)})^{-1}(u)\big)$. Then

$$\frac{\mathrm{d}z^{(N)}}{\mathrm{d}u}-2M(z^{(N)})^{\frac{q}{2}}+2e^u=\frac{N-1}{r^{(N)}}\sqrt{z^{(N)}}$$

for $N>1$, and

$$\frac{\mathrm{d}z^{(1)}}{\mathrm{d}u}-2M(z^{(1)})^{\frac{q}{2}}+2e^u=0,$$

we obtain that

$$\frac{z^{(1)}-z^{(N)}}{\mathrm{d}u}=2M\big((z^{(1)})^{\frac{q}{2}}-(z^{(N)})^{\frac{q}{2}}\big)-\frac{N-1}{r^{(N)}}\sqrt{z^{(N)}}<2M\big((z^{(1)})^{\frac{q}{2}}-(z^{(N)})^{\frac{q}{2}}\big). \tag{99}$$

We have also

$$u_a^{(j)}(r)=a+\int_0^r s^{1-j}\int_0^s\big(M|u_{a\,r}^{(j)}|^q-e^{u_a^{(j)}}\big)t^{j-1}\,\mathrm{d}t\,\mathrm{d}s, \tag{100}$$

hence

$$u_a^{(j)}(r)=-\frac{re^a}{j}\big(1+o(1)\big)\quad\text{as } r\to 0.$$

By integration we obtain in particular $u_a^{(N)}(r)>u_a^{(1)}(r)$ near $r=0$, and

$$z^{(1)}(u)-z^{(N)}(u)=\frac{e^{2a}(N^2-1)}{N^2}\big(1+o(1)\big)\quad\text{as } u\to a.$$

Let $\widehat{a}$ be the infimum of the $u\in(b,a)$ such that $z^{(1)}(u)-z^{(N)}(u)>0$. If $\widehat{a}>b$, then $z^{(1)}(\widehat{a})-z^{(N)}(\widehat{a})=0$ and $\frac{\mathrm{d}(z^{(1)}-z^{(N)})}{\mathrm{d}r}(\widehat{a})\geq 0$, which contradicts (99), hence $z^{(1)}(u)>z^{(N)}(u)$ on (b,a). By assumption $u^{(N)}(\rho_{a,b}^N)=b$. This implies that $(u_a^{(N)}-u_a^{(1)})_r>0$ on the interval $\big(0,\min\{\rho_{a,b}^{(N)},\rho_{a,b}^{(1)}\}\big)$. By integration $u_a^{(N)}>u_a^{(1)}$ on this interval, which implies $\rho_{a,b}^{(N)}>\rho_{a,b}^{(1)}$ and $u_a^{(N)}>u_a^{(1)}$ on $(0,\rho_{a,b}^{(1)}]$, which is the claim.

Step 2: Convergence of the regular solutions. Because the upper estimates of Theorem 1(1) and Theorem 3 hold independently of n, one can extract a subsequence $\{n_j\}$ such that $\{u_{n_j}^N\}$ converges in the $C^1_{\mathrm{loc}}\big((0,\rho_{*,b}^{(1)})\big)$-topology to a function $U_b^{*(N)}>b$ which is a solution of (17) on $(0,\rho_{*,b}^{(1)})$, is larger than the 1-dimensional solution $U_b^{*(1)}>b$. Thus it is singular and by Theorem 18 it satisfies

$$\lim_{r\to 0}r^q e^{U_b^{*(N)}(r)}=Mq^q\qquad\text{and}\qquad \lim_{r\to 0}rU_{b\,r}^{*(N)}(r)=-q.$$

This solution can be extended as a global solution on $(0,\infty)$ by Proposition 6. This ends the proof. □

Remark 25. The uniqueness of the singular solution of eikonal type when $N\geq 2$ is a challenging question. The remarkable fact is that all the solutions of this type have the same asymptotic expansion up to any order. This result is proved in the appendix.

Remark 26. Step 1 of our proof is an adaptation to equation (17) of a method introduced by Voirol [25, Proposition 1.7] dealing with the Chipot–Weissler equation

$$-u_{rr}-\frac{N-1}{r}u_r+|u_r|^q-u^p=0. \tag{101}$$

5.3. *Existence of singular solutions of Hamilton–Jacobi type*

In this section we prove the existence of singular solutions with the same behaviour as the one of the solutions of the Hamilton–Jacobi equation.

Theorem 27. *Let $N=1$ and $q>2$. Then for any $u_0\in\mathbb{R}$ there exists at least one solution u of* (17) *in* $(0,\infty)$ *satisfying*

$$u(r)=u_0-\frac{q-1}{q-2}\left(\frac{1}{M(q-1)}\right)^{\frac{1}{q-1}}r^{\frac{q-2}{q-1}}\big(1+o(1)\big)\quad when\ r\to 0. \tag{102}$$

Furthermore u is decreasing on $(0,\infty)$.

Proof. We still use system (96). For any $w\in\mathbb{R}$, we denote by (w,c_w^*) the intersection of $\mathscr{T}_*$ with the axis $u=w$. For $k_0>c_w^*$ the trajectory $\mathscr{T}_{k_0,w}$ going through (w,k_0) is singular and differs from $\mathscr{T}_*$. By Theorem 18 $\mathscr{T}_{k_0,w}$ has a vertical asymptote $u=u_0$ for some $u_0>w$. Because two trajectories cannot intersect, the mapping $k_0\mapsto f(k_0)=u_0$ is decreasing from (c_w^*,∞) to (w,∞). It is certainly continuous because of the non-intersection condition and onto from (c_w^*,∞) to (w,∞). This implies the claim for any $u_0>w$ and for any $w\in\mathbb{R}$. □

Theorem 28. *Let $N\geq 2$ and $q>2$. Then for any $u_0\in\mathbb{R}$ there exists at least one solution u of* (17) *in* $(0,\infty)$ *satisfying*

$$u(r)=u_0+\frac{q-1}{q-2}\left(\frac{N(q-1)-N}{M(q-1)}\right)^{\frac{1}{q-1}}r^{\frac{q-2}{q-1}}\big(1+o(1)\big)\quad when\ r\to 0. \tag{103}$$

The function u is increasing near 0.

Proof. We use the system (29) with $Z=-\frac{re^u}{u_r}$, $V=r|u_r|^{q-1}$ and $\Phi=-ru_r$. The solution u we look for satisfies $\lim_{r\to 0}u(r)=u_0$ and $\lim_{r\to 0}r^{\frac{1}{q-1}}u_r(r)=k$ for some real number k. Since we search increasing solutions, $k\geq 0$, Φ and Z are non-positive and the system becomes

$$\begin{cases} Z_t=Z(N-\Phi-MV-Z)\\ V_t=V\big(N-(N-1)q+(q-1)(Z+MV)\big)\\ \Phi_t=\Phi\big(2-N+Z+MV\big).\end{cases} \tag{104}$$

Step 1: The linearised problem. In this system the relation $V=e^{(2-q)t}|\Phi|^{q-1}$ holds and thus

$$Z=e^{(2-q)t}\frac{X}{\Phi},\quad X=\frac{|\Phi|^{q-1}\Phi}{V}\quad\text{and}\quad e^{qt}X=ZV^{\frac{2}{q-2}}|\Phi|^{-\frac{q}{q-2}}.$$

The system (104) admits $P_0=(0,V_0,0)$ for stationary point with $V_0=\frac{(N-1)q-N}{M(q-1)}$. Setting $V=V_0+\overline{V}$ the linearised system is

$$\begin{cases} Z_t=\frac{q}{q-1}Z\\ \overline{V}_t=(q-1)V_0(M\overline{v}+Z)\\ \Phi_t=\frac{q-2}{q-1}\Phi.\end{cases} \tag{105}$$

The eigenvalues of the linearised system $(Z,\overline{V},\Phi)$ at $(0,0,0)$ are $\left(\frac{q}{q-1},(N-1)q-N,\frac{q-2}{q-1}\right)$. They are all positive. Hence there exists a neighbourhood $\mathscr{V}$ of P_0 such that all trajectories with a point in $\mathscr{V}$ converge to P_0 when $t\to-\infty$.

Step 2: End of the proof. We recall that u is a solution of (17) if and only if (X,Φ) defined by (19) satisfies the system (21) but this system is not equivalent to (29). Let $\sigma>0$ such that the ball $B_\sigma(P_0)$ is in the attractive basin of P_0 when $t\to-\infty$. If $P^*:=(Z^*,V^*,\Phi^*)\in B_\sigma(P_0)$ with $Z^*,\Phi^*<0$ we denote by $\mathcal{T}_{P^*}$ the backward trajectory of P^* and we set

$$H(t)=\frac{V^{\frac{1}{q-1}}}{\Phi}e^{\frac{q-2}{q-1}t}.$$

Then

$$\begin{aligned}(q-1)\frac{H_t}{H}&=q-2+\frac{V_t}{V}-(q-1)\frac{\Phi_t}{\Phi}\\&=q-2+N-q(N-1)+q-1)(MV+Z)-(q-1)(2-N+MV+Z)\\&=q-2+N-(N-1)q+(q-1)(N-2)\\&=0.\end{aligned}$$

Hence the function H is constant. This implies that $V=be^{(2-q)t}|\Phi|^{q-1}$ for some $b>0$. It is easy to check by computation that $X=\frac{Z|\Phi|^{q-1}\Phi}{V}=b^{-1}e^{(q-2)t}Z\Phi$ satisfies the following system,

$$\begin{cases}X_t=(q-\Phi)X\\ \Phi_t=-(N-2)\Phi+be^{(2-q)t}\big(X-M|\Phi|^q\big).\end{cases}$$

We define $a=\frac{1}{2-q}\ln b$, $\tau=t+a$, $X^{(a)}(\tau)=X(\tau-a)$ and $\Phi^{(a)}(\tau)=\Phi(\tau-a)$. Then $\Phi^{(a)}(\tau)\sim -V_0^{\frac{1}{q-1}}e^{\frac{q-2}{q-1}t}$ as $\tau\to-\infty$ and $(X^{(a)},\Phi^{(a)})$ satisfies the system

$$\begin{cases}X^{(a)}_t=(q-\Phi^{(a)})X^{(a)}\\ \Phi^{(a)}_t=-(N-2)\Phi^{(a)}+e^{(2-q)\tau}\big(X^{(a)}-M|\Phi^{(a)}|^q\big).\end{cases}$$

By Lemma 8, equivalently the function $\rho\mapsto u^{(a)}(\rho)=\ln(\rho^{-q}X^{(a)})(\ln\rho)$ satisfies (17), with $\rho u^{(a)}_\rho=-\Phi^{(a)}(\tau)$ and $\rho(u^{(a)}_\rho)^{q-1}=e^{(2-q)\tau}(-\Phi^{(a)})^{q-1}$, thus $\lim_{\rho\to0}\rho\big(u^{(a)}_\rho(\rho)\big)^{q-1}=V_0$. Since $q>2$ the function $u^{(a)}_\rho$ is integrable near 0 and there exists some u_0 such that $\lim_{\rho\to0}u^{(a)}(\rho)=u_0$. This implies that

$$u(r)-u_0=\frac{q-1}{q-2}V_0^{\frac{1}{q-1}}r^{\frac{q-2}{q-1}}\big(1+o(1)\big)\quad\text{as } r\to0.$$

Since $D\mathcal{G}(P_0)$ is invertible the flow of $\mathcal{G}$ is conjugate to the one of $D\mathcal{G}(P_0)$ in a neighbourhood of P_0. hence there exists a C^2 diffeomorphism Θ from $B_\sigma(P_0)$ (up to reducing σ in order $B_\sigma(P_0)$ is a subset of the basin of attraction of P_0) to a neighbourhood $\mathcal{V}$ of 0 such that $\mathcal{G}-\mathcal{G}(P_0)=\mathcal{V}\circ D\mathcal{F}(P_0)$, and the trajectories of (104) in $B_a(P_0)$ (for some $a>0$) are in one to one correspondence via $\mathcal{V}$ with the trajectories of

$$(Z_t,V_t,\Phi_t)=D\mathcal{F}(P_0)\big((Z,V,\Phi)\big).\tag{106}$$

Since all the trajectories of (106) converge to 0 when $t\to-\infty$, all the trajectories of (104) issued from $B_\sigma(P_0)$ converge to P_0 when $t\to-\infty$. This ends the proof. Given arbitrary coefficients C_j ($j=1,2,3$) with $C_3>0$, the solutions of the linearized system (106) are expressed by

$$\begin{cases}Z(t)=C_1e^{\frac{qt}{q-1}}\\ \overline{V}(t)=C_2e^{(N(q-1)-N)t}+\frac{(q-2)C_1}{M(q-1)}e^{\frac{qt}{q-1}}\\ \Phi(t)=C_3e^{\frac{(q-2)t}{q-1}},\end{cases}$$

in general, with a standard modification if $N=3$ and $q=\frac{3+\sqrt3}{2}$ or $N=2$ and $q=2+\sqrt2$ since in these two cases $\frac{q}{q-1}=(N-1)q-N$. If (V,Φ) satisfies (29), then the equivalence of trajectories yields

$$V(t)=be^{(2-q)t}\Phi^{q-1}(t)\sim be^{(2-q)t}C_3^{q-1}e^{(q-2)t}=bC_3^{q-1},$$

and clearly $b = V_0 C_3^{1-q} = e^{(2-q)a}$ with the previous notations. Hence $e^{qa} = (V_0 C_3^{1-q})^{\frac{q}{2-q}}$, from which equality it follows that

$$u_0 = (V_0 C_3^{1-q})^{-\frac{q}{q-2}} V_0^{\frac{2}{q-2}} C_1 C_3^{\frac{2}{q-2}} = V_0^{-1} C_1 C_3^{-\frac{q(q-1)+2}{q-2}}. \tag{107}$$

This implies that u_0 can take any value. □

6. Behaviour of solutions in an exterior domain

In this section we consider radial solutions of (1) defined in $B^c_{r_0}$. If u is such a solution, it satisfies (11) that we recall

$$e^{u(r)} \le C r^{-\min\{2,q\}} \quad \text{for all } r \ge 2r_0. \tag{108}$$

Equivalently

$$e^{u(r)} \le C \begin{cases} r^{-2} & \text{if } q > 2, \\ r^{-q} & \text{if } 1 < q < 2. \end{cases} \tag{109}$$

Applying Theorem 3 in $B_{\frac{|x|}{2}}(x)$ for $x \in B^c_{r_0}$ we obtain the following result.

Theorem 29. *Let $N \ge 1$ and $q > 1$. If $u \in C(B^c_{r_0})$ is a radial solution of* (1) *in $B^c_{r_0}$, there holds in $B^c_{2r_0}$, for some positive constant $\widetilde{C} = \widetilde{C}(N,q,M)$,*

$$\left|u_r(r)\right| \le \widetilde{C} \begin{cases} r^{-\frac{1}{q-1}} & \text{if } q > 2, \\ r^{-1} & \text{if } 1 < q < 2. \end{cases} \tag{110}$$

We will see later on that in both cases the estimate is of the form

$$\left|u_r(r)\right| \le \widetilde{C} r^{-1}. \tag{111}$$

Theorem 30. *Let $N \ge 3$ and $q > 2$. If $u \in C(B^c_{r_0})$ is a radial solution of* (1) *in $B^c_{r_0}$, it satisfies*

$$\lim_{r\to\infty} r^2 e^{u(r)} = 2(N-2) \qquad \text{and} \qquad \lim_{r\to\infty} r u_r(r) = -2. \tag{112}$$

Proof. As in Theorem 15 we use the systems (20) and (27) with variables $x(t) = r^2 e^{u(r)}$, $\Phi = -r u_r(r)$ and $\Theta(t) = e^{(2-q)t}$. Here, since $q > 2$ and $t = \ln r \to \infty$, we have that $\Theta(t) \to 0$ when $t \to \infty$. The system in the variable t admits $O = (0,0,0)$ and $P_0 = \big(2(N-2),2,0\big)$ for equilibria. The eigenvalues of the linearised operator at P_0 given by (44) are $\lambda_1 = 2-q < 0$ and λ_2, λ_3 are negative too if $N \ge 11$, double with value $2-N$ if $N = 10$ or non-real with negative real part $\mathscr{R}e(\lambda_j) = 2-N$ if $3 \le N \le 9$. Therefore P_0 is a sink with a domain of attraction which contains some ball $B_a(P_0)$. The eigenvalues of the linearised operator at O

$$\begin{cases} x_t = 2x \\ \Phi_t = x - (N-2)\Phi \\ \Theta_t = (2-q)\Theta, \end{cases}$$

are 2, $2-q$ and $2-N$, hence O is a saddle point with a 2-dimensional stable manifold $\mathscr{M}$ and an unstable trajectory $\mathscr{T}_0$.

We claim that all the solutions in $B^c_{r_0}$ belong to the domain of attraction of P_0 and behave as in (112). Indeed (x,Φ) satisfies (20) that is

$$\begin{cases} x_t = (2-\Phi)x \\ \Phi_t = x + (2-N)\Phi - Me^{(2-q)t}|\Phi|^q. \end{cases}$$

Since $e^{u(r)} \to 0$ from (109), u is decreasing, thus $u_r(r) < 0$ (the inequality is strict from the equation), then $\Phi(t) > 0$ and $x(t)$ is bounded from (109). Suppose now that Φ is unbounded.

- Either Φ is monotone, thus $\Phi(t) \to \infty$. From (20) we have $\Phi_t < \frac{N-2}{2}\Phi$ for t large enough, which implies that $t \mapsto e^{\frac{N-2}{2}t}\Phi(t)$ is decreasing. Therefore it is bounded which is contradictory.
- Or there exists a sequence $\{t_n\}$ tending to infinity such that $\Phi(t_n)$ is a local maximum of Φ and $\Phi(t_n) \to \infty$. Since $\Phi_t(t_n) = 0$ we obtain from (20) that $x(t_n) = (N-2)\Phi(t_n) + Me^{(2-q)t_n}\Phi(t_n)^q \to \infty$ which contradicts the boundedness of $x(t)$.

Therefore $\Phi(t)$ remains bounded. This implies that the system (20) is an exponentially small perturbation of the system associated to Emden–Chandrasekhar equation (4)

$$\begin{cases} x_t = (2-\Phi)x \\ \Phi_t = x + (2-N)\Phi. \end{cases} \tag{113}$$

By Proposition 13 the omega-limit set of any trajectory of (27) is a compact connected subset invariant for (113). This system is well studied (e.g. [5,12]). It is known that the solutions of (113) converge to an equilibrium when $t \to \infty$ and this equilibrium is either $\big(2(N-2), 2\big)$ or $(0,0)$. In the first case we obtain (112). In the second case we use system (27) and we analyse the eventual convergence of a solution (x, ϕ, Θ) to $O = (0,0,0)$. This would imply that this trajectory belongs to the stable manifold $\mathcal{M}$, therefore

$$\big|\Phi(t)\big| = O(e^{(2-N)t}). \tag{114}$$

If we plug this estimate into (20), we obtain that

$$x(t) = x(t_0)e^{\int_{t_0}^t (2-\Phi(s))\,ds}.$$

Set $A(t_0, t) = e^{-\int_{t_0}^t (\Phi(s))\,ds}$, then there exists $\theta_1 > \theta_2 > 0$ such that $\theta_2 \le A(t_0, t) \le \theta_1$, for all $t > t_0 > 0$. This implies

$$\big|x(t)\big| \ge \big|x(t_0)\big| e^{2(t-t_0)}\theta_2 \quad \text{for all } t \ge t_0.$$

Consequently, if the omega limit set of the trajectory $\big(x(t), \Phi(t)\big)$ of (20) is $(0,0)$ there must hold $x(t_0) = 0$. As a consequence this trajectory must be contained in the plane $x = 0$. In such a case Φ satisfies the equation

$$\Phi_t = (2-N)\Phi - Me^{(2-q)t}|\Phi|^q. \tag{115}$$

This equation is explicitly integrable and we obtain

$$\big|\Phi(t)\big|^{-q}\Phi(t) = t^{(q-1)(N-2)}\big|\Phi(1)\big|^{-q}\Phi(1) + \frac{M}{(q-1)(N-2)}(t^{(q-1)(N-2)} - t).$$

This relation is not compatible with (114). Hence the omega limit set of the trajectory is reduced to P_0, which ends the proof. □

In the case $1 < q < 2$ we show that all the solutions behave at infinity like the solution of the eikonal equation.

Theorem 31. *Let $N \ge 3$ and $1 < q < 2$. If $u \in C(B^c_{r_0})$ is a radial solution of* (1) *in $B^c_{r_0}$, it satisfies*

$$\lim_{r\to\infty} r^q e^{u(r)} = Mq^q \qquad \text{and} \qquad \lim_{r\to\infty} r u_r(r) = -q. \tag{116}$$

Note that this behaviour is satisfied both by any regular solution or by the singular solution we have constructed.

Proof. We use the system (21) with variables $X(t) = r^q e^{u(r)}$ and $\Phi(t) = -r u_r(r)$, always with $t = \ln r$, that is

$$\begin{cases} X_t = X(q-\Phi) \\ \Phi_t = (2-N)\Phi - e^{(2-q)t}\big(-M|\Phi|^q + X\big). \end{cases}$$

Note that $u(r)$ cannot have local minimum, and since it tends to $-\infty$ by (109) it is decreasing, thus $\Phi(t) \ge 0$. Here X is bounded by (109) and also Φ is bounded by (110).

We claim that X is bounded from below. We still use the function G defined at (80), that is

$$G(t)=Mq^q\frac{X^2}{2}-\frac{X^3}{3}+e^{(q-2)t}\Big((N-2)q\frac{X}{2}-\frac{X_t^2}{2}\Big).$$

Then from (81),

$$G_t(t)=MX^{1-q}(qX-Y)(q^qX^q-Y^q)+e^{(q-2)t}\Psi(t),$$

where

$$\Psi(t)=(q-2)\Big((N-2)q\frac{X}{2}-\frac{X_t^2}{2}\Big)+\frac{(N-2)q}{2}X_t-\frac{X_t^3}{X}+(N-2)(X_t-qX)X_t.$$

Since X and Φ are bounded the same holds with X_t from the equation (21), and because $\frac{X_t^3}{X}=X^2(q-\Phi)^3$, we obtain that Ψ is also bounded by some constant $C>0$. Because $(qX-Y)(q^qX^q-Y^q)\geq 0$ we obtain that $G_t\geq -Ce^{(q-2)t}$ from what it follows that the function $t\mapsto G(t)+\frac{C}{q-2}e^{(q-2)t}$ is increasing. Since it is bounded, it converges to some limit ℓ when $t\to\infty$. Hence the function $t\mapsto Mq^q\frac{X^2(t)}{2}-\frac{X^(t)}{3}$ tends to ℓ which implies that $X(t)$ admits a limit λ when $t\to\infty$, and λ satisfies $Mq^q\frac{\lambda^2}{2}-\frac{\lambda^3}{3}=\ell$. If $\Phi(t)$ is not monotone, then at each extremum t_n of $\Phi(t)$ we have

$$X(t_n)-M\Phi^q(t_n)=(N-2)e^{(q-2)t_n}\Phi(t_n)\quad\Longrightarrow\quad\lim_{n\to\infty}\Phi^q(t_n)=\frac{\lambda}{M}.$$

This implies that

$$\limsup_{t\to\infty}\Phi(t)=\Big(\frac{\lambda}{M}\Big)^{\frac{1}{q}}=\liminf_{t\to\infty}\Phi(t)=\lim_{t\to\infty}\Phi(t).$$

If Φ is monotone, then it admits also a limit. In any case we set $L=\lim_{t\to\infty}\Phi(t)$. From the equation (21) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(e^{(N-2)t}\Phi(t)\big)=e^{(N-q)t)}\big(X(t)-M\Phi^q(t)\big).$$

Hence

$$\Phi(t)=e^{(2-N)(t-t_0)}\Phi(t)+e^{(2-N)t}\int_{t_0}^t e^{(N-q)s}\big(X(s)-M\Phi^q(s)\big)\,\mathrm{d}s. \tag{117}$$

If $\lambda-ML^q\neq 0$ we obtain a contradiction, since the integration of (117) implies

$$\Phi(t)=\frac{e^{(2-q)t}}{N-q}(\lambda-ML^q)\big(1+o(1)\big).$$

Therefore

$$\lim_{t\to\infty}X(t)=M\Big(\lim_{t\to\infty}\Phi(t)\Big)^q.$$

Again from (21),

$$X(t)=X(t_0)e^{\int_{t_0}^t(q-\Phi(s)\,\mathrm{d}s}.$$

Then, if $L<q$, $e^{\int_{t_0}^t(q-\Phi(s)\,\mathrm{d}s}\to\infty$, contradiction. If $L>q$, $e^{\int_{t_0}^t(q-\Phi(s)\,\mathrm{d}s}\to 0$, then $X(t)\to 0$, then $\lambda=L=0$. We use again the first equation in (117) and we have $X_t=\tilde q(t)X(t)$ with $\tilde q=q-\Phi(t)\to q$ when $t\to\infty$. Then for any $t_0>0$ we have

$$X(t)=X(t_0)e^{\int_{t_0}^t\tilde q(s)\,\mathrm{d}s}.$$

Thus $X(t)\to\infty$, contradiction. Hence

$$\lim_{t\to\infty}\Phi(t)=q\qquad\text{and}\qquad\lim_{t\to\infty}X(t)=Mq^q. \tag{118}$$

This ends the proof. □

7. Globality of the solutions when $1<q<2$

Theorem 32. *Let $1<q<2$ and $N\geq 1$. Then the maximal interval of a solution u of (17) is $(0,\infty)$ if $N\geq 2$ and $(-\infty,\infty)$ if $N=1$. In that case u is symmetric with respect to some $a\in\mathbb{R}$.*

Proof. For $N\geq 1$ we consider any solution u defined on a maximal open interval $I_u=(\rho,\eta)$ and let $r_0\in I_u$. From Remark 7 if $u_r(r_0)\leq 0$, then $\eta=\infty$. If $u_r(r_0)>0$, either u has a unique maximum at some point $r_1\in(r_0,\eta)$, or $u_r(r_0)>0$.

(i). We first prove that $\eta=\infty$. From Remark 7, if $u_r(r_0)\leq 0$, then $\eta=\infty$. If $u_r(r_0)>0$ has a maximum at r_1, then u is decreasing on $(r_1\eta)$ and thus $\eta=\infty$ by the same remark.

Hence $u_r>0$ on (r_1,η) and we can encounter two possibilities: either $\eta=\infty$ and $u(r)\to L\in(0,\infty)$ when $r\to\infty$, which contradicts the upper estimate (11), or $\eta<\infty$, and we are left with this case, with again two possibilities.

If $\lim_{r\to\eta}u(r)=L<\infty$, then

$$(-r^{N-1}u_r)_r=r^{N-1}(e^u-u_r^q)\leq k=\eta^{N-1}e^L.$$

Therefore the function $r\mapsto r^{N-1}u_r+kr$ is increasing and it tends to ∞ since the solution is the maximal one and u remains bounded. This implies that $u_r(r)\to\infty$ and $e^{u(r)}=o(u_r^q)$ when $r\to\eta$. Hence the equation becomes $-u_{rr}+Mu_r^q\big(1+o(1)\big)$. Setting $v=u_r$ we obtain by integration that

$$v(r)=u_r(r)=\big(M(q-1)(\eta-r)\big)^{-\frac{1}{q-1}}\big(1+o(1)\big)\quad\text{as } r\to\eta,$$

but this is not possible since $1<q<2$ and thus u_r is not integrable near η, which contradicts $L<\infty$. Consequently there must hold $L=\infty$. Again u_r is monotone near η since at any extremal point r where $u_{rr}(r)=0$, we have that $u_{rrr}(r)=\big(\frac{N-1}{r^2}-e^u\big)u_r$. Since $\frac{N-1}{r^2}-e^u\to-\infty$ when $r\to\eta$, then u_{rrr} keeps a constant sign. Therefore u_r is non-decreasing and clearly

$$u_{rr}=Mu_r^q-\frac{N-1}{r}u_r-e^u\geq 0.$$

Then $e^u\leq Mu_r^q\big(1+o(1)\big)$, and by integration we deduce that $e^u\leq C(\eta-r)^{-q}$. Now we set $u(r)=v\big(h(r)\big)$ where $h(r)=\eta-r$ and consider the system in the variables

$$X=h^qe^v,\quad \Phi=-hv_r(h),\quad h=e^\tau, \tag{119}$$

which is

$$\begin{cases} X_\tau=X(q-\Phi)\\ \Phi_\tau=\Phi+e^{(2-q)\tau}(X-M\Phi^q)-\frac{N-1}{\eta-e^\tau}\Phi,\end{cases} \tag{120}$$

and $\tau\to-\infty$ when $r\to\eta$. We claim now that Φ is bounded. If it is not the case and $\Phi(\tau)$ is not monotone, at any $\tilde\tau$ where $\Phi_\tau(\tilde\tau)=0$ we obtain

$$\Phi(\tilde\tau)\big(1+o(1)\big)=e^{(2-q)\tilde\tau}\big(M\Phi^q(\tilde\tau)-X(\tilde\tau)\big)\leq e^{(2-q)\tilde\tau}M\Phi^q(\tilde\tau),$$

and thus $M\Phi^{q-1}(\tilde\tau)\geq e^{-(2-q)\tilde\tau}\big(1+o(1)\big)$. Since this is valid in particular at the local minima of Φ, we deduce that $\Phi(\tau)\to\infty$ when $\tau\to-\infty$, with the additional information that $\Phi_{\tau\tau}(\tilde\tau)\geq 0$. Since

$$\Phi_{\tau\tau}(\tilde\tau)=e^{(2-q)\tilde\tau}\big(2X(\tilde\tau)-(X\Phi)(\tilde\tau)-M(2-q)\Phi^q(\tilde\tau)\big)-\frac{N-1}{\eta-e^{\tilde\tau}}e^{\tilde\tau}\Phi(\tilde\tau)\left(1+\frac{e^\tau}{\eta-e^{\tilde\tau}}\right)\geq 0,$$

which yields

$$2X(\tilde\tau)-(X\Phi)(\tilde\tau)-M(2-q)\Phi^q(\tilde\tau)\geq e^{(q-1)\tilde\tau}\Phi(\tilde\tau)\left(1+\frac{e^\tau}{\eta-e^{\tilde\tau}}\right),$$

which in turn implies that $\Phi(\tilde\tau)$ is bounded, contradiction. As a consequence Φ is monotone and tends to ∞ at $-\infty$. From the first equation in (120), $X=-X\Phi\big(1+o(1)\big)$ when $\tau\to-\infty$, which is impossible if X remains bounded. As a consequence the function Φ is bounded and the system (120) is an exponential perturbation of the system

$$\begin{cases} X_\tau = X(q-\Phi) \\ \Phi_\tau = \Phi. \end{cases} \tag{121}$$

This implies that $\big(X(\tau),\Phi(\tau)\big)$ converges to the unique stationary point of (121) which is $(0,0)$.

Returning to the variable u, we get that $u_r = o\big((\eta-r)^{-1}\big)$ near $r=\eta$, then

$$u_{rr} = M|u_r|^q - \frac{N-1}{r}u_r - e^u = o\big((\eta-r)^{-q}\big),$$

which implies that $u_r(r) = o\big((\eta-r)^{1-q}\big)$. Thus u_r is integrable near η and $u(r)$ admits a finite limit when $r\to\eta$, which is a contradiction. As a consequence we deduce the following.

(ii). If $N=1$ then the solution is defined on whole $\mathbb{R}$, indeed $r\mapsto u(-r)$ is also a solution, thus $I_u = (\rho,\eta) = (-\infty,\infty)$.

(iii). If $N\geq 2$, we claim that $\rho = 0$. We proceed again by contradiction, assuming that $\rho>0$. Clearly u is monotone near ρ.

If u is increasing near ρ it has a finite limit L at ρ because

$$-(r^{N-1}u_r)_r = r^{N-1}(e^u - Mu_r^q) \leq C \quad \text{as } r\to\rho, \tag{122}$$

which implies that $r^{N-1}u_r + Cr$ is increasing. Then u_r admits a finite limit at ρ. Combined with the fact that $u(r)\to L$, we see that ρ cannot be the infimum of I_u. Consequently $u(r)$ decreases to $-\infty$ when $r\to\rho$. In such a case $e^{u(r)}\to 0$. Moreover u_r is also monotone near ρ: indeed at each local extremum $\tilde{r}$ of u_r we have $u_{rr}(\tilde{r}) = 0$, we have

$$u_{rrr}(\tilde{r}) = \left(\frac{N-1}{\tilde{r}^2}\right)u_r(\tilde{r}) \longrightarrow \frac{N-1}{\rho^2} > 0 \quad \text{as } \tilde{r}\to\rho.$$

As a consequence these local extrema are local minima of u_r; necessarily $u_r(r)$ cannot oscillate and it tends to ∞ when $r\to\rho$. Again this fact implies that $u_{rr} = Mu_r^q\big(1+o(1)\big)$ and by integration we encounter a contradiction since $1<q<2$.

If u is decreasing near ρ, then $u(r)\to L<\infty$ when $r\to\rho$. By (122), $-(r^{N-1}u_r)_r\leq C$, hence $r\mapsto -r^{N-1}u_r + Cr$ is increasing and thus $\lim_{r\to\rho} u_r(r) = -\infty$ since ρ is an endpoint I_u. We have again that u_r is monotone and u is convex. Therefore $0\leq u_{rr} = M|u_r|^q - \frac{N-1}{r}u_r - e^u$. Hence $e^u \leq M|u_r|^q\big(1+o(1)\big)$ and by integration we deduce that $e^{u(r)}\leq C(r-\rho)^{-q}$. As in (i) we set $h(r) = r-\rho\ e^\tau = r-\rho$ and define $X(\tau)$ and $\Phi(\tau)$ as in (119). Since (X,Φ) satisfies

$$\begin{cases} X_\tau = X(q-\Phi) \\ \Phi_\tau = \Phi + e^{(2-q)\tau}(X - M\Phi^q) - \frac{N-1}{\rho+e^\tau}\Phi, \end{cases}$$

we obtain a contradiction as in this case. This ends the proof. □

Appendix

In Lemma 21, formula (83) gives an expansion of u near zero at the order 1. We show below that $u(r)$ satisfies a *unique* expansion of order n of the form

$$u(r) = \ln\frac{Mq^q}{r^q} + a_1 r^{q-2} + a_2 r^{2(q-2)} + \cdots + r^{n(q-2)}\big(a_n + o(1)\big) \quad \text{as } r\to 0, \tag{123}$$

where the a_n can be computed by induction.

Theorem 33. *Let $N \geq 2$ and $q > 2$. There exists a unique sequence of real numbers $\{a_n\}_{n\geq 1}$ such that any radial function u satisfying* (125) *in $B_{r_0} \setminus \{0\}$ and* (70) *there holds for any $n \in \mathbb{N}^*$,*

$$u(r) = \ln \frac{q^q}{r^q} + a_1 r^{q-2} + a_2 r^{2(q-2)} + \cdots + a_n r^{n(q-2)}\big(1 + d_n(r)\big), \tag{124}$$

where $d_n(r) \to 0$ when $r \to 0$.

Proof. For the sake of simplicity we consider equation (17) with $M = 1$. Actually, this is not a restriction since if u^* satisfies (1), the function $x \mapsto u(x) = u^*(M^{\frac{1}{q-2}} x) + M^{\frac{2}{q-2}}$ satisfies

$$-\Delta u + |\nabla u|^q - e^u = 0. \tag{125}$$

Since any solution is decreasing, we consider u_r as a function of u and set

$$u_r = -f(u).$$

Then $u_{rr} = -f(u)\frac{\mathrm{d}f}{\mathrm{d}u} = f^q(u) - e^u + \frac{N-1}{r} f(u)$. Thus we obtain the system

$$\begin{cases} \dfrac{\mathrm{d}f}{\mathrm{d}u} = \dfrac{N-1}{r} + \dfrac{f^q(u) - e^u}{f(u)} \\[2mm] \dfrac{\mathrm{d}r}{\mathrm{d}u} = -\dfrac{1}{f(u)}. \end{cases} \tag{126}$$

Using (70) we have

$$f(u) = e^{\frac{u}{q}}\big(1 + o(1)\big) \quad \text{and} \quad r(u) = q e^{-\frac{u}{q}}\big(1 + o(1)\big) \quad \text{as } u \to \infty.$$

We set

$$\varpi = e^{-\frac{u}{q}} f(u), \quad V = \varpi - 1 \quad \text{and} \quad \theta = \frac{q-2}{q}. \tag{127}$$

Notice that $e^{\theta u} \to \infty$ since $q > 2$. The new system in (ϖ, r) is

$$\begin{cases} \dfrac{\mathrm{d}\varpi}{\mathrm{d}u} = -\dfrac{\varpi}{q} + \dfrac{N-1}{r e^{\frac{u}{q}}} + \dfrac{\varpi^q - 1}{\varpi} e^{\theta u} \\[2mm] \dfrac{\mathrm{d}r}{\mathrm{d}u} = -\dfrac{e^{-\frac{u}{q}}}{\varpi}. \end{cases} \tag{128}$$

Step 1: Development of order 1. We look for A_1 such that

$$\varpi(u) = 1 + A_1 e^{-\theta u}\big(1 + o(1)\big) \quad \text{as } u \to \infty.$$

Equivalently $V(u) = A_1 e^{-\theta u}\big(1 + o(1)\big)$. The system in V and r is

$$\begin{cases} \dfrac{\mathrm{d}V}{\mathrm{d}u} = -\dfrac{1+V}{q} + \dfrac{N-1}{r e^{\frac{u}{q}}} + \dfrac{(1+V)^q - 1}{1+V} e^{\theta u} \\[2mm] \dfrac{\mathrm{d}r}{\mathrm{d}u} = -\dfrac{e^{-\frac{u}{q}}}{1+V}. \end{cases} \tag{129}$$

We set $\psi(u) = e^{\theta u} V(u)$, then $\psi(u) = o(e^{\theta u})$ when $u \to \infty$. Hence

$$e^{-\theta u}\left(\frac{\mathrm{d}\psi}{\mathrm{d}u} - \frac{q-2}{q}\psi\right) = -\frac{1 + e^{-\theta u}\psi}{q} + \frac{N-1}{r e^{\frac{u}{q}}} + e^{\theta u} \frac{(1 + e^{-\theta u})^q \psi - 1}{1 + e^{-\theta u}\psi},$$

$$\frac{\mathrm{d}r}{\mathrm{d}u} = \frac{e^{-\frac{u}{q}}}{1 + e^{-\theta u}\psi}.$$

As a consequence

$$\begin{aligned} \frac{\mathrm{d}\psi}{\mathrm{d}u} &= \frac{q-2}{q}\psi - \frac{e^{\theta u}}{q} - \frac{\psi}{q} + \frac{N-1}{r e^{\frac{u}{q}}} e^{\theta u} + q e^{\theta u}\psi\big(1 + o(1)\big) \\ &= q e^{\theta u}\psi\big(1 + o(1)\big) + e^{\theta u}\left(-\frac{1}{q} + \frac{N-1}{q}\big(1 + o(1)\big)\right), \end{aligned}$$

that we can write under the following form

$$\frac{d\psi}{du} - qe^{\theta u}\psi\big(1+o(1)\big) = \frac{N-2}{q}e^{\theta u}\big(1+o(1)\big). \tag{130}$$

- Either ψ is not monotone for large u. At each u where $\frac{d\psi}{du} = 0$, we get that $\psi(u) = \frac{2-N}{q^2}\big(1+o(1)\big)$. These implies $\lim_{u\to\infty}\psi(u) = \frac{2-N}{q^2}$.
- Or ψ is monotone and there holds $\lim_{u\to\infty}\psi(u) = \pm\infty$. Therefore $\frac{d\psi}{du} = qe^{\theta u}\psi\big(1+o(1)\big)$, which implies $\frac{d}{du}\big(\ln|\psi| - e^{(q-2)u}\big) > 0$ and thus $\big|\psi(u)\big| \geq Ke^{e^{(q-2)u}}$. Since $\psi(u) = o(e^{\theta u})$ we obtain a contradiction.
- Or ψ is monotone and $\psi(u) \to \ell$ when $u \to \infty$ for some ℓ. If $\ell \neq \frac{2-N}{q^2}$, we obtain the following relation $\frac{d\psi}{du} = \big(q\ell - \frac{N-2}{q}\big)\big(1+o(1)\big)$ which is not compatible with $\psi(u) \to \ell$.

In all the cases we obtain

$$\lim_{u\to\infty}\psi(u) = \frac{2-N}{q^2}. \tag{131}$$

This yields the first term of our expansion

$$\varpi = 1 + V = 1 + A_1 e^{-\theta u}\big(1+o(1)\big) \quad \text{with } A_1 = \frac{2-N}{q^2}. \tag{132}$$

as a consequence

$$\frac{dr}{du} = -\frac{e^{-\frac{u}{q}}}{1 + A_1 e^{-\theta u}\big(1+o(1)\big)} = -e^{-\frac{u}{q}}\Big(1 - A_1 e^{-\theta u}\big(1+o(1)\big)\Big) = -e^{-\frac{u}{q}} + A_1 e^{\frac{1-q}{q}u}\big(1+o(1)\big). \tag{133}$$

Integrating this relation and using that $r(u) \to 0$ when $u \to \infty$, we obtain

$$\begin{aligned} r(u) &= -\int_u^\infty \frac{dr}{ds}\,ds \\ &= \int_u^\infty e^{-\frac{s}{q}}\,ds - A_1\int_u^\infty e^{\frac{1-q}{q}s}\big(1+\epsilon(s)\big)\,ds \\ &= qe^{-\frac{u}{q}} + \frac{qA_1}{1-q}e^{\frac{1-q}{q}u}\big(1+o(1)\big) \\ &= qe^{-\frac{u}{q}} + \frac{N-2}{q(q-1)}e^{\frac{1-q}{q}u}\big(1+o(1)\big), \end{aligned} \tag{134}$$

since for any $\epsilon_0 > 0$and $\big|\epsilon(s)\big| \leq \epsilon_0$ there exists u_{ϵ_0} such that for any $u \geq u_{\epsilon_0}$ there holds

$$\left|\int_u^\infty e^{\frac{1-q}{q}s}\big(1+\epsilon(s)\big)\,ds - \int_u^\infty e^{\frac{1-q}{q}s}\,ds\right| \leq \frac{q\epsilon_0}{q-1}.$$

Step 2: Development of order n. We proceed by induction assuming that we have already obtained the development of ϖ at the order $n-1$

$$\varpi(u) = 1 + V = 1 + A_1 e^{-\theta u} + A_2 e^{-2\theta u} + \cdots + e^{-(n-1)\theta u}\big(A_{n-1} + \epsilon_{n-1}(u)\big), \tag{135}$$

where the A_j depend on N and q and $\epsilon_{n-1}(u) \to 0$ when $u \to \infty$. Consequently

$$\begin{aligned} \frac{dr}{du} &= -\frac{e^{-\frac{u}{q}}}{1+V} \\ &= -\frac{e^{-\frac{u}{q}}}{1 + A_1 e^{-\theta u} + A_2 e^{-2\theta u} + \cdots + e^{-(n-1)\theta u}\big(A_{n-1} + \epsilon_{n-1}(u)\big)} \\ &= e^{-\frac{u}{q}}\Big(1 + B_1 e^{-\theta u} + B_2 e^{-2\theta u} + \cdots + e^{-(n-1)\theta u}\big(B_{n-1} + \widetilde{\epsilon}_{n-1}(u)\big)\Big), \end{aligned} \tag{136}$$

where $\widetilde{\epsilon}_{n-1}(u) \to 0$ as $u \to \infty$, with $B_1 = -A_1$ and the other coefficients can be made explicit through a lengthy but explicit computation. Set

$$\varpi = 1 + A_1 e^{-\theta u} + A_2 e^{-2\theta u} + \cdots + A_{n-1}e^{-(n-1)\theta u} + e^{-n\theta u}\Phi. \tag{137}$$

We do not know if Φ is bounded but only that $\Phi = \widetilde{\epsilon}_{n-1} e^{\theta u}$, thus $e^{-\theta u}\Phi(u) \to 0$ when $u \to \infty$. Thus

$$\frac{\mathrm{d}V}{\mathrm{d}u} = -\theta A_1 e^{-\theta u} - 2\theta A_2 e^{-2\theta u} - \cdots - (n-1)\theta A_{n-1} e^{-(n-1)\theta u} - n\theta\Phi e^{-n\theta u} + \frac{\mathrm{d}\Phi}{\mathrm{d}u} e^{-n\theta u}. \tag{138}$$

We recall the expression

$$\frac{\mathrm{d}V}{\mathrm{d}u} = -\frac{1+V}{q} + \frac{N-1}{re^{\frac{u}{q}}} + e^{\theta u}\frac{(1+V)^q - 1}{1+V}, \tag{139}$$

obtained by using the following expansion at the order $n-1$ (actually, valid at any order)

$$\frac{(1+V)^q - 1}{1+V} = qV\Big(1 + g_1 V + \cdots + g_k V^k + \cdots + g_{n-1} V^{n-1}\big(1 + \eta_{n-1}(V)\big)\Big), \tag{140}$$

where $g_1 = \frac{q-3}{2}$ and the g_k are polynomials in k and q and also $\eta_{n-1}(V) \to 0$ when $V \to 0$.

Moreover, by integration of (136), we obtain

$$\begin{aligned} r(u) &= -\int_u^\infty \frac{\mathrm{d}r}{\mathrm{d}s}\,\mathrm{d}s \\ &= -\int_u^\infty e^{-\frac{s}{q}}\,\mathrm{d}s + B_1\int_u^\infty e^{\frac{1-q}{q}s} + \cdots + \int_u^\infty e^{-\frac{1+(n-1)(q-2)}{q}s}(B_{n-1} + \widehat{\epsilon}_{n-1}s)\,\mathrm{d}s \\ &= qe^{-\frac{u}{q}}\left(1 + \frac{B_1}{q-1}e^{-\theta u} + \cdots + \frac{B_{n-1}}{1+(n-1)(q-2)}e^{-(n-1)\theta u} + \widehat{\epsilon}_{n-1}(u)e^{-(n-1)\theta u}\right), \end{aligned}$$

therefore

$$\begin{aligned} \frac{N-1}{re^{\frac{u}{q}}} &= \frac{N-1}{q}\,\frac{1}{1 + \frac{B_1}{q-1}e^{-\theta u} + \cdots + \frac{B_{n-1}}{1+(n-1)(q-2)}e^{-(n-1)\theta u} + \widetilde{\epsilon}_{n-1}(u)e^{-(n-1)\theta u}} \\ &= \frac{N-1}{q}\big(1 + C_1 e^{-\theta u} + \cdots + C_{n-1}e^{-(n-1)\theta u} + \epsilon^*_{n-1}(u)e^{-(n-1)\theta u}\big), \end{aligned}$$

where $C_1 = -\frac{B}{q-1}$ and C_k is a polynomial in q and k. Then using (139) and (140),

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}u} &= -\theta A_1 e^{-\theta u} - 2\theta A_2 e^{-2\theta u} - \cdots - (n-1)\theta A_{n-1}e^{-(n-1)\theta u} - n\theta e^{-n\theta u}\Phi + e^{-n\theta}\frac{\mathrm{d}\Phi}{\mathrm{d}u} \\ &= \frac{N-1}{q}\big(1 + C_1 e^{-\theta u} + \cdots + C_{n-1}e^{-(n-1)\theta u} + \epsilon^*_{n-1}(u)e^{-(n-1)\theta u}\big) \\ &\quad - \frac{1+V}{q} + qVe^{\theta u}\Big(1 + g_1 V + \cdots + g_k V^k + \cdots + g_{n-1}V^{n-1}\big(1 + \eta_{n-1}(V)\big)\Big), \end{aligned} \tag{141}$$

and, by the definition of Φ, from (137) there holds

$$Ve^{\theta u} = A_1 + A_2 e^{-\theta u} + \cdots + A_{n-1}e^{-(n-2)\theta u} + e^{-(n-1)\theta u}\Phi, \tag{142}$$

and

$$V = A_1 e^{-\theta u} + A_2 e^{-2\theta u} + \cdots + A_{n-1}e^{-(n-1)\theta u} + e^{-(n-1)\theta u}\widetilde{\epsilon}_{n-1}(u). \tag{143}$$

Then, using (142) we compute the expression $1 + g_1 V + g_2 V^2 + \cdots + g_n V^n\big(1 + \eta_{n-1}(V)\big)$. Since $\big|\epsilon_{n-1}(u)\big| \ll 1$, we write for any $k = 1, \ldots, n-1$,

$$V^k = (a+\tau)^k \quad \text{with} \quad a = A_1 e^{-\theta u} + A_2 e^{-2\theta u} + \cdots + A_{n-1}e^{-(n-1)\theta u} \quad \text{and} \quad \tau = e^{-(n-1)\theta u}\widetilde{\epsilon}_{n-1}(u).$$

Then

$$\begin{aligned} \big|(a+\tau)^k - a^k\big| &\leq k|\tau|\big(|a| + |\tau|\big)^k \\ &\leq k|\tau|\Big(|A_1| + \cdots + |A_{n-1}| + \big|\widetilde{\epsilon}_{n-1}(u)\big|\Big)^{k-1}\big|\widetilde{\epsilon}_{n-1}(u)\big| \\ &\leq k|A_{n-1}|\big(|A_1| + \cdots + |A_{n-1}| + 1\big)^k e^{-(n-1)u}\big|\widetilde{\epsilon}_{n-1}(u)\big| \\ &= c_k\big|\widetilde{\epsilon}_{n-1}(u)\big|e^{-(n-1)u}; \end{aligned}$$

now

$$g_k V^k = g_k\big(A_1 e^{-\theta u} + A_2 e^{-2\theta u} + \cdots + A_{n-1}e^{-(n-1)\theta u}\big)^k + \delta_{n-1,k}(u),$$

with $\left|\delta_{n-1,k}(u)\right| \le c_k|g_k|\left|\epsilon_{n-1}(u)\right|$. Therefore

$$\begin{aligned}1+g_1V+\dots+g_kV^k+\dots+g_{n-1}V^{n-1}\big(1+\eta_{n-1}(V)\big)\\ =1+g_1\big(A_1e^{-\theta u}+A_2e^{-2\theta u}+\dots+A_{n-1}e^{-(n-1)\theta u}\big)+\cdots\\ +g_k\big(A_1e^{-\theta u}+A_2e^{-2\theta u}+\dots+A_{n-1}e^{-(n-1)\theta u}\big)^k\\ +g_n\big(A_1e^{-\theta u}+A_2e^{-2\theta u}+\dots+A_{n-1}e^{-(n-1)\theta u}\big)^n+e^{-(n-1)\theta u}\delta_{n-1}(u),\end{aligned}$$

then

$$\begin{aligned}1+g_1V+\dots+g_kV^k+\dots+g_{n-1}V^{n-1}\big(1+\eta_{n-1}(V)\big)\\ =1+D_1e^{-\theta u}+D_2e^{-2\theta u}+\dots+D_{n-1}e^{-(n-1)\theta u}+e^{-(n-1)\theta u}\delta_{n-1}(u)\end{aligned}\tag{144}$$

with $\left|\delta_{n-1}(u)\right| \le C_{n-1}\left|\tilde\epsilon_{n-1}(u)\right|$ and the coefficients D_j can be explicitly computed. Next we compute from the expression of ϖ, (138)–(143) and (144),

$$\begin{aligned}&e^{-n\theta}\left(\left(\frac{1}{q}-n\theta\right)\Phi+\frac{\mathrm{d}\Phi}{\mathrm{d}u}\right)\\ &=\frac{\mathrm{d}V}{\mathrm{d}u}+\theta A_1e^{-\theta u}+2\theta A_2e^{-2\theta u}+\dots+(n-1)\theta A_{n-1}e^{-(n-1)\theta u}+\frac{1}{q}e^{-n\theta}\Phi\\ &=\frac{\mathrm{d}V}{\mathrm{d}u}+\left(\theta-\frac{1}{q}\right)A_1e^{-\theta u}+\left(2\theta-\frac{1}{q}\right)A_2e^{-2\theta u}+\dots+\left((n-1)\theta-\frac{1}{q}\right)A_{n-1}e^{-(n-1)\theta u}\\ &=\frac{N-1}{q}\Big(1+C_1e^{-\theta u}+\dots+C_ke^{-k\theta u}+\dots+\big(C_{n-1}+\tilde\epsilon_{n-1}(u)\big)e^{-(n-1)\theta u}\Big)\\ &\quad-\frac{1+V}{q}+qVe^{\theta u}\Big(1+g_1V+\dots+g_kV^k+\dots+g_{n-1}V^{n-1}\big(1+\eta_{n-1}(V)\big)\Big)\\ &=\frac{N-1}{q}\Big(1+C_1e^{-\theta u}+\dots+C_ke^{-k\theta u}+\dots+\big(C_{n-1}+\tilde\epsilon_{n-1}(u)\big)e^{-(n-1)\theta u}\Big)\\ &\quad+q\big(A_1+A_2e^{-\theta u}+\dots+A_{n-1}e^{-(n-2)\theta u}+e^{-(n-1)}\Phi\big)\\ &\quad\times\big(1+D_1e^{-\theta u}+D_2e^{-2\theta u}+\dots+D_{n-1}e^{-(n-1)\theta u}+e^{-(n-1)\theta u}\delta_{n-1}(u)\big)\\ &\quad+\left(\theta-\frac{1}{q}\right)A_1e^{-\theta u}+\left(2\theta-\frac{1}{q}\right)A_2e^{-2\theta u}+\dots+\left((n-1)\theta-\frac{1}{q}\right)A_{n-1}e^{-(n-1)\theta u}\\ &=\frac{N-2}{q}+qA_1+E_1e^{-\theta u}+E_2e^{-2\theta u}+\cdots\\ &\quad+E_{n-1}e^{-(n-1)\theta u}\big(1+\delta_{n-1}(u)\big)+qe^{-(n-1)\theta u}\Phi\big(1+\tilde\delta_n(u)\big),\end{aligned}\tag{145}$$

where $\tilde\delta_n(u)\to 0$.

Because of (132) $\frac{N-2}{q}+qA_1=0$. *We claim that* $E_2=E_3=\dots=E_{n-2}=0$. If this does not hold, let $k\in[1,n-2]$ be the smaller integer such that $E_k\neq 0$. Then

$$\left(\frac{1}{q}-n\theta-qe^{\theta u}\big(1+\tilde\delta_n(u)\big)\right)\Phi+\frac{\mathrm{d}\Phi}{\mathrm{d}u}=E_ke^{(n-k)\theta u}.$$

Since $\Phi(u)=o(e^{\theta u})$, then $\big(\frac{1}{q}-n\theta-qe^{\theta u}\big(1+\tilde\delta_n(u)\big)\big)\Phi=o(e^{2\theta u})$. Thus

$$\frac{\mathrm{d}\Phi}{\mathrm{d}u}=E_ke^{(n-k)\theta u}\big(1+o(1)\big),$$

which implies $|\Phi|\ge Ce^{(n-k)\theta u}$, contradiction. Therefore we are led to the following relation

$$\left(\frac{1}{q}-n\theta-qe^{\theta u}\big(1+\tilde\delta_n(u)\big)\right)\Phi+\frac{\mathrm{d}\Phi}{\mathrm{d}u}=E_{n-1}e^{\theta u}\big(1+\tilde\delta_n(u)\big),\tag{146}$$

which yields

$$-qe^{\theta u}\big(1+\delta'_n(u)\big)\Phi+\frac{\mathrm{d}\Phi}{\mathrm{d}u}=E_{n-1}e^{\theta u}\big(1+\widetilde{\delta}_n(u)\big). \tag{147}$$

Finally we conclude as in Step 1, considering the three possibilities for Φ:

- either non-monotone near infinity which implies that $\lim_{u\to\infty}=-\frac{E_{n-1}}{q}:=A_n$;
- or Φ is monotone and tends to $\pm\infty$ and this would lead to a contradiction with $\Phi(u)=o(e^{\theta u})$ when $u\to\infty$;
- or Φ is monotone and bounded, thus it admits a limit which is necessarily $-\frac{E_{n-1}}{q}$.

Thus we have obtained the development at the order n of the function $\varpi(u)$. This yields a development of $\frac{\mathrm{d}r}{\mathrm{d}u}$ by the second equation in system (129) under the form

$$\frac{\mathrm{d}r}{\mathrm{d}u}=e^{-\frac{u}{q}}\Big(1+B_1e^{-\theta u}+B_2e^{-2\theta u}+\cdots+e^{-n\theta u}\big(B_n+\widetilde{\epsilon}_n(u)\big)\Big).$$

From this we obtain the expansion of $r(u)$ by integration and finally obtain algebraically the expansions in the variable r

$$e^{\frac{u(r)}{q}}=\frac{q}{r}\Big(1+b_1r^{q-2}+\cdots+\big(b_n+o(1)\big)r^{n(q-2)}\Big),$$

and

$$u(r)=\ln\frac{q^q}{r^q}+a_1r^{q-2}+\cdots+r^{n(q-2)}\big(a_n+o(1)\big).$$

It is noticeable that all the terms of order n in these expansions are polynomials in N,n,q. □

Remark 34. The fact that a solution of eikonal type can be expressed formally by a series in the variable r^{q-2} is a strong presumption for uniqueness. However it appears very difficult to prove that the radius of convergence of this series is positive. This type of question is similar to the existence and uniqueness problem encountered in the study of the singular solutions of the capillary equation as shown in [15,16]. However in this problem, the formal series obtained in the study of singular solutions can be proved to have a null radius of convergence.

Remark 35. If $N=2$ there exists an explicit solution of (125), namely $u^*(x)=\ln\frac{q^q}{|x|^q}$ and clearly it satisfies (70). As a consequence all the terms D_j in the expansion of another radial solution of (125) satisfying (70) are zero, which means that for any n, there holds

$$u(r)=\ln\frac{q^q}{r^q}+o(r^{n(q-2)})\quad\text{as } r\to 0.$$

This situation is reminiscent of a result of Brezis and Nirenberg [10, Theorem 2] dealing with the equation

$$-\Delta u+|\nabla u|^2=h^2(u).$$

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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