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Irregularity of some toroidal compactifications

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Abstract. We compute the irregularity of certain 2-dimensional toroidal compactifications.

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1. Introduction

1.1.

The main objects of study of the present note are *toroidal compactifications* S of a complex ball quotient $U := \mathbb{B}/\Gamma$ (see e.g. [7] for foundations). Recall that S is a minimal (i.e. without (-1)-curves) projective surface and $U \subset S$ is a Zariski open subset with the following auxiliary data:

- $\mathbb{B} \subset \mathbb{CP}^2$ is the unit ball and $\Gamma \subset \mathrm{PSU}(2,1)$ is a discrete subgroup[1] acting freely and properly discontinuously on $\mathbb{B}$;
- the boundary divisor $D := S \setminus U$ is a disjoint union of smooth elliptic curves (note that their self-intersection numbers are all negative).

We are interested in computing the *irregularity* $q(S) := h^1(S, \mathscr{O}_S)$ of the surface S — under certain assumptions on the latter (see Section 1.2 and the discussion after Theorem 1 below for motivations).

[1] For all surfaces S, considered further, Γ will be an *arithmetic lattice*. This implies in particular that the rank of the homology group $H_1(U, \mathbb{Z})$ is finite (see also [9, Corollary 4.5] or [2, Theorem 1.3]).

1.2.

Consider an Abelian surface A and $\mathrm{Bl}(A) :=$ [the blowup of A at a finite set of distinct points]. Let $h\colon \mathrm{Bl}(A) \to A$ be the blowup map of distinct points on A with the (-1)-curves $C_i \subset \mathrm{Bl}(A)$. We will require these to satisfy the following two properties:

(1) there exists a *finite* morphism $f\colon S \to \mathrm{Bl}(A)$ (hence $q(S) \geq 2$) such that $f\big|_U$ is étale;
(2) all preimages in $\mathbb{B}$ of the curves $f^{-1}(C_i) \cap U$ under the quotient morphism $\mathbb{B} \to \mathbb{B}/\Gamma$ have *infinitely many* connected components.

Let us now formulate our main result.

Theorem 1. *Under the assumptions* (1) *and* (2) *we have* $q(S) = 2$.

Note that setting $S := Y_1$ in Theorem 1 one recovers the result from [15, p. 2.2]. These surfaces Y_1 constitute a special case of examples from [5] (cf. [6]) and *all* the latter satisfy our preceding assumptions.

It is also interesting to compare the result of Theorem 1 with the case of *compact* ball quotients. Namely, in [8] several smooth surfaces of general type, satisfying $c_1^2 = 3c_2 = 225$ and having irregularity > 2, have been constructed. All these surfaces turned out to be the minimal resolutions of cyclic coverings of $\mathbb{P}^2$ ramified in configurations of lines (cf. [4]).

As for the non-compact ball quotients, let us recall that in [14, Theorem 1] (cf. [14, Theorem 5]) the author constructs a sequence $\{D_j\}$ of arithmetic quotients of $\mathbb{B}$, having finite volumes $\mathrm{Vol}\, D_j$ for all j (so that D_j are Kähler hyperbolic by the discussion in Section 2.1 below), arbitrarily large number of cusps (in fact their number grows like $(\mathrm{Vol}\, D_j)^{5/8}$) and topological Betti numbers $b_1(D_j) \gg (\mathrm{Vol}\, D_j)^{3/8}$ for $j \to \infty$ (compare with [1]). All these D_j arise from the so-called *Deligne–Mostow orbifolds* (see [14, Section 4] for the precise constructions).

Remark 2. Note that in [12, Theorem 1.1], assuming only that the Kodaira dimension $\mathrm{Kod.dim}(S)$ is less than 2, the equality $q(S) = 2$ is claimed for *any* toroidal compactification S. We should indicate however that some of the intermediate technical assertions stated in [12] are wrong. Moreover, as it was pointed out in [2], the main result of [12] is actually false because there exist such S satisfying all the stated assumptions, but having $q(S) = 1$. Yet, it would still be interesting to repair (if possible) the method of [12] in order to investigate those toroidal S, for which $\mathrm{Kod.dim}(S) < 2$ and $q(S) > 2$ (cf. Section 2.3 below).

The proof of Theorem 1 is given in Sections 2.3 and 2.4 by bringing the assumption $q(S) > 2$ to contradiction. Our argument relies on two ingredients: the vanishing $\mathscr{H}^1(U) = 0$ of L^2-cohomology (see Section 2.2) and an explicit embedding $\mathbb{B} \subset \mathbb{C}^q$ via the Albanese map (cf. (3)).

2. Proof of Theorem 1

2.1. *Kähler hyperbolicity*

We retain the notation of Section 1. Let ω be a Kähler form on S with at most logarithmic poles along D (e.g. the one that represents the class of $c_1\big(\mathcal{O}_S(1)\big) + D$ for some projective embedding $S \subset \mathbb{P}^N$ and $\mathcal{O}_S(1) := \mathcal{O}_{\mathbb{P}^N}(1)\big|_S$). Then we have the following.

Lemma 3. $\mathrm{Vol}\, U := \int_U \omega \wedge \overline{\omega} < \infty$.

Proof. Locally near every (smooth) component of D one may set U to coincide with $\mathbb{B}$ and $\omega\big|_U$ be induced by the Poincaré metric. The claim now follows via the standard partition of unity argument (cf. [13, Section 1]). □

From Lemma 3 we readily obtain that U is *complete* (with respect to the metric corresponding to $\omega\big|_U$) — for all compact balls in U have uniformly bounded volumes and so any diverging sequence of points $u_i \in U$ can have a limit only on the boundary $\partial U = D$. In particular, replacing U by $\mathbb{B}$ near D (cf. the proof of Lemma 3) and using the completeness of Poincaré metric, we obtain that the sequence of u_i is not Cauchy. Hence U is complete.

Following [3] we call U *Kähler hyperbolic* — for the lift $\widetilde{\omega}$ of the Kähler form $\omega\big|_U$ to $\mathbb{B}$ is *d(bounded)* (cf. [3, 0.1.B, 0.2.A, 0.3.A(b)]). What makes this property interesting to us is Theorem 4 below. The latter will be used in estimating the irregularity $q(S)$ of the surfaces satisfying the assumptions from Theorem 1.

2.2. L^2-*cohomology*

Next we consider the spaces of L^2-*cohomology* $\mathscr{H}^p(U)$ consisting of harmonic (w.r.t. $\omega\big|_U$) holomorphic L^2-forms on U of degree $p \ge 0$. One identifies these $\mathscr{H}^p(U)$ with the subspaces of similarly defined (w.r.t. $\widetilde{\omega}$) Γ-invariant forms on $\mathbb{B}$.

Here is the Lefschetz-type result we will need.

Theorem 4 (see [3, 1.2.B, 2.5, 2.5.A]). $\mathscr{H}^1(U) = 0$.

In addition, if $L^2\Omega^p$ denotes the sheaf of holomorphic L^2-forms on U of degree p, then the following Hodge decomposition takes place (see [3, p. 1.1.C]):

$$L^2\Omega^p = \mathscr{H}^p(U) \oplus \overline{\mathrm{d}L^2\Omega^{p-1}} \oplus \overline{\delta L^2\Omega^{p+1}}. \tag{1}$$

Here d is the usual Kähler differential, $\delta := \pm * \mathrm{d} *$, $*$ is the Hodge star operator acting on $H^\bullet(U)$, and the last two direct summands in (1) are the L^2-closures of respective images. A similar decomposition (with (1) being its Γ-invariant part) also holds for the L^2-forms on $\mathbb{B}$.

2.3. *Universal cover of U when* $q(S) > 2$

From this moment on we will assume that $q := q(S) > 2$ (recall that initially $q(S) \ge 2$ by the discussion in Section 1.2).

Let alb: $S \to \mathrm{Alb}(S)$ be the Albanese morphism. Consider the preimage Δ of $\mathrm{alb}(D)$ in the universal cover $\mathbb{C}^q$ of $\mathrm{Alb}(S)$.

Lemma 5. $\pi_1(\mathbb{C}^q \setminus \Delta) = \{1\}$.

Proof. Note that Δ is a countable union $\bigcup_i L_i$ of *affine lines* $L_i \subset \mathbb{C}^q$. Indeed, this follows from the fact that D is a disjoint union of elliptic curves, as we have assumed in Section 1.1.

Further, write $\mathbb{C}^q \setminus \Delta = \bigcap_k \big(\mathbb{C}^q \setminus \bigcup_{i=1}^k L_i\big)$, which implies (by definition of the fundamental group and universal cover) that

$$\pi_1(\mathbb{C}^q \setminus \Delta) = \varprojlim_k G_k. \tag{2}$$

Here the groups $G_k := \pi_1\big(\mathbb{C}^q \setminus \bigcup_{i=1}^k L_i\big)$ form a projective system with respect to the natural surjections $G_{k+1} \twoheadrightarrow G_k$ for all $k \ge 1$ (cf. [10, Proposition 2.10, (2.10.1)]).

Finally, recall that according to [10, Theorem 2.14, (2.14.1)] if X is a normal quasi-projective variety with a pencil $|H|$ of hypersurfaces and a base point $x \in X$, whose general member $H_t \ni x$ is connected, then the natural homomorphism $\pi_1(H_t) \to \pi_1(X)$ is surjective. Let us apply this result to show that $G_k = \{1\}$ for all k. Namely, put $X := \mathbb{C}^q \setminus \bigcup_{i=1}^k L_i$ and $H_t := X \cap \mathbb{C}^{q-1}$, where $\mathbb{C}^{q-1} \subset \mathbb{C}^q$ is a general affine hyperplane. We have $H_t = \mathbb{C}^{q-1} \setminus$ [a finite set of points $L_i \cap \mathbb{C}^{q-1}$], which implies that $\pi_1(H_t) = \pi_1(\mathbb{C}^{q-1})$, since $q > 2$ by assumption. Thus we get that $\pi_1(H_t) = \{1\}$ surjects onto $\pi_1(X) = G_k$. Then $\pi_1(\mathbb{C}^q \setminus \Delta) = \{1\}$ as well according to (2). $\square$

Identify the origin $0 \in \mathbb{B}$ with its image in U under the universal covering $\mathbb{B} \to U = \mathbb{B}/\Gamma$.

Let $\gamma \in H_1(U, \mathbb{C})$ be any element, which we treat as a loop on U passing through 0, and let $\widetilde{\gamma}$ be some lift of γ to $\mathbb{B}$. Let also $\omega_1, \ldots, \omega_q$ be Γ-invariant closed holomorphic 1-forms on $\mathbb{B}$ that give a basis for $H^1(S, \mathbb{C})$ when restricted to $U \subset S$. Notice that by construction all ω_i are *bounded* 1-forms on $\mathbb{B}$.

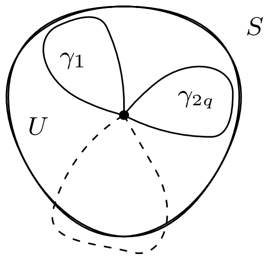

Figure 1. $2q$ 1-cycles on U.

Choose a splitting for the natural surjection $H_1(U, \mathbb{Z}) \twoheadrightarrow H_1(S, \mathbb{Z}) = \mathbb{Z}^{2q} \oplus \text{torsion}$ (cf. Figure 1). Then from Lemma 5 and definition of alb we obtain the following *analytic* morphism:

$$x \in \mathbb{B} \longmapsto \left(\int_0^x \omega_1, \ldots, \int_0^x \omega_q \right) \in \mathbb{C}^q_\Delta := \mathbb{C}^q \setminus \Delta \tag{3}$$

(here the ring of holomorphic functions on $\mathbb{C}^q_\Delta \subset \mathbb{C}^q$ is generated by the affine coordinates coming from $\mathbb{C}^q$ because $\pi_1(\mathbb{C}^q_\Delta) = \{1\}$). More precisely, the integrals $\int_0^x \omega_i$ along a path $[0, x]$ joining 0 and a given point x are defined modulo various $\int_{\widetilde{\gamma}} \omega_i$, with $\gamma \in H_1(S, \mathbb{Z}) \subseteq H_1(U, \mathbb{Z})$. But the latter integrals are all zero for $\pi_1(\mathbb{B}) = \{1\}$ and ω_i being bounded along the $[0, x]$.

We observe next that the morphism

$$\mathrm{alb}\big|_U \colon U \longrightarrow \mathbb{C}^q_\Delta / H_1(S, \mathbb{Z}) = \mathrm{Alb}(S) \setminus \mathrm{alb}(D)$$

fits into a commutative diagram of analytic spaces and morphisms

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{\mathrm{alb}_{\mathbb{B}}} & \mathbb{C}^q_\Delta \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathrm{Alb}(S) \setminus \mathrm{alb}(D). \end{array} \tag{4}$$

Here the vertical arrows are the universal coverings of U and $\mathrm{Alb}(S) \setminus \mathrm{alb}(D)$, whereas the horizontal ones are (3) and $\mathrm{alb}\big|_U$, respectively.

Remark 6. Note that the morphism $U \to \mathrm{Alb}(S) \setminus \mathrm{alb}(D)$ in (4) depends on the splitting of $H_1(U, \mathbb{Z}) \twoheadrightarrow H_1(S, \mathbb{Z})$ and need not extend to S (i.e. coincide with $\mathrm{alb}\big|_U$) if the codimension of $\mathrm{alb}(D)$ is 1 (and hence $q = 2$). Indeed, we have $\pi_1(\mathbb{C}^q_\Delta) \neq \{1\}$ in this case, so that (3) is not analytic — the ring of holomorphic functions on $\mathbb{C}^2_\Delta$ is *larger* than that on $\mathbb{C}^2$.

Proposition 7. *The induced morphism* $\mathrm{alb}\big|_U \colon U \to \mathrm{alb}(U)$ *is finite and smooth.*

Proof. Note that the morphism $h \circ f$ factors through alb by functoriality (cf. Section 1.2). Hence it suffices to show that none of the curves $f^{-1}(C_i) \subset S$ is contracted by alb. Suppose the contrary. Then, restricting $\mathrm{alb}_{\mathbb{B}}$ to the preimages of $f^{-1}(C_i) \cap U$ in $\mathbb{B}$, we find a set $\mathfrak{S} :=$ [disjoint union of infinitely many curves in $\mathbb{B}$] such that all the (holomorphic) components f_i of $\mathrm{alb}_{\mathbb{B}}$ are constant on $\mathfrak{S}$, $1 \leq i \leq q$.

On the other hand, it follows from the construction (see (3)) that $\mathrm{alb}_{\mathbb{B}}$ extends to the closure $\overline{\mathbb{B}} \subset \mathbb{P}^2$, with $\mathfrak{S}$ having accumulation points on the boundary $\overline{\mathbb{B}} \setminus \mathbb{B}$. In particular, we find that all the f_i must be identically constant, which is impossible because $\dim \mathrm{alb}_{\mathbb{B}}(\mathbb{B}) = \dim \mathrm{alb}(U) = 2$.

The proof of Proposition 7 is complete. □

Corollary 8. $\mathrm{alb}\big|_U$ *is an isomorphism (onto its image).*

Proof. This follows from Proposition 7 and (4). Indeed, the morphism $\mathrm{alb}_{\mathbb{B}}$ must also be finite and smooth, which implies that $\mathrm{alb}_{\mathbb{B}}(\mathbb{B}) \subset \mathbb{C}^q$ is a *contractible subvariety* (cf. (3)). Hence $\mathrm{alb}_{\mathbb{B}}$ (and $\mathrm{alb}\big|_U$ as well) is of topological degree 1. □

2.4. *Final step: the use of vanishing $\mathscr{H}^1(U) = 0$*

It follows from (the proof of) Corollary 8 that $\mathrm{alb}_{\mathbb{B}}$ is an isomorphism onto its image. Thus we may identify $\mathbb{B}$ (resp. U) with an analytic subset in $\mathbb{C}^q$ (resp. with $\mathrm{alb}(U) \subset \mathrm{Alb}(S)$). Note also that $\mathbb{B} = \mathrm{alb}_{\mathbb{B}}(\mathbb{B}) \subset \mathbb{C}^q$ is a *bounded* subset by construction of $\mathrm{alb}_{\mathbb{B}}$. In particular, for appropriate *affine* coordinates $x_1, \ldots, x_q$ on $\mathbb{C}^q \supset \mathbb{B}$ one may put x_1, x_2 to be analytic coordinates on $\mathbb{B}$, so that $x_i = x_i(x_1, x_2)$ are some holomorphic functions in x_1, x_2 for all $i \geq 3$.

Further, since $\pi_1(U) = \Gamma$ (cf. Section 1.1), it follows from (4) that the group $\Gamma \subseteq H_1(S, \mathbb{Z})$ acts on $\mathbb{B}$ via the $\mathbb{Z}$-shifts,

$$\gamma\colon x_i \longmapsto x_i + c_i(\gamma), \quad \gamma \in \Gamma, \tag{5}$$

with certain $c_i(\gamma) \in \mathbb{Z}$ and all i. We can also assume $c_1(\gamma) \neq 0$ for some $\gamma \in \Gamma$ because x_1, x_2 are coordinates on $\mathbb{B}$, where Γ acts *non-trivially*. Let us bring this to contradiction (compare with [11]).

Consider the 1-form $v := \mathrm{d}x_1$. It is Γ-invariant by (5) and its descent to U is L^2 w.r.t. the metric corresponding to $\omega\big|_U$ (the L^2-norm of v equals $\mathrm{Vol}\, U$ by definition). Furthermore, since $\mathrm{d}v = 0$ and $\mathscr{H}^1(U) = 0$ by Theorem 4, from (1) we obtain that $v = \lim_{i\to\infty} \mathrm{d}f_i$ for some Γ-invariant holomorphic L^2-functions f_i on $\mathbb{B}$. In particular, since $v = \mathrm{d}x_1$, we obtain that $x_1 = \lim_{i\to\infty} f_i$ is Γ-invariant, which contradicts $c_1(\gamma) \neq 0$.

The proof of Theorem 1 is finished.

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Declaration of interests

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