ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus

Mathématique

Alain Haraux

The universal bound property

Volume 363 (2025), p. 1499-1506

Online since: 12 December 2025

https://doi.org/10.5802/crmath.796

CENTRE
MERSENNE

The Comptes Rendus. Mathématique *are a member of the*
Mersenne Center for open scientific publishing
www.centre-mersenne.org — e-ISSN : 1778-3569

ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus. Mathématique
2025, Vol. 363, p. 1499-1506
https://doi.org/10.5802/crmath.796

Review article
Dynamical systems

The universal bound property

Alain Haraux [a]

[a] Laboratoire Jacques-Louis Lions, Sorbonne University, Pierre and Marie Curie Campus, 4 Pl. Jussieu, 75005 Paris, France
E-mail: alain.haraux@sorbonne-universite.fr

This short survey paper is dedicated to the memory of the great nonlinear analyst and lover of global methods Haim Brezis

Abstract. This survey paper is devoted to the question of universal bounds, independent of the initial state, for the trajectories of some nonlinear semi-groups and even more general processes. In the case of second order ODEs, rather surprisingly, it turns out that dissipation alone is not enough to produce such a property, and nonlinear elastic forces result in universal boundedness only when they dominate the damping in a very precise sense.

Keywords. Evolution equations, second order, damping term, restoring force, energy bounds.

2020 Mathematics Subject Classification. 34A40, 34D23, 34E10, 34G20, 35L71, 37L15.

Note. Article submitted by invitation.

Manuscript received 12 February 2025, revised and accepted 18 September 2025.

1. Introduction

This survey paper is mainly devoted to the property of universal bounds for a semi-group of continuous operators $S(t)$ defined on some metric space X. The property of universal bound means that $S(t)X$ is bounded for $t \geq t_0$. This property is of course only of interest if X is not compact. And when it happens, in practice we can take t_0 arbitrarily small.

To start with a simple familiar example, let us imagine, neglecting for the moment the weight, that we launch horizontally at $t = 0$ a point ball with initial velocity v_0 and we look for the value $v(t)$ of the velocity at time $t > 0$. The ODE satisfied by $v(t)$ is

$$v' = -av - b|v|v \tag{1}$$

with a the viscosity of air and b the hydrodynamic resistance. For t large, we all know that the damping (assuming that at least a or b is not zero), drives $v(t)$ to 0. For t small one might imagine to linearize the equation around v_0, but this would be a big mistake, since if v_0 is very large, even for t small, the quadratic term prevails: the relevant approximate equation is therefore rather

$$v' = -b|v|v, \tag{2}$$

even though, of course, here the complete equation is not difficult to integrate. Again, everybody knows the result: assuming $v_0 > 0$ (which is equivalent to choose an orientation of the axis) we have $v(t) > 0$ and

$$(1/v)' = b$$

hence

$$v(t) = \frac{1}{bt + \frac{1}{v_0}}.$$

As a consequence, no matter how large the initial velocity, we have $v(1) \le 1/b$. You can spend all the energy you have to send the ball as fast as possible, even if you use a cannon, at $t = 1$ the velocity will be less than $1/b$. Of course, if you hit the ball very hard, it will reach the target before the velocity becomes small. But the information $v(1) \le 1/b$ is not without interest: no matter how much energy you use to send the projectile, you have at $t = 1$ a universal bound of the kinetic energy, assuming that the projectile survived both launching shock and heating due to the damping...

This kind of property is frequent for nonlinear scalar first order ODEs, and it is a really nonlinear feature. In this note, we shall be mainly concerned by second order ODEs of the form

$$u'' + f(u) + g(u') = 0$$

and hyperbolic semi-linear PDEs of the form

$$u_{tt} - \Delta u + f(u) + g(u_t) = 0,$$

the main question being whether or not both f and g need to be non-linear for such a property to happen. This question has been studied thoroughly in the sequence of papers [3–5,8,9]. We shall report on the main results of these papers and indicate only the main steps of the proofs, the details being given in the papers cited above.

The plan of the paper is as follows: in Section 2, we give a simple sufficient condition for the system defined by a first order ODE to have the universal bound property. In Section 3, we give a case of second order strongly dissipative scalar equation which was the starting point of our study and for which there is no universal bound. Actually, universal bounds require both damping and restoring force to be nonlinear, and even we need the restoring force to be more nonlinear than the damping, as explained in Section 4. Finally Section 5 is devoted to the infinite dimensional case of wave equations with nonlinear damping and a restoring force growing faster than the damping, in a sense which is made clear.

2. First order equations

This part is supposed to help making the connection between the simple example of the introduction and the main point concerning second order ODEs. The equations (1) and (2) are first order scalar ODEs with respect to v, and it is natural to ask about conditions for such an equation to display the universal bound property. So if we consider more generally

$$v' = -g(v) \tag{3}$$

what would be a sufficient condition on g? A simple sufficient condition is that for some positive constants M, γ, δ, we have

$$|v| \ge M \quad \Longrightarrow \quad g(v)v \ge \gamma |v|^{1+\delta}. \tag{4}$$

Indeed, from this inequality we first deduce that if for some time $t_0 \ge 0$, we have $|v(t_0)| \le M$, then $|v(t)| \le M$ for all $t \ge t_0$. On the other hand, if $|v(t)| > M$ for all $t \ge 0$, then setting $w := v^2$ we deduce from (4) that

$$w' \le -2\gamma w^{1+\frac{\delta}{2}}.$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}(w^{-\frac{\delta}{2}}) \ge \gamma\delta,$$

yielding

$$|v(t)| \le (\gamma\delta t)^{-\frac{1}{\delta}},$$

and finally in all cases we find

$$\left|v(t)\right| \le \max\{M, (\gamma\delta t)^{-\frac{1}{\delta}}\}. \tag{5}$$

This result is for instance applicable when $g(v)$ is a polynomial of odd degree with a positive leading coefficient. As a consequence of the maximum principle, a similar property occurs for the semilinear heat equation

$$u_t - \Delta u + \delta |u|^r u = 0$$

with either Dirichlet or Neumann homogeneous BCs, all solutions lie in a fixed bounded subset of L^∞ for each $t > 0$. Actually the same property was found long ago by J. Simon [7] in the more delicate case of some quasilinear equations.

3. A case without universal bound

Coming back to the second order case, let us now consider the slightly more complicated case of a mass attached to a spring vertically suspended in presence of gravity. We produce oscillations by pulling the mass down and then letting it go. We consider for simplicity only the hydrodynamic damping. The equation of motion is

$$u'' + ku + b|v|v = -g,$$

where $v = u'$ and on replacing u by $u + \frac{g}{k}$ we are reduced to

$$u'' + ku + b|u'|u' = 0.$$

By a suitable simultaneous rescaling in (t, u) we even obtain

$$u'' + u + |u'|u' = 0. \tag{6}$$

In the plane $\mathbb{R}^2$ with coordinates (u, u'), this equation classically generates a non linear contraction semi-group $S(t)$, the action of which involves both elastic restoring force and damping. The damping tends to reduce the velocity, and this results, by the action of the elastic force, in a reduction of the total energy

$$E = \frac{1}{2}(u'^2 + u^2).$$

A natural simple question (raised in the author's state/habilitation thesis in 1978) is whether $S(t)\mathbb{R}^2$ is bounded for $t > 0$. However, it is only in 1992 that the author got the idea of looking at the backward equation

$$u'' + u = |u'|u'. \tag{7}$$

Without the restoring force, all constants are solutions, and any non-constant solution blows up in finite time. When the u term is added, the line of (constant) global solutions gets duplicated in two global trajectories, more precisely

$$u(t) = \pm\left(\frac{1}{4}t^2 - \frac{1}{2}\right), \qquad u'(t) = \pm\frac{t}{2}.$$

As a consequence of the existence of *unbounded* global solutions to the backward equation and the autonomous character of the equation, it turns out that $S(t)\mathbb{R}^2$ is unbounded for every $t > 0$. In 1995, Philippe Souplet [8] established that the backward equation has no other global solutions than those already found and their time translates.

4. Nonlinear restoring force and nonlinear damping

Let us now consider the equation

$$u'' + |u'|^{\alpha} u' + |u|^{\beta} u = 0. \tag{8}$$

In [8] also, Philippe Souplet proved that if $\alpha \geq \beta > 0$, the backward equation has two opposite global solutions, so that $S(t)\mathbb{R}^2$ is unbounded for all $t > 0$. After this result, nobody tried to obtain a universal bound for a second order ODE. However, if $0 < \alpha < \beta$, Philippe Souplet observed that the backward equation has no non-trivial global solution. A careful examination of his proof reveals that the life-time of solutions tends to 0 when the initial energy tends to infinity. This could have led to the conclusion that $S(t)\mathbb{R}^2$ is bounded, and it is indeed the case. We do not discuss here well-posedness of the initial-value problem for equation (8) which is completely standard. The energy defined by the formula

$$E(t) = \frac{1}{2} u'(t)^2 + \frac{1}{\beta+2} \left|u(t)\right|^{\beta+2} \tag{9}$$

is a good way of measuring the size of the solution at time t. By multiplying equation (8) by u', we find

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = -\left|u'(t)\right|^{\alpha+2} \leq 0, \tag{10}$$

hence the energy of any solution is non-increasing. Universal boundedness of the dynamical system generated by equation (8) is equivalent to the existence of a bound of the energy independent of the solution. By using adapted Lyapunov functions, in the joint paper [3] we obtained the following very precise result.

Theorem 1. *Assuming* $0 < \alpha < \beta$, *there is a constant* C *independent of* $E(0)$ *such that*

$$\forall\ t > 0, \quad E(t) \leq C \max\left\{ t^{-\frac{2}{\alpha}},\ t^{-\frac{(\alpha+1)(\beta+2)}{\beta-\alpha}} \right\}. \tag{11}$$

The proof is rather involved and uses different Lyapunov functions according to the relative positions of α and $\alpha_0 := \frac{\beta}{\beta+2}$, cf. [3].

Remark 2. The estimate (11) is optimal. For instance for $\alpha_0 \leq \alpha < \beta$ the decay estimate is sharp for all nontrivial solutions, while the growth for t tending to 0 is sharp for some special solutions of the backward equation found by Souplet [8] in 1995. On the other side, when $0 < \alpha \leq \alpha_0$, all non-trivial solutions of the backward equation blow-up exactly at the rate allowed by the estimate, while the decay estimate is sharp for the “slow-decaying” solutions; for the details, cf. [3,6].

Remark 3. It is clear that the last theorem can be extended under relaxed assumptions on the non-linearities, and for some kind of vector equations, cf. e.g. [1,2] for the relevant framework. More interesting is the case of semilinear hyperbolic equations which we shall study now.

5. The case of hyperbolic PDEs

We shall now explain what happens in the regime $0 < \alpha < \beta$ for wave equations such as

$$u_{tt} - \Delta u + |u|^{\beta} u + |u_t|^{\alpha} u_t = 0, \tag{12}$$

with either Dirichlet or Neumann boundary conditions, or for analogous plate equations where the Laplacian is replaced by a bi-Laplacian. The issue seems to be non-trivial because there is no such maximum principle as in the parabolic case, and because without the nonlinear term $|u|^{\beta} u$ the universal boundedness does not take place. Nevertheless, a natural slight change of the method of the scalar case, involving a power of the total energy, gives the result for a large class of equations that fit into a natural functional framework. For these equations, we can prove a universal bound for all positive times, and a universal decay at infinity under slightly stronger assumptions.

More generally, following [4], let H be a Hilbert space, and let V be another Hilbert space continuously imbedded into H as a dense subspace. If we identify H with its dual H', we obtain a classical Hilbert triple $V \subseteq H \subseteq V'$. We denote norms by double bars, and scalar products and duality pairings by angle brackets. In the time interval $[0,T]$ or $(0,T)$ we consider evolution equations of the form

$$u''(t) + \nabla F\big(u(t)\big) + g\big(t, u'(t)\big) = 0, \tag{13}$$

where F and g satisfy the following assumptions:

(F1) the function $F\colon V \to \mathbb{R}$ is of class C^1, and $\nabla F \in C^0(V, V')$ is its gradient;

(G1) the function $g\colon (0,T) \times V \to V'$ is such that for every $v \in L^\infty\big((0,T), V\big)$ the function $t \to g\big(t, v(t)\big)$ belongs to $L^1\big((0,T), V'\big)$.

Under these assumptions we can introduce a notion of strong solutions to (13).

Definition 4 (Strong solutions). *A* strong solution *to* (13) *is a function*

$$u \in W^{1,\infty}\big((0,T), V\big) \cap W^{2,1}\big((0,T), V'\big)$$

for which (13) *holds true as an equality in* $L^1\big((0,T), V'\big)$.

Remark 5. Every strong solution belongs in particular to the class

$$C^0\big((0,T], V\big) \cap C^1\big([0,T], H\big),$$

and as a consequence the pointwise values $u(t) \in V$ and $u'(t) \in H$ are well defined for every $t \in [0,T]$ (endpoints included). Moreover, the classical energy

$$E_0(t) := \frac{1}{2}\|u'(t)\|_H^2 + F\big(u(t)\big) \tag{14}$$

belongs to $W^{1,\infty}\big((0,T), \mathbb{R}\big)$ and

$$E_0'(t) = -\big\langle g\big(t, u'(t)\big), u'(t)\big\rangle_{V',V} \tag{15}$$

for almost every $t \in (0,T)$.

Now we assume that α and β are two positive real numbers, and that X and Y are two Banach spaces that extend the original Hilbert triple $V \subseteq H \subseteq V'$ to a chain of seven spaces with continuous imbeddings

$$V \subseteq Y \subseteq X \subseteq H \subseteq X' \subseteq Y' \subseteq V'. \tag{16}$$

The following additional assumptions on F and g are needed in our abstract result concerning the uniform bound property.

(F2) There exist real numbers $\delta_1 > 0$ and $C_1 \geq 0$ such that

$$F(u) \geq \delta_1 \|u\|_Y^{\beta+2} - C_1, \quad \forall\, u \in V. \tag{17}$$

(F3) There exist real numbers $\delta_2 > 0$ and $C_2 \geq 0$ such that

$$\big\langle \nabla F(u), u \big\rangle_{V',V} \geq \delta_2 F(u) - C_2, \quad \forall\, u \in V. \tag{18}$$

(G2) There exist real numbers $\delta_3 > 0$ and $C_3 \geq 0$ such that

$$\big\langle g(t,v), v \big\rangle_{V',V} \geq \delta_3 \|v\|_X^{\alpha+2} - C_3 \tag{19}$$

for every $v \in V$ and almost every $t \in (0,T)$.

(G3) For every $(t,v) \in (0,T)\times V$, we have $g(t,v)\in X'$, and there exist real numbers $C_4 \geq 0$ and $D_4 > 0$ such that
$$\left\| g(t,v)\right\|_{X'} \leq D_4 \|v\|_X^{\alpha+1} + C_4 \tag{20}$$
for every $v \in V$ and almost every $t \in (0,T)$.

Theorem 6 (Universal bound property [4]). *Let us consider the chain of functional spaces* (16)*, and let F and g be two functions satisfying assumptions* (F1)–(F2)–(F3) *and* (G1)–(G2)–(G3)*. Let $T>0$ be a real number, and let $u\colon (0,T)\to V$ be a strong solution to* (13)*. Let us assume in addition that $0<\alpha<\beta$, and let us set*
$$\gamma := \min\left\{\frac{\alpha}{2}, \frac{\beta-\alpha}{(\alpha+1)(\beta+2)}\right\}. \tag{21}$$
Then there exist two real numbers Γ and Γ_* such that
$$\left\|u'(t)\right\|_H^2 + F\big(u(t)\big) \leq \Gamma\, t^{-1/\gamma} + \Gamma_*, \quad \forall\ t\in(0,T). \tag{22}$$
The constants Γ and Γ_* depend on the immersions (16)*, and on the constants that appear in* (17) *through* (20)*, but they are independent of T and u.*

Principle of the proof. To begin with, we introduce the energy
$$E(t) := \frac{1}{2}\left\|u'(t)\right\|_H^2 + F\big(u(t)\big) + C_1 + 1. \tag{23}$$
This energy coincides with the energy $E_0(t)$ defined in (14) up to an additive constant, and therefore its time-derivative is again given by the right-hand side of (15). Due to assumption (17), this new energy is bounded from below by 1, and therefore we can define the modified energy
$$\Phi(t) := E(t) + \varepsilon E(t)^{\gamma}\left\langle u(t), u'(t)\right\rangle_H,$$
where γ is defined by (21), and $\varepsilon>0$ is a small parameter. We claim that, for every $\varepsilon>0$ small enough, the modified energy Φ has the following two properties:

- it is a small perturbation of E in the sense that
$$\frac{1}{2}E(t) \leq \Phi(t) \leq \frac{3}{2}E(t), \quad \forall\ t\in(0,T); \tag{24}$$
- it satisfies the differential inequality
$$\Phi'(t) \leq -\varepsilon\frac{\delta_2}{8}\left(\frac{2}{3}\right)^{\gamma+1}\Phi(t)^{\gamma+1} + \frac{3C_3}{2} + 2 \tag{25}$$
for almost every $t\in(0,T)$, where δ_2 and C_3 are the constants that appear in (18) and (19), respectively.

The smallness of ε depends only on the norms of the continuous imbeddings (16), and on all the constants in (17) through (20), but it depends neither on T nor on u.

As an application, let us consider, in a cylinder $(0,T)\times\Omega$ or $(0,+\infty)\times\Omega$, with Ω a bounded open subset of $\mathbb{R}^N$, semilinear wave equations of the form
$$u_{tt} - \Delta u + b|u|^{\beta}u - \lambda u + c|u_t|^{\alpha}u_t - \mu u_t = h, \tag{26}$$
where α, β, b, c are positive real parameters, λ, μ are real parameters and $h = h(t,x)$ is bounded with values in $L^2(\Omega)$. Let us add initial conditions
$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \tag{27}$$
and either homogeneous Dirichlet boundary conditions
$$u(t,x) = 0 \quad \text{in } (0,T)\times\partial\Omega,$$

or homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial n}(t,x)=0 \quad \text{in } (0,T)\times\partial\Omega.$$

If we assume that

$$(N-2)\beta \le 2, \tag{28}$$

then $H^1(\Omega)$ is continuously imbedded into $L^{2\beta+2}(\Omega)$. In this case the initial-boundary-value problem is classically well-posed for initial data $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ in the case of Dirichlet boundary conditions, and for initial data $(u_0,u_1)\in H^1(\Omega)\times L^2(\Omega)$ in the case of Neumann boundary conditions. Here "well-posedness" refers to weak solutions, while strong solutions exist under additional regularity assumptions on the initial data and on the forcing term h. The problem fits in the abstract framework if we set

$$F(u):=\frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)}-\frac{\lambda}{2}\|u\|^2_{L^2(\Omega)}+\frac{b}{\beta+2}\|u\|^{\beta+2}_{L^{\beta+2}(\Omega)},$$

and

$$\left[g(t,v)\right](t,x):=c\left|v(x)\right|^{\alpha}v(x)-\mu v(x)-h(t,x), \tag{29}$$

and we choose the functional spaces

$$H:=L^2(\Omega), \qquad X:=L^{\alpha+2}(\Omega), \qquad Y:=L^{\beta+2}(\Omega), \tag{30}$$

with $V:=H_0^1(\Omega)$ in the case of Dirichlet boundary conditions, and $V:=H^1(\Omega)$ in the case of Neumann boundary conditions. The verification of (F1) and (G1) is quite straightforward with this choice of the functional spaces.

Assumption (F2): Inequality (17) holds true, both in the Neumann and in the Dirichlet case, because of the super-quadratic power $\beta+2$.

Assumption (F3): Inequality (18) is always true with $\delta_2=2$ and $C_2=0$, for every admissible value of the parameters.

Assumption (G2): From (29) it follows that

$$\begin{aligned}\left\langle g(t,v),v\right\rangle_{V',V}&=c\|v\|^{\alpha+2}_{L^{\alpha+2}(\Omega)}-\mu\|v\|^2_{L^2(\Omega)}-\int_\Omega h(t,x)v(x)\,\mathrm{d}x\\ &\ge c\|v\|^{\alpha+2}_{L^{\alpha+2}(\Omega)}-\mu\|v\|^2_{L^2(\Omega)}-\left\|h(t,x)\right\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}.\end{aligned}$$

At this point inequality (19) follows from the imbedding $L^{\alpha+2}(\Omega)\subseteq L^2(\Omega)$, and from the super-quadratic growth of the power $\alpha+2$.

Assumption (G3): Setting for simplicity $\sigma:=(\alpha+2)/(\alpha+1)$, we observe that $X'=L^\sigma(\Omega)$.

From (29) we deduce that

$$\begin{aligned}\left|\left[g(t,v)\right](t,x)\right|&\le c\left|v(x)\right|^{\alpha+1}+|\mu|\cdot\left|v(x)\right|+\left|h(t,x)\right|\\ &\le K_1\left|v(x)\right|^{\alpha+1}+K_2+\left|h(t,x)\right|.\end{aligned} \tag{31}$$

Since $\sigma\le 2$, when we compute the norm in $L^\sigma(\Omega)$ we obtain that

$$\left\|K_2+\left|h(t,x)\right|\right\|_{L^\sigma(\Omega)}\le K_3+\|h\|_{L^\sigma(\Omega)}\le K_3+K_4\|h\|_{L^2(\Omega)}\le K_5,$$

and

$$\left\|K_1\left|v(x)\right|^{\alpha+1}\right\|_{L^\sigma(\Omega)}=K_1\|v\|^{\alpha+1}_{L^{\alpha+2}(\Omega)}.$$

Plugging these two estimates into (31) we obtain (20). Under the assumptions described above, and in particular in the regime $0<\alpha<\beta$, with β satisfying (28), in both Neumann and Dirichlet cases, there exist two positive real numbers Γ and Γ_* such that any weak solution in $(0,T)$ satisfies

$$\|u_t\|^2_{L^2(\Omega)}+\|\nabla u\|^2_{L^2(\Omega)}+\|u\|^{\beta+2}_{L^{\beta+2}(\Omega)}\le\Gamma\,t^{-1/\gamma}+\Gamma_*, \quad \forall\ t\in(0,T),$$

where γ is defined by (21).

Remark 7. When $\alpha \geq \beta$, it is very likely, considering the results of Souplet [8] that no universal bound exists for any finite value $t > 0$. This is clear in the case of Neumann BC, and less obvious in the case of Dirichlet BC. In both cases, it would be interesting to study the shape of $S(t)X$ where $X = V \times H$, since this set is likely to shrink for large values of t, and even for all $t > 0$ in a sense to be specified.

Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

References

[1] M. Abdelli, M. Anguiano and A. Haraux, "Existence, uniqueness and global behavior of the solutions to some nonlinear vector equations in a finite dimensional Hilbert space", *Nonlinear Anal., Theory Methods Appl.* **161** (2017), pp. 157–181.

[2] M. Abdelli and A. Haraux, "Global behavior of the solutions to a class of nonlinear, singular second order ODE", *Nonlinear Anal., Theory Methods Appl.* **96** (2014), pp. 18–37.

[3] M. Abdelli and A. Haraux, "The universal bound property for a class of second order ODEs", *Port. Math.* **76** (2019), no. 1, pp. 49–56.

[4] M. Ghisi, M. Gobbino and A. Haraux, "Universal bounds for a class of second order evolution equations and applications", *J. Math. Pures Appl. (9)* **142** (2020), pp. 184–203.

[5] A. Haraux, "Remarks on the wave equation with a nonlinear term with respect to the velocity", *Port. Math.* **49** (1992), no. 4, pp. 447–454.

[6] A. Haraux, "Sharp decay estimates of the solutions to a class of nonlinear second order ODE's", *Anal. Appl., Singap.* **9** (2011), no. 1, pp. 49–69.

[7] J. Simon, "Quelques propriétés de solutions d'équations et d'inéquations d'évolution paraboliques non linéaires", *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4)* **2** (1975), no. 4, pp. 585–609.

[8] P. Souplet, "Existence of exceptional growing-up solutions for a class of non-linear second order ordinary differential equations", *Asymptotic Anal.* **11** (1995), no. 2, pp. 185–207.

[9] P. Souplet, "Critical exponents, special large-time behavior and oscillatory blow-up in non-linear ODE's", *Differ. Integral Equ.* **11** (1998), no. 1, pp. 147–167.