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A new proof of a Liouville theorem for the one dimensional Gross–Pitaevskii equation

Une nouvelle preuve d'un théorème de Liouville pour l'équation de Gross–Pitaevskii en dimension un

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In the memory of Haim Brezis, with respect and admiration

Abstract. The asymptotic stability of the black and dark solitons of the one-dimensional Gross–Pitaevskii equation was proved by Béthuel, Gravejat and Smets (*Ann. Sci. Éc. Norm. Supér.*, 2015), and Gravejat and Smets (*Proc. Lond. Math. Soc.*, 2015), using a rigidity property in the vicinity of solitons. We provide an alternate proof of the Liouville theorems in those two articles, using a factorization identity for the linearized operator which trivializes the spectral analysis.

Résumé. La stabilité asymptotique des solitons de l'équation de Gross–Pitaevskii en dimension un a été démontrée par Béthuel, Gravejat et Smets (*Ann. Sci. Éc. Norm. Supér.*, 2015), et Gravejat et Smets (*Proc. Lond. Math. Soc.*, 2015), à l'aide d'une propriété de rigidité dans le voisinage d'un soliton. On donne une nouvelle démonstration des théorèmes de Liouville contenus dans ces articles, utilisant une identité de factorisation pour l'opérateur linéarisé qui rend triviale l'analyse spectrale du problème.

Keywords. Solitons, Gross–Pitaevskii equation, asymptotic stability.

Mots-clés. Solitons, équation de Gross–Pitaevskii, stabilité asymptotique.

2020 Mathematics Subject Classification. 35L71, 35B40, 37K40.

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1. Introduction

We consider the one-dimensional Gross–Pitaevskii equation

$$\mathrm{i}\partial_t\psi + \partial_x^2\psi + \psi\big(1-|\psi|^2\big) = 0, \quad (t,x)\in\mathbb{R}^2, \tag{GP}$$

for a function $\psi\colon (t,x)\in\mathbb{R}^2 \mapsto \psi(t,x)\in\mathbb{C}$, with the condition $\lim_{|x|\to\infty}\big|\psi(t,x)\big| = 1$. For a solution of (GP), the Hamiltonian is formally conserved

$$E(\psi) = \frac{1}{2}\int(\partial_x\psi)^2 + \frac{1}{4}\int\big(1-|\psi|^2\big)^2.$$

Using the notation $\eta = 1-|\psi|^2$, it is natural to define the energy space as follows

$$\mathscr{E} = \big\{\psi\in\mathscr{C}(\mathbb{R};\mathbb{C}) : \psi'\in L^2(\mathbb{R}) \text{ and } \eta\in L^2(\mathbb{R})\big\}.$$

We denote

$$\|f\| := \|f\|_{L^2}, \qquad \|f\|_\rho = \|\rho^{\frac{1}{2}} f\|, \qquad \|f\|_{\mathscr{H}} = \big(\|f'\|^2 + \|f\|_\rho^2\big)^{\frac{1}{2}}, \qquad \rho(x) = \operatorname{sech}(x).$$

Following [3], we equip the energy space $\mathscr{E}$ with the distance

$$d(\psi_1,\psi_2) = \big(\|\psi_1-\psi_2\|_{\mathscr{H}}^2 + \|\eta_1-\eta_2\|^2\big)^{\frac{1}{2}}$$

so that $(\mathscr{E}, d)$ is a complete metric space. Recall from [2,3,8] that the Cauchy problem is globally well-posed in $\mathscr{E}$: for any $\psi_0\in\mathscr{E}$, there exists a unique global solution $\psi\in\mathscr{C}(\mathbb{R},\mathscr{E})$ of (GP) with $\psi(0) = \psi_0$.

It is well-known that for any velocity $c\in(-\sqrt{2},\sqrt{2})$, there exists a nontrivial traveling wave solution $\psi(t,x) = U_c(t-cx)$ to this problem where $U_c(x)$ is the solution of

$$-\mathrm{i}cU_c' + U_c'' + U_c\big(1-|U_c|^2\big) = 0 \quad \text{on } \mathbb{R}$$

explicitly given by the formula

$$U_c = R_c + \mathrm{i}I_c, \qquad R_c(x) = \sqrt{\frac{2-c^2}{2}}\tanh\left(\frac{\sqrt{2-c^2}}{2}x\right), \qquad I_c = \frac{c}{\sqrt{2}}.$$

In case $c\neq 0$, the traveling wave solution is called dark soliton and in case $c=0$, it is called black soliton. The orbital stability of both kinds of solitons was proved in a satisfactory functional setting; we refer to [1, Theorem 1] and [3, Theorem 1].

We are interested in the question of asymptotic stability of the family of traveling waves in the framework developed by Béthuel, Gravejat, Smets [1], and Gravejat, Smets [3]. From those articles, their approach relies on the following Liouville theorem for smooth solutions of (GP) that are close to a traveling wave and uniformly localized in space.

Theorem 1 ([1,3]). *Let $c_0\in(-\sqrt{2},\sqrt{2})$. Let $M>0$ and $\gamma>0$. There exists $\alpha_0>0$ such that if a solution $\psi\in\mathscr{C}(\mathbb{R};\mathscr{E})$ of* (GP) *satisfies $\psi\in\mathscr{C}^\infty(\mathbb{R}\times\mathbb{R})$, $d(\psi(0),U_{c_0})\leq\alpha_0$ and*

$$\inf_{a\in\mathbb{R}}\left\{\big|\eta(t,x+a)\big| + \sum_{k=1,2,3}\big|\partial_x^k\psi(t,x+a)\big|\right\}\leq Me^{-\gamma|x|}$$

then there exist $c_1\in(-\sqrt{2},\sqrt{2})$ and $a_1\in\mathbb{R}$ such that $\psi(t,x) = U_{c_1}(x-c_1t+a_1)$ on $\mathbb{R}\times\mathbb{R}$.

In this note, we provide an alternate proof of Theorem 1, inspired by recent articles on the asymptotic stability of solitons for the nonlinear Klein–Gordon equation [5] and the nonlinear Schrödinger equation [6]. As in those articles, we introduce a factorization identity of the linearized operator around a traveling wave which leads to a transformed problem with trivial spectral properties. We refer to Lemma 3 for the identity and to Remark 4 for heuristics. The identity obtained in Lemma 3 is certainly related to the integrability of the model. However, as shown in [6,7] for nonlinear Schrödinger models, it is possible to extend the arguments to less specific situations. Therefore, we hope that the factorization will be useful elsewhere.

Finally, we expect that our approach will enable a direct proof of the asymptotic stability for any $c \in (-\sqrt{2}, \sqrt{2})$, as in [4–6].

2. Stability and modulation

For $c \in (-\sqrt{2}, \sqrt{2})$, we set $\beta = \sqrt{2-c^2} > 0$ and we define

$$Q_c = 1 - |U_c|^2 = \sqrt{2} R_c' = \frac{\beta^2}{2} \operatorname{sech}^2\left(\frac{\beta x}{2}\right), \qquad Q_c'' - \beta^2 Q_c + 3Q_c^2 = 0.$$

Lemma 2. *Under the assumptions of Theorem 1, for $\alpha_0 > 0$ sufficiently small, there exist functions $a \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$, $c \in \mathscr{C}^\infty\big(\mathbb{R}, (-\sqrt{2}, \sqrt{2})\big)$, $\theta \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$ such that*

$$\psi(t,x) = e^{\mathrm{i}\theta(t)}\Big(U_{c(t)}\big(x - a(t)\big) + \varepsilon\big(t, x - a(t)\big)\Big)$$

where $\varepsilon \in \mathscr{C}^\infty(\mathbb{R}\times\mathbb{R})$ satisfies the orthogonality conditions

$$\int \Re(U_c' \bar\varepsilon) = \int \Re(\mathrm{i} U_c' \bar\varepsilon) = \int \Re(\mathrm{i} R_c Q_c \bar\varepsilon) = 0$$

and the equation

$$\mathrm{i}\partial_t \varepsilon + \partial_x^2 \varepsilon - \mathrm{i}c\partial_x \varepsilon + q = \Omega$$

with

$$q = \big(1 - |U_c + \varepsilon|^2\big)(U_c + \varepsilon) - \big(1 - |U_c|^2\big) U_c,$$
$$\Omega = \mathrm{i}(\dot a - c)(U_c' + \partial_x \varepsilon) - \mathrm{i}\dot c \partial_c U_c + \dot\theta (U_c + \varepsilon).$$

Moreover, setting $\zeta(t,x) = Q_c(x) - \eta\big(t, x + a(t)\big)$, there exists $\gamma \in (0, \beta/2)$ such that, for all $t \in \mathbb{R}$,

$$\big\|\varepsilon(t)\big\|_{\mathscr{H}} + \big\|\zeta(t)\big\| + \big|c(t) - c_0\big| \lesssim \alpha_0,$$
$$\big|\dot a(t) - c(t)\big|^2 + \big|\dot c(t)\big| + \big|\dot\theta(t)\big|^2 \lesssim \int \rho^\gamma \big|\varepsilon(t)\big|^2,$$
$$\big|\zeta(t)\big| + \sum_{k=1,2,3} \big|\partial_x^k \varepsilon(t)\big| \lesssim \rho^\gamma.$$

We shall not reproduce here the proof of Lemma 2. See [1,3]. The bound on the time derivative of the parameters a, c and θ is deduced from the proofs in [1,3]. Note that only $|\dot c|$ is quadratic in ε.

We denote $\varepsilon = \varepsilon_1 + \mathrm{i}\varepsilon_2$, $q = q_1 + \mathrm{i} q_2$ and $\Omega = \Omega_1 + \mathrm{i}\Omega_2$. The orthogonality conditions become

$$\int Q_c \varepsilon_1 = \int Q_c \varepsilon_2 = \int R_c Q_c \varepsilon_2 = 0.$$

We observe from its definition $\zeta = Q_c - \big(1 - |U_c + \varepsilon|^2\big)$ that

$$\zeta = Q_c - \big(1 - |U_c|^2 - 2\Re(U_c \varepsilon) - |\varepsilon|^2\big) = 2R_c \varepsilon_1 + c\sqrt{2}\varepsilon_2 + |\varepsilon|^2.$$

It holds $q_1 = Q_c \varepsilon_1 - R_c \zeta - \zeta \varepsilon_1$ and $q_2 = Q_c \varepsilon_2 - \frac{c}{\sqrt{2}}\zeta - \zeta\varepsilon_2$. Thus

$$\partial_t \varepsilon_1 = -\partial_x^2 \varepsilon_2 + c\partial_x \varepsilon_1 - Q_c \varepsilon_2 + \frac{c}{\sqrt{2}}\zeta + \zeta\varepsilon_2 + \Omega_2,$$
$$\partial_t \varepsilon_2 = \partial_x^2 \varepsilon_1 + c\partial_x \varepsilon_2 + Q_c \varepsilon_1 - R_c \zeta - \zeta \varepsilon_1 - \Omega_1.$$

Define

$$L_+ = -\partial_x^2 + \beta^2 - 3Q_c, \qquad L_- = -\partial_x^2 + c^2 - Q_c,$$
$$S_c = \partial_x + \sqrt{2} R_c = Q_c \cdot \partial_x \cdot Q_c^{-1}, \qquad S_c^\star = -\partial_x + \sqrt{2} R_c = -Q_c^{-1} \cdot \partial_x \cdot Q_c.$$

Using $2R_c^2 = \beta^2 - 2Q_c$, we obtain

$$\begin{cases} \partial_t \varepsilon_1 = L_- \varepsilon_2 + cS_c \varepsilon_1 + N_2 + \Omega_2 \\ \partial_t \varepsilon_2 = -L_+ \varepsilon_1 - cS_c^\star \varepsilon_2 - N_1 - \Omega_1 \end{cases}$$

where

$$N_1 = -R_c|\varepsilon|^2 - 2R_c\varepsilon_1^2 - \sqrt{2}c\varepsilon_1\varepsilon_2 - \varepsilon_1|\varepsilon|^2,$$
$$N_2 = \frac{c}{\sqrt{2}}|\varepsilon|^2 + 2R_c\varepsilon_1\varepsilon_2 + \sqrt{2}c\varepsilon_2^2 + \varepsilon_2|\varepsilon|^2,$$

and

$$\Omega_1 = -(\dot a - c)\partial_x\varepsilon_2 + \frac{\dot c}{\sqrt{2}} + \dot\theta(R_c+\varepsilon_2),$$
$$\Omega_2 = (\dot a - c)(R_c' + \partial_x\varepsilon_1) - \dot c\partial_c R_c + \dot\theta\left(\frac{c}{\sqrt{2}} + \varepsilon_2\right).$$

3. The transformed problem

3.1. *Factorization*

We state the key lemma of this note.

Lemma 3. *For any $c\in(-\sqrt{2},\sqrt{2})$, it holds $L_+ = S_c^\star S_c$ and $S_c(L_- - c^2)S_c^\star = (\partial_x^2-\beta^2)\partial_x^2$.*

Remark 4. To illustrate heuristically the interest of the above identities, observe that for $c=0$ (to simplify the exposition), setting $w_1 = S_0\varepsilon_1$ and $\varepsilon_2 = S_0^\star w_2$, one obtains formally

$$\begin{cases}\partial_t\varepsilon_1 = L_-\varepsilon_2\\ \partial_t\varepsilon_2 = -L_+\varepsilon_1\end{cases} \iff \begin{cases}\partial_t w_1 = (\partial_x^2-2)\partial_x^2 w_2\\ \partial_t w_2 = -w_1.\end{cases}$$

Being without any potential, the system for (w_1,w_2) is simpler; see Lemma 11.

Proof. First, the identity $L_+ = S_c^\star S_c$ is standard and related to the fact that $L_+Q_c = 0$. Second, we recall that $L_- - c^2 = -\partial_x^2 - \sqrt{2}R_c'$ and so, using $R_c'' = -\sqrt{2}R_c'R_c$,

$$\begin{aligned}(L_- - c^2)S_c^\star &= (-\partial_x^2 - \sqrt{2}R_c')(-\partial_x + \sqrt{2}R_c)\\ &= \partial_x^3 - \sqrt{2}R_c\partial_x^2 - 2\sqrt{2}R_c'\partial_x - \sqrt{2}R_c'' + \sqrt{2}R_c'\partial_x - 2R_c'R_c = A\partial_x\end{aligned}$$

where $A = \partial_x^2 - \sqrt{2}R_c\partial_x - \sqrt{2}R_c'$. Thus,

$$S_c(L_- - c^2)S_c^\star = S_cA\partial_x.$$

We calculate, using $2R_c^2 = \beta^2 - 2\sqrt{2}R_c'$ and $R_c'' = -\sqrt{2}R_c'R_c$,

$$\begin{aligned}S_cA &= (\partial_x + \sqrt{2}R_c)(\partial_x^2 - \sqrt{2}R_c\partial_x - \sqrt{2}R_c')\\ &= \partial_x^3 - 2\sqrt{2}R_c'\partial_x - \sqrt{2}R_c'' - 2R_c^2\partial_x - 2R_cR_c' = (\partial_x^2-\beta^2)\partial_x\end{aligned}$$

which finishes the proof of the second identity. □

In view of the definition of ε_2 in Remark 4, we will need to invert the operator $S_c^\star$.

Lemma 5. *For any $f\in W^{1,\infty}(\mathbb{R})$ such that $\int fQ_c = 0$, the function $g\in W^{2,\infty}(\mathbb{R})$ defined by*

$$g(x) = \frac{1}{Q_c(x)}\int_x^\infty fQ_c = -\frac{1}{Q_c(x)}\int_{-\infty}^x fQ_c$$

satisfies $S_c^\star g = f$. Moreover,

$$|g(x)| + |g'(x)| \lesssim \sup_{|y|\ge|x|}|f(y)|, \qquad |g'(x)| + |g''(x)| \lesssim \sup_{|y|\ge|x|}|f'(y)| + Q_c(x)|f(x)|$$

and, for any $0<\kappa<2\beta$,

$$\|\rho^\kappa g\| + \|\rho^\kappa g'\| \lesssim \|\rho^{\frac{\kappa}{2}} f\|.$$

Remark 6. If $S_c^\star g = 0$, then $Q_c g = C$, where C is a constant. Thus, for a given $f\in W^{1,\infty}(\mathbb{R})$, the function g defined in Lemma 5 is the only bounded solution of $S_c^\star g = f$.

Proof. Using $S_c^\star = -Q_c^{-1}\cdot\partial_x\cdot Q_c$, we check that $S_c^\star g = f$. Moreover, the estimate $\left|g(x)\right| + \left|g'(x)\right| \lesssim \sup_{|y|\geq|x|}\left|f(y)\right|$ follows easily from the definition of g and

$$g' = -\frac{Q_c'}{Q_c^2}\int_x^\infty fQ_c - f.$$

Integrating by parts, we also get for $x>0$,

$$g'(x) = -\frac{Q_c'}{Q_c^2}\int_x^\infty f'(y)\int_y^\infty Q_c \mathrm{d}y + k_c(x)f(x) \quad \text{where} \quad k_c(x) = -\frac{Q_c'}{Q_c^2}\int_x^\infty Q_c - 1.$$

Replacing Q_c by its explicit expression, we obtain $k_c(x) = -e^{-\beta x}$. Proceeding similarly for $x<0$, we obtain the pointwise estimate on g'. To obtain the pointwise estimate on g'', it suffices to differentiate the expression for g' and to use similar estimates.

Now, we observe that for $0\lesssim\kappa<2\beta$ and for all $x>0$,

$$\begin{aligned}\rho^{2\kappa}g^2+\rho^{2\kappa}(g')^2 &\lesssim \frac{\rho^{2\kappa}}{Q_c^2}\left(\int_x^\infty fQ_c\right)^2+\rho^{2\kappa}f^2\\ &\lesssim \frac{\rho^{2\kappa}}{Q_c^2}\left(\int_x^\infty \rho^{-\kappa}Q_c^2\right)\left(\int \rho^{\kappa}f^2\right)+\rho^{2\kappa}f^2\\ &\lesssim \rho^{\kappa}\left(\int\rho^\kappa f^2\right)+\rho^{2\kappa}f^2.\end{aligned}$$

This, together with the analogue estimate for $x<0$, implies $\|\rho^\kappa g\|+\|\rho^\kappa g'\|\lesssim\|\rho^{\frac{\kappa}{2}}f\|$ by integrating in x. □

3.2. *First change of variables*

We set $v_1 = S_c\varepsilon_1$. Then, we use Lemma 5 and the orthogonality $\int\varepsilon_2 Q_c = 0$ to define a smooth and bounded function w_2 such that $S_c^\star w_2 = \varepsilon_2$. We determine the equations for v_1 and w_2. First,

$$\partial_t v_1 = S_cL_-S_c^\star w_2 + cS_cv_1 + P_2 + \Theta_2, \qquad P_2 = S_cN_2 + \sqrt{2}\dot{c}\varepsilon_1\partial_cR_c, \qquad \Theta_2 = S_c\Omega_2.$$

From $S_c^\star w_2 = \varepsilon_2$ and $L_+ = S_c^\star S_c$, we find

$$\begin{aligned}\partial_tS_c^\star w_2 &= S_c^\star\partial_t w_2 + \sqrt{2}\dot{c}w_2\partial_cR_c\\ &= -L_+\varepsilon_1 - cS^\star\varepsilon_2 - N_1 - \Omega_1\\ &= -S_c^\star(v_1 + cS_c^\star w_2) - N_1 - \Omega_1,\end{aligned}$$

hence, rearranging

$$S_c^\star(\partial_t w_2 + v_1 + cS_c^\star w_2) = -W - N_1 - \Omega_1, \qquad W = \sqrt{2}\dot{c}w_2\partial_cR_c.$$

Since the left-hand side of the above identity is orthogonal to Q_c, we have

$$-W - N_1 - \Omega_1 = -W^\perp - N_1^\perp - \Omega_1^\perp,$$

where $f^\perp = f - \frac{Q_c}{\|Q_c\|^2}\int fQ_c$. Moreover, by Lemma 5 we can define P_1 and Θ_1 such that

$$S_c^\star P_1 = -W^\perp - N_1^\perp, \qquad S_c^\star\Theta_1 = -\Omega_1^\perp.$$

We obtain

$$\begin{aligned}\partial_t v_1 &= S_cL_-S_c^\star w_2 + cS_cv_1 + P_2 + \Theta_2,\\ \partial_t w_2 &= -v_1 - cS_c^\star w_2 + P_1 + \Theta_1.\end{aligned}$$

3.3. *Second change of variables*

We define

$$w_1 = v_1 + cS_c w_2 + \frac{1}{\sqrt{2}}|\varepsilon|^2.$$

Since $S_c^\star w_2 = \varepsilon_2$, we have $S_c w_2 = \varepsilon_2 + 2\partial_x w_2$ and thus, using $\zeta = 2R_c\varepsilon_1 + c\sqrt{2}\varepsilon_2 + |\varepsilon|^2$,

$$w_1 = v_1 + c\varepsilon_2 + 2c\partial_x w_2 + \frac{1}{\sqrt{2}}|\varepsilon|^2 = \partial_x\varepsilon_1 + 2c\partial_x w_2 + \frac{\zeta}{\sqrt{2}}.$$

By Lemma 2, the last expression shows that the function w_1 has exponential decay at infinity, unlike the function v_1 (see Lemma 10 below).

First, we compute using the definition of w_1 and then Lemma 5

$$\begin{aligned}\partial_t w_1 &= \partial_t v_1 + \dot{c} S_c w_2 + \sqrt{2}c\dot{c}w_2\partial_c R_c + cS_c\partial_t w_2 + \frac{1}{\sqrt{2}}\partial_t|\varepsilon|^2\\ &= S_c L_- S_c^\star w_2 - c^2 S_c S_c^\star w_2 + F_2\\ &= (\partial_x^2 - \beta^2)\partial_x^2 w_2 + F_2\end{aligned}$$

where

$$F_2 = P_2 + \Theta_2 + \dot{c}S_c w_2 + \sqrt{2}c\dot{c}w_2\partial_c R_c + cS_c P_1 + cS_c\Theta_1 + \frac{1}{\sqrt{2}}\partial_t|\varepsilon|^2.$$

Second, using $v_1 = w_1 - cS_c w_2 - \frac{1}{\sqrt{2}}|\varepsilon|^2$ and $S_c - S_c^\star = 2\partial_x$, we obtain

$$\partial_t w_2 = -w_1 + 2c\partial_x w_2 + F_1 \quad \text{where} \quad F_1 = P_1 + \Theta_1 + \frac{1}{\sqrt{2}}|\varepsilon|^2.$$

Summarizing, the transformed problem is defined by

$$\begin{cases}\partial_t w_1 = (\partial_x^2 - \beta^2)\partial_x^2 w_2 + F_2\\ \partial_t w_2 = -w_1 + 2c\partial_x w_2 + F_1.\end{cases}$$

4. Technical lemmas

Lemma 7. *For any $\kappa > 0$ and any $f \in \mathscr{E}$,*

$$\|f\rho^\kappa\|_{L^\infty} \lesssim \|f\|_{\mathscr{H}}$$

and for any $A > 0$,

$$\int_0^A f^2\rho^\kappa \lesssim \int_0^1 f^2 + \left(\int_0^A (f')^2\right)^{\frac{1}{2}}\left(\int_0^A f^2\rho^{2\kappa}\right)^{\frac{1}{2}}.$$

Proof. For $x, y \in \mathbb{R}$, $f(x) = f(y) + \int_y^x f'$ and thus by the Cauchy–Schwarz inequality

$$f^2(x) \lesssim f^2(y) + \big(|x| + |y|\big)\int (f')^2.$$

Multiplying by $\rho(y)$ and integrating in y, then multiplying by $\rho^{2\kappa}(x)$ and taking the supremum in x, we obtain the first inequality.

For the second inequality, we fix $A \geq 1$. For $0 \leq y \leq 1 \leq x \leq A$, we write

$$f^2(x) = f^2(y) + 2\int_y^x f'(z)f(z)\,\mathrm{d}z$$

so that multiplying by $\rho^\kappa(x)$ and integrating in $x\in[1,A]$,

$$\int_1^A f^2\rho^\kappa \lesssim f^2(y)+\int_1^A\left(\int_0^x |f'(z)||f(z)|\,\mathrm{d}z\right)\rho^\kappa(x)\,\mathrm{d}x$$
$$\lesssim f^2(y)+\int_0^A |f'(z)||f(z)|\left(\int_z^A \rho^\kappa(x)\,\mathrm{d}x\right)\mathrm{d}z$$
$$\lesssim f^2(y)+\int_0^A |f'||f|\rho^\kappa.$$

Thus, by the Cauchy–Schwarz inequality,

$$\int_1^A f^2\rho^\kappa \lesssim f^2(y)+\left(\int_0^A (f')^2\right)^{\frac12}\left(\int_0^A f^2\rho^{2\kappa}\right)^{\frac12}.$$

Integrating in $y\in[0,1]$ yields the estimate for $\int_1^A f^2\rho^\kappa$. The estimate for $\int_0^1 f^2\rho^\kappa$ is clear. □

Now, we define

$$\mathcal{N}=\left(\int w_1^2+\int(\partial_x w_2)^2+\int(\partial_x^2 w_2)^2\right)^{\frac12}.$$

Lemma 8. *For any* $\kappa>0$,

$$\int(\partial_x\varepsilon_2)^2+\int\varepsilon_2^2\rho^\kappa+\|\varepsilon_2^2\rho^\kappa\|_{L^\infty}+\int w_2^2\rho^\kappa\lesssim\mathcal{N}^2.$$

Moreover,

$$\left|\partial_x w_2(x)\right|+\left|\partial_x^2 w_2(x)\right|\lesssim\sup_{|y|\ge|x|}\left|\partial_x\varepsilon_2(y)\right|+Q_c(x)\left|\varepsilon_2(x)\right|\lesssim\rho^\gamma.$$

Lemma 9. *For any* $\kappa>0$,

$$\int(\partial_x\varepsilon_1)^2\rho^\kappa+\int\varepsilon_1^2\rho^\kappa+\|\varepsilon_1^2\rho^\kappa\|_{L^\infty}\lesssim\mathcal{N}^2.$$

Moreover,

$$\left|w_1(x)\right|\lesssim\left|\partial_x\varepsilon_1(x)\right|+\sup_{|y|\ge|x|}\left|\partial_x\varepsilon_2(y)\right|+Q_c(x)\left|\varepsilon_2(x)\right|+\left|\zeta(x)\right|\lesssim\rho^\gamma.$$

Lemma 10. *It holds*

$$|\dot a-c|^2+|\dot c|+|\dot\theta|^2\lesssim\mathcal{N}^2,\qquad \|\rho^{\frac{\gamma}{4}}\partial_x F_1\|\lesssim\mathcal{N}^2.$$

Moreover, $F_2=F_{2,1}+\partial_x F_{2,2}$ *where*

$$\|\rho^{\frac{\gamma}{4}}F_{2,1}\|+\|\rho^{\frac{\gamma}{4}}F_{2,2}\|\lesssim\mathcal{N}^2.$$

Proof of Lemma 8. The orthogonality $\int\varepsilon_2 Q_c=0$ was already used to construct w_2 in Section 3.2. Now, we use the second orthogonality relation on ε_2, which is $\int\varepsilon_2 R_c Q_c=0$. Indeed, using $Q_c\cdot S_c^\star=-\partial_x\cdot Q_c$, it implies that

$$0=\int\varepsilon_2 R_c Q_c=\int(S_c^\star w_2)R_cQ_c=-\int R_c\partial_x(w_2Q_c)=\frac{1}{\sqrt2}\int Q_c^2 w_2.$$

From this orthogonality relation, it is standard to prove the inequality $\int w_2^2\rho^\kappa\lesssim\int(\partial_x w_2)^2$.

From $\varepsilon_2=S_c^\star w_2$, we obtain $|\varepsilon_2|\lesssim|\partial_x w_2|+|w_2|$ and so

$$\int\varepsilon_2^2\rho^\kappa\lesssim\int\left((\partial_x w_2)^2+w_2^2\right)\rho^\kappa\lesssim\int(\partial_x w_2)^2.$$

Moreover, differentiating, we have $\partial_x\varepsilon_2=-\partial_x^2 w_2+\sqrt2 R_c\partial_x w_2+Q_c w_2$, and so

$$\int(\partial_x\varepsilon_2)^2\lesssim\int(\partial_x^2 w_2)^2+\int(\partial_x w_2)^2+\int w_2^2\rho^\kappa.$$

It is standard to obtain the L^∞ bound from the above.

The second estimate of the lemma follows from Lemmas 2 and 5. □

Proof of Lemma 9. By the definition of w_1 and $v_1 = S_c\varepsilon_1$, we have

$$S_c\varepsilon_1 = Q_c\partial_x(\varepsilon_1/Q_c) = -\frac{1}{\sqrt{2}}\varepsilon_1^2 + h \quad \text{where} \quad h = w_1 - cS_c w_2 - \frac{1}{\sqrt{2}}\varepsilon_2^2.$$

By integration on $[0,x]$, we have

$$\varepsilon_1 = bQ_c + Q_c\int_0^x \frac{\varepsilon_1^2}{Q_c} + Q_c\int_0^x \frac{h}{Q_c}$$

for some integration constant b. Using the orthogonality relation $\int \varepsilon_1 Q_c = 0$, we obtain

$$b\int Q_c^2 = -\int Q_c^2\int_0^x \frac{\varepsilon_1^2}{Q_c} - \int Q_c^2\int_0^x \frac{h}{Q_c}$$

and so by the Fubini theorem and the Cauchy–Schwarz inequality

$$|b| \lesssim \int \varepsilon_1^2 Q_c + \left(\int h^2 Q_c^{\frac{3}{2}}\right)^{\frac{1}{2}}.$$

Using Lemma 7, since $\int(\partial_x\varepsilon_1)^2 \lesssim \alpha_0$ and $\int_0^1 \varepsilon_1^2 \lesssim \int \varepsilon_1^2 Q_c^2 \lesssim \alpha_0$ (Lemma 2), we obtain

$$b^2 \lesssim \alpha_0\int \varepsilon_1^2 Q_c^2 + \int h^2 Q_c^{\frac{3}{2}}.$$

Then we estimate

$$\int \varepsilon_1^2\rho^\kappa \lesssim b^2\int \rho^\kappa Q_c^2 + \int \rho^\kappa Q_c^2\left(\int_0^x \frac{\varepsilon_1^2}{Q_c}\right)^2 + \int \rho^\kappa Q_c^2\left(\int_0^x \frac{h}{Q_c}\right)^2.$$

For the second term on the right-hand side, we use Lemma 7 to obtain

$$\begin{aligned}
\int \rho^\kappa Q_c^2\left(\int_0^x \frac{\varepsilon_1^2}{Q_c}\right)^2 &= \int \rho^\kappa Q_c^2\left(\int_0^x \frac{\varepsilon_1^2\rho^{\frac{\kappa}{4}}}{Q_c\rho^{\frac{\kappa}{4}}}\right)^2 \\
&\le \int \rho^{\frac{\kappa}{2}}\left(\int_0^x \varepsilon_1^2\rho^{\frac{\kappa}{4}}\right)^2 \\
&\lesssim \left(\int \varepsilon_1^2\rho^\kappa\right)^2 + \left(\int(\partial_x\varepsilon_1)^2\right)\int \rho^{\frac{\kappa}{2}}\left|\int_0^x \varepsilon_1^2\rho^{\frac{\kappa}{2}}\right| \\
&\lesssim \alpha_0\int \varepsilon_1^2\rho^\kappa.
\end{aligned}$$

Besides,

$$\int \rho^\kappa Q_c^2\left(\int_0^x \frac{h}{Q_c}\right)^2 \lesssim \int h^2\rho^{\frac{\kappa}{2}}.$$

Combining the previous estimates, assuming $\kappa \le \beta/2$, we have obtained

$$\int \varepsilon_1^2\rho^\kappa \lesssim \alpha_0\int \varepsilon_1^2\rho^\kappa + \int h^2\rho^{\frac{\kappa}{2}}$$

and thus for α_0 small enough, we have proved $\int \varepsilon_1^2\rho^\kappa \lesssim \int h^2\rho^{\frac{\kappa}{2}}$. Using the expression of h,

$$\int \varepsilon_1^2\rho^\kappa \lesssim \int w_1^2 + \int(\partial_x w_2)^2 + \int w_2^2\rho^{\frac{\kappa}{2}} + \int \varepsilon_2^4\rho^{\frac{\kappa}{2}}.$$

Using Lemma 7, we have $\|\varepsilon_2\rho^{\kappa/8}\|_{L^\infty} \lesssim \|\varepsilon_2\|_{\mathcal{H}} \lesssim 1$ and so by the proof of Lemma 8, we obtain

$$\int \varepsilon_1^2\rho^\kappa \lesssim \int w_1^2 + \int(\partial_x w_2)^2.$$

Using $\partial_x\varepsilon_1 = -w_1 + 2c\partial_x w_2 + \sqrt{2}R_c\varepsilon_1 + c\varepsilon_2 + \frac{1}{\sqrt{2}}|\varepsilon|^2$ we obtain

$$\int(\partial_x\varepsilon_1)^2\rho^\kappa \lesssim \int w_1^2 + \int(\partial_x w_2)^2.$$

To prove the second estimate, we recall that $w_1 = \partial_x \varepsilon_1 + 2c\partial_x w_2 + \frac{\zeta}{\sqrt{2}}$ so that

$$|w_1(x)| \lesssim |\partial_x \varepsilon_1(x)| + |\partial_x w_2(x)| + Q_c(x)|\varepsilon_2(x)| + |\zeta(x)|.$$

We finish the proof using Lemmas 2 and 8. □

Proof of Lemma 10. From Lemmas 2, 8 and 9, one has

$$|\dot{a} - c|^2 + |\dot{c}| + |\dot{\theta}|^2 \lesssim \int |\varepsilon|^2 \rho^\gamma \lesssim \mathcal{N}^2.$$

Estimate of $\partial_x F_1$. By the definition of F_1

$$\partial_x F_1 = \partial_x P_1 + \sqrt{2}(\partial_x \varepsilon_1)\varepsilon_1 + \sqrt{2}(\partial_x \varepsilon_2)\varepsilon_2 + \partial_x \Theta_1.$$

By the definition of P_1 and Lemma 5,

$$\|\rho^{\frac{\gamma}{4}} \partial_x P_1\| \lesssim \|\rho^{\frac{\gamma}{8}} W^\perp\| + \|\rho^{\frac{\gamma}{8}} N_1^\perp\| \lesssim \|\rho^{\frac{\gamma}{8}} W\| + \|\rho^{\frac{\gamma}{8}} N_1\|.$$

From Lemma 8, one has $\|\rho^{\frac{\gamma}{8}} w_2\| \lesssim \mathcal{N}$, and so by the definition of W, $|\partial_c R_c| \lesssim 1$ and $|\dot{c}| \lesssim \mathcal{N}^2$, it holds $\|\rho^{\frac{\gamma}{8}} W\| \lesssim \mathcal{N}^3$. Moreover, $|N_1| \lesssim |\varepsilon|^2$ and thus from Lemmas 8 and 9, it holds

$$\|\rho^{\frac{\gamma}{8}} N_1\| \lesssim \|\rho^{\frac{\gamma}{16}} \varepsilon\| \|\rho^{\frac{\gamma}{16}} \varepsilon\|_{L^\infty} \lesssim \mathcal{N}^2.$$

Therefore, $\|\rho^{\frac{\gamma}{4}} \partial_x P_1\| \lesssim \mathcal{N}^2$ is proved. Similarly, we see that

$$\left\|\rho^{\frac{\gamma}{4}} (\partial_x \varepsilon_1)\varepsilon_1\right\| + \left\|\rho^{\frac{\gamma}{4}} (\partial_x \varepsilon_2)\varepsilon_2\right\| \lesssim \mathcal{N}^2.$$

Now, we deal with $\partial_x \Theta_1$. We decompose $\Omega_1^\perp = \Omega_{1,1}^\perp + \Omega_{1,2}^\perp = -S_c^\star \Theta_{1,1} - S_c^\star \Theta_{1,2}$ where

$$\Omega_{1,1} = -(\dot{a} - c)\partial_x \varepsilon_2 + \frac{\dot{c}}{\sqrt{2}} + \dot{\theta}\varepsilon_2, \qquad \Omega_{1,2} = \Omega_{1,2}^\perp = \dot{\theta} R_c.$$

The term $\Theta_{1,2}$ could be problematic since $\Omega_{1,2} = \dot{\theta} R_c$ is linear in $\mathcal{N}$. However, since $S_c^\star 1 = \sqrt{2} R_c$, we have $\Theta_{1,2} = -\frac{1}{\sqrt{2}}\dot{\theta}$. Thus, $\partial_x \Theta_{1,2} = 0$ and this term actually has no contribution to $\partial_x F_1$. The terms in $\Omega_{1,1}$ are quadratic and as before, by Lemma 5, we have

$$\|\rho^{\frac{\gamma}{4}} \partial_x \Theta_{1,1}\| \lesssim \|\rho^{\frac{\gamma}{8}} \Omega_{1,1}^\perp\| \lesssim \|\rho^{\frac{\gamma}{8}} \Omega_{1,1}\| \lesssim \mathcal{N}^2.$$

Estimate of F_2. In the definition of F_2, we replace P_2 by its expression and we insert the expressions of $\partial_t \varepsilon_1$ and $\partial_t \varepsilon_2$:

$$\begin{aligned} F_2 = S_c N_2 + \sqrt{2}\dot{c}\varepsilon_1 \partial_c R_c + \Theta_2 + \dot{c} S_c w_2 + \sqrt{2} c\dot{c} w_2 \partial_c R_c + c S_c P_1 + c S_c \Theta_1 \\ + 2\sqrt{2}\varepsilon_1(-\partial_x^2 \varepsilon_2 + c\partial_x \varepsilon_1 - Q_c \varepsilon_2 + \frac{c}{\sqrt{2}}\zeta - \zeta\varepsilon_2 + \Omega_2) \\ + 2\sqrt{2}\varepsilon_2(\partial_x^2 \varepsilon_1 + c\partial_x \varepsilon_2 + Q_c \varepsilon_1 - R_c \zeta - \zeta\varepsilon_1 - \Omega_1). \end{aligned}$$

Then, replacing Θ_2 and $\Theta_{1,2}$ by their definitions, we split $F_2 = F_{2,1} + \partial_x F_{2,2}$ where

$$\begin{aligned} F_{2,1} = S_c N_2 + \sqrt{2}\dot{c}\varepsilon_1 \partial_c R_c + \dot{c} S_c w_2 + \sqrt{2} c\dot{c} w_2 \partial_c R_c + c S_c P_1 \\ - (\dot{a} - c)\sqrt{2} R_c \partial_x \varepsilon_1 - \dot{c} S_c \partial_c R_c + \dot{\theta} S_c \varepsilon_2 + c S_c \Theta_{1,1} \\ + \varepsilon_1(2c\zeta + 2\sqrt{2} c\zeta\varepsilon_2 + 2\sqrt{2}\Omega_2) - 2\sqrt{2}\varepsilon_2(R_c \zeta + \zeta\varepsilon_1 + \Omega_1) \end{aligned}$$

and

$$F_{2,2} = -(\dot{a} - c)\partial_x \varepsilon_1 + 2\sqrt{2}(\varepsilon_2 \partial_x \varepsilon_1 - \varepsilon_1 \partial_x \varepsilon_2 + c\varepsilon_1 \varepsilon_2).$$

Observe that for the terms Θ_2 and $c S_c \Theta_1$, we have used $S_c R_c' = 0$ and a cancellation of two terms in $\dot{\theta}$, exactly as for F_1. From the expressions of $F_{2,1}$ and $F_{2,2}$ above, in which all the terms are quadratic, we obtain similarly as before $\|\rho^{\frac{\gamma}{4}} F_{2,1}\| + \|\rho^{\frac{\gamma}{4}} F_{2,2}\| \lesssim \mathcal{N}^2$. □

5. Virial identity for the transformed problem

Set

$$\mathcal{I} = \int x(\partial_x w_2)(w_1 - c\partial_x w_2).$$

Note that by Lemmas 8 and 9, the functional $\mathcal{I}$ is well defined and uniformly bounded.

Lemma 11. *It holds* $\dot{\mathcal{I}} = \frac{1}{2}\mathcal{Q} + \mathcal{R}$ *where*

$$\mathcal{Q} = \int (w_1 - 2c\partial_x w_2)^2 + \beta^2 \int (\partial_x w_2)^2 + 3\int (\partial_x^2 w_2)^2,$$
$$\mathcal{R} = \int x(\partial_x w_2)F_2 + \int x(\partial_x F_1)(w_1 - 2c\partial_x w_2) - \dot{c}\int x(\partial_x w_2)^2.$$

Proof. We compute from the system for (w_1, w_2)

$$\begin{aligned}
\dot{\mathcal{I}} &= \int x(\partial_x\partial_t w_2)(w_1 - c\partial_x w_2) + \int x(\partial_x w_2)(\partial_t w_1 - c\partial_x\partial_t w_2) - \dot{c}\int x(\partial_x w_2)^2\\
&= \int x\big(\partial_x(-w_1 + 2c\partial_x w_2 + F_1)\big)(w_1 - c\partial_x w_2) + \int x(\partial_x w_2)\big((\partial_x^2 - \beta^2)\partial_x^2 w_2 + F_2\big)\\
&\quad - c\int x(\partial_x w_2)\partial_x(-w_1 + 2c\partial_x w_2 + F_1) - \dot{c}\int x(\partial_x w_2)^2\\
&= \mathcal{Q} + \mathcal{R}
\end{aligned}$$

where $\mathcal{R}$ is defined as in the statement of the lemma and

$$\begin{aligned}
\mathcal{Q} &= -2\int x\big(\partial_x(w_1 - 2c\partial_x w_2)\big)(w_1 - 2c\partial_x w_2) + 2\int x(\partial_x w_2)(\partial_x^2 - \beta^2)\partial_x^2 w_2\\
&= \int (w_1 - 2c\partial_x w_2)^2 + \beta^2\int (\partial_x w_2)^2 + 3\int (\partial_x^2 w_2)^2
\end{aligned}$$

using integrations by parts. □

Lemma 12. *It holds*

$$\mathcal{Q} \geq \frac{\beta^2}{16}\mathcal{N}^2.$$

Proof. Let $b = \sqrt{1 + 7c^2/2}$. We rewrite

$$\mathcal{Q} = \frac{\beta^2}{2b}\int w_1^2 + \frac{\beta^2}{2}\int (\partial_x w_2)^2 + 3\int (\partial_x^2 w_2)^2 + \int \left(\frac{2c}{b} w_1 - b\partial_x w_2\right)^2$$

which is sufficient to prove the result. □

6. Proof of the Liouville theorem

Lemma 13. *It holds*

$$|\mathcal{R}| \lesssim \mathcal{N}^3.$$

Remark 14. Formally, $\mathcal{R}$ contains quadratic terms and should satisfy an estimate of the form $|\mathcal{R}| \lesssim \mathcal{N}^4$. However, some loss is necessary to recover space decay by Lemmas 8 and 9.

Proof. We set $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3$, where

$$\mathcal{R}_1 = \int x(\partial_x w_2)F_2,$$
$$\mathcal{R}_2 = \int x(\partial_x F_1)(w_1 - 2c\partial_x w_2),$$
$$\mathcal{R}_3 = -\dot{c}\int x(\partial_x w_2)^2.$$

Estimate of $\mathscr{R}_1$. We use the decomposition $F_2 = F_{2,1} + \partial_x F_{2,2}$ from Lemma 10 to write

$$\mathscr{R}_1 = \int x(\partial_x w_2)F_{2,1} - \int (\partial_x w_2)F_{2,2} - \int x(\partial_x^2 w_2)F_{2,2}.$$

Thus, by the Cauchy–Schwarz inequality

$$|\mathscr{R}_1| \lesssim \|\rho^{\frac{\gamma}{4}} F_{2,1}\| \|x\rho^{-\frac{\gamma}{4}}\partial_x w_2\| + \|\rho^{\frac{\gamma}{4}} F_{2,2}\| \big(\|\rho^{-\frac{\gamma}{4}}\partial_x w_2\| + \|x\rho^{-\frac{\gamma}{4}}\partial_x^2 w_2\|\big).$$

By the Cauchy–Schwarz inequality and then Lemma 8

$$\left|\int x^2\rho^{-\frac{\gamma}{2}}(\partial_x w_2)^2\right| \lesssim \mathscr{N}\left(\int x^4\rho^{-\gamma}(\partial_x w_2)^2\right)^{\frac{1}{2}} \lesssim \mathscr{N}\left(\int x^2\rho^{\gamma}\right)^{\frac{1}{2}} \lesssim \mathscr{N}$$

and similarly $\|\rho^{-\frac{\gamma}{4}}\partial_x w_2\| + \|x\rho^{-\frac{\gamma}{4}}\partial_x^2 w_2\| \lesssim \mathscr{N}^{\frac{1}{2}}$. Thus, the estimate for $\mathscr{R}_1$ follows from Lemma 10.

Estimate of $\mathscr{R}_2$. We have

$$|\mathscr{R}_2| \lesssim \|\rho^{\frac{\gamma}{4}}\partial_x F_1\| \big(\|\rho^{-\frac{\gamma}{4}} x w_1\| + \|\rho^{-\frac{\gamma}{4}} x\partial_x w_2\|\big).$$

As before, using Lemmas 8 and 9, $\|\rho^{-\frac{\gamma}{4}} x w_1\| + \|\rho^{-\frac{\gamma}{4}} x\partial_x w_2\| \lesssim \mathscr{N}^{\frac{1}{2}}$. Thus, Lemma 10 implies the result for $\mathscr{R}_2$.

Estimate of $\mathscr{R}_3$. By Lemma 8, $\int |x|(\partial_x w_2)^2 \lesssim \mathscr{N}$. The estimate of $\mathscr{R}_3$ follows from Lemma 10. □

From Lemmas 11, 12 and 13, for α_0 sufficiently small, which implies that $\mathscr{N}$ is small, it holds $\dot{\mathscr{I}} \gtrsim \mathscr{N}^2$. Thus, by integration in $t \in (-\infty, +\infty)$, and the bound on $\mathscr{I}$, we have $\int_{-\infty}^{+\infty} \mathscr{N}^2 < +\infty$. Thus, there exists a sequence $t_n \to +\infty$, such that $\lim_{n\to+\infty} \mathscr{N}(t_n) = 0$. From Lemmas 8 and 9, we obtain $\lim_{n\to+\infty} \|\varepsilon(t_n)\|_{\mathscr{H}} = 0$. Using also the pointwise decay estimate of η in Lemma 2, we have $\lim_{n\to+\infty} \|\eta(t_n) - Q_{c(t_n)}\| = 0$. This implies that

$$\lim_{n\to+\infty} d\Big(\psi(t_n), e^{\mathrm{i}\theta(t_n)} U_{c(t_n)}\big(x - a(t_n)\big)\Big) = 0.$$

By the stability statement Lemma 2, we obtain that ψ is exactly a soliton.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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