ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus
Mathématique

Stefano Buccheri and Augusto C. Ponce

Recovering functions via doubly homogeneous nonlocal gradients

Volume 363 (2025), p. 1377-1406

Online since: 3 November 2025

https://doi.org/10.5802/crmath.800

The Comptes Rendus. Mathématique *are a member of the*
Mersenne Center for open scientific publishing
www.centre-mersenne.org — e-ISSN : 1778-3569

ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus. Mathématique
2025, Vol. 363, p. 1377-1406
https://doi.org/10.5802/crmath.800

Research article / *Article de recherche*
Partial differential equations / *Équations aux dérivées partielles*

Recovering functions via doubly homogeneous nonlocal gradients

Reconstruction de fonctions à l'aide de gradients non locaux doublement homogènes

Stefano Buccheri [a] and **Augusto C. Ponce** [b]

[a] Università degli Studi di Napoli Federico II, Dipartimento di Matematica e Applicazioni R. Caccioppoli, Via Vicinale dell'Infermeria 58, 80125 Napoli, Italy

[b] Université catholique de Louvain, Institut de Recherche en Mathématique et Physique, Chemin du cyclotron 2, L7.01.02, 1348 Louvain-la-Neuve, Belgium

E-mails: stefano.buccheri@unina.it, augusto.ponce@uclouvain.be

To the memory of Haïm Brezis, with deep admiration and gratitude

Abstract. We investigate a class of nonlocal gradients featuring distinct homogeneities at zero and infinity. We establish a representation formula for such doubly homogeneous operators and derive associated Sobolev-type inequalities. We also propose open questions linked to our results, suggesting directions for future research inspired by the work of Haïm Brezis.

Résumé. Nous étudions une classe de gradients non locaux présentant des homogénéités distinctes en zéro et à l'infini. Nous établissons une formule de représentation pour ces opérateurs doublement homogènes et en déduisons des inégalités de Sobolev associées. Nous proposons également des questions ouvertes liées à nos résultats, suggérant des directions de recherche inspirées par les travaux de Haïm Brezis.

Keywords. Riesz fractional gradient, nonlocal gradient, representation formula, Sobolev inequality, Fourier transform, Brezis-type problems.

Mots-clés. Gradient fractionnaire de Riesz, gradient non local, formule de représentation, inégalité de Sobolev, transformée de Fourier, problèmes de type Brezis.

2020 Mathematics Subject Classification. 26A33, 47G20, 42A38, 47A67.

Funding. The first author received support from the Fonds de la Recherche scientifique (FNRS, grant CR 40006150), from the Austrian Science Fund (FWF, grant 10.55776/ESP9), and from the GNAMPA-INdAM Project 2025 "Local and nonlocal equations with lower order terms" (CUP E5324001950001). The second author received partial support from the Fonds de la Recherche scientifique (FNRS, grant J.0020.18).

Note. Article submitted by invitation.

Note. Article soumis sur invitation.

Manuscript received 15 June 2025, revised 22 September 2025, accepted 7 October 2025.

1. Introduction

We study a class of *nonlocal gradients* whose canonical example associated to a function $u \in C_c^\infty(\mathbb{R}^d)$ in any dimension $d \geq 1$ is the Riesz fractional gradient of order $0 < s < 1$:

$$\nabla^s u(x) = \int_{\mathbb{R}^d} \big(u(x) - u(y)\big) \frac{x - y}{|x - y|^{d+s+1}} \,\mathrm{d}y,$$

which reduces to the Riesz transform for $s = 0$. When $s = 1$ one should not expect to obtain the classical gradient since as observed by Bourgain, Brezis and Mironescu [7] one has

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\big|u(x) - u(y)\big|}{|x - y|^{d+1}} \,\mathrm{d}x \,\mathrm{d}y = \infty,$$

unless u is a constant. This is due to a defect in the definition of ∇^s that does not take into consideration its behavior as $s \to 1$. In fact, the *correct* operator that allows one to recover ∇ should be instead $(1-s)\nabla^s$, as one verifies that

$$(1-s)\nabla^s u \longrightarrow \frac{\sigma_d}{d} \nabla u \quad \text{when } s \to 1, \tag{1}$$

both uniformly and in $L^1(\mathbb{R}^d)$, where σ_d denotes the area of the unit sphere ∂B_1 in $\mathbb{R}^d$.

The fractional gradient ∇^s introduced in [21] within singular integral theory has attracted significant attention in both applied and theoretical mathematics. For example, in nonlocal continuum mechanics [25] and image processing [17], see also [35] for further references, but also in theoretical developments including Bourgain–Brezis–Mironescu-type results [26], variational nonlocal elliptic operators [33,34], and distributional fractional calculus [13,14,35].

Generalizations of ∇^s have also emerged in elasticity and peridynamics [2–4], where the singular kernel is modified to define broader classes of nonlocal gradients. In this paper, we advance this direction by focusing on *doubly homogeneous gradients* — a flexible class of nonlocal operators — and establishing results for their associated function spaces and inversion theory.

The nonlocal gradients we consider are defined as follows.

Definition 1. *Let $g \in C^\infty\big(\mathbb{R}^d \setminus \{0\}\big)$ satisfy the integrability condition*

$$\int_{B_1} |x| \big|\nabla g(x)\big| \,\mathrm{d}x + \int_{\mathbb{R}^d \setminus B_1} \big|\nabla g(x)\big| \,\mathrm{d}x < \infty. \tag{2}$$

*The $\mathscr{G}$-*fractional gradient *of a function $u \in C_c^\infty(\mathbb{R}^d)$ is defined pointwise as*

$$\mathscr{G}u(x) = -\int_{\mathbb{R}^d} \big(u(x) - u(y)\big) \nabla g(x - y) \,\mathrm{d}y. \tag{3}$$

The choice $g(x) = |x|^{-(d-1+s)}$ with $0 < s < 1$ recovers, up to a multiplicative constant, the Riesz fractional gradient $\nabla^s u$. For this case, there exists a vector field $V_s(x) = Ax/|x|^{d-s+1}$, for some constant A depending on s and d, such that one has the following inversion formula [29, Proposition 15.8]:

$$u = V_s * \nabla^s u \quad \text{in } \mathbb{R}^d. \tag{4}$$

This serves as a fractional counterpart to the Fundamental Theorem of Calculus.

The identity (4) demonstrates that smooth, compactly supported functions u can be fully recovered from their Riesz fractional gradients. Such representation formulas are powerful tools for proving embedding theorems via convolution estimates. For instance, Sobolev- and Hardy-type inequalities for $\|\nabla^s u\|_{L^p}$ with $p > 1$ were established by Shieh and Spector in [33]. The endpoint case $p = 1$ presents additional subtleties: as shown by Schikorra, Spector and Van Schaftingen [32], the inequality

$$\|u\|_{L^{\frac{d}{d-s}}(\mathbb{R}^d)} \leq C \|\nabla^s u\|_{L^1(\mathbb{R}^d)} \tag{5}$$

holds for $d \geq 2$, with a proof that relies on duality and the curl-free property of the fractional gradient.

In Section 3 we provide a streamlined proof of (5) that follows the strategy of [32], based on an idea from [37], while eliminating the need for L^p-bounds on the Riesz transform. A careful inspection of that proof gives a constant in (5) that is compatible with the limit behavior of $(1-s)\nabla^s$ as $s \to 1$ given by (1). More precisely, the following result holds.

Theorem 2. *Let $d \geq 2$. There exists a constant $\widetilde{C} > 0$ such that, for every $u \in C_c^\infty(\mathbb{R}^d)$ and every $1/2 \leq s < 1$,*

$$\|u\|_{L^{\frac{d}{d-s}}(\mathbb{R}^d)} \leq \widetilde{C}(1-s)\|\nabla^s u\|_{L^1(\mathbb{R}^d)}. \tag{6}$$

As a consequence, one recovers the classical Sobolev inequality as $s \to 1$ in the spirit of analogous results of Bourgain, Brezis and Mironescu [8,9] concerning the behavior of the constant with respect to the fractional parameter. Note that

$$\|\nabla^s u\|_{L^1(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|}{|x-y|^{d+s}}\,\mathrm{d}x\,\mathrm{d}y,$$

where the right-hand side is the Gagliardo seminorm associated to the fractional Sobolev space $W^{s,1}(\mathbb{R}^d)$. Hence, the estimate (6) gives a stronger version of the fractional Sobolev inequality that is stable in the limit as $s \to 1$ and had been obtained using various methods [1,8,16,23,24,30].

In light of the representation formula (4), it is therefore natural to investigate what conditions on the nonlocal operator $\mathscr{G}$ permit a representation formula, and what analytical insights such a formula might provide. We focus on kernels g exhibiting *double homogeneity*, with distinct scaling behaviors near the origin and at infinity. We do not focus on the cases of dimensions $d = 1$ and 2 to avoid technical details specific to these dimensions. Three representative examples illustrate our framework:

(a) *localized fractional gradient*: for $0 < r < R$ and $0 < s < 1$,

$$g(x) = \begin{cases} 1/|x|^{d-1+s} & \text{if } |x| \leq r, \\ 0 & \text{if } |x| \geq R, \end{cases} \tag{7}$$

which restricts the fractional derivative to a bounded region;

(b) *integrable tail*: for $\alpha > d$ and $a, b > 0$,

$$g(x) = \begin{cases} a/|x|^{d-1+s} & \text{if } |x| \leq r, \\ b/|x|^{\alpha} & \text{if } |x| \geq R, \end{cases} \tag{8}$$

ensuring integrability at infinity while preserving fractional behavior locally;

(c) *two-scale fractional kernel*: for $0 < s, t < 1$,

$$g(x) = \begin{cases} a/|x|^{d-1+s} & \text{if } |x| \leq r, \\ b/|x|^{d-1+t} & \text{if } |x| \geq R, \end{cases} \tag{9}$$

interpolating between different fractional regimes.

We shall also suppose that

$$\begin{cases} g \in C^\infty\big(\mathbb{R}^d \setminus \{0\}\big) \text{ is radial and} \\ \rho \in (0,\infty) \mapsto \rho^{d-1} g(\rho x) \text{ is non-increasing and convex for any } x \neq 0. \end{cases} \tag{10}$$

These assumptions ensure that the Fourier transform of g is well-defined and positive, which is a natural request within our approach, see (14) and also Sections 5 and 6 below.

For such kernels, we establish a general representation formula as follows.

Theorem 3. *Let $d \geq 3$ and suppose that g given by* (7), (8), *or* (9) *satisfies* (10). *Then, there exists a smooth vector field $V : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d$ such that, for every $u \in C_c^\infty(\mathbb{R}^d)$,*

$$u = V * \mathscr{G}u \quad \textit{in } \mathbb{R}^d. \tag{11}$$

Under the assumption (7) such a representation formula has been obtained by Bellido, Cueto and Mora-Corral [3], see also [4].

The proof of (14) also provides precise estimates on the behavior of V and its derivatives. For the kernels (7) and (8), we obtain for every multi-index ν:

$$\left|\partial^\nu V(x)\right| \leq \begin{cases} C/|x|^{d-s+|\nu|} & \text{if } |x| \leq 1, \\ C/|x|^{d-1+|\nu|} & \text{if } |x| \geq 1. \end{cases} \tag{12}$$

This shows that both compactly supported kernels and those with integrable tails induce identical decay for V at infinity. It may be of some interest to notice that at infinity the same behaviour of the classical local representation formula is recovered. In contrast, for the doubly homogeneous kernel (9) the estimates reflect its distinct scaling regimes:

$$\left|\partial^\nu V(x)\right| \leq \begin{cases} C/|x|^{d-s+|\nu|} & \text{if } |x| \leq 1, \\ C/|x|^{d-t+|\nu|} & \text{if } |x| \geq 1. \end{cases} \tag{13}$$

The core idea to prove Theorem 3 is to find $\omega \in C^\infty(\mathbb{R}^d)$ satisfying:

$$\omega * g(z) = \frac{1}{|z|^{d-2}} \quad \text{for all } z \neq 0. \tag{14}$$

Most of this paper is devoted to the solution of this convolution equation. This approach requires a careful analysis of the Fourier transform of the homogeneous function $1/|x|^{d-\alpha}$ which does not belong to the usual L^1 or L^2 settings (Sections 4 and 5), the computation of the Fourier transform of g (Section 7), and the inversion of the transformed solution (Section 8).

Defining

$$V = -\frac{1}{\sigma_d(d-2)}\nabla\omega$$

and applying Proposition 12 then yields:

$$V * \mathscr{G}u = V * g * \nabla u = \left(\frac{1}{\sigma_d}\frac{z}{|z|^d}\right) * \nabla u = u.$$

In the proof of Theorem 3, we strongly rely on the explicit formulas of g near 0 and infinity. It would be interesting to have a weaker assumption that relies mostly on the behavior of g in these regions. A common roof that could collect the assumptions (7) and (8) would be the following.

Open problem 4. *Assume that $g \in C^\infty\big(\mathbb{R}^d \setminus \{0\}\big)$ is a radial function such that*

$$\lim_{x \to 0} |x|^{d-1+s} g(x) \in (0, \infty) \tag{15}$$

and

$$\left|g(x)\right| \leq C/|x|^\alpha \quad \textit{for } |x| \geq R, \tag{16}$$

where $0 < s < 1$ and $\alpha > d$. Does there exist a smooth vector field $V : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d$ such that the representation formula (11) *holds for every $u \in C_c^\infty(\mathbb{R}^d)$?*

An immediate consequence of (11) is that if $u \in C_c^\infty(\mathbb{R}^d)$ is such that $\mathscr{G}u = 0$ in $\mathbb{R}^d$, then $u = 0$. In the spirit of H. Brezis' work [10] that characterizes measurable functions that are constant almost everywhere based on the condition

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\left|u(x) - u(y)\right|}{|x-y|^{d+1}}\,\mathrm{d}x\,\mathrm{d}y < \infty,$$

see also [11, Section 6.2] for alternative elementary proofs, one could extend the fractional derivative $\mathscr{G}u$ and investigate an analogous question to functions that are not necessarily smooth. Indeed, if $u\colon \mathbb{R}^d \to \mathbb{R}$ is a measurable function that satisfies

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big|u(x)-u(y)\big|\big|\nabla g(x-y)\big| \, \mathrm{d}x \, \mathrm{d}y < \infty, \tag{17}$$

then, by Tonelli's theorem, the formula (3) for $\mathscr{G}u(x)$ is well-defined for almost every $x \in \mathbb{R}^d$.

Open problem 5. Assume that $g \in C^\infty\big(\mathbb{R}^d \setminus \{0\}\big)$ is radial and satisfies (15) and (16). If u is a measurable function in $\mathbb{R}^d$ that verifies (17) and $\mathscr{G}u = 0$ almost everywhere in $\mathbb{R}^d$, is u a constant in $\mathbb{R}^d$?

Note that for a given function g the representation formula may fail, but still the fractional gradient could characterize the constant functions. A further challenging question would be the following.

Open problem 6. Identify functions $g \in C^\infty\big(\mathbb{R}^d \setminus \{0\}\big)$ for which the condition $\mathscr{G}u = 0$ characterizes all constant measurable functions u that satisfy (17). For example, is this the case if the Fourier transform of g is well-defined and almost everywhere nonzero?

The next results concern fractional Sobolev inequalities in our doubly homogeneous setting. Clearly, one does not expect to have an estimate for u in a single Lebesgue space, but rather in a suitable space that accommodates the difference between homogeneity rates. The sum of Lebesgue spaces represents a valid alternative to this issue, as we show in the next couple of results.

Theorem 7. *Take $d \geq 3$ and $1 \leq p < d/s$ with $0 < s < 1$. Assume that g is given by* (7) *or* (8) *and satisfies* (10)*. Then, for every $u \in C_c^\infty(\mathbb{R}^d)$,*

$$\|u\|_{L^{\frac{pd}{d-sp}} + L^{\frac{pd}{d-p}}} \leq C\|\mathscr{G}u\|_{L^p(\mathbb{R}^d)}.$$

Unless otherwise stated, $C > 0$ is an absolute constant that may depend on the parameters d, s, t, p and on the function g but that does not depend on the function u. Let us recall that u belongs to $(L^m + L^q)(\mathbb{R}^d)$ whenever there exist $u_1 \in L^m(\mathbb{R}^d)$ and $u_2 \in L^q(\mathbb{R}^d)$ so that $u = u_1 + u_2$, in which case $\|u\|_{L^m + L^q}$ is the infimum of the sum

$$\|u_1\|_{L^m(\mathbb{R}^d)} + \|u_2\|_{L^q(\mathbb{R}^d)}$$

over all such decompositions of u.

In the same spirit as Theorem 7 we have the following result.

Theorem 8. *Take $d \geq 3$ and $1 \leq p < \min\{d/s, d/t\}$ with $0 < s, t < 1$. Assume that g is given by* (9) *and satisfies* (10)*. Then, for every $u \in C_c^\infty(\mathbb{R}^d)$,*

$$\|u\|_{L^{\frac{pd}{d-sp}} + L^{\frac{pd}{d-tp}}} \leq C\|\mathscr{G}u\|_{L^p(\mathbb{R}^d)}.$$

The first step in the proof of Theorem 8 is to write the vector field V as $V = V_1 + V_2$ so that V_1 behaves as $1/|x|^{d-s}$ and V_2 as $1/|x|^{d-t}$. Therefore, the composition formula provided by Theorem 3 becomes

$$u = V_1 * \mathscr{G}u + V_2 * \mathscr{G}u.$$

The second step is to treat the two contributions on the right-hand side above. While for $p > 1$ it is enough to rely on classical estimates for the Riesz potential, the borderline case $p = 1$ requires an adaptation of the more delicate proof of the already mentioned [32, Theorem A], see Section 3 below.

Actually, for $p > 1$ we may improve Theorems 7 and 8 by both refining the embedding on the Lorentz scale and having a better description of the influence of the different homogeneities of g.

Theorem 9. *Take $d \geq 3$ and $1 < p < d/s$ with $0 < s < 1$. Assume that g is given by* (7) *or* (8) *and satisfies* (10). *Then, for every $u \in C_c^\infty(\mathbb{R}^d)$, there exists $k \geq 0$ depending on u such that*

$$\left\|G_k(u)\right\|_{L^{\frac{pd}{d-sp},p}(\mathbb{R}^d)} + \left\|T_k(u)\right\|_{L^{\frac{pd}{d-p},p}(\mathbb{R}^d)} \leq C\|\mathscr{G}u\|_{L^p(\mathbb{R}^d)}.$$

Here,

$$u = G_k(u) + T_k(u)$$

where the truncations G_k and T_k are explicitly defined for every $\tau \in \mathbb{R}$ as

$$T_k(\tau) = \max\left\{-k, \min\{\tau, k\}\right\} \quad \text{and} \quad G_k(\tau) = \tau - T_k(\tau).$$

In the same spirit, concerning the two-scale fractional kernel, the following holds.

Theorem 10. *Take $d \geq 3$ and $1 < p < \min\left\{d/s, d/t\right\}$ with $0 < s, t < 1$. Assume that g is given by* (9) *and satisfies* (10). *Then, for every $u \in C_c^\infty(\mathbb{R}^d)$, there exists $k \geq 0$ depending on u such that*

$$\left\|G_k(u)\right\|_{L^{\frac{pd}{d-sp},p}(\mathbb{R}^d)} + \left\|T_k(u)\right\|_{L^{\frac{pd}{d-tp},p}(\mathbb{R}^d)} \leq C\|\mathscr{G}u\|_{L^p(\mathbb{R}^d)}.$$

We observe in the previous statement that the estimate of $G_k(u)$ in the Lorentz spaces $L^{\frac{pd}{d-sp},p}(\mathbb{R}^d)$ tells us that the local integrability of u is related to the singularity at zero of the kernel g. Similarly, the estimate of $T_k(u)$ expresses the integrability at infinity of u that depends on the tail of g.

The level at which we slice the function u in our proof is taken as $k = \|\mathscr{G}u\|_{L^p(\mathbb{R}^d)}$. Let us point out that the level of truncation k cannot be chosen independently of u. Indeed, if this was the case, taking the limit as k goes to 0 or infinity, one would obtain an estimate where the influence of one of the two homogeneities of g is lost.

For the Riesz fractional gradient, namely when $g(x) = c/|x|^{d-1+s}$, Spector [36] showed that Theorem 10 is valid for $p = 1$ and $s = t$. This leads to the following question.

Open problem 11. Given $g \in C^\infty\left(\mathbb{R}^d \setminus \{0\}\right)$ of the form (9) that satisfies (10), is it true that for every $u \in C_c^\infty(\mathbb{R}^d)$ there exists a truncation level $k = k(u)$ such that

$$\left\|G_k(u)\right\|_{L^{\frac{d}{d-s},1}(\mathbb{R}^d)} + \left\|T_k(u)\right\|_{L^{\frac{d}{d-t},1}(\mathbb{R}^d)} \leq C\|\mathscr{G}u\|_{L^1(\mathbb{R}^d)}\ ?$$

2. Relation between $\mathscr{G}u$ and ∇u

We begin by showing that the fractional gradient can be seen as a convolution of the classical gradient.

Proposition 12. *If $g \in C^\infty\left(\mathbb{R}^d \setminus \{0\}\right)$ satisfies* (2), *then $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and, for every $u \in C_c^\infty(\mathbb{R}^d)$, we have*

$$\mathscr{G}u = g * \nabla u \quad \textit{in } \mathbb{R}^d.$$

This identity, implicit for example in [15,26], was highlighted by Brezis and Mironescu [12] as a key ingredient behind nonlocal approximations of the gradient in the spirit of (1). The composition in the right-hand side is well-defined since ∇u is a bounded function with compact support and g is locally summable. That the latter holds follows from the next result.

Lemma 13. *If $g \in C^\infty\left(\mathbb{R}^d \setminus \{0\}\right)$ satisfies* (2), *then*

$$\int_{B_1} |g| + \int_{\mathbb{R}^d \setminus B_1} |g|^{\frac{d}{d-1}} < \infty.$$

In particular, $g \in (L^1 + L^{\frac{d}{d-1}})(\mathbb{R}^d)$.

Proof of Lemma 13. Take a smooth function $\psi\colon \mathbb{R}^d \to \mathbb{R}$ with $\psi = 1$ in $\mathbb{R}^d \setminus B_1$ and $\psi = 0$ in $B_{1/2}$. Since $g\psi \in W^{1,1}(\mathbb{R}^d)$, by the Sobolev inequality we have $g\psi \in L^{\frac{d}{d-1}}(\mathbb{R}^d)$ and then

$$\biggl(\int_{\mathbb{R}^d\setminus B_1} |g|^{\frac{d}{d-1}}\biggr)^{\frac{d-1}{d}} \le \|g\psi\|_{L^{\frac{d}{d-1}}} \le C_1 \bigl\|\nabla(g\psi)\bigr\|_{L^1} < \infty.$$

Next, by the Fundamental Theorem of Calculus, for every $0 < r < 1$ and every $y \in \partial B_1$ we have

$$\bigl|g(ry)\bigr| = \biggl|g(y) - \int_r^1 \frac{\mathrm{d}}{\mathrm{d}t} g(ty)\,\mathrm{d}t\biggr| \le \bigl|g(y)\bigr| + \int_r^1 \bigl|\nabla g(ty)\bigr|\,\mathrm{d}t.$$

We then multiply both sides by r^{d-1} and integrate with respect to r. Applying Tonelli's theorem,

$$\int_0^1 \bigl|g(ry)\bigr| r^{d-1}\,\mathrm{d}r \le \frac{1}{d}\bigl|g(y)\bigr| + \int_0^1 \biggl(\int_0^t \bigl|\nabla g(ty)\bigr| r^{d-1}\,\mathrm{d}r\biggr)\mathrm{d}t = \frac{1}{d}\bigl|g(y)\bigr| + \frac{1}{d}\int_0^1 \bigl|\nabla g(ty)\bigr| t^d\,\mathrm{d}t.$$

Integrating with respect to y and applying the integration formula in polar coordinates, we obtain

$$\int_{B_1} |g| = \int_{\partial B_1} \biggl(\int_0^1 \bigl|g(ry)\bigr| r^{d-1}\,\mathrm{d}r\biggr)\mathrm{d}\sigma(y) \le \frac{1}{d}\int_{\partial B_1} |g|\,\mathrm{d}\sigma + \frac{1}{d}\int_{B_1} \bigl|\nabla g(x)\bigr||x|\,\mathrm{d}x,$$

which gives the conclusion. □

Proof of Proposition 12. Since u is a bounded smooth function and g satisfies (2), we have

$$\mathscr{G}u(x) = \lim_{\substack{r\to 0\\ R\to\infty}} \int_{B_R(x)\setminus B_r(x)} \bigl(u(x) - u(y)\bigr)\nabla_y g(x-y)\,\mathrm{d}y.$$

Since g is locally summable and ∇u has compact support, we also have

$$g * \nabla u(x) = -\lim_{\substack{r\to 0\\ R\to\infty}} \int_{B_R(x)\setminus B_r(x)} \nabla_y\bigl(u(x) - u(y)\bigr) g(x-y)\,\mathrm{d}y.$$

For every $0 < r < R$ and $j \in \{1, \dots, d\}$, by the Divergence Theorem we have

$$\begin{aligned}
&\int_{B_R(x)\setminus B_r(x)} \nabla_y\bigl(u(x) - u(y)\bigr) g(x-y)\,\mathrm{d}y - \int_{B_R(x)\setminus B_r(x)} \bigl(u(x) - u(y)\bigr)\nabla_y g(x-y)\,\mathrm{d}y\\
&\qquad = \int_{B_R(x)\setminus B_r(x)} \nabla_y\Bigl[\bigl(u(x) - u(y)\bigr) g(x-y)\Bigr]\,\mathrm{d}y\\
&\qquad = \int_{\partial B_R(x)} \bigl(u(x) - u(y)\bigr) g(x-y)\nu(y)\,\mathrm{d}\sigma(y) - \int_{\partial B_r(x)} \bigl(u(x) - u(y)\bigr) g(x-y)\nu(y)\,\mathrm{d}\sigma(y).
\end{aligned} \tag{18}$$

We estimate

$$\begin{aligned}
\biggl|\int_{\partial B_R(x)} \bigl(u(x) - u(y)\bigr) g(x-y)\nu(y)\,\mathrm{d}\sigma(y)\biggr| &\le 2\|u\|_{L^\infty} \int_{\partial B_R(x)} \bigl|g(x-y)\bigr|\,\mathrm{d}\sigma(y)\\
&= 2\|u\|_{L^\infty} \int_{\partial B_R} |g|\,\mathrm{d}\sigma
\end{aligned}$$

and

$$\begin{aligned}
\biggl|\int_{\partial B_r(x)} \bigl(u(x) - u(y)\bigr) g(x-y)\nu(y)\,\mathrm{d}y\biggr| &\le \|\nabla u\|_{L^\infty} \int_{\partial B_r(x)} |x-y|\bigl|g(x-y)\bigr|\,\mathrm{d}\sigma(y)\\
&= \|\nabla u\|_{L^\infty}\, r \int_{\partial B_r} |g|\,\mathrm{d}\sigma.
\end{aligned}$$

We now show that there exist sequences of positive numbers $(r_j)_{j\in\mathbb{N}}$ and $(R_j)_{j\in\mathbb{N}}$ such that $r_j \to 0$ and $R_j \to \infty$ with

$$\lim_{j\to\infty} r_j \int_{\partial B_{r_j}} |g|\,\mathrm{d}\sigma = \lim_{j\to\infty} \int_{\partial B_{R_j}} |g|\,\mathrm{d}\sigma = 0.$$

To this end, we apply the integration formula in polar coordinates to get

$$\int_0^1 t\biggl(\int_{\partial B_t} |g|\,\mathrm{d}\sigma\biggr)\frac{\mathrm{d}t}{t} = \int_{B_1} |g| < \infty,$$

which gives the existence of the sequence $r_j \to 0$ since $\int_0^1 \mathrm{d}t/t = \infty$. For the other sequence, by Hölder's inequality we also have

$$\int_1^\infty \left(\int_{\partial B_t} |g|\,\mathrm{d}\sigma\right)^{\frac{d}{d-1}} \frac{\mathrm{d}t}{t} \le C_2 \int_1^\infty t\left(\int_{\partial B_t} |g|^{\frac{d}{d-1}}\,\mathrm{d}\sigma\right) \frac{\mathrm{d}t}{t} = C_2 \int_{\mathbb{R}^d\setminus B_1} |g|^{\frac{d}{d-1}} < \infty.$$

Since $\int_1^\infty \mathrm{d}t/t = \infty$, there exists a sequence $R_j \to \infty$ with the required property. Taking $R = R_j$ and $r = r_j$ in (18) and letting $j \to \infty$, we get

$$-\nabla u * g(x) + \mathscr{G}u(x) = 0,$$

from which the conclusion follows. □

Corollary 14. *Let $g \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be a function that satisfies* (2)*. If there exists $V \in (L^1 + L^d)(\mathbb{R}^d)$ such that, for every $z \in \mathbb{R}^d \setminus \{0\}$,*

$$V * g(z) = \frac{1}{\sigma_d} \frac{z}{|z|^d},$$

where σ_d is the area surface of the sphere ∂B_1, then, for every $u \in C_c^\infty(\mathbb{R}^d)$,

$$u = V * \mathscr{G}u \quad \text{in } \mathbb{R}^d.$$

Proof. Since $V \in (L^1 + L^d)(\mathbb{R}^d)$ and $g \in (L^1 + L^{\frac{d}{d-1}})(\mathbb{R}^d)$, we have $|V| * |g| \in (L^1 + L^\infty)(\mathbb{R}^d)$. By Proposition 12 and Fubini's theorem,

$$V * \mathscr{G}u = V * (\nabla u * g) = \nabla u * (V * g).$$

By the assumption on $V * g$ and the classical representation formula involving the gradient, we then have for every $x \in \mathbb{R}^d$,

$$V * \mathscr{G}u(x) = \frac{1}{\sigma_d} \int_{\mathbb{R}^d} \nabla u(x-y) \frac{y}{|y|^d}\,\mathrm{d}y = u(x). \qquad \square$$

We denote the jth component of $\mathscr{G}u$ with respect to the canonical basis $e_1, \dots, e_d$ by $\mathscr{G}_j u$, so that

$$\mathscr{G}u = \sum_{j=1}^d \mathscr{G}_j u\, e_j.$$

It follows from the relation with the classical gradient that the fractional gradient is also curl free in the following sense.

Corollary 15. *Let $g \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be a function that satisfies* (2)*. If $u \in C_c^\infty(\mathbb{R}^d)$, then*

$$\partial_l(\mathscr{G}_j u) = \partial_j(\mathscr{G}_l u) \quad \text{for every } j, l \in \{1, \dots, d\}.$$

Proof. By Proposition 12, we have $\mathscr{G}_j u = \partial_j u * g$. Then, interchanging the order of integration and differentiation,

$$\partial_l(\mathscr{G}_j u) = \partial_l(\partial_j u * g) = (\partial_l \partial_j u) * g.$$

By smoothness of u, we may exchange the order of the derivatives and obtain the identity in the statement. □

3. Nonlocal Sobolev inequalities in Lebesgue spaces

We present the strategy of the proof of the fractional Sobolev inequality from [32] in a way that makes it easier to track the dependence of the constants involved. Our aim is to justify the multiplicative factor $1-s$ in (6). The heart of the matter can be summarized in the next inequality.

Proposition 16. *Let $d \geq 2$ and $0 < s < d$. If $v \in C^\infty\big(\mathbb{R}^d \setminus \{0\}\big)$ is such that, for every $z \in \mathbb{R}^d \setminus \{0\}$,*

$$\big|v(z)\big| + |z|\big|\nabla v(z)\big| \leq \frac{C}{|z|^{d-s}}, \tag{19}$$

then there exists a constant $C' > 0$, depending on C and d/s, such that

$$\|v * F\|_{L^{\frac{d}{d-s}}} \leq C' \|F\|_{L^1}$$

for each map $F \in L^1(\mathbb{R}^d;\mathbb{R}^d)$ such that, for every $l, j \in \{1,\dots,d\}$,

$$\partial_l F_j = \partial_j F_l \quad \text{in the sense of distributions in } \mathbb{R}^d. \tag{20}$$

The conclusion fails without the curl-free condition (20). This key-type of assumption — which unlocks new elliptic estimates — was first brought to light and beautifully explored by Bourgain and Brezis [5,6] in their foundational work on div-curl estimates.

We begin with the following standard estimate.

Lemma 17. *Let $d \geq 1$, $0 < s < d$ and $\rho \in C_c^\infty(B_1)$. Then, for every $v \in C^\infty\big(\mathbb{R}^d \setminus \{0\}\big)$ that satisfies* (19) *and for every $x \in \mathbb{R}^d$, we have*

$$\big|v * \nabla\rho_r(x)\big| + \left|v * \frac{\mathrm{d}\rho_r}{\mathrm{d}r}(x)\right| \leq \frac{C''}{\big(r + |x|\big)^{d-s+1}},$$

where

$$\rho_r(z) := \frac{1}{r^d}\rho\Big(\frac{z}{r}\Big).$$

Proof of Lemma 17. Since ρ is smooth and ρ_r is supported in B_r, we have

$$|\nabla\rho_r| + \left|\frac{\mathrm{d}\rho_r}{\mathrm{d}r}\right| \leq \frac{C_1}{r^{d+1}}\chi_{B_r},$$

where χ_{B_r} denotes the characteristic function of the ball B_r. For $|x| \leq 2r$, it then follows by the pointwise assumption of v that

$$\big|v * \nabla\rho_r(x)\big| + \left|v * \frac{\mathrm{d}\rho_r}{\mathrm{d}r}(x)\right| \leq \frac{C_2}{r^{d+1}}\int_{B_r(x)}\big|v(z)\big|\,\mathrm{d}z \leq \frac{C_3}{r^{d-s+1}}. \tag{21}$$

Since ρ_r is supported by B_r, by the Divergence Theorem we have

$$\big|v * \nabla\rho_r(x)\big| \leq \int_{B_r}\big|\nabla v(x-y)\big|\rho_r(y)\,\mathrm{d}y.$$

We next observe that

$$\frac{\mathrm{d}\rho_r}{\mathrm{d}r}(y) = -\frac{1}{r}\operatorname{div}\big(y\rho_r(y)\big).$$

Another application of the Divergence Theorem also gives

$$\left|v * \frac{\mathrm{d}\rho_r}{\mathrm{d}r}(x)\right| \leq \int_{B_r}\big|\nabla v(x-y)\big|\rho_r(y)\,\mathrm{d}y.$$

Since $\rho_r \leq C_4/r^d$, by the pointwise assumption of ∇v we get for $|x| \geq 2r$,

$$\big|v * \nabla\rho_r(x)\big| + \left|v * \frac{\mathrm{d}\rho_r}{\mathrm{d}r}(x)\right| \leq \frac{C_4}{r^d}\int_{B_r(x)}\big|\nabla v(z)\big|\,\mathrm{d}z \leq \frac{C_5}{|x|^{d-s+1}}. \tag{22}$$

The conclusion then follows from (21) and (22). □

Proof of Proposition 16. We first assume that $F = (F_1,\dots,F_d)$ is smooth, every ∇F_l is summable in $\mathbb{R}^d$ and the linear condition (20) holds pointwise in $\mathbb{R}^d$. We prove in this case that

$$\left|\int_{\mathbb{R}^d} v * F_1 \cdot \varphi\right| \leq C_1\|F\|_{L^1}\|\varphi\|_{L^{\frac{d}{s}}} \quad \text{for every } \varphi \in C_c^\infty(\mathbb{R}^d).$$

The estimate for the other components of F follows along the same lines. Note that

$$\int_{\mathbb{R}^d} v * F_1 \cdot \varphi = \int_{\mathbb{R}^d} F_1 \cdot \widetilde{v} * \varphi,$$

where $\widetilde{v}(z) := v(-z)$. Given a mollifier ρ supported in B_1 and $\epsilon > 0$, for $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $x_d \in \mathbb{R}$, following an idea of Van Schaftingen [37, proof of Theorem 1.5] we estimate

$$\begin{aligned}\left|\int_{\mathbb{R}^{d-1}} F_1(x', x_d) \cdot \widetilde{v} * \varphi(x', x_d)\,\mathrm{d}x'\right| \\ \le \left|\int_{\mathbb{R}^{d-1}} F_1(x', x_d) \cdot \widetilde{v} * (\varphi - \rho_\epsilon * \varphi)(x', x_d)\,\mathrm{d}x'\right| + \left|\int_{\mathbb{R}^{d-1}} F_1(x', x_d) \cdot \widetilde{v} * \rho_\epsilon * \varphi(x', x_d)\,\mathrm{d}x'\right|.\end{aligned}$$

We show that

$$\left|\int_{\mathbb{R}^{d-1}} F_1(x', x_d) \cdot \widetilde{v} * (\varphi - \rho_\epsilon * \varphi)(x', x_d)\,\mathrm{d}x'\right| \le C_2 \epsilon^{\frac{s}{d}} \left\|F_1(\cdot, x_d)\right\|_{L^1(\mathbb{R}^{d-1})} \mathscr{M}\Phi(x_d) \tag{23}$$

and

$$\left|\int_{\mathbb{R}^{d-1}} F_1(x', x_d) \cdot \widetilde{v} * \rho_\epsilon * \varphi(x', x_d)\,\mathrm{d}x'\right| \le \frac{C_3}{\epsilon^{\frac{d-s}{d}}} \|F_d\|_{L^1(\mathbb{R}^d)} \mathscr{M}\Phi(x_d), \tag{24}$$

where $\Phi\colon \mathbb{R} \to [0, \infty)$ is given by

$$\Phi(x_d) := \left(\int_{\mathbb{R}^{d-1}} \left|\varphi(x', x_d)\right|^{\frac{d}{s}} \mathrm{d}x'\right)^{\frac{s}{d}}$$

and $\mathscr{M}\Phi\colon \mathbb{R} \to [0, \infty]$ is the maximal function associated to Φ,

$$\mathscr{M}\Phi(t) := \sup_{r>0} \frac{1}{2r} \int_{t-r}^{t+r} \Phi.$$

We begin with (23). For every $y \in \mathbb{R}^d$,

$$\rho_\epsilon * \varphi(y) - \varphi(y) = \int_0^\epsilon \frac{\mathrm{d}\rho_r}{\mathrm{d}r} * \varphi(y)\,\mathrm{d}r.$$

Thus, by Fubini's theorem,

$$\left|\widetilde{v} * (\rho_\epsilon * \varphi - \varphi)(x)\right| \le \int_0^\epsilon \left|\widetilde{v} * \frac{\mathrm{d}\rho_r}{\mathrm{d}r} * \varphi(x)\right| \mathrm{d}r. \tag{25}$$

By Lemma 17, the pointwise assumptions on v and ∇v yield, for every $z \in \mathbb{R}^d$,

$$\left|\widetilde{v} * \frac{\mathrm{d}\rho_r}{\mathrm{d}r}(z)\right| + \left|\widetilde{v} * \nabla\rho_r(z)\right| \le \frac{C_4}{\left(r + |z|\right)^{d-s+1}},$$

which by application of Hölder's inequality implies, for every $r > 0$ and $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$,

$$\left|\widetilde{v} * \frac{\mathrm{d}\rho_r}{\mathrm{d}r} * \varphi(x)\right| + \left|\widetilde{v} * \nabla\rho_r * \varphi(x)\right| \le \frac{C_5}{r^{\frac{d-s}{d}}} \mathscr{M}\Phi(x_d). \tag{26}$$

A combination of (25) and (26) gives

$$\left|\widetilde{v} * (\rho_\epsilon * \varphi - \varphi)(x)\right| \le C_5 \int_0^\epsilon \frac{\mathrm{d}r}{r^{\frac{d-s}{d}}} \mathscr{M}\Phi(x_d) = C_6 \epsilon^{\frac{s}{d}} \mathscr{M}\Phi(x_d),$$

which implies (23).

We now show (24). Since $d \ge 2$ and we are assuming that $\partial_d F_1 \in L^1(\mathbb{R}^d)$, for almost every $x' \in \mathbb{R}^{d-1}$ we have $\partial_d F_1(x', \cdot) \in L^1(\mathbb{R})$ and then by the Fundamental Theorem of Calculus, for every $x_d \in \mathbb{R}$ we have

$$F_1(x', x_d) = -\int_{x_d}^\infty \partial_d F_1(x', t)\,\mathrm{d}t.$$

Since $\partial_d F_1 = \partial_1 F_d$, we get by Fubini's theorem and integration by parts,

$$\int_{\mathbb{R}^{d-1}} F_1(x', x_d) \cdot \tilde{v} * \rho_\epsilon * \varphi(x', x_d)\,\mathrm{d}x' = -\int_{\mathbb{R}^{d-1}} \left(\int_{x_d}^{\infty} \partial_1 F_d(x', t) \cdot \tilde{v} * \rho_\epsilon * \varphi(x', x_d)\,\mathrm{d}t \right) \mathrm{d}x'$$
$$= \int_{x_d}^{\infty} \left(\int_{\mathbb{R}^{d-1}} F_d(x', t) \cdot \tilde{v} * \partial_1 \rho_\epsilon * \varphi(x', x_d)\,\mathrm{d}x' \right) \mathrm{d}t.$$

By (26), we have

$$\left| \tilde{v} * \partial_1 \rho_\epsilon * \varphi(x', x_d) \right| \le \frac{C_5}{\epsilon^{\frac{d-s}{d}}} \mathcal{M}\Phi(x_d),$$

which implies (24).

Combining (23) and (24), we get for every $\epsilon > 0$,

$$\left| \int_{\mathbb{R}^{d-1}} F_1(x', x_d) \cdot \tilde{v} * \varphi(x', x_d)\,\mathrm{d}x' \right| \le \left(C_2 \epsilon^{\frac{s}{d}} \left\| F_1(\cdot, x_d) \right\|_{L^1(\mathbb{R}^{d-1})} + \frac{C_3}{\epsilon^{\frac{d-s}{d}}} \|F_d\|_{L^1(\mathbb{R}^d)} \right) \mathcal{M}\Phi(x_d).$$

Then, optimizing the right-hand side with respect to ϵ, we get for every $x_d \in \mathbb{R}$,

$$\left| \int_{\mathbb{R}^{d-1}} F_1(x', x_d) \cdot \tilde{v} * \varphi(x', x_d)\,\mathrm{d}x' \right| \le C_7 \left\| F_1(\cdot, x_d) \right\|_{L^1(\mathbb{R}^{d-1})}^{\frac{d-s}{d}} \|F_d\|_{L^1(\mathbb{R}^d)}^{\frac{s}{d}} \mathcal{M}\Phi(x_d).$$

Integrating with respect to x_d and applying Hölder's inequality, we then get

$$\left| \int_{\mathbb{R}^d} v * F_1 \cdot \varphi \right| = \left| \int_{\mathbb{R}^d} F_1 \cdot \tilde{v} * \varphi \right| \le C_7 \|F_1\|_{L^1(\mathbb{R}^d)}^{\frac{d-s}{d}} \|F_d\|_{L^1(\mathbb{R}^d)}^{\frac{s}{d}} \|\mathcal{M}\Phi\|_{L^{\frac{d}{s}}(\mathbb{R})}.$$

Since $s < d$, we have the strong type estimate for $\mathcal{M}\Phi$,

$$\|\mathcal{M}\Phi\|_{L^{\frac{d}{s}}(\mathbb{R})} \le C_8 \|\Phi\|_{L^{\frac{d}{s}}(\mathbb{R})} = \|\varphi\|_{L^{\frac{d}{s}}(\mathbb{R}^d)},$$

the conclusion follows from the Riesz Representation Theorem by taking the supremum with respect to all $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\|\varphi\|_{L^{\frac{d}{s}}(\mathbb{R}^d)} \le 1$.

In the case where F merely belongs to $L^1(\mathbb{R}^d; \mathbb{R}^d)$, one may apply the inequality thus obtained with $\rho_r * F$ for any $r > 0$. Note that in this case that (20) holds pointwise by $\rho_r * F$ in $\mathbb{R}^d$ and, for every $l \in \{1, \dots, d\}$, since $F \in L^1(\mathbb{R}^d; \mathbb{R}^d)$, the gradient

$$\nabla(\rho_r * F_l) = (\nabla \rho_r) * F$$

is summable in $\mathbb{R}^d$. We then have, for every $r > 0$ and $u \in C_c^\infty(\mathbb{R}^d)$,

$$\left\| v * (\rho_r * F) \right\|_{L^{\frac{d}{d-s}}} \le C' \|\rho_r * F\|_{L^1} \le C' \|F\|_{L^1}.$$

As $r \to 0$, the conclusion then follows from Fatou's lemma. □

The constant $C' > 0$ in the statement of Proposition 16 is under control as long as s/d stays away from 0 and 1. In particular, it can be taken independently of s when $1/2 \le s \le 1$.

Proof of Theorem 2. In dimension $d \ge 2$, the representation formula (4) holds for every $u \in C_c^\infty(\mathbb{R}^d)$. We rely on the explicit formula of the constant $A = A(s, d)$ such that $V_s = Ax/|x|^{d-s+1}$ that can be obtained using the Fourier transform of homogeneous functions. In fact, one finds that

$$A = K(s, d)(1 - s) \quad \text{where} \quad K(s, d) := \frac{1}{\sigma_d \pi^{d/2}} \frac{\Gamma\left(\frac{d+1+s}{2}\right)\Gamma\left(\frac{d+1-s}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{3-s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}.$$

We next apply Proposition 16 with $F = \nabla^s u$ and v given by the components of V_s/A. Estimate (19) is then verified with a constant $C > 0$ that is independent of $1/2 \le s < 1$ and we find

$$\frac{1}{A} \|u\|_{L^{\frac{d}{d-s}}} \le \sum_{j=1}^{d} \left\| V_{s,j} * \nabla^s u \right\|_{L^{\frac{d}{d-s}}} \le dC' \|\nabla^s u\|_{L^1}.$$

Hence, by the form of A,

$$\|u\|_{L^{\frac{d}{d-s}}} \le dC' K(s,d)(1-s)\|\nabla^s u\|_{L^1},$$

which gives the conclusion since $K(s,d)$ is bounded by a constant independent of $1/2 \le s < 1$. $\square$

4. Characterization of radial homogeneous distributions

We say that a distribution T is represented in an open set Ω by a function $f \in L^1_{\mathrm{loc}}(\Omega;\mathbb{C})$ whenever

$$\langle T,\varphi\rangle = \int_{\mathbb{R}^d} f\varphi \quad \text{for every } \varphi \in C_c^\infty(\Omega). \tag{27}$$

We then use the notation $[T] := f$ in Ω. The aim of this section is to identify radial homogeneous distributions that are represented in $\mathbb{R}^N$ by a locally integrable function.

Definition 18. *Let T be a distribution in $\mathbb{R}^d \setminus \{0\}$. We say that T is* radial *whenever, for each orthogonal transformation $R \in O(d)$,*

$$\langle T,\varphi\circ R\rangle = \langle T,\varphi\rangle \quad \textit{for every } \varphi \in C_c^\infty\big(\mathbb{R}^d \setminus \{0\}\big).$$

Given $\lambda \in \mathbb{R}$, we say that T is homogeneous of order λ *whenever, for each $t > 0$,*

$$\langle T,\varphi_t\rangle = t^\lambda \langle T,\varphi\rangle \quad \textit{for every } \varphi \in C_c^\infty\big(\mathbb{R}^d \setminus \{0\}\big),$$

where φ_t is the function defined by

$$\varphi_t(x) = \frac{1}{t^d}\varphi\Big(\frac{x}{t}\Big). \tag{28}$$

Note, for example, that the distribution associated to $|x|^\lambda$ in $\mathbb{R}^d \setminus \{0\}$ is radial and homogeneous of order $\lambda \in \mathbb{R}$. These definitions have immediate counterparts for distributions in $\mathbb{R}^d$ and $|x|^\lambda$ defines a distribution in $\mathbb{R}^d$ for $\lambda > -d$.

We present a proof of the following proposition based on an unpublished note by E. Y. Jaffe [22].

Proposition 19. *Let $d \ge 2$. If T is a radial homogeneous distribution of order 0 in $\mathbb{R}^d \setminus \{0\}$, then T is constant. More precisely, there exists a constant $C \in \mathbb{R}$ such that*

$$\langle T,\varphi\rangle = C\int_{\mathbb{R}^d\setminus\{0\}} \varphi \quad \textit{for every } \varphi \in C_c^\infty\big(\mathbb{R}^d \setminus \{0\}\big).$$

We begin with a counterpart of Euler's homogeneous condition for distributions of order zero.

Lemma 20. *If T is a distribution of order 0 in $\mathbb{R}^d \setminus \{0\}$, then*

$$\sum_{j=1}^d x_j \partial_j T = 0 \quad \textit{in } \mathbb{R}^d \setminus \{0\}.$$

Proof of Lemma 20. Let $\varphi \in C_c^\infty\big(\mathbb{R}^d \setminus \{0\}\big)$ and φ_t be given by (28). Since T is homogeneous of order 0, for every $t > 0$ we have

$$\langle T,\varphi_t\rangle = \langle T,\varphi\rangle. \tag{29}$$

Observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x)\bigg|_{t=1} = -d\varphi(x) - \nabla\varphi(x)\cdot x = -\operatorname{div}\big(\varphi(x)x\big).$$

Thus, differentiating both sides of (29) with respect to t at 1, one gets

$$\Big\langle \sum_{j=1}^d x_j\partial_j T,\varphi\Big\rangle = -\big\langle T_x, \operatorname{div}\big(\varphi(x)x\big)\big\rangle = 0. \qquad \square$$

We now prove that a radial distribution has zero derivative with respect to vector fields that are tangential to spheres centered at 0.

Lemma 21. *If T is a radial distribution in $\mathbb{R}^d \setminus \{0\}$, then for every smooth functions $c_1, \dots, c_d$ such that $\sum_{j=1}^{d} c_j(x) x_j = 0$ in $\mathbb{R}^d \setminus \{0\}$, we have*

$$\sum_{j=1}^{d} c_j \partial_j T = 0 \quad \text{in } \mathbb{R}^d \setminus \{0\}.$$

Proof of Lemma 21. Let $(\rho_\epsilon)_{\epsilon>0}$ be a family of radial mollifiers in $C_c^\infty(\mathbb{R}^d)$ such that ρ_ϵ is supported in the ball B_ϵ. Given $r > 0$, for every $x \in \mathbb{R}^d \setminus \overline{B}_r$ and $0 < \epsilon < r$ the function $y \mapsto \rho_\epsilon(x - y)$ is supported in $\mathbb{R}^d \setminus \{0\}$ and then the convolution

$$\rho_\epsilon * T(x) := \langle T_y, \rho_\epsilon(x - y) \rangle$$

is well-defined and smooth in $\mathbb{R}^d \setminus \overline{B}_r$. Given $R \in O(d)$, by radiality of T and ρ we have

$$\rho_\epsilon * T(Rx) = \langle T_y, \rho_\epsilon(Rx - y) \rangle = \langle T_y, \rho_\epsilon(Rx - Ry) \rangle = \langle T_y, \rho_\epsilon(x - y) \rangle = \rho_\epsilon * T(x).$$

Thus, $\rho_\epsilon * T$ is a radial function in $\mathbb{R}^d \setminus \overline{B}_r$.

By the assumption on the functions $c_1, \dots, c_d$, the vector $v := \big(c_1(x), \dots, c_d(x)\big)$ belongs to the tangent plane of the sphere $\partial B_{|x|}$ at x. Thus, there exists a smooth curve $\gamma \colon (-1, 1) \to \partial B_{|x|}$ such that $\gamma(0) = x$ and $\gamma'(0) = v$. In particular, $(\rho_\epsilon * T) \circ \gamma$ is constant and applying the chain rule we get

$$\sum_{j=1}^{d} c_j(x) \partial_j (\rho_\epsilon * T)(x) = \frac{\mathrm{d}}{\mathrm{d}t} (\rho_\epsilon * T) \circ \gamma(t) \Big|_{t=0} = 0.$$

As $\epsilon \to 0$, the family of functions $\sum_{j=1}^{d} c_j \partial_j (\rho_\epsilon * T)$ converges weakly in the sense of distributions in $\mathbb{R}^d \setminus \overline{B}_r$ to $\sum_{j=1}^{d} c_j \partial_j T$. Hence,

$$\sum_{j=1}^{d} c_j \partial_j T = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B}_r.$$

Since this property holds for any $r > 0$, the conclusion follows. □

Proof of Proposition 19. Let $k \in \{1, \dots, d\}$. There exist smooth functions $p \colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ and $w \colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d$ such that, for every $x \neq 0$,

$$e_k = p(x) \sum_{j=1}^{d} x_j e_j + w(x) \quad \text{and} \quad w(x) \cdot x = 0.$$

Indeed, it suffices to take $p(x) := x \cdot e_k / |x|^2$. Writing $w = \sum_{j=1}^{d} c_j e_j$ for smooth functions $c_1, \dots, c_d$ in $\mathbb{R}^d \setminus \{0\}$, then by Lemmas 20 and 21 we have

$$\sum_{j=1}^{d} x_j \partial_j T = 0 \quad \text{and} \quad \sum_{j=1}^{d} c_j \partial_j T = 0 \quad \text{in } \mathbb{R}^d \setminus \{0\}.$$

Thus, for every $k \in \{1, \dots, d\}$,

$$\partial_k T = p \sum_{j=1}^{d} x_j \partial_j T + \sum_{j=1}^{d} c_j \partial_j T = 0 \quad \text{in } \mathbb{R}^d \setminus \{0\}.$$

Since $d \geq 2$, the set $\mathbb{R}^d \setminus \{0\}$ is connected and we conclude that T is constant. □

Observe that when f is a homogeneous function of order $\gamma > -d$, then $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and one may simply take as T the distribution associated to f by integration, more precisely, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\langle T, \varphi \rangle = \int_{\mathbb{R}^d} f \varphi.$$

Note in this case that $[T] = f$ in $\mathbb{R}^d$.

Corollary 22. *Let $d \geq 2$ and $\lambda > -d$. If T is a radial homogeneous distribution of order λ in $\mathbb{R}^d$, then T is represented by the function $C|x|^\lambda$ in $\mathbb{R}^d$ for some constant $C \in \mathbb{R}$, that is,*

$$[T] = C|x|^\lambda \quad \text{in } \mathbb{R}^d.$$

Proof. Since the function $|x|^{-\lambda}$ is smooth in $\mathbb{R}^d \setminus \{0\}$, $|x|^{-\lambda} T$ is well-defined as a distribution in $\mathbb{R}^d \setminus \{0\}$. This distribution is radial and homogeneous of order 0, whence by Proposition 19 it is represented by a constant in $\mathbb{R}^d \setminus \{0\}$. We deduce that T is represented by $C|x|^\lambda$ in $\mathbb{R}^d \setminus \{0\}$. Since $\lambda > -d$, we have $C|x|^\lambda \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and then this function represents a distribution S in $\mathbb{R}^d$. It thus follows that the distribution $T - S$ is supported by $\{0\}$, whence it is a finite combination of a Dirac mass and its derivatives. Since these are all homogeneous distributions of order less than or equal to $-d$ and $T - S$ is homogeneous of order λ greater than $-d$, we deduce that $T - S = 0$ in $\mathbb{R}^d$. □

5. Fourier transform of radial homogeneous distributions

The Fourier transform of $f \in L^1(\mathbb{R}^d)$ is the bounded continuous function $\widehat{f} \colon \mathbb{R}^d \to \mathbb{C}$ defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \imath x \cdot \xi} f(x)\,\mathrm{d}x$$

and the inverse Fourier transform $\check{f} \colon \mathbb{R}^d \to \mathbb{C}$ is

$$\check{f}(x) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi \imath x \cdot \xi} f(\xi)\,\mathrm{d}\xi.$$

More generally, for a tempered distribution T in $\mathbb{R}^d$, we denote by $\mathscr{F} T$ the distributional Fourier transform defined for every Schwartz function η in $\mathbb{R}^d$ by

$$\langle \mathscr{F} T, \eta \rangle = \langle T, \widehat{\eta} \rangle$$

and, by analogy, one defines the inverse Fourier transform $\mathscr{F}^{-1} T$ using $\check{\varphi}$. Observe that $\widehat{\varphi}$ and $\check{\varphi}$ are Schwartz functions in $\mathbb{R}^d$, whence $\mathscr{F} T$ and $\mathscr{F}^{-1} T$ are well-defined.

Example 23. If a tempered distribution T is represented by a function in $\mathbb{R}^d$ and $[T] \in L^1(\mathbb{R}^d)$, then it follows from Fubini's theorem that $\mathscr{F} T$ is represented by the Fourier transform of $[T]$,

$$[\mathscr{F} T] = \widehat{[T]} \quad \text{in } \mathbb{R}^d.$$

For $[T] \in L^2(\mathbb{R}^d)$, not necessarily summable, $[\mathscr{F} T]$ is the L^2 Fourier–Plancherel transform of T. More generally, assume that $[T]$ is merely a locally integrable function in $\mathbb{R}^d$ with at most polynomial growth at infinity. In this case, one can compute $[\mathscr{F} T]$ by approximation of $[T]$ as follows,

$$\langle \mathscr{F} T, \eta \rangle = \lim_{r \to \infty} \int_{\mathbb{R}^d} \widehat{[T]\chi_{B_r}}\, \eta, \tag{30}$$

where η is any Schwartz function. Indeed, since T is a tempered distribution and $[T]$ has at most polynomial growth at infinity, we may approximate η by smooth functions with compact support and deduce the companion identity to (27) for tempered distributions,

$$\langle T, \eta \rangle = \int_{\mathbb{R}^d} [T]\, \eta.$$

As a result, replacing in this identity η by $\widehat{\eta}$, by the Dominated Convergence Theorem we get

$$\langle \mathscr{F} T, \eta \rangle = \langle T, \widehat{\eta} \rangle = \int_{\mathbb{R}^d} [T]\, \widehat{\eta} = \lim_{r \to \infty} \int_{\mathbb{R}^d} [T]\chi_{B_r} \widehat{\eta},$$

which immediately implies (30) by a standard property of the Fourier transform of a function in $L^1(\mathbb{R}^d)$.

Observe that if T is a tempered radial distribution in $\mathbb{R}^d$ that is homogeneous of degree $\alpha - d$, then $\mathscr{F}T$ is a tempered radial distribution in $\mathbb{R}^d$ that is homogeneous of degree $-\alpha$. We may thus rely on a large class of homogeneous functions in $\mathbb{R}^d$ to construct distributions whose Fourier transforms can be easily identified by Corollary 22.

Example 24. Given $0 < \alpha < d$, consider the homogeneous function f_α of order $\alpha - d$ defined for $x \neq 0$ by

$$f_\alpha(x) = \frac{1}{|x|^{d-\alpha}}, \tag{31}$$

which belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and defines a tempered distribution T_α in $\mathbb{R}^d$. Note that T_α is radial and homogeneous of order $\alpha - d$. Hence, $\mathscr{F}T_\alpha$ is radial and homogeneous of order $-\alpha$ in $\mathbb{R}^d$. Therefore, by Corollary 22, we deduce that

$$[\mathscr{F}T_\alpha](\xi) = \frac{c_\alpha}{|\xi|^\alpha} \quad \text{in } \mathbb{R}^d. \tag{32}$$

One then shows that $c_\alpha = \pi^{\frac{d}{2}-\alpha}\Gamma\big(\frac{\alpha}{2}\big)/\Gamma\big(\frac{d-\alpha}{2}\big)$, see [31, p. 490].

If $\alpha \le 0$ in (31), then f_α does not belong to $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. However, let us recall a remarkable property of homogeneous functions in $\mathbb{R}^d \setminus \{0\}$ proved by Hörmander.

Theorem 25 ([20, Theorems 3.2.3 and 3.2.4]). *For every homogeneous function $f \in C^\infty\big(\mathbb{R}^d \setminus \{0\}\big)$, there exists a tempered distribution T in $\mathbb{R}^d$ that is represented by f in $\mathbb{R}^d \setminus \{0\}$, in other words,*

$$[T] = f \quad \text{in } \mathbb{R}^d \setminus \{0\}.$$

The proof of Theorem 25 is based on the explicit construction of the tempered distribution and one can follow the argument to construct suitable T_α with $\alpha \le 0$, see the examples below. Concerning the Fourier transform of a tempered distribution, we also recall the following result.

Theorem 26 ([20, Theorem 7.1.18]). *If a distribution T in $\mathbb{R}^d$ is represented in $\mathbb{R}^d \setminus \{0\}$ by a smooth homogeneous function $f \colon \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$, then T is a tempered distribution and $\mathscr{F}T$ is also represented in $\mathbb{R}^d \setminus \{0\}$ by a smooth homogeneous function $h \colon \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$.*

Note that Theorem 26 does not provide a clear relation between the functions that represent T and $\mathscr{F}T$. Here we rely on the characterization of the previous section to compute the Fourier transform of T_α for $\alpha < 0$ that is not an even integer.

Example 27. When $\alpha \le 0$ is not an integer, let T_α be the tempered distribution defined for every Schwartz function η by

$$\langle T_\alpha, \eta \rangle = \int_{\mathbb{R}^d} \frac{1}{|x|^{d-\alpha}} \big(\eta(x) - P_0^k \eta(x)\big)\, \mathrm{d}x, \tag{33}$$

where $P_0^k \eta$ is the Taylor polynomial of η of order $k \in \mathbb{N}$ at 0. We take k such that $k + 1 + \alpha > 0$ to ensure summability in a neighborhood of the origin and $k + \alpha < 0$ for summability near infinity. For (33) we refer to the already mentioned proof of [20, Theorem 3.2.4] or [31, equation (25.22)]. Note that T_α is represented in $\mathbb{R}^d \setminus \{0\}$ by f_α given by (31). This distribution T_α is radial and homogeneous of order $\alpha - d$. Its Fourier transform $\mathscr{F}T_\alpha$ is a radial homogeneous distribution of order $-\alpha$. To see why it is homogeneous, one first observes that

$$\widehat{\eta_t}(x) = \widehat{\eta}\Big(\frac{x}{t}\Big) \quad \text{and} \quad P_0^k \widehat{\eta_t}(x) = P_0^k \widehat{\eta}\Big(\frac{x}{t}\Big).$$

By Corollary 22, $\mathscr{F}T_\alpha$ is therefore represented in $\mathbb{R}^d$ by the L^1_{loc} function (32).

Example 28. Assume that $\alpha \le 0$ is an integer and take $k := -\alpha$. When $k = -\alpha$ is odd, we may take (33) as the definition of T_α by considering the integral as a principal value, more precisely,

$$\langle T_\alpha, \eta \rangle = \lim_{r \to \infty} \int_{B_r} \frac{1}{|x|^{d-\alpha}} \big(\eta(x) - P_0^k \eta(x)\big)\, \mathrm{d}x.$$

Summability near infinity is then ensured by the fact that the term of order k in the Taylor polynomial is an odd function and therefore its principal value integral is equal to zero. Since $\mathscr{F}T_\alpha$ is radial and homogeneous of order $-\alpha$, it is also represented in $\mathbb{R}^d$ by the L^1_{loc} function (32).

When $k=-\alpha$ is even, it is not possible to define a homogeneous tempered distribution in $\mathbb{R}^d$ that can be represented by $1/|x|^{d-\alpha}$ in $\mathbb{R}^d\setminus\{0\}$ and one has to waive the homogeneity property. An alternative in this case is to take T_α defined for every Schwartz function η by

$$\langle T_\alpha,\eta\rangle=\frac{-1}{k!}\left[\int_0^\infty \log\rho\,\overline{\eta}^{(k+1)}(\rho)\,\mathrm{d}\rho+\overline{\eta}^{(k)}(0)\sum_{j=1}^{k}\frac{1}{j}\right],$$

where

$$\overline{\eta}(\rho):=\int_{\partial B_1}\eta(\rho y)\,\mathrm{d}\sigma(y).$$

The distribution T_α is not homogeneous but is represented in $\mathbb{R}^d\setminus\{0\}$ by f_α given by (31). In fact, if $\varphi\in C_c^\infty\big(\mathbb{R}^d\setminus\{0\}\big)$ then, integrating by parts, one gets

$$\langle T_\alpha,\varphi\rangle=-\frac{1}{k!}\int_0^\infty \log\rho\,\overline{\varphi}^{(k+1)}(\rho)\,\mathrm{d}\rho=\int_{\mathbb{R}^d}\frac{\varphi(x)}{|x|^{d+k}}\,\mathrm{d}x.$$

The Fourier transform of such a distribution can be explicitly evaluated for every Schwartz function η as

$$\langle\mathscr{F}T_\alpha,\eta\rangle=\int_{\mathbb{R}^d}\big(-A_k\log|\xi|+\lambda_k\big)|\xi|^k\eta(\xi)\,\mathrm{d}\xi, \tag{34}$$

where

$$A_k:=(-1)^{\frac{k}{2}}\frac{2\pi^{k+\frac{d}{2}}}{\Gamma\big(\frac{N+k}{2}\big)(k/2)!}\quad\text{and}\quad \lambda_k:=\frac{\mathrm{d}}{\mathrm{d}\alpha}\big((\alpha+k)c_\alpha\big)\bigg|_{\alpha=-k}. \tag{35}$$

We refer to [31, Lemma 25.2 and Remark 25.2] for a proof of (34).

6. Positivity of the Fourier transform

The following result concerning the positivity of the Fourier transform is based on a standard argument, see e.g. [3, Section 5.1]. We present a proof for the convenience of the reader.

Proposition 29. *Let T be a tempered distribution represented in $\mathbb{R}^d$ by a radial function $[T]\in C^\infty\big(\mathbb{R}^d\setminus\{0\}\big)$ that is locally integrable in $\mathbb{R}^d$ and is such that, for any $x\neq 0$, the function*

$$\rho\in(0,\infty)\longmapsto\rho^{d-1}[T](\rho x)$$

is non-increasing, convex, and converges to zero as $\rho\to\infty$. If $\mathscr{F}T$ is represented by a function $[\mathscr{F}T]$ in $\mathbb{R}^d$ which is smooth in $\mathbb{R}^d\setminus\{0\}$ and $[T]$ is not identically zero, then $[\mathscr{F}T]>0$ in $\mathbb{R}^d\setminus\{0\}$.

We rely on the following elementary property.

Lemma 30. *Let Λ be a locally integrable function in $[0,\infty)$. If Λ is nonnegative, non-increasing and converges to zero at infinity, then, for every $r>0$,*

$$\int_0^r \sin(2\pi\tau)\Lambda(\tau)\,\mathrm{d}\tau\geq 0 \tag{36}$$

and the the integral converges in the extended interval $[0,\infty]$ as $r\to\infty$.

Proof of Lemma 30. Since Λ is nonnegative, for $r\in(k,k+1/2]$ with $k\in\mathbb{N}$ we have

$$\int_0^r \sin(2\pi\tau)\Lambda(\tau)\,\mathrm{d}\tau\geq\int_0^k \sin(2\pi\tau)\Lambda(\tau)\,\mathrm{d}\tau,$$

whereas for $r\in(k+1/2,k+1]$ with $k\in\mathbb{N}$ we have

$$\int_0^r \sin(2\pi\tau)\Lambda(\tau)\,\mathrm{d}\tau\geq\int_0^{k+1} \sin(2\pi\tau)\Lambda(\tau)\,\mathrm{d}\tau.$$

By additivity of the integral, it thus suffices to show that, for every $j \in \mathbb{N}$,

$$\int_j^{j+1} \sin(2\pi\tau)\Lambda(\tau)\,\mathrm{d}\tau \geq 0.$$

Since Λ is nonnegative and non-increasing,

$$\int_j^{j+1} \sin(2\pi\tau)\Lambda(\tau)\,\mathrm{d}\tau \geq \Lambda(j+1/2)\int_j^{j+1} \sin(2\pi\tau)\,\mathrm{d}\tau = 0.$$

Thus, the sequence of integrals in (36) with $r = j \in \mathbb{N}$ is nondecreasing, whence its limit

$$\lim_{j\to\infty}\int_0^j \sin(2\pi\tau)\Lambda(\tau)\,\mathrm{d}\tau$$

exists in $[0,\infty]$. To conclude the existence of the limit of the integrals in (36) as $r \to \infty$, it suffices to observe that for every $r \in [j, j+1]$ we have

$$\left|\int_j^r \sin(2\pi\tau)\Lambda(\tau)\,\mathrm{d}\tau\right| \leq (r-j)\Lambda(j) \leq \Lambda(j)$$

and that by assumption the right-hand side converges to zero as $j \to \infty$. □

Proof of Proposition 29. By (30), we have

$$\langle \mathscr{F}T, \eta\rangle = \lim_{r\to\infty}\int_{\mathbb{R}^d} \widehat{f_r}\,\eta, \tag{37}$$

where $f := [T]$ and $f_r := f\chi_{B_r}$. Since f_r is radial, and in particular even, applying the integration formula in polar coordinates we get

$$\widehat{f_r}(\xi) = \int_{B_r} \cos(2\pi x\cdot\xi) f(x)\,\mathrm{d}x = \int_{\partial B_1}\left(\int_0^r \cos(2\pi\rho y\cdot\xi)\vartheta(\rho)\rho^{d-1}\,\mathrm{d}\rho\,\mathrm{d}\sigma(y)\right),$$

where $\vartheta \in C^\infty(0,\infty)$ is such that $\vartheta(|x|) = f(x)$ for every $x \neq 0$.

We show that the innermost integral is positive for every $y \in \partial B_1$ and $\xi \in \mathbb{R}^d$. It is enough to consider the case where $\alpha := y\cdot\xi \neq 0$. Denote $\Theta(\rho) = -\rho^{d-1}\vartheta(\rho)$, so that

$$\int_0^r \cos(2\pi\rho y\cdot\xi)\vartheta(\rho)\rho^{d-1}\,\mathrm{d}\rho = -\int_0^r \cos(2\pi\rho\alpha)\Theta(\rho)\,\mathrm{d}\rho.$$

Integrating by parts, for every $0 < \epsilon < r$ we get

$$\int_\epsilon^r \cos(2\pi\rho y\cdot\xi)\vartheta(\rho)\rho^{d-1}\,\mathrm{d}\rho = -\frac{1}{2\pi|\alpha|}\sin\big(2\pi\rho|\alpha|\big)\Theta(\rho)\Big|_{\rho=\epsilon}^{r} + \frac{1}{2\pi|\alpha|}\int_\epsilon^r \sin\big(2\pi\rho|\alpha|\big)\Theta'(\rho)\,\mathrm{d}\rho.$$

Note that by assumption the function Θ' is nonnegative, whence by the Fundamental Theorem of Calculus it is locally integrable in $[0,\infty)$. On the other hand, since f is locally integrable, we have $\int_0^1 \Theta(\rho)\rho\,\mathrm{d}\rho/\rho < \infty$. Thus, there exists a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive numbers in $(0,1)$ that converges to zero and satisfies $\Theta(\epsilon_n)\epsilon_n \to 0$. Taking $\epsilon = \epsilon_n$ and letting $n \to \infty$, we then get

$$\int_0^r \cos(2\pi\rho y\cdot\xi)\vartheta(\rho)\rho^{d-1}\,\mathrm{d}\rho = -\frac{1}{2\pi|\alpha|}\sin\big(2\pi r|\alpha|\big)\Theta(r) + \frac{1}{2\pi|\alpha|}\int_0^r \sin\big(2\pi\rho|\alpha|\big)\Theta'(\rho)\,\mathrm{d}\rho.$$

By assumption, Θ converges to zero at infinity. Hence, the first term in the right-hand side converges to zero as $r \to \infty$, uniformly with respect to α. Moreover, by Lemma 30, the second term is nonnegative and converges as $r \to \infty$. Hence, by the Dominated Convergence Theorem and Fatou's lemma, for every nonnegative Schwartz function η we get

$$\lim_{r\to\infty}\int_{\mathbb{R}^d}\widehat{f_r}\,\eta \geq \int_{\mathbb{R}^d}\left(\int_{\partial B_1}\frac{1}{2\pi|y\cdot\xi|}\int_0^\infty \sin\big(2\pi\rho|y\cdot\xi|\big)\Theta'(\rho)\,\mathrm{d}\rho\,\mathrm{d}y\right)\eta(\xi)\,\mathrm{d}\xi.$$

Since (37) holds and $\mathscr{F}T$ is represented by a smooth function in $\mathbb{R}^d\setminus\{0\}$, we deduce that

$$[\mathscr{F}T](\xi) \geq \int_{\partial B_1}\frac{1}{2\pi|y\cdot\xi|}\int_0^\infty \sin\big(2\pi\rho|y\cdot\xi|\big)\Theta'(\rho)\,\mathrm{d}\rho\,\mathrm{d}y \geq 0.$$

Since the function Θ' is non-trivial, an inspection of the proof of Lemma 30 shows that the innermost integral above with respect to ρ is positive and the conclusion follows. □

7. Fourier transform of radial doubly homogeneous functions

In this section, we consider tempered distributions that are represented by functions that are homogeneous in a neighborhood of the origin and at infinity, but the homogeneity rates need not be the same.

Proposition 31. *Take $0 < \alpha < d$ and $\beta < d$ with $\beta \notin -2\mathbb{N}$. Let $g \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be a radial function such that*

$$g(x) = \begin{cases} a/|x|^{d-\alpha} & \text{if } |x| \le r, \\ b/|x|^{d-\beta} & \text{if } |x| \ge R, \end{cases}$$

where $0 < r < R$ and $a, b > 0$. Then, there exists a tempered distribution S_g in $\mathbb{R}^d$ that is represented in $\mathbb{R}^d$ by g and its Fourier transform $\mathscr{F} S_g$ is represented in $\mathbb{R}^d$ by a radial function $[\mathscr{F} S_g] \in C^\infty(\mathbb{R}^d \setminus \{0\})$ that satisfies

$$[\mathscr{F} S_g](\xi) = \begin{cases} bc_\beta/|\xi|^\beta + \eta(\xi) & \text{for } |\xi| \le r, \\ ac_\alpha/|\xi|^\alpha + \zeta(\xi) & \text{for } |\xi| \ge R, \end{cases}$$

where η and ζ are Schwartz functions.

We rely on the following result.

Lemma 32. *If a tempered distribution T in $\mathbb{R}^d$ is represented in $\mathbb{R}^d \setminus \{0\}$ by a homogeneous function $f \in C^\infty(\mathbb{R}^d \setminus \{0\})$, then, for every $\varphi, \psi \in C_c^\infty(\mathbb{R}^d)$ that equal 1 in a neighborhood of 0, both distributions*

$$\mathscr{F}(\varphi T) - (1-\psi)\mathscr{F} T \quad \text{and} \quad \mathscr{F}\big((1-\varphi)T\big) - \psi \mathscr{F} T \tag{38}$$

are represented in $\mathbb{R}^d$ by Schwartz functions.

Proof of Lemma 32. Denote the distributions in (38) by L_1 and L_2, respectively. Observe that $L_1 + L_2 = 0$. Since φT is a distribution with compact support, $\mathscr{F}(\varphi T)$ can be represented by a smooth function in $\mathbb{R}^d$; see [18, Theorem 2.3.21]. Moreover, by homogeneity of f, it follows from Theorem 26 that $\mathscr{F} T$ satisfies

$$\langle \mathscr{F} T, \phi\rangle = \int_{\mathbb{R}^d} h\phi \quad \text{for every } \phi \in C_c^\infty(\mathbb{R}^d \setminus \{0\}),$$

for some smooth function $h \colon \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$. Since $1 - \varphi = 0$ in a neighborhood of 0, we deduce that $(1-\varphi)\mathscr{F} T$ is represented in $\mathbb{R}^d$ by the smooth function $(1-\psi)h$. Hence, by linearity, L_1 is represented in $\mathbb{R}^d$ by a smooth function $[L_1]$. As $L_2 = -L_1$, the same holds for the distribution L_2 and for the sequel we write in $\mathbb{R}^d \setminus \{0\}$,

$$[L_2] = \big[\mathscr{F}\big((1-\varphi)T\big)\big] - \psi[\mathscr{F} T].$$

Since the function $\psi[\mathscr{F} T]$ is smooth in $\mathbb{R}^d \setminus \{0\}$ and has compact support in $\mathbb{R}^d$, to show that $[L_2]$ is a Schwartz function, it suffices to show that

$$p := \big[\mathscr{F}\big((1-\varphi)T\big)\big] = \big[\mathscr{F}\big((1-\varphi)h\big)\big]$$

and its derivatives have a fast decay at infinity. More precisely, for every multi-index ν and every $j \in \mathbb{N}$ sufficiently large, we have

$$\big|\partial^\nu p(\xi)\big| \le C_1 |\xi|^{-2j} \quad \text{for every } |\xi| \ge 1, \tag{39}$$

where $C_1 > 0$ is some constant depending on ν and j. Since $(1-\varphi)T$ is represented in $\mathbb{R}^d$ by the smooth function $(1-\varphi)h$, for any $j \in \mathbb{N}$ we have

$$\big(2\pi|\xi|\big)^{2j}\partial^\nu p(\xi) = [\mathscr{F}P_{j,\nu}],$$

where $P_{j,\nu}(x) := \Delta^j\big((-2\pi i x)^\nu(1-\varphi)h\big)$ and Δ denotes the Laplacian. Thanks to the Leibniz rule, the fact that φ has compact support and the homogeneity of h, we deduce that

$$\int_{\mathbb{R}^d}|P_{j,\nu}| \le C_2\int_{\partial B_1} h\,\mathrm{d}\sigma\int_1^\infty \rho^{\alpha-1-|\nu|}\,\mathrm{d}\rho,$$

where $\alpha \in \mathbb{R}$ is the degree of homogeneity of h. When $2j > d-\alpha+|\nu|$ we obtain $P_{j,\nu} \in L^1(\mathbb{R}^d)$ and its Fourier transform is a bounded continuous function. Then, given a multi-index ν, taking any j as above we deduce that p satisfies (39). It follows that $[L_2]$ is a Schwartz function, which completes the proof. □

Proof of Proposition 31. We may assume for simplicity that $a = b = 1$, $r = 1$ and $R = 2$. Given a radial function $\varphi_1 \in C_c^\infty(\mathbb{R}^d)$ equal to 1 in the ball $B_1(0)$ and 0 in the complement of the ball $B_2(0)$, we decompose g as follows

$$g = g_1 + g_2 + g_3, \tag{40}$$

where

$$g_1 := \varphi_1 g = \varphi_1 f_\alpha \quad\text{and}\quad g_2 := (1-\varphi_2)g = (1-\varphi_2)f_\beta$$

and $f_t(x) = 1/|x|^{d-t}$ with $t = \alpha,\beta$. Note that $g_3 = g - g_1 - g_2$ is smooth, vanishes in a neighborhood of 0 and has compact support. We may thus associate to g_3 a distribution S_{g_3} defined by integration with respect to g_3. Then, by definition S_{g_3} is represented by g_3 in $\mathbb{R}^d$ and its Fourier transform $\mathscr{F}S_{g_3}$ is represented in $\mathbb{R}^d$ by

$$[\mathscr{F}S_{g_3}] = \widehat{g_3}.$$

Since $0 < \alpha < d$, by Example 24 the tempered distribution T_α is represented by f_α in $\mathbb{R}^d$. We then have that the tempered distribution $S_{g_1} := \varphi_1 T_\alpha$ is represented by $g_1 = \varphi_1 f_\alpha$ in $\mathbb{R}^d$. We then deduce from Lemma 32 applied with $\varphi = \psi = \varphi_1$ that there exists a Schwartz function η_1 representing a distribution S_{η_1} such that

$$\mathscr{F}S_{g_1} = (1-\varphi_1)\mathscr{F}T_\alpha + S_{\eta_1}.$$

Since $\mathscr{F}T_\alpha$ is represented in $\mathbb{R}^d$ by $c_\alpha/|\xi|^\alpha$, we deduce that $\mathscr{F}S_{g_1}$ is represented in $\mathbb{R}^d$ by

$$[\mathscr{F}S_{g_1}] = (1-\varphi_1)\frac{c_\alpha}{|\xi|^\alpha} + \eta_1. \tag{41}$$

Finally, since $\beta < d$ and $\beta \neq -2\ell$ for any $\ell \in \mathbb{N}$, by Examples 24, 27 and 28 the tempered distribution T_β is represented by f_β in $\mathbb{R}^d\setminus\{0\}$. Since $1-\varphi_1$ vanishes in a neighborhood of 0, we then have that the tempered distribution $S_{g_2} := (1-\varphi_1)T_\beta$ is represented by $g_2 = (1-\varphi_1)f_\beta$ in $\mathbb{R}^d$. Moreover, from Lemma 32 there exists a Schwartz function η_2 representing a distribution S_{η_2} such that

$$\mathscr{F}S_{g_2} = \varphi_1\mathscr{F}T_\beta + S_{\eta_2}.$$

Observe that $\mathscr{F}T_\beta$ is represented in $\mathbb{R}^d$ by $c_\beta/|\xi|^\beta$. We deduce that $\mathscr{F}S_{g_2}$ is represented in $\mathbb{R}^d$ by

$$[\mathscr{F}S_{g_2}] = \varphi_1\frac{c_\beta}{|\xi|^\beta} + \eta_2.$$

We conclude that the tempered distribution $S_g := S_{g_1} + S_{g_2} + S_{g_3}$ is represented by g in $\mathbb{R}^d$ and its Fourier transform is represented in $\mathbb{R}^d$ by a function of the form in the statement. □

Let us now consider the case β equal to a negative even integer.

Proposition 33. *Take* $0 < \alpha < d$ *and* $\beta = -2\ell$ *for some* $\ell \in \mathbb{N}$. *Let* $g \in C^\infty\big(\mathbb{R}^d \setminus \{0\}\big)$ *be a radial function such that*

$$g(x) = \begin{cases} a/|x|^{d-\alpha} & \textit{if } |x| \le r, \\ b/|x|^{d-\beta} & \textit{if } |x| \ge R, \end{cases}$$

where $0 < r < R$ *and* $a, b > 0$. *Then, there exists a tempered distribution* S_g *in* $\mathbb{R}^d$ *that is represented in* $\mathbb{R}^d$ *by* g *and its Fourier transform* $\mathscr{F} S_g$ *is represented in* $\mathbb{R}^d$ *by a radial function* $[\mathscr{F} S_g] \in C^\infty\big(\mathbb{R}^d \setminus \{0\}\big)$ *that satisfies*

$$[\mathscr{F} S_g](\xi) = \begin{cases} \big(-A_{2\ell} \log|\xi| + \lambda_{2\ell}\big)|\xi|^{2\ell} + \eta(\xi) & \textit{for } |\xi| \le r, \\ a c_\alpha/|\xi|^\alpha + \zeta(\xi) & \textit{for } |\xi| \ge R, \end{cases}$$

where η *and* ζ *are Schwartz functions.*

Proof. Following the same notation of the proof of Proposition 31, let us decompose g as in (40). The treatment of g_1 and g_3 is the same as before. To deal with g_2 in the case $\beta = -2\ell$, for some $\ell \in \mathbb{N}$, notice that by Example 28, T_β is again represented by f_β in $\mathbb{R}^d \setminus \{0\}$. Therefore, the tempered distribution $S_{g_2} := (1-\varphi_1) T_\beta$ is represented by $g_2 = (1-\varphi_1) f_\beta$ in $\mathbb{R}^d$ and, again by Lemma 32, there exists a Schwartz function η_2 representing a distribution S_{η_2} such that

$$\mathscr{F} S_{g_2} = \varphi_1 \mathscr{F} T_\beta + S_{\eta_2}.$$

Since in this case $\mathscr{F} T_\beta$ is represented in $\mathbb{R}^d$ by $\big(-A_\ell \log|\xi| + \lambda_\ell\big)|\xi|^{2\ell}$, we deduce that $\mathscr{F} S_{g_2}$ is represented in $\mathbb{R}^d$ by

$$[\mathscr{F} S_{g_2}] = \varphi_1 \big(-A_\ell \log|\xi| + \lambda_\ell\big)|\xi|^{2\ell} + \eta_2. \qquad \square$$

8. Existence of the representation formula

We prove in this section the representation formula in Theorem 3 and then apply it to establish the nonlocal Sobolev inequality associated to the sum of Lebesgue spaces (Theorems 7 and 8). We rely on the next proposition with $\gamma = 2$, which is possible as we assume that $d \ge 3$.

Proposition 34. *Take* $0 < \alpha < \gamma < d$ *and* $\beta < \min\{\gamma, 1\}$ *with* $\beta \ne 0$. *If* $g \in C^\infty\big(\mathbb{R}^d \setminus \{0\}\big)$ *is a positive radial function that satisfies* (10) *and such that*

$$g(x) = \begin{cases} a/|x|^{d-\alpha} & \textit{if } |x| \le r, \\ b/|x|^{d-\beta} & \textit{if } |x| \ge R, \end{cases}$$

where $0 < r < R$ *and* $a > 0, b \ge 0$, *then there exists a radial function* $\omega \in C^\infty\big(\mathbb{R}^d \setminus \{0\}\big)$ *such that, for every* $x \in \mathbb{R}^d \setminus \{0\}$, *we have* $|\omega| * g(x) < \infty$ *and*

$$\omega * g(x) = \frac{1}{|x|^{d-\gamma}}. \tag{42}$$

Moreover, if $b > 0$, *for every multi-index* ν,

$$\big|\partial^\nu \omega(x)\big| \le \begin{cases} C/|x|^{d-(\gamma-\alpha)+|\nu|} & \textit{if } |x| \le 1, \\ C/|x|^{d-(\gamma-\beta^+)+|\nu|} & \textit{if } |x| \ge 1, \end{cases} \tag{43}$$

where $\beta_+ = \max\{\beta, 0\}$, *while, if* $b = 0$,

$$\big|\partial^\nu \omega(x)\big| \le \begin{cases} C/|x|^{d-(\gamma-\alpha)+|\nu|} & \textit{if } |x| \le 1, \\ C/|x|^{d-\gamma+|\nu|} & \textit{if } |x| \ge 1. \end{cases} \tag{44}$$

Before proving Proposition 34, we need some technical lemmas.

Lemma 35. *Let $0 < \delta < d$ and $\phi \in C_c^\infty(\mathbb{R}^d)$ be a radial function. If $S\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is the function defined by*

$$S(\xi) = \frac{\phi(\xi)}{|\xi|^\delta},$$

then, for every multi-index ν and every $x \in \mathbb{R}^d$,

$$\left|\partial^\nu \check{S}(x)\right| \le \frac{C'}{\left(1+|x|\right)^{d-\delta+|\nu|}}.$$

Proof of Lemma 35. We write ϕ for every $\xi \in \mathbb{R}^d$ as $\phi(\xi) = h(|\xi|)\psi(\xi)$, where $h \in C^\infty(\mathbb{R})$ is even and $\psi \in C_c^\infty(\mathbb{R}^d)$ is a nonnegative radial function equal to one in B_r and supported in B_{2r}, for some $r > 0$ large enough. By symmetry of h, the 2nd order Taylor polynomial of h at zero has the form $\lambda_0 + \lambda_1 t^2$. We then write S as

$$S(\xi) = P(\xi) + R(\xi),$$

where

$$P(\xi) := \frac{\lambda_0 + \lambda_1|\xi|^2}{|\xi|^\delta}\psi(\xi) \quad \text{and} \quad R(\xi) := \frac{h(|\xi|) - \left(\lambda_0 + \lambda_1|\xi|^2\right)}{|\xi|^\delta}\psi(\xi).$$

By Example 24 and Lemma 32 applied to both homogeneous terms of P, we conclude that

$$\left|\partial^\nu \check{P}(x)\right| \le \frac{C_1}{\left(1+|x|\right)^{d-\delta+|\nu|}}, \tag{45}$$

for every $x \in \mathbb{R}^m$ and every multi-index ν. Since h is smooth and even, for any multi-indexes μ and ν the remainder R satisfies

$$\xi^\nu \partial^\mu R(\xi) = O\left(|\xi|^{-\delta+4+|\nu|-|\mu|}\right) \quad \text{as } \xi \to 0.$$

As R has compact support, we deduce that for every multi-index μ such that $|\mu| < d - \delta + 4 + |\nu|$, we have

$$\int_{\mathbb{R}^d} |\xi|^{|\nu|}|\partial^\mu R|\,\mathrm{d}\xi < \infty.$$

This in turn implies that $\check{R}$ is smooth and

$$\left|\partial^\nu \check{R}(x)\right| \le \frac{C_2}{\left(1+|x|\right)^m}, \tag{46}$$

for every $x \in \mathbb{R}^d$ and every nonnegative integer $m < d - \delta + 4 + |\nu|$. We may take in particular $m = d - \delta + |\nu|$, and the conclusion then follows from the combination of (45) and (46). □

Lemma 36. *Let $0 < \delta < d$, $\theta > 0$, $a, b \in C^\infty(\mathbb{R}^d)$ be two positive radial functions and $\varphi \in C_c^\infty(\mathbb{R}^d)$ be a radial function. If $H\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is the function*

$$H(\xi) := \frac{\varphi(\xi)}{|\xi|^\delta\left(a(\xi) + b(\xi)|\xi|^\theta\right)},$$

then, for every multi-index ν,

$$\left|\partial^\nu \check{H}(x)\right| \le \frac{C}{\left(1+|x|\right)^{d-\delta+|\nu|}}.$$

Proof of Lemma 36. Let k be the smallest integer such that $k > 1/\theta$ and consider the following finite sum

$$H_k(\xi) = \frac{1}{a(\xi)|\xi|^\delta}\sum_{j=1}^{k}\left(-\frac{b(\xi)}{a(\xi)}\right)^{j-1}|\xi|^{(j-1)\theta}\varphi(\xi). \tag{47}$$

The generic term of (47) is in the form

$$S_j(\xi) = \phi(\xi)|\xi|^{(j-1)\theta-\delta},$$

with ϕ a smooth radially symmetric function with compact support. Then we can apply Lemma 35 to deduce that, for every multi-index ν,

$$\left|\partial^\nu \check{S}_j(x)\right| \le \frac{C_1}{\left(1+|x|\right)^{d+(j-1)\theta-\delta+|\nu|}}.$$

The leading term is the one corresponding to $j=1$. Looking at the definition (47) and using the linearity of the Fourier transform we conclude that

$$\left|\partial^\nu \check{H}_k(x)\right| \le \frac{C_2}{\left(1+|x|\right)^{d-\delta+|\nu|}}. \tag{48}$$

Claim. *The remainder $B_k \coloneqq H - H_k$ satisfies*

$$B_k(\xi) = \left(-\frac{b(\xi)}{a(\xi)}\right)^k |\xi|^{k\theta-\delta} \frac{\varphi(\xi)}{a(\xi)+b(\xi)|\xi|^\theta} \quad \textit{for } k = 1,2,\dots \tag{49}$$

Proof of the claim. We proceed by induction. For $k=1$ we have that

$$\begin{aligned}
B_1(\xi) &= \frac{\varphi(\xi)}{|\xi|^\delta}\frac{1}{a(\xi)+b(\xi)|\xi|^\theta} - \frac{\varphi(\xi)}{a(\xi)|\xi|^\delta}\\
&= \frac{\varphi(\xi)}{|\xi|^\delta}\left(\frac{1}{a(\xi)+b(\xi)|\xi|^\theta} - \frac{1}{a(\xi)}\right)\\
&= -\frac{b(\xi)}{a(\xi)}|\xi|^{\theta-\delta}\frac{\varphi(\xi)}{a(\xi)+b(\xi)|\xi|^\theta}.
\end{aligned}$$

Take now $k \ge 1$ and assume that (49) holds. Then,

$$\begin{aligned}
B_{k+1}(\xi) &= H - H_k - \frac{1}{a(\xi)}\left(-\frac{b(\xi)}{a(\xi)}\right)^k |\xi|^{k\theta-\delta}\varphi(\xi)\\
&= \left(-\frac{b(\xi)}{a(\xi)}\right)^k |\xi|^{k\theta-\delta}\frac{\varphi(\xi)}{a(\xi)+b(\xi)|\xi|^\theta} - \frac{1}{a(\xi)}\left(-\frac{b(\xi)}{a(\xi)}\right)^k |\xi|^{k\theta-\delta}\varphi(\xi)\\
&= \left(-\frac{b(\xi)}{a(\xi)}\right)^k |\xi|^{k\theta-\delta}\varphi(\xi)\left(\frac{1}{a(\xi)+b(\xi)|\xi|^\theta} - \frac{1}{a(\xi)}\right)\\
&= \left(-\frac{b(\xi)}{a(\xi)}\right)^{k+1} |\xi|^{(k+1)\theta-\delta}\frac{\varphi(\xi)}{a(\xi)+b(\xi)|\xi|^\theta},
\end{aligned}$$

and the claim is proved. □

The fact that B_k is integrable with compact support implies that $\check{B}_k \in C^\infty(\mathbb{R}^d)$. To estimate the decay of $\check{B}_k(x)$ as $x \to \infty$, we may rely on the integrability of derivatives of B_k. For any multi-indices μ and ν, it follows that

$$\xi^\nu \partial^\mu B_k(\xi) = O\left(|\xi|^{k\theta-\delta+|\nu|-|\mu|}\right) \quad \text{for } |\xi| \le 1.$$

Using again that B_k has compact support, we deduce that, if $|\mu| < k\theta + d - \delta + |\nu|$, then $\xi^\nu\partial^\mu B_k \in L^1(\mathbb{R}^d)$. This in turn implies that

$$\left|\partial^\nu \check{B}_k(x)\right| \le \frac{C_3}{|x|^m},$$

for any non-negative integer $m < k\theta - \delta + |\nu| + d$. Since $k\theta \ge 1$, we can take $m = d - [\delta] + |\nu|$ and, recalling that $\check{B}_k$ is smooth, we conclude that

$$\left|\partial^\nu \check{B}_k(x)\right| \le \frac{C_4}{\left(1+|x|\right)^{d-[\delta]+|\nu|}} \quad \text{in } \mathbb{R}^d. \tag{50}$$

Thanks to (48), (50) and the fact that $H = H_k + B_k$, it follows that

$$\left|\partial^\nu \check{H}(x)\right| \le \frac{C_5}{\left(1+|x|\right)^{d-\delta+|\nu|}} \quad \text{in } \mathbb{R}^d. \qquad \square$$

Proof of Proposition 34. Assume for simplicity that $a = b = 1$, $r = 1$ and $R = 2$, and let us start dealing with the case where $\beta \notin -2\mathbb{N}$. By Proposition 31, there exists a tempered distribution S_g that is represented by g in $\mathbb{R}^d$ such that its Fourier transform $\mathscr{F} S_g$ in $\mathbb{R}^d$ by

$$[\mathscr{F} S_g](\xi) = \begin{cases} c_\beta/|\xi|^\beta + \zeta_1(\xi) & \text{for } |\xi| \le 1, \\ c_\alpha/|\xi|^\alpha + \zeta_2(\xi) & \text{for } |\xi| \ge 2, \end{cases}$$

where ζ_1 and ζ_2 are Schwartz functions. Moreover, Proposition 29 assures us that $[\mathscr{F} S_g](\xi) > 0$ for all $\xi \in \mathbb{R}^d$.

Let us define H as

$$H(\xi) := \frac{c_\gamma}{|\xi|^\gamma [\mathscr{F} S_g](\xi)}. \tag{51}$$

Now we claim that H is a function that represents a tempered distribution S_H in $\mathbb{R}^d$ and that

$$\omega := [\mathscr{F}^{-1} S_H]$$

satisfies (43).

As before, let us decompose H as

$$H_1 := H\varphi_1, \quad H_2 := H(1-\varphi_2) \quad \text{and} \quad H_3 := H - H_1 - H_2.$$

We handle each of these functions separately. Since H_3 is smooth with compact support, its Fourier transform is a Schwartz function. To deal with H_2, we set

$$D(\xi) := H_2(\xi) - \frac{1}{c_\alpha |\xi|^{\gamma-\alpha}}\big(1-\varphi_2(\xi)\big) = \frac{1}{c_\alpha |\xi|^{\gamma-2\alpha}} \frac{\zeta_2(\xi)}{c_\alpha + \zeta_2(\xi)|\xi|^\alpha}\big(1-\varphi_2(\xi)\big).$$

Thanks to the fast decay of $\zeta_2(\xi)$ at infinity and the fact that $D(\xi)$ is zero in a neighborhood of the origin, we deduce that $\xi^\nu \partial^\mu D(\xi) \in L^1(\mathbb{R}^d)$ for any multi-indexes μ and ν. Therefore, $\check{D}$ is a Schwartz function and we can apply Lemma 32 to deduce that

$$[\mathscr{F}^{-1} H_2](x) = \frac{C_1}{|x|^{d+\alpha-\gamma}} \varphi + \eta_1,$$

where η_1 is a Schwartz function. Let us focus now on the term H_1. If $0 < \beta < d$, then H_1 can be written as

$$H_1(\xi) = \frac{\varphi_1(\xi)}{|\xi|^{\gamma-\beta}\big(c_\beta + \zeta_1(\xi)|\xi|^\beta\big)},$$

while, if $\beta < 0$ and different from an even integer, we have that

$$H_1(\xi) = \frac{\varphi_1(\xi)}{|\xi|^\gamma\big((c_\beta|\xi|^{-\beta} + \zeta_1(\xi)\big)}.$$

In both cases we can apply Lemma 36 to deduce that for every multi-index ν,

$$\big|\partial^\nu \check{H}_1(x)\big| \le \frac{C_2}{\big(1+|x|\big)^{d+\beta^+-\gamma+|\nu|}}. \tag{52}$$

Thanks to the linearity of the Fourier transform, the proof of the claim follows by summing up the information obtained for H_1, H_2 and H_3. To prove that the function ω satisfies (42), notice that by construction it follows that

$$H(\xi)[\mathscr{F} S_g] = \frac{c_\gamma}{|\xi|^\gamma}.$$

Thanks to the properties of H_1, H_2, H_3, we have that

$$T_\gamma = \mathscr{F}^{-1} Y_1 + \mathscr{F}^{-1} Y_2 + \mathscr{F}^{-1} Y_3,$$

where Y_i is defined as

$$\langle Y_i, \eta\rangle := \int_{\mathbb{R}^d} H_i [\mathscr{F} S_g] \eta.$$

Recalling that H_3 is a Schwartz function we have that

$$\langle \mathscr{F}^{-1} Y_3, \eta \rangle = \langle Y_3, \check{\eta} \rangle = \int_{\mathbb{R}^d} [\mathscr{F} S_g] H_3 \check{\eta} = \langle \mathscr{F} S_g, H_3 \check{\eta} \rangle = \langle S_g, \widehat{H_3} * \eta \rangle = \int_{\mathbb{R}^d} g \widehat{H_3} * \eta = \int_{\mathbb{R}^d} \widecheck{H_3} * g\eta. \quad (53)$$

To handle the first piece, notice that

$$\langle \mathscr{F}^{-1} Y_1, \eta \rangle = \langle Y_1, \check{\eta} \rangle = \int_{\mathbb{R}^d} [\mathscr{F} S_g] H_1 \check{\eta} = \lim_{\epsilon \to 0} \int_{\mathbb{R}^d} [\mathscr{F} S_g] (H_1 \check{\eta}) * \rho_\epsilon,$$

where the limit is justified by the fact that convolution in the integral converges in $L^q(\mathbb{R}^d)$ for all $q < \frac{d}{\gamma - \beta^+}$, that $[\mathscr{F} S_g] \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for any $p < \frac{d}{\beta^+}$, that $q' < \frac{d}{\beta^+}$, and that H_1 has compact support. Moreover, we have

$$\int_{\mathbb{R}^d} [\mathscr{F} S_g] (H_1 \check{\eta}) * \rho_\epsilon = \big\langle S_g, \big[(H_1 \check{\eta}) * \rho_\epsilon\big]^\wedge \big\rangle = \int_{\mathbb{R}^d} g \widehat{H_1} * \eta \widehat{\rho_\epsilon}.$$

Therefore,

$$\langle \mathscr{F}^{-1} Y_1, \eta \rangle = \int_{\mathbb{R}^d} \widecheck{H_1} * g\eta. \quad (54)$$

Finally, we have that

$$\begin{aligned} \langle \mathscr{F}^{-1} Y_2, \eta \rangle &= \int_{\mathbb{R}^d} [\mathscr{F} S_g] H_2 \check{\eta} \\ &= \big\langle S_g, (H_2 \check{\eta})^\wedge \big\rangle \\ &= \int_{\mathbb{R}^d} g (H_2 \check{\eta})^\wedge \\ &= \int_{\mathbb{R}^d} g [\mathscr{F} H_2] * \eta \\ &= \int_{\mathbb{R}^d} [\mathscr{F}^{-1} H_2] * g\eta, \end{aligned} \quad (55)$$

where we have used that $H_2 \check{\eta}$ belongs to the Schwartz class, that $(H_2 \check{\eta})^\wedge = [\mathscr{F} H_2] * \eta$ and $[\mathscr{F} H_2](-\xi) = \big[\mathscr{F}^{-1}(H_2)\big](\xi)$. Putting together (53), (54) and (55), we conclude that the previously defined ω satisfies (42).

Let us address now the choice $b = 0$ in the definition of g. Using the same notation as in the proof of Proposition 31, we see that $g = g_1$, where g_1 has been defined in (40). Therefore, recalling (41), it follows that

$$[\mathscr{F} S_g] = [\mathscr{F} S_{g_1}] = (1 - \varphi_1) \frac{c_\alpha}{|\xi|^\alpha} + \zeta_1$$

for a suitable radial Schwartz function ζ_1. Now we proceed as before, defining H as in (51) and decomposing it in H_1, H_2, and H_3. Recalling that $[\mathscr{F} S_g] > 0$, it follows that

$$H_1(\xi) = \frac{\varphi_1(\xi)}{|\xi|^\gamma \zeta_1(\xi)} = \frac{\widetilde{\varphi}(\xi)}{|\xi|^\gamma},$$

where $\widetilde{\varphi} = \varphi_1 / \zeta_1$ is smooth, radial, and with compact support. Therefore (52) holds true and the rest of the proof follows as for $b > 0$ and $\beta < 0$ different from a negative even number.

Finally let us go back to the case $\beta = -2\ell$ for some $\ell = 1, 2, \ldots$ Recalling (35), the expression for $[\mathscr{F} S_g](\xi)$ becomes

$$[\mathscr{F} S_g](\xi) = \begin{cases} \big(-A_k \log|\xi| + \lambda_k\big)|\xi|^k + \eta(\xi) & \text{for } |\xi| \le 1, \\ c_\alpha / |\xi|^\alpha + \eta(\xi) & \text{for } |\xi| \ge 2. \end{cases}$$

The strategy of the proof is the same as in the previous case, we keep the same notation, and only highlight the differences. In particular we have that $H_1(\xi)$ modifies as follows

$$H_1(\xi) = \frac{\varphi_1(\xi)}{|\xi|^\gamma} \frac{1}{\big(-A_k \log|\xi| + \lambda_k\big)|\xi|^k + \eta(\xi)}.$$

We further decompose H_1 as follows

$$H_1(\xi) = S(\xi) + B(\xi),$$

where

$$S(\xi) := \frac{1}{\eta(\xi)|\xi|^\gamma}\varphi_1(\xi)$$

and

$$B(\xi) = -\frac{\big(-A_k\log|\xi| + \lambda_k\big)|\xi|^{k-\gamma}}{\eta(\xi)\big[\eta(\xi) + \big(-A_k\log|\xi| + \lambda_k\big)|\xi|^k\big]}\varphi_1(\xi).$$

Using Lemma 35, we deduce that

$$\big|\partial^\nu \check{S}(x)\big| \le \frac{C_3}{\big(1+|x|\big)^{d-\gamma+|\nu|}} \quad \text{in } \mathbb{R}^d.$$

To deal with $\check{B}$, notice that, for any multi-indices μ, ν, it follows that

$$\big|\xi^\nu \partial^\mu B(\xi)\big| \le C_4\big|\log|\xi|\big|\,|\xi|^{k-\gamma+|\nu|-|\mu|} \quad \text{for } |\xi| \le 1/2.$$

Together with the fact that B has compact support, this implies that, for every nonnegative integer $m < k - \gamma + |\nu| + d$,

$$\big|\partial^\nu \check{B}(x)\big| \le \frac{C_5}{|x|^m}.$$

Choosing $m = d - \lfloor\gamma\rfloor + |\nu|$ and using the linearity of the Fourier transform, we conclude that again that, for any multi-index ν,

$$\big|\partial^\nu \check{H}_1(x)\big| \le \frac{C_6}{\big(1+|x|\big)^{d-\gamma+|\nu|}} \quad \text{in } \mathbb{R}^d.$$

The remaining of the proof follows along the same lines. □

Proof of Theorem 3. Consider at first the case (7) and let us apply Proposition 34 with $\gamma = 2$, $\alpha = 1 - s$, $a = 1$ and $b = 0$. We deduce that there exists $\omega \in C^\infty\big(\mathbb{R}^d \setminus \{0\}\big)$ such that $\omega * g(x) = 1/|x|^{d-2}$. Estimate (44) implies that the vector valued function $V = -\frac{1}{(N-2)\sigma_d}\nabla\omega$ belongs to $L^1 + L^d$. Moreover, a straightforward computation shows that

$$V * g(x) = \frac{1}{\sigma_d}\frac{x}{|x|^d}.$$

Therefore, Corollary 14 implies the desired result. To deal with the remaining cases (8) and (9), we have just to apply Proposition 34 with the appropriate values of the considered parameters, the rest of the proof being the same. □

Proof of Theorem 7. Theorem 3 assures us that

$$u = V * \mathscr{G}u.$$

Moreover, under the considered assumptions, Proposition 34 implies that $V = c\nabla\omega$ satisfies estimate (12). Therefore, decomposing V as

$$V = V\varphi_1 + V(1-\varphi_1) =: V_1 + V_2,$$

we obtain that

$$\big|V_1(z)\big| + |z|\big|\nabla V_1(z)\big| \le \frac{C_1}{|z|^{d-s}} \quad \text{and} \quad \big|V_2(z)\big| + |z|\big|\nabla V_2(z)\big| \le \frac{C_2}{|z|^{d-1}}.$$

When $p = 1$, Proposition 16 implies that

$$\big\|V_1 * \mathscr{G}u\big\|_{L^{\frac{d}{d-s}}} \le C_3\|\mathscr{G}u\|_{L^1} \quad \text{and} \quad \big\|V_2 * \mathscr{G}u\big\|_{L^{\frac{d}{d-1}}} \le C_4\|\mathscr{G}u\|_{L^1}.$$

Thanks to the linearity of the convolution, we obtain the desired result.

To deal with the case $1 < p < d/s$, it is enough to notice that the estimates

$$\left\| V_1 * \mathscr{G}u \right\|_{L^{\frac{pd}{d-sp}}} \le C_5 \|\mathscr{G}u\|_{L^p} \quad \text{and} \quad \left\| V_2 * \mathscr{G}u \right\|_{L^{\frac{pd}{d-p}}} \le C_6 \|\mathscr{G}u\|_{L^p},$$

follow by classical results on convolution operators, see for instance [27, Theorem, p. 139]. □

The proof of Theorem 8 follows the same structure as the proof of Theorem 7, now applying (13) rather than (12) and shall be omitted.

9. Nonlocal Sobolev inequalities in Lorentz spaces

Before proving Theorems 9 and 10, let us first recall for the convenience of the reader the definition of the Lorentz spaces that are involved. We need to introduce the decreasing rearrangement of a measurable function and some related notation, see [18,27,28,38] for details.

Definition 37. *The* decreasing rearrangement *of a measurable function $v \colon \mathbb{R}^d \to \mathbb{R}$ is defined for every $0 < y < \infty$ as*

$$v^*(y) = \inf\{\tau \ge 0 \,:\, A(\tau) < y\},$$

where $A(t) := \left|\{|v| > t\}\right|$.

Taking

$$v^{**}(y) = \frac{1}{y}\int_0^y v^*(\tau)\,\mathrm{d}\tau,$$

the measurable function v belongs to the Lorentz space $L^{p,q}(\mathbb{R}^d)$ whenever

$$\|v\|_{L^{p,q}(\mathbb{R}^d)} := \left(\int_0^\infty \left(\tau^{\frac{1}{p}} v^{**}(\tau)\right)^q \frac{\mathrm{d}\tau}{\tau}\right)^{\frac{1}{q}} < \infty \quad \text{for } 1 < p < \infty \text{ and } 1 \le q < \infty,$$

and

$$\|v\|_{L^{p,\infty}(\mathbb{R}^d)} := \sup_{\tau > 0} \tau^{\frac{1}{p}} v^{**}(\tau) < \infty \quad \text{for } 1 \le p \le \infty \text{ and } q = \infty.$$

We begin with the following estimate.

Proposition 38. *Let $1 < p < \infty$, let $0 < s < 1$ with $sp < d$, and let $0 < t \le 1$ with $tp < d$. Take $F \in L^p(\mathbb{R}^d, \mathbb{R}^d)$ and a measurable function $v \colon \mathbb{R}^d \to \mathbb{R}^d$ such that, for almost every $z \in \mathbb{R}^d$,*

$$\left|v(z)\right| \le \begin{cases} C/|z|^{d-s} & \text{if } |z| \le 1, \\ C/|z|^{d-t} & \text{if } |z| > 1. \end{cases} \tag{56}$$

Then, the convolution $h := v * F$ belongs to $(L^{\frac{pd}{d-sp},p} + L^{\frac{pd}{d-tp},p})(\mathbb{R}^d)$ and

$$\int_0^1 \tau^{\frac{d-sp}{d}} h^{**}(\tau)\frac{\mathrm{d}\tau}{\tau} + \int_1^\infty \tau^{\frac{d-tp}{d}} h^{**}(\tau)\frac{\mathrm{d}\tau}{\tau} \le C' \|F\|^p_{L^p(\mathbb{R}^d)}.$$

Proof. Writing $v = v\chi_{B_1} + v\chi_{B_1^c} = v_1 + v_2$, we can use assumption (56) to deduce that

$$\left|v_1(z)\right| \le c/|z|^{d-s} \quad \text{and} \quad \left|v_2(z)\right| \le c/|z|^{d-t}.$$

Therefore, the embedding of the Riesz potential between Lorentz spaces, see e.g. [27, Theorem, p. 139], ensures that $h := v_1 * F + v_2 * F$ is well-defined and belongs to $L^{\frac{pd}{d-sp},p} + L^{\frac{pd}{d-tp},p}$. Let us now set, with an abuse of notation, $v^* = (|v|)^*$ and $v^{**} = (|v|)^{**}$. A direct computation shows that

$$v^{**}(\tau) \le \mathrm{v}(\tau) := \begin{cases} c\tau^{-1+s/d} & \text{if } \tau \le 1, \\ c\tau^{-1+t/d} & \text{if } \tau > 1, \end{cases} \tag{57}$$

for a suitable constant $c > 0$. Thanks to the estimate for convolution operators provided in [27, Lemma 1.6], we deduce that

$$h^{**}(\tau) \le \int_\tau^\infty v^{**}(y) F^{**}(y)\,\mathrm{d}y. \tag{58}$$

Therefore,

$$\begin{aligned}\int_1^\infty \tau^{\frac{d-tp}{d}} h^{**}(\tau)^p \frac{\mathrm{d}\tau}{\tau} &\le \int_1^\infty \tau^{-\frac{tp}{d}} \left(\int_\tau^\infty v^{**}(y)F^{**}(y)\,\mathrm{d}y\right)^p \mathrm{d}\tau\\ &\le c^p \int_1^\infty \tau^{-\frac{tp}{d}} \left(\int_\tau^\infty y^{-1+\frac{t}{d}} F^{**}(y)\,\mathrm{d}y\right)^p \mathrm{d}\tau\\ &\le C_1 \|F^{**}\|^p_{L^p(\mathbb{R})},\end{aligned}$$

where the second inequality follows from the estimates on v^{**} and the last one by Hardy's inequality where we use that $t < d/p$; see [19, equation (9.9.10), p. 246]. On the other hand, integration by parts leads us to

$$\begin{aligned}B &:= \int_0^1 \tau^{-\frac{sp}{d}} \left(\int_\tau^\infty v^{**}(y)F^{**}(y)\,\mathrm{d}y\right)^p \mathrm{d}\tau\\ &= C_2\left(\int_1^\infty v^{**}(y)F^{**}(y)\,\mathrm{d}y\right)^p + C_3\int_0^1 \tau^{1-\frac{sp}{d}}\left(\int_\tau^\infty v^{**}(y)F^{**}(y)\,\mathrm{d}y\right)^{p-1} v^{**}(\tau)F^{**}(\tau)\,\mathrm{d}\tau\\ &\le C_2\left(\int_1^\infty v^{**}(y)^{p'}\,\mathrm{d}y\right)^{p-1}\int_1^\infty F^{**}(y)^p\,\mathrm{d}y + C_4\int_0^1 \tau^{1-\frac{sp}{d}} v^{**}(\tau)^p F^{**}(\tau)^p\,\mathrm{d}\tau + \frac{B}{2},\end{aligned}$$

where we have used Hölder and Young inequalities in the third line. Thanks to the assumptions on v, we conclude that

$$\int_0^1 \tau^{\frac{d-sp}{d}} h^{**}(\tau)^p \frac{\mathrm{d}\tau}{\tau} \le B \le C_5\|F^{**}\|^p_{L^p(\mathbb{R})}.$$

Combining both estimates for h^{**} and using the fact that the norms $\|F^{**}\|_{L^p(\mathbb{R})}$ and $\|F\|_{L^p(\mathbb{R}^d)}$ are comparable, we obtain the desired result. □

We now deduce the following estimates involving the truncations G_k and T_k.

Proposition 39. *Let p, s and t be as in the statement of Proposition 38. For every $F \in L^p(\mathbb{R}^d,\mathbb{R}^d)$ and every measurable function $v\colon \mathbb{R}^d \to \mathbb{R}^d$ that satisfies* (56)*, the convolution $h = v * F$ satisfies*

$$\left\|G_{\bar{k}}(h)\right\|_{L^{\frac{pd}{d-ps},p}(\mathbb{R}^d)} + \left\|T_{\bar{k}}(h)\right\|_{L^{\frac{pd}{d-pt},p}(\mathbb{R}^d)} \le C\|F\|_{L^p(\mathbb{R}^d)},$$

with $\bar{k} := \|F\|_{L^p(\mathbb{R}^d)}$.

Proof. Let us first notice that, using (58),

$$h^{**}(\tau) \le \int_\tau^\infty v^{**}(y)F^{**}(y)\,\mathrm{d}y \le \left(\int_\tau^\infty \mathrm{v}(y)^{p'}\,\mathrm{d}y\right)^{\frac{1}{p'}} \|F\|_{L^p} =: \mathrm{I}(\tau)\|F\|_{L^p}, \tag{59}$$

where v is given in (57). By construction, the real function $\mathrm{I} : (0,\infty) \to (0,\infty)$ defined above is invertible. Therefore,

$$A(k) = \left|\left\{|h| > k\right\}\right| = \left|\left\{h^* > k\right\}\right| \le \mathrm{I}^{-1}\big(k/\|F\|_{L^p}\big). \tag{60}$$

Set

$$\bar{k} = \|F\|_{L^p}, \quad h_1 = G_{\bar{k}}(h) \quad \text{and} \quad h_2 = T_{\bar{k}}(h).$$

Since $h_1^* \le h^*\chi_{(0,A(\bar{k}))}$, it follows that

$$\begin{aligned}\|h_1\|_{L^{\frac{pd}{d-ps},p}} &\le C_1\left(\int_0^1 \tau^{\frac{d-sp}{d}} h^{**}(\tau)^p \frac{\mathrm{d}\tau}{\tau}\right)^{\frac{1}{p}} + C_2\left(\int_1^{\max\{A(\bar{k}),1\}} \tau^{\frac{d-sp}{d}} h^{**}(\tau)^p \frac{\mathrm{d}\tau}{\tau}\right)^{\frac{1}{p}}\\ &\le C_3\|F\|_{L^p} + h^{**}(1)\big(A(\bar{k})-1\big)_+^{\frac{1}{p}}\\ &\le C_4\|F\|_{L^p},\end{aligned}$$

where we have used Proposition 38 to estimate the first integral, and formula (59) with $\tau = 1$ and (60) with $k = \|F\|_{L^p(\mathbb{R}^d)}$ to estimate the second one. To deal with h_2, notice that $h_2^* = T_{\overline{k}}(h^*)$. Therefore,

$$\begin{aligned}\|h_2\|_{L^{\frac{pd}{d-pt},p}} &\leq \left(\int_0^1 \tau^{\frac{d-tp}{d}}\,\overline{k}^p \frac{\mathrm{d}\tau}{\tau}\right)^{\frac{1}{p}} + \left(\int_1^\infty \tau^{\frac{d-tp}{d}}\,h^{**}(\tau)^p \frac{\mathrm{d}\tau}{\tau}\right)^{\frac{1}{p}}\\ &\leq C_5\overline{k} + \left(\int_1^\infty \tau^{\frac{d-tp}{d}}\,h^{**}(\tau)^p \frac{\mathrm{d}\tau}{\tau}\right)^{\frac{1}{p}}\\ &\leq C_6\|F\|_{L^p},\end{aligned}$$

where the last inequality follows by the choice of $\overline{k}$ and Proposition 38. □

Proof of Theorem 9. Under the considered assumptions, Proposition 34 implies that $V := c\nabla\omega$ satisfies estimate (12) and that $u = V * \mathscr{G}u$ for all $u \in C_c^\infty(\mathbb{R}^d)$. Therefore, we apply Proposition 39 to $h = u = V * \mathscr{G}u$, with $0 < s < 1$ and $t = 1$, to get the desired result. □

The proof of Theorem 10 follows along the lines of the proof of Theorem 9, by applying (13) rather than (12).

Acknowledgments

The authors thank D. Spector for bringing to their attention the discussion of the Fourier transform in the book [31], and the referee for helpful comments.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

References

[1] F. J. Almgren Jr. and E. H. Lieb, "Symmetric decreasing rearrangement is sometimes continuous", *J. Am. Math. Soc.* **2** (1989), no. 4, pp. 683–773.

[2] S. Almi, M. Caponi, M. Friedrich and F. Solombrino, "A fractional approach to strain-gradient plasticity: beyond core-radius of discrete dislocations", *Math. Ann.* **391** (2025), no. 3, pp. 4063–4115.

[3] J. C. Bellido, J. Cueto and C. Mora-Corral, "Non-local gradients in bounded domains motivated by continuum mechanics: fundamental theorem of calculus and embeddings", *Adv. Nonlinear Anal.* **12** (2023), no. 1, article no. 20220316 (48 pages).

[4] J. C. Bellido, C. Mora-Corral and H. Schönberger, "Nonlocal gradients: fundamental theorem of calculus, Poincaré inequalities and embeddings", *J. Lond. Math. Soc. (2)* **112** (2025), no. 2, article no. e70277 (53 pages).

[5] J. Bourgain and H. Brezis, "On the equation div $Y = f$ and application to control of phases", *J. Am. Math. Soc.* **16** (2003), no. 2, pp. 393–426.

[6] J. Bourgain and H. Brezis, "New estimates for elliptic equations and Hodge type systems", *J. Eur. Math. Soc.* **9** (2007), no. 2, pp. 277–315.

[7] J. Bourgain, H. Brezis and P. Mironescu, "Another look at Sobolev spaces", in *Optimal control and partial differential equations* (J. L. Menaldi, E. Rofman and A. Sulem, eds.), IOS Press, 2001, pp. 439–455.

[8] J. Bourgain, H. Brezis and P. Mironescu, “Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications”, *J. Anal. Math.* **87** (2002), pp. 77–101.

[9] J. Bourgain, H. Brezis and P. Mironescu, “$H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg–Landau equation”, *Publ. Math., Inst. Hautes Étud. Sci.* (2004), no. 99, pp. 1–115.

[10] H. Brezis, “How to recognize constant functions. A connection with Sobolev spaces”, *Russ. Math. Surv.* **57** (2002), no. 4, pp. 693–708.

[11] H. Brezis and P. Mironescu, *Sobolev maps to the circle — From the perspective of analysis, geometry, and topology*, Progress in Nonlinear Differential Equations and their Applications, vol. 96, Birkhäuser/Springer, 2021.

[12] H. Brezis and P. Mironescu, “Non-local approximations of the gradient”, *Confluentes Math.* **15** (2023), pp. 27–44.

[13] E. Bruè, M. Calzi, G. E. Comi and G. Stefani, “A distributional approach to fractional Sobolev spaces and fractional variation: asymptotics II”, *Comptes Rendus. Mathématique* **360** (2022), pp. 589–626.

[14] G. E. Comi and G. Stefani, “A distributional approach to fractional Sobolev spaces and fractional variation: existence of blow-up”, *J. Funct. Anal.* **277** (2019), no. 10, pp. 3373–3435.

[15] Q. Du, M. Gunzburger, R. B. Lehoucq and K. Zhou, “A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws”, *Math. Models Methods Appl. Sci.* **23** (2013), no. 3, pp. 493–540.

[16] R. L. Frank and R. Seiringer, “Non-linear ground state representations and sharp Hardy inequalities”, *J. Funct. Anal.* **255** (2008), no. 12, pp. 3407–3430.

[17] G. Gilboa and S. Osher, “Nonlocal operators with applications to image processing”, *Multiscale Model. Simul.* **7** (2008), no. 3, pp. 1005–1028.

[18] L. Grafakos, *Classical Fourier analysis*, 3rd edition, Graduate Texts in Mathematics, vol. 249, Springer, 2014.

[19] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Mathematical Library, Cambridge University Press, 1988.

[20] L. Hörmander, *The analysis of linear partial differential operators I. Distribution theory and Fourier analysis*, Classics in Mathematics, Springer, 2003.

[21] J. Horváth, “On some composition formulas”, *Proc. Am. Math. Soc.* **10** (1959), pp. 433–437.

[22] E. Y. Jaffe, “Fourier transform of homogeneous radial distributions”. Online at https://r-grande.github.io/Expository/Fourier_transform_of_homogeneous_distributions.pdf. Unpublished notes.

[23] V. I. Kolyada and A. K. Lerner, “On limiting embeddings of Besov spaces”, *Stud. Math.* **171** (2005), no. 1, pp. 1–13.

[24] V. Maz’ya and T. Shaposhnikova, “On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces”, *J. Funct. Anal.* **195** (2002), no. 2, pp. 230–238. Erratum: *J. Funct. Anal.* **201** (2003), 298–300.

[25] M. M. Meerschaert, J. Mortensen and S. W. Wheatcraft, “Fractional vector calculus for fractional advection-dispersion”, *Phys. A: Stat. Mech. Appl.* **367** (2006), pp. 181–190.

[26] T. Mengesha and D. E. Spector, “Localization of nonlocal gradients in various topologies”, *Calc. Var. Partial Differ. Equ.* **52** (2015), no. 1–2, pp. 253–279.

[27] R. O’Neil, “Convolution operators and $L(p,q)$ spaces”, *Duke Math. J.* **30** (1963), pp. 129–142.

[28] E. T. Oklander, *Interpolación, espacios de Lorentz y teorema de Marcinkiewicz*, Cursos y Seminarios de Matemática, vol. 20, Universidad de Buenos Aires, 1965.

[29] A. C. Ponce, *Elliptic PDEs, measures and capacities*, EMS Tracts in Mathematics, vol. 23, European Mathematical Society, 2016.

[30] A. C. Ponce and D. E. Spector, "A boxing inequality for the fractional perimeter", *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)* **20** (2020), no. 1, pp. 107–141.

[31] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives. Theory and applications*, Gordon and Breach Science Publishers, 1993.

[32] A. Schikorra, D. E. Spector and J. Van Schaftingen, "An L^1-type estimate for Riesz potentials", *Rev. Mat. Iberoam.* **33** (2017), no. 1, pp. 291–303.

[33] T.-T. Shieh and D. E. Spector, "On a new class of fractional partial differential equations", *Adv. Calc. Var.* **8** (2015), no. 4, pp. 321–336.

[34] T.-T. Shieh and D. E. Spector, "On a new class of fractional partial differential equations II", *Adv. Calc. Var.* **11** (2018), no. 3, pp. 289–307.

[35] M. Šilhavý, "Fractional vector analysis based on invariance requirements (critique of coordinate approaches)", *Contin. Mech. Thermodyn.* **32** (2020), no. 1, pp. 207–228.

[36] D. E. Spector, "An optimal Sobolev embedding for L^1", *J. Funct. Anal.* **279** (2020), no. 3, article no. 108559 (26 pages).

[37] J. Van Schaftingen, "Estimates for L^1-vector fields", *Comptes Rendus. Mathématique* **339** (2004), no. 3, pp. 181–186.

[38] W. P. Ziemer, *Weakly differentiable functions*, Graduate Texts in Mathematics, vol. 120, Springer, 1989.