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Mathématique

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Volume 363 (2025), p. 1429-1455

Online since: 28 November 2025

https://doi.org/10.5802/crmath.802

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www.centre-mersenne.org — e-ISSN : 1778-3569

ACADÉMIE DES SCIENCES
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Comptes Rendus. Mathématique
2025, Vol. 363, p. 1429-1455
https://doi.org/10.5802/crmath.802

Research article
Functional analysis

Characterizations of the Sobolev norms and the total variation via nonlocal functionals, and related problems

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Dedicated to Haïm Brezis with admiration, gratitude, and memories

Abstract. We briefly discuss the contribution of Haïm Brezis and his co-authors on the characterizations of the Sobolev norms and the total variation using non-local functionals. Some ideas of the analysis are given and new results are presented.

Keywords. Sobolev norms, total variations, Gamma-convergence, inequalities, non-local functionals.

2020 Mathematics Subject Classification. 26B15, 26B25, 26B30, 42B25.

Note. Article submitted by invitation.

Manuscript received 1 July 2025, revised 15 September 2025 and 29 September 2025, accepted 9 October 2025.

1. Introduction

In this paper, we first briefly discuss the contribution of Haïm Brezis and his co-authors on the characterizations of the Sobolev norms and the total variation using non-local functionals, and related problems. Some ideas of the analysis are also given. We then present several new results by developing these ideas.

We begin with the BBM formula due to Bourgain, Brezis, and Mironescu [5] (see also [10,22]). To this end, for $N \ge 1$, $p \ge 1$, and $u \in L^p(\mathbb{R}^N)$, it is convenient to denote

$$\Phi(u) = \begin{cases} \|\nabla u\|^p_{L^p(\mathbb{R}^N)} \,\mathrm{d}x & \text{if } p > 1, \\ \|\nabla u\|_{\mathcal{M}(\mathbb{R})} & \text{if } p = 1. \end{cases} \tag{1}$$

Recall that, for $f \in L^1(\mathbb{R}^N)$,

$$\|\nabla f\|_{\mathcal{M}(\mathbb{R}^N)} := \sup\left\{\left|\int_{\mathbb{R}^N} f \operatorname{div}\varphi\right|;\ \varphi \in C^\infty_c(\mathbb{R}^N) \text{ with } \|\varphi\|_{L^\infty(\mathbb{R}^N)} \le 1\right\}.$$

In what follows, a sequence of functions $(\rho_n)_{n\ge1} \subset L^1(0,+\infty)$ is called a sequence of non-negative mollifiers if the following properties hold:

$$\rho_n \ge 0,$$

$$\lim_{n\to\infty} \int_\tau^\infty \rho_n(r) r^{N-1}\, dr = 0 \quad \forall\, \tau > 0, \quad \text{and} \quad \int_0^{+\infty} \rho_n(r) r^{N-1}\, dr = 1.$$

ISSN (electronic): 1778-3569

Here is the BBM formula.

Theorem 1 (BBM formula, Bourgain & Brezis & Mironescu). *Let $N \geq 1$, $1 \leq p < +\infty$, and let $(\rho_n)_{n\geq 1}$ be a sequence of non-negative mollifiers. Then, for $u \in L^p(\mathbb{R}^N)$,*

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\big|u(x)-u(y)\big|^p}{|x-y|^p}\rho_n\big(|x-y|\big)\,dx\,dy \leq C_{N,p}\Phi(u), \tag{2}$$

and

$$\lim_{n\to\infty} \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\big|u(x)-u(y)\big|^p}{|x-y|^p}\rho_n\big(|x-y|\big)\,dx\,dy = K_{N,p}\Phi(u), \tag{3}$$

where $K_{N,p}$ is defined by

$$K_{N,p} = \int_{\mathbb{S}^{N-1}} |e\cdot\sigma|^p\,d\sigma, \tag{4}$$

for any $e \in \mathbb{S}^{N-1}$, the unit sphere of $\mathbb{R}^N$.

Here and in what follows, $C_{N,p}$ denotes a positive constant depending only on N and p, and might change from one place to another.

As a convention, the RHS of (2) or (3) is infinite if $u \not\in W^{1,p}(\mathbb{R}^N)$ for $p > 1$ and $u \not\in \mathrm{BV}(\mathbb{R}^N)$ for $p = 1$.

Theorem 1 was established by Bourgain, Brezis, and Mironescu [5] (see also [10]) in the case $p > 1$. In the case $p = 1$, they also showed there that the lim inf and the lim sup of the LHS of (2) as $n \to +\infty$ is of the order of the RHS of (2) instead of (3). The proof of (3) in the case $p = 1$ and $u \in \mathrm{BV}(\mathbb{R}^N)$ is due to Davila [22].

We next briefly discuss some ideas of the proof. We first deal with (2) under the additional assumption that $\Phi(u) < +\infty$. We first consider the case where $u \in C_c^\infty(\mathbb{R}^N)$. Using the fact, for $x, y \in \mathbb{R}^N$,

$$u(y) - u(x) = \int_0^1 \nabla u\big(x + t(y-x)\big)\cdot(y-x)\,dt,$$

and Jensen's inequality, one can prove that

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\big|u(x)-u(y)\big|^p}{|x-y|^p}\rho_n\big(|x-y|\big)\,dx\,dy \leq \iint_{\mathbb{R}^N\times\mathbb{R}^N}\int_0^1 \big|\nabla u\big(x+t(y-x)\big)\big|^p\,dt\,\rho_n\big(|y-x|\big)\,dx\,dy. \tag{5}$$

By a change of variables $(x,z) = (x, x-y)$ and by applying Fubini's theorem, we obtain (2) from (5). We next deal with (2) for which $\Phi(u) < +\infty$. The proof in this case follows from the previous case by considering a sequence $(u_k) \subset C_c^\infty(\mathbb{R}^N)$ such that $\Phi(u_k) \to \Phi(u)$ and $u_k \to u$ for almost every $x \in \mathbb{R}^N$, as $k \to +\infty$. By Fatou's lemma, one has, for $n \geq 1$,

$$\liminf_{k\to+\infty} \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\big|u_k(x)-u_k(y)\big|^p}{|x-y|^p}\rho_n\big(|x-y|\big)\,dx\,dy \geq \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\big|u(x)-u(y)\big|^p}{|x-y|^p}\rho_n\big(|x-y|\big)\,dx\,dy. \tag{6}$$

Assertion (2) now follows from (6) by applying (5) to u_k and then letting $k \to +\infty$.

We now address (3) for u such that $\Phi(u) < +\infty$. Using the properties of the mollifier sequence (ρ_n), in particular, the mass of ρ_n concentrates around 0, and a Taylor expansion, one can prove (3) if $u \in C_c^\infty(\mathbb{R}^N)$ in addition. Combining this with (2), one can derive (3) for u satisfying $\Phi(u) < +\infty$ after using the fact that, for $u_k, u \in L^p(\mathbb{R}^N)$,

$$\left| \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\left|u_k(x) - u_k(y)\right|^p}{|x-y|^p} \rho_n\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y - \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\left|u(x) - u(y)\right|^p}{|x-y|^p} \rho_n\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y \right| \\ \le C_p \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\left|(u_k - u)(x) - (u_k - u)(y)\right|^p}{|x-y|^p} \rho_n\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y.$$

It remains to prove that if the $\liminf_{n\to+\infty}$ of the LHS of (2) is finite then $\Phi(u) < +\infty$. One way to prove this fact is to use the convexity of t^p and the convolution suggested by Stein and presented in [10] as follows. Let $(\varphi_k)_{k\ge 1}$ be a smooth non-negative sequence of approximations to the identity such that $\operatorname{supp}\varphi_k \subset B_{1/k}$. Set

$$u_k = \varphi_k * u \quad \text{for } k \ge 1.$$

Using Jensen's inequality, we derive that, for $n \ge 1$,

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\left|u_k(x) - u_k(y)\right|^p}{|x-y|^p} \rho_n\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y \le \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\left|u(x) - u(y)\right|^p}{|x-y|^p} \rho_n\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y \quad \text{for } k \ge 1.$$

Since $u_k \in C^\infty(\mathbb{R}^N)$, one can show that, for $R > 2$ and $0 < r < 1$,

$$\lim_{n\to+\infty} \iint_{x \in B_R;\, |y-x|<r} \frac{\left|u_k(x) - u_k(y)\right|^p}{|x-y|^p} \rho_n\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y \ge K_{N,p} \int_{B_R} |\nabla u_k|^p\,\mathrm{d}x. \tag{7}$$

Here and in what follows, B_R denotes the open ball in $\mathbb{R}^N$ centered at 0 and of radius R for $R > 0$. This implies

$$\lim_{n\to+\infty} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\left|u_k(x) - u_k(y)\right|^p}{|x-y|^p} \rho_n\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y \ge K_{N,p} \int_{\mathbb{R}^N} |\nabla u_k|^p\,\mathrm{d}x.$$

We thus obtain

$$+\infty > \liminf_{n\to+\infty} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\left|u(x) - u(y)\right|^p}{|x-y|^p} \rho_n\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y \ge K_{N,p} \int_{\mathbb{R}^N} |\nabla u_k|^p\,\mathrm{d}x \quad \text{for } k \ge 1. \tag{8}$$

Since $k \ge 1$ is arbitrary, we obtain $\Phi(u) < +\infty$. The details of these arguments can be found in [5] and [10].

We next give a useful consequence of Theorem 1.

Proposition 2. *Let $N \ge 1$, $1 \le p < +\infty$, and $r > 0$, and let $u \in L^p(\mathbb{R}^N)$. Then*

$$\lim_{\varepsilon\to 0} \iint_{\substack{\mathbb{R}^N \times \mathbb{R}^N \\ |x-y|<r}} \frac{\left|u(x) - u(y)\right|^p}{|x-y|^p} \frac{\varepsilon}{|x-y|^{N-\varepsilon}}\,\mathrm{d}x\,\mathrm{d}y = K_{N,p}\Phi(u). \tag{9}$$

Consequently, if $u \in L^p(\mathbb{R}^N)$ satisfies

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\left|u(x) - u(y)\right|^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y < +\infty,$$

then u is constant.

The reader can find many other interesting examples in [5,10,17] on the way to determine whether or not a given function is constant. One of the motivations for determining whether or not a function is a constant is from the study of the Ginzburg–Landau equations, see, e.g., [4]. Further properties related to the BBM formula can be found in [14,46] (see also [27]) and the references therein.

Remark 3. The limit of the term in the LHS of (9) for $\varepsilon \to 1_-$ was studied by Maz'ya and Shaposhnikova [30].

We next discuss a related result of Theorem 1 due to Nguyen [31], and Bourgain and Nguyen [9].

Theorem 4 (Bourgain & Nguyen). *Let $N \geq 1$ and $1 \leq p < +\infty$. The following two facts hold.*

(i) *For $p > 1$ and $u \in W^{1,p}(\mathbb{R}^N)$, we have*

$$\iint\limits_{\substack{\mathbb{R}^N \times \mathbb{R}^N \\ |u(x)-u(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}} \,\mathrm{d}x\,\mathrm{d}y \leq C_{N,p}\Phi(u) \quad \text{for } \delta > 0, \tag{10}$$

and

$$\lim_{\delta \to 0_+} \iint\limits_{\substack{\mathbb{R}^N \times \mathbb{R}^N \\ |u(x)-u(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}} \,\mathrm{d}x\,\mathrm{d}y = \frac{1}{p} K_{N,p}\Phi(u), \tag{11}$$

where $K_{N,p}$ is defined by (4).

(ii) *If $u \in L^p(\mathbb{R}^N)$ and $p \geq 1$, then*

$$\Phi(u) \leq C_{N,p} \liminf_{\delta \to 0_+} \iint\limits_{\substack{\mathbb{R}^N \times \mathbb{R}^N \\ |u(x)-u(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}} \,\mathrm{d}x\,\mathrm{d}y. \tag{12}$$

In particular, $u \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$ and $u \in \mathrm{BV}(\mathbb{R}^N)$ if $p = 1$ if the RHS of (12) is finite.

Remark 5. The quantity given in the LHS of (10) has its roots in estimates for the topological degree of continuous maps from a sphere into itself, which is due to Bourgain, Brezis, and Nguyen [8] (see also [6,7,33,37]).

Assertion (10) is based on the theory of maximal functions. As in the analysis of (2), it suffices to consider the case $u \in C_c^\infty(\mathbb{R}^N)$. Using spherical coordinates, we have

$$\iint\limits_{\substack{\mathbb{R}^N \times \mathbb{R}^N \\ |u(x)-u(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}} \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \int_0^\infty_{|u(x+h\sigma)-u(x)|>\delta} \frac{\delta^p}{h^{p+1}} \,\mathrm{d}h\,\mathrm{d}\sigma\,\mathrm{d}x. \tag{13}$$

By a change of variables, we obtain

$$\int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \int_0^\infty_{|u(x+h\sigma)-u(x)|>\delta} \frac{\delta^p}{h^{p+1}} \,\mathrm{d}h\,\mathrm{d}\sigma\,\mathrm{d}x = \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \int_0^\infty_{\frac{1}{\delta h}|u(x+\delta h\sigma)-u(x)|h>1} \frac{1}{h^{p+1}} \,\mathrm{d}h\,\mathrm{d}\sigma\,\mathrm{d}x, \tag{14}$$

Combining (13) and (14) yields

$$\iint\limits_{\substack{\mathbb{R}^N \times \mathbb{R}^N \\ |u(x)-u(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}} \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \int_0^\infty_{\frac{1}{\delta h}|u(x+\delta h\sigma)-u(x)|h>1} \frac{1}{h^{p+1}} \,\mathrm{d}h\,\mathrm{d}\sigma\,\mathrm{d}x. \tag{15}$$

Since

$$\frac{1}{\delta}\left|u(x+\delta h\sigma) - u(x)\right| = h\left|\int_0^1 \nabla u(x + t\delta h\sigma)\cdot\sigma \,\mathrm{d}t\right| \leq M(\nabla u, \sigma)(x) h,$$

where

$$M(\nabla u,\sigma)(x) := \sup_{t>0} \fint_0^t \left|\nabla u(x+s\sigma)\cdot\sigma\right| \mathrm{d}s,$$

it follows (15) that

$$\begin{aligned}\iint\limits_{\substack{\mathbb{R}^N\times\mathbb{R}^N\\ |u(x)-u(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y &\le \int_{\mathbb{R}^N}\int_{\mathbb{S}^{N-1}}\int_{\substack{0\\ M(\nabla u,\sigma)(x)h>1}}^{\infty} \frac{1}{h^{p+1}}\,\mathrm{d}h\,\mathrm{d}\sigma\,\mathrm{d}x\\ &= \frac{1}{p}\int_{\mathbb{S}^{N-1}}\int_{\mathbb{R}^N} \left|M(\nabla u,\sigma)\right|^p \mathrm{d}x\,\mathrm{d}\sigma.\end{aligned} \tag{16}$$

Applying the theory of maximal functions, see, e.g., [47], we have

$$\int_{\mathbb{R}^N} \left|M(\nabla u,\sigma)\right|^p \mathrm{d}x = \int_{\mathbb{R}_\sigma^\perp}\int_{\mathbb{R}_\sigma} \left|M(\nabla u,\sigma)\right|^p \mathrm{d}x \le c_p \int_{\mathbb{R}_\sigma^\perp}\int_{\mathbb{R}_\sigma} |\nabla u\cdot\sigma|^p\,\mathrm{d}x \le c_p\int_{\mathbb{R}^N}|\nabla u|^p. \tag{17}$$

Here, for $\sigma\in\mathbb{S}^{N-1}$,

$$\mathbb{R}_\sigma = \{t\sigma;\ t\in\mathbb{R}\} \quad\text{and}\quad \mathbb{R}_\sigma^\perp = \{x\in\mathbb{R}^{N-1};\ x\cdot\sigma = 0\}.$$

Assertion (10) for $u\in C_c^\infty(\mathbb{R}^N)$ now follows from (16) and (17).

The proof of (11) for $u\in C_c^\infty(\mathbb{R}^N)$ follows from (15) and a Taylor expansion. The arguments used to prove (11) in the general case follow from this case and (10) as in the analysis of the BBM formula.

Remark 6. Part (i) of Theorem 4 was also obtained independently by Ponce and Van Schaftingen with a different proof as mentioned in [31].

We next briefly discuss the proof of (12) under the stronger assumption in which the lim inf is replaced by the lim sup, i.e.,

$$\Phi(u) \le C_{N,p}\limsup_{\delta\to 0} \iint\limits_{\substack{\mathbb{R}^N\times\mathbb{R}^N\\ |u(x)-u(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y. \tag{18}$$

The ideas of the proof of (12) under the assumption stated in Theorem 4 will be later given in Section 7.

To be able to apply the arguments used in the proof of the BBM formula, one arranges to gain some convexity so that the arguments involving the convolution can be used. This can be done by an appropriate integration with respect to δ as suggested by the author in [31]. Indeed, we first assume that $u\in L^\infty(\mathbb{R}^N)$ and let a be a positive constant which is greater than $2\|u\|_{L^\infty(\mathbb{R}^N)}+1$. We have, by using Fubini's theorem,

$$\int_0^a \varepsilon\delta^{-1+\varepsilon} \iint\limits_{\substack{\mathbb{R}^N\times\mathbb{R}^N\\ |u(x)-u(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\varepsilon}{p+\varepsilon}\frac{\left|u(x)-u(y)\right|^{p+\varepsilon}}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y. \tag{19}$$

Let $(\varphi_k)_{k\ge 1}$ be a smooth non-negative sequence of approximations to the identity such that $\operatorname{supp}\varphi_k\subset B_{1/k}$. Set

$$u_k = \varphi_k * u \quad \text{for } k\ge 1.$$

Since, by Jensen's inequality,

$$\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\varepsilon}{p+\varepsilon}\frac{\left|u_k(x)-u_k(y)\right|^{p+\varepsilon}}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y \le \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\varepsilon}{p+\varepsilon}\frac{\left|u(x)-u(y)\right|^{p+\varepsilon}}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y,$$

it follows from (19) that

$$\int_0^a \varepsilon\delta^{-1+\varepsilon} \iint\limits_{\substack{\mathbb{R}^N\times\mathbb{R}^N\\ |u(x)-u(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y \geq \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\varepsilon}{p+\varepsilon}\frac{\left|u_k(x)-u_k(y)\right|^{p+\varepsilon}}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y. \tag{20}$$

Using the fact that $u_k \in C^\infty(\mathbb{R}^N)$, we obtain, as in the proof of (8),

$$\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\varepsilon}{p+\varepsilon}\frac{\left|u_k(x)-u_k(y)\right|^{p+\varepsilon}}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y \geq C_{N,p}\int_{\mathbb{R}^N}|\nabla u_k|^p\,\mathrm{d}x,$$

which yields, by (20),

$$\limsup_{\delta\to 0_+} \iint\limits_{\substack{\mathbb{R}^N\times\mathbb{R}^N\\ |u(x)-u(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y \geq C_{N,p}\int_{\mathbb{R}^N}|\nabla u_k|^p\,\mathrm{d}x.$$

Since $k \geq 1$ is arbitrary, the conclusion follows.

The proof in the general case, without the assumption $u \in L^\infty(\mathbb{R}^N)$, follows from this case after noting that

$$\iint\limits_{\substack{\mathbb{R}^N\times\mathbb{R}^N\\ |u(x)-u(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y \geq \iint\limits_{\substack{\mathbb{R}^N\times\mathbb{R}^N\\ |u_A(x)-u_A(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y, \tag{21}$$

where

$$u_A = \min\{\max\{u,-A\},A\}.$$

Indeed, since $u_A \in L^\infty(\mathbb{R}^N)$, we have

$$\limsup_{\delta\to 0_+} \iint\limits_{\substack{\mathbb{R}^N\times\mathbb{R}^N\\ |u_A(x)-u_A(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y \geq C_{N,p}\Phi(u_A). \tag{22}$$

Combining (21) and (22) yields

$$\iint\limits_{\substack{\mathbb{R}^N\times\mathbb{R}^N\\ |u(x)-u(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y \geq C_{N,p}\Phi(u_A),$$

which implies (18) after letting $A \to +\infty$.

The setting considered in Theorem 4 has been extended in several directions. See, e.g., [15,18,19,34] where a more general functional was considered, see, e.g., [38,43] where a magnetic field was involved (related properties for the BBM formula were also considered, see, e.g., [38,40,43]).

One of these extensions, recently proposed by Brezis, Seeger, Van Schaftingen, and Yung [18], will be now discussed. Set, for a measurable function u defined in $\mathbb{R}^N$,

$$Q_b u(x,y) = \frac{u(x)-u(y)}{|x-y|^{1+b}} \quad \text{for } x,y \in \mathbb{R}^N,$$

and, for a measurable set E of $\mathbb{R}^N \times \mathbb{R}^N$,

$$\nu_\gamma(E) = \iint_E |x-y|^{-N+\gamma}\,\mathrm{d}x\,\mathrm{d}y,$$

and

$$E_{\lambda,b}(u) = \left\{(x,y);\ \left|Q_b u(x,y)\right| > \lambda\right\}.$$

Given $\gamma \in \mathbb{R}$ and $p \geq 1$, set

$$\Phi_\lambda(u) = \nu_\gamma\left(E_{\lambda,\gamma/p}(u)\right) \quad \text{for } \lambda > 0.$$

Given a measurable subset Ω of $\mathbb{R}^N$ and a measurable function u defined in Ω, we also denote

$$E_{\lambda,b,\Omega}(u) = \Big\{(x,y) \in \Omega \times \Omega;\ \big|Q_b u(x,y)\big| > \lambda\Big\}.$$

and

$$\Phi_{\lambda,\Omega}(u) = \lambda^p \nu_\gamma\Big(E_{\lambda,\gamma/p,\Omega}(u)\Big).$$

One has the following result.

Theorem 7 (Brezis & Seeger & Van Schaftingen & Yung). *Let $N \ge 1$, $1 \le p < +\infty$, $\gamma \in \mathbb{R} \setminus \{0\}$. The following facts hold.*

(i) *For $p > 1$ and $u \in W^{1,p}(\mathbb{R}^N)$, we have*

$$\Phi_\lambda(u) \le C_{N,p}\Phi(u) \quad \textit{for } \lambda > 0, \tag{23}$$

and

$$\lim_{\lambda \to 0_+} \Phi_\lambda(u) = \frac{1}{|\gamma|} K_{N,p}\Phi(u) \quad \textit{if } \gamma < 0, \tag{24}$$

and

$$\lim_{\lambda \to +\infty} \Phi_\lambda(u) = \frac{1}{|\gamma|} K_{N,p}\Phi(u) \quad \textit{if } \gamma > 0, \tag{25}$$

where $K_{N,p}$ is defined by (4)*.*

(ii) *If $u \in L^p(\mathbb{R}^N)$ and $p \ge 1$, then*

$$\Phi(u) \le C \sup_{\lambda>0} \Phi_\lambda(u). \tag{26}$$

In particular, $u \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$ and $u \in \mathrm{BV}(\mathbb{R}^N)$ if $p = 1$ if the RHS of (26) *is finite.*

Remark 8. For $\gamma = -p$, one has

$$\Phi_\delta(u) = \iint\limits_{\substack{\mathbb{R}^N \times \mathbb{R}^N \\ |u(x)-u(y)|>\delta}} \frac{\delta^p}{|x-y|^{N+p}} \,\mathrm{d}x\,\mathrm{d}y,$$

the quantity considered in Theorem 4. For $\gamma = N$, the quantity Φ_λ was previously considered by Brezis, Van Schaftingen, and Yung [19]. A generalization in this case to one-parameter families of operators was considered by Domínguez and Milman [23].

Remark 9. It is worth noting that (24) and (25) fail for $p = 1$ and for $u \in \mathrm{BV}(\mathbb{R}^N)$ or even for $u \in W^{1,1}(\mathbb{R}^N)$. This has been noted by Brezis, Seeger, Van Schaftingen, and Yung [18]. This phenomenon in the case $\gamma = -1$ was previously observed by Ponce, see also the work of Brezis and Nguyen [15].

We next briefly discuss the proof of part (i) of Theorem 7. The proof is closely related to the proof of Theorem 4 mentioned previously and is different from the one given in [18]. We only discuss the case $N = 1$. The general case can be established as in the proof of Theorem 4. We begin with the proof of assertion (23). As in the proof of (4), it suffices to consider only the case $u \in C_c^\infty(\mathbb{R})$. We have

$$\iint\limits_{\substack{\mathbb{R}\times\mathbb{R} \\ \frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}}>\lambda}} |x-y|^{-1+\gamma}\,\mathrm{d}x\,\mathrm{d}y = 2 \int_{\mathbb{R}}\int_0^\infty\limits_{\frac{|u(x+h)-u(h)|}{h^{1+\frac{\gamma}{p}}}>\lambda} h^{-1+\gamma}\,\mathrm{d}h\,\mathrm{d}x. \tag{27}$$

Since

$$\big|u(x+h) - u(x)\big| = \left|\int_x^{x+h} u'(s)\,\mathrm{d}s\right| \le \int_x^{x+h} \big|u'(s)\big|\,\mathrm{d}s \le M(u')(x)h,$$

where $M(u')(x) = \sup_{h>0} \fint_x^{x+h} \big|u'(s)\big| \, ds$, it follows that

$$\int_{\mathbb{R}} \int_{0 \atop \frac{|u(x+h)-u(x)|}{h^{1+\frac{\gamma}{p}}} > \lambda}^{\infty} h^{-1+\gamma} \, dh \, dx \le \int_{\mathbb{R}} \int_{0 \atop \frac{M(u')(x)}{h^{\frac{\gamma}{p}}} > \lambda}^{\infty} h^{-1+\gamma} \, dh \, dx. \tag{28}$$

By considering $\gamma > 0$ and $\gamma < 0$ separately, one can show that

$$\lambda^p \int_{\mathbb{R}} \int_{0 \atop \frac{M(u')(x)}{h^{\frac{\gamma}{p}}} > \lambda}^{\infty} h^{-1+\gamma} \, dh \, dx \le C_\gamma \int_{\mathbb{R}} \big|M(u')(x)\big|^p \, dx. \tag{29}$$

Since, by the theory of maximal functions,

$$\int_{\mathbb{R}} \big|M(u')(x)\big|^p \, dx \le c_p \int_{\mathbb{R}} |u'|^p \, dx,$$

assertion (23) follows for $u \in C_c^\infty(\mathbb{R})$.

To prove (24) and (25), one notes that, by a change of variables, with $\delta^{-\frac{\gamma}{p}} = \lambda$,

$$\lambda^p \int_{\mathbb{R}} \int_{0 \atop \frac{|u(x+h)-u(x)|}{h^{1+\frac{\gamma}{p}}} > \lambda}^{\infty} h^{-1+\gamma} \, dh \, dx = \int_{\mathbb{R}} \int_{0 \atop \frac{|u(x+\delta h)-u(x)|}{\delta h} h^{-\frac{\gamma}{p}} > 1}^{\infty} h^{-1+\gamma} \, dh \, dx,$$

which yields, by (27), with $\delta^{-\frac{\gamma}{p}} = \lambda$,

$$\Phi_\lambda(u) = \int_{\mathbb{R}} \int_{0 \atop \frac{|u(x+\delta h)-u(x)|}{\delta h} h^{-\frac{\gamma}{p}} > 1}^{\infty} h^{-1+\gamma} \, dh \, dx. \tag{30}$$

The proof of (24) and (25) for $u \in C_c^\infty(\mathbb{R})$ follows from (30) and a Taylor expansion. The arguments in the general case follow from this case and (23) as in the analysis of the BBM formula.

The proof of part (ii) of Theorem 7 given in [18] is based on the BBM formula and the Lorentz duality, which in turn involves rearrangement properties. Later, we state and prove a stronger version of part (ii) (see Section 2). Our proof is in the spirit of the one of Theorem 4 and thus different from [18].

Brezis, Seeger, Van Schaftingen, and Yung [18] also proved that, for $p = 1$, the following result.

Proposition 10 (Brezis & Seeger & Van Schaftingen & Yung). *Let $N \ge 1$, $p = 1$, and $\gamma \in \mathbb{R} \setminus \{0\}$. Then* (24) *holds for $u \in C_c^1(\mathbb{R}^N)$ if $\gamma \notin [-1,0)$ and for $u \in W^{1,1}(\mathbb{R}^N)$ if $\gamma > 0$. We also have, for $\gamma \notin [-1,0]$*

$$\Phi_\lambda(u) \le C_{N,1} \|\nabla u\|_{\mathcal{M}}, \quad \forall \, \lambda > 0, \tag{31}$$

Remark 11. Brezis & Seeger & Van Schaftingen & Yung [18] also showed that (31) fails for $\gamma \in [-1,0)$.

Remark 12. Assertion (23) for $\gamma = N$ was obtained by Brezis, Van Schaftingen, and Yung [20]. They also discussed there variants of Gagliardo & Nirenberg interpolation inequalities for functions in $W^{1,1}(\mathbb{R}^N)$.

The proof of (31) is based on the Vitali covering lemma in the case $\gamma > 0$ and involves a clever way to estimate double integrals in the case $\gamma < -1$ (and $N = 1$), see [18].

Viewing Theorem 7 and Proposition 10, the following questions are proposed, in the spirit of Theorem 4.

Open question 13. *Let $N \ge 1$, $p \ge 1$, and $\gamma \in \mathbb{R} \setminus \{0\}$, and let $u \in L^p(\mathbb{R}^N)$. Is it true that, for some positive constant $C_{N,p}$,*

$$\limsup_{\lambda \to 0_+} \Phi_\lambda(u) \ge C_{N,p}\Phi(u) \quad \text{for } \gamma < 0$$

and

$$\limsup_{\lambda \to +\infty} \Phi_\lambda(u) \ge C_{N,p}\Phi(u) \quad \text{for } \gamma > 0?$$

A stronger version which has been proposed by Brezis & Seeger & Van Schaftingen & Yung [18] is the following question.

Open question 14. *Let $N \ge 1$, $p \ge 1$, and $\gamma \in \mathbb{R} \setminus \{0\}$, and let $u \in L^p(\mathbb{R}^N)$. Is it true that, for some positive constant $C_{N,p}$,*

$$\liminf_{\lambda \to 0_+} \Phi_\lambda(u) \ge C_{N,p}\Phi(u) \quad \text{for } \gamma < 0$$

and

$$\liminf_{\lambda \to +\infty} \Phi_\lambda(u) \ge C_{N,p}\Phi(u) \quad \text{for } \gamma > 0?$$

If the answer to Question 14 or Question 13 is positive, one then improves (23) and (31). Various results related to these questions will be stated in the next sections, Sections 2–7. In Section 8, we discuss about the Gamma-convergence of Φ_λ, and in Section 9 we discuss various inequalities related to Φ_λ.

2. Characterizations of the Sobolev norms and the total variations

In this section, we present various positive and negative results related to Questions 13 and 14. These results particularly improve part (ii) of Theorem 7. Here are the main results in this direction. We begin with the case $\gamma < 0$.

Theorem 15. *Let $N \ge 1$, $1 \le p < +\infty$, and $\gamma < -p$, and let $u \in L^p(\mathbb{R}^n)$. We have*

$$\frac{K_{N,p}}{|\gamma|}\Phi(u) \le \limsup_{\lambda \to 0_+} \Phi_\lambda(u). \tag{32}$$

In particular, $u \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$ and $u \in \mathrm{BV}(\mathbb{R}^N)$ if $p = 1$ if the RHS of (32) is finite.

The proof of Theorem 15 is in the spirit of the one of (18) and is given in Section 3.

Theorem 16. *Let $N \ge 1$, $1 \le p < +\infty$, and $-p \le \gamma \le -1$, and let $u \in L^p(\mathbb{R}^N)$. We have*

$$C_{N,p}\Phi(u) \le \liminf_{\lambda \to 0_+} \Phi_\lambda(u) \tag{33}$$

for some positive constant $C_{N,p}$. In particular, $u \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$ and $u \in \mathrm{BV}(\mathbb{R}^N)$ if $p = 1$ if the RHS of (33) is finite.

As mentioned previously, the case $\gamma = -p$ is due to Bourgain and Nguyen [9]. The proof of Theorem 15 is in the same spirit and is presented in Section 7.

The case $-1 < \gamma < 0$ is quite special. We have the following result whose proof is given in Section 6.

Proposition 17. *Let $N \ge 1$, $1 \le p < +\infty$, and $-1 < \gamma < 0$. There exists a non-zero function $u \in \mathrm{BV}(\mathbb{R}^N)$ with compact support such that*

$$\lim_{\lambda \to 0_+} \Phi_\lambda(u) = 0.$$

Proposition 17 gives a negative answer to Questions 13 and 14 in the case $-1 < \gamma < 0$. Nevertheless, we can prove the following result which improves part (ii) of Theorem 7 in the case $-1 < \gamma < 0$.

Theorem 18. *Let $N \ge 1$, $1 \le p < +\infty$, and $-1 < \gamma < 0$, and let $u \in L^p(\mathbb{R}^n)$. Then*

$$\frac{K_{N,p}}{\gamma + 2p}\Phi(u) \le \limsup_{\lambda \to 0_+} \Phi_\lambda(u) + \limsup_{\lambda \to +\infty} \Phi_\lambda(u). \tag{34}$$

In particular, $u \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$ and $u \in \mathrm{BV}(\mathbb{R}^N)$ if $p = 1$ if the RHS of (34) *is finite.*

The proof of Theorem 18 is in the spirit of the one of Theorem 4 mentioned in the introduction and given in Section 5.

We next deal with the case $\gamma > 0$.

Theorem 19. *Let $N \ge 1$, $1 \le p < +\infty$, and $\gamma > 0$, and let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then*

$$\frac{K_{N,p}}{\gamma + p}\Phi(u) \le \limsup_{\lambda \to +\infty} \Phi_\lambda(u). \tag{35}$$

In particular, $u \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$ and $u \in \mathrm{BV}(\mathbb{R}^N)$ if $p = 1$ if $u \in L^p(\mathbb{R}^N)$ and the RHS of (35) *is finite.*

The proof of Theorem 19 is in the spirit of the one of Theorem 4 mentioned in the introduction and given in Section 4.

Theorem 19 is known in the case $p > 1$ and partially known in the case $p = 1$. In the case $\gamma = N$ and $p \ge 1$, Theorem 19 was established by Poliakovsky [44] with $C_{N,p}$ instead of $\frac{K_{N,p}}{\gamma+p}$ in the LHS of (35). A stronger result of Theorem 19 with $C_{N,p,\gamma}$, blowing up as $\gamma \to 0_+$, instead of $\frac{K_{N,p}}{\gamma+p}$ in the LHS of (35) but with lim inf instead of lim sup on the RHS was obtained by Gobbino and Picenni [25]. Their results do not imply ours in the case $p = 1$ since (25) does not hold for $p = 1$. Theorem 19 for $p = 1$ is sharp in the sense that the equality holds in (35) when u is the characteristic function of a bounded convex domain with smooth boundary, see [18, Lemma 3.6] (see also [42]).

3. Proof of Theorem 15

The proof is based on the ideas of the convexity and the BBM formula as mentioned in the proof of (18). We first assume that $u \in L^\infty(\mathbb{R}^N)$ and let $m \ge 1$ be such that

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le m. \tag{36}$$

Fix $a > 0$ sufficiently small. Then

$$\limsup_{\lambda \to 0_+} \Phi_\lambda(u) \ge \limsup_{\varepsilon \to 0_+} \int_0^a \frac{\varepsilon}{\lambda^{1-\varepsilon}} \Phi_\lambda(u)\, \mathrm{d}\lambda. \tag{37}$$

We have, for $\varepsilon > 0$,

$$\begin{aligned}
\int_0^a \frac{\varepsilon}{\lambda^{1-\varepsilon}} \Phi_\lambda(u)\, \mathrm{d}\lambda &= \int_0^a \frac{\varepsilon}{\lambda^{1-\varepsilon}} \lambda^p \iint\limits_{\frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}} > \lambda} |x-y|^{-N+\gamma}\, \mathrm{d}x\, \mathrm{d}y \\
&= \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^{-N+\gamma}\, \mathrm{d}x\, \mathrm{d}y \int_0^a \varepsilon \lambda^{p-1+\varepsilon} \mathbb{1}_{\lambda < \frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}}}\, \mathrm{d}\lambda \\
&\ge \iint\limits_{\frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}} < a} |x-y|^{-N+\gamma} \frac{\varepsilon}{p+\varepsilon} \frac{|u(x)-u(y)|^{p+\varepsilon}}{|x-y|^{p+\varepsilon+\frac{\gamma(p+\varepsilon)}{p}}}\, \mathrm{d}x\, \mathrm{d}y.
\end{aligned}$$

We thus obtain

$$\int_0^a \frac{\varepsilon}{\lambda^{1-\varepsilon}} \Phi_\lambda(u)\,\mathrm{d}\lambda \geq \iint\limits_{\frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}} < a} \frac{1}{p+\varepsilon} \frac{\big|u(x)-u(y)\big|^{p+\varepsilon}}{|x-y|^{p+\varepsilon}} \frac{\varepsilon}{|x-y|^{N+\frac{\gamma\varepsilon}{p}}}\,\mathrm{d}x\,\mathrm{d}y. \tag{38}$$

Set

$$\beta = 1 + \frac{\gamma}{p}.$$

Since $\gamma < -p$, it follows that $\beta < 0$. Denote

$$r = (2m/a)^{1/\beta}.$$

Since $\beta < 0$, it follows that if $|x-y| < r$ then $\frac{|u(x)-u(y)|}{|x-y|^\beta} < \frac{2m}{r^\beta} = a$. We derive from (38) that

$$\int_0^a \frac{\varepsilon}{\lambda^{1-\varepsilon}} \Phi_\gamma(u)\,\mathrm{d}\lambda \geq \iint\limits_{|x-y|<r} \frac{1}{p+\varepsilon} \frac{\big|u(x)-u(y)\big|^{p+\varepsilon}}{|x-y|^{p+\varepsilon}} \frac{\varepsilon}{|x-y|^{N+\frac{\gamma\varepsilon}{p}}}\,\mathrm{d}x\,\mathrm{d}y. \tag{39}$$

Let (φ_k) be a smooth non-negative sequence of approximations to the identity such that $\operatorname{supp}\varphi_k \subset B_{1/k}$. Set

$$u_k = \varphi_k * u \quad \text{for } k \geq 1.$$

Since $p+\varepsilon > 1$, we have, by Jensen's inequality,

$$\begin{aligned} &\iint\limits_{|x-y|<r} \frac{1}{p+\varepsilon} \frac{\big|u(x)-u(y)\big|^{p+\varepsilon}}{|x-y|^{p+\varepsilon}} \frac{\varepsilon}{|x-y|^{N+\frac{\gamma\varepsilon}{p}}}\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad\qquad \geq \iint\limits_{|x-y|<r} \frac{1}{p+\varepsilon} \frac{\big|u_k(x)-u_k(y)\big|^{p+\varepsilon}}{|x-y|^{p+\varepsilon}} \frac{\varepsilon}{|x-y|^{N+\frac{\gamma\varepsilon}{p}}}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{40}$$

Combining (39) and (40) yields, for $k \geq 1$,

$$\int_0^a \frac{\varepsilon}{\lambda^{1-\varepsilon}} \Phi_\gamma(u)\,\mathrm{d}\lambda \geq \iint\limits_{|x-y|<r} \frac{1}{p+\varepsilon} \frac{\big|u_k(x)-u_k(y)\big|^{p+\varepsilon}}{|x-y|^{p+\varepsilon}} \frac{\varepsilon}{|x-y|^{N+\frac{\gamma\varepsilon}{p}}}\,\mathrm{d}x\,\mathrm{d}y. \tag{41}$$

By letting $\varepsilon \to 0_+$, and noting that $\gamma < 0$ and using (37), we obtain, as in the proof of (8),

$$\limsup_{\lambda\to 0} \Phi_\lambda(u) \geq \frac{1}{|\gamma|} K_{N,p} \int_{\mathbb{R}^N} |\nabla u_k|^p\,\mathrm{d}x \tag{42}$$

since, for $\gamma < 0$,

$$\lim_{\varepsilon\to 0_+} \int_0^r \frac{\varepsilon}{s^{1+\frac{\gamma\varepsilon}{p}}}\,\mathrm{d}s = \frac{p}{|\gamma|}.$$

By taking $k \to +\infty$, we reach the conclusion when $u \in L^\infty(\mathbb{R}^N)$. The proof in the general case follows from the case where $u \in L^\infty(\mathbb{R}^N)$ as in the proof of Theorem 4 after noting that

$$\Phi_\lambda(u) \geq \Phi_\lambda(u_A),$$

where

$$u_A = \min\big\{\max\{u, -A\}, A\big\}.$$

The details are omitted.

4. Proof of Theorem 19

The proof is again based on the ideas of the convexity and the BBM formula as mentioned in the proof of (18) related to Theorem 4. We first assume that

$$\|u\|_{L^\infty} \le m.$$

Let $a > 0$ be sufficiently large. Then

$$\limsup_{\lambda\to+\infty} \Phi_\lambda(u) \ge \limsup_{\varepsilon\to 0_+} \int_a^\infty \frac{\varepsilon}{\lambda^{1+\varepsilon}} \Phi_\lambda(u)\,\mathrm{d}\lambda. \tag{43}$$

We have

$$\begin{aligned}\int_a^\infty \frac{\varepsilon}{\lambda^{1+\varepsilon}} \Phi_\lambda(u)\,\mathrm{d}\lambda &= \int_a^\infty \frac{\varepsilon}{\lambda^{1+\varepsilon}} \lambda^p \iint\limits_{\frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}}>\lambda} |x-y|^{-N+\gamma}\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} |x-y|^{-N+\gamma}\,\mathrm{d}x\,\mathrm{d}y \int_a^\infty \varepsilon\lambda^{p-1-\varepsilon} \mathbb{1}_{\lambda<\frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}}}\,\mathrm{d}\lambda.\end{aligned}$$

It follows that, for every $R > 2$ and $0 < r < 1$,

$$\begin{aligned}\int_a^\infty \frac{\varepsilon}{\lambda^{1+\varepsilon}} \Phi_\lambda(u)\,\mathrm{d}\lambda &\ge \iint\limits_{x\in B_R;\ |x-y|<r} |x-y|^{-N+\gamma}\,\mathrm{d}x\,\mathrm{d}y \int_a^\infty \varepsilon\lambda^{p-1-\varepsilon} \mathbb{1}_{\lambda<\frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}}}\,\mathrm{d}\lambda \\ &\ge \iint\limits_{x\in B_R;\ |x-y|<r} |x-y|^{-N+\gamma}\left(\frac{\varepsilon}{p-\varepsilon}\frac{\big|u(x)-u(y)\big|^{p-\varepsilon}}{|x-y|^{p-\varepsilon+\frac{\gamma(p-\varepsilon)}{p}}} - \frac{\varepsilon}{p-\varepsilon}a^{p-\varepsilon}\right)\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{p-\varepsilon}\iint\limits_{x\in B_R;\ |x-y|<r} \frac{\big|u(x)-u(y)\big|^{p-\varepsilon}}{|x-y|^{p-\varepsilon}} \frac{\varepsilon}{|x-y|^{N-\frac{\varepsilon\gamma}{p}}}\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad - \frac{\varepsilon a^{p-\varepsilon}}{p-\varepsilon}\iint\limits_{x\in B_R;\ |x-y|<r} |x-y|^{-N+\gamma}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{44}$$

We have

$$\begin{aligned}&\frac{1}{p-\varepsilon}\iint\limits_{x\in B_R;\ |x-y|<r} \frac{\big|u(x)-u(y)\big|^{p-\varepsilon}}{|x-y|^{p-\varepsilon}} \frac{\varepsilon}{|x-y|^{N-\frac{\varepsilon\gamma}{p}}}\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad \ge \frac{1}{p-\varepsilon}\frac{1}{|2m|^\varepsilon}\iint\limits_{x\in B_R;\ |x-y|<r} \frac{\big|u(x)-u(y)\big|^{p}}{|x-y|^{p}} \frac{\varepsilon}{|x-y|^{N-\frac{\varepsilon\gamma}{p}-\varepsilon}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{45}$$

Let $(\varphi_k)_{k\ge1}$ be a smooth non-negative sequence of approximations to the identity such that $\operatorname{supp}\varphi_k \subset B_{1/k}$. Set

$$u_k = \varphi_k * u \quad \text{for } k \ge 1.$$

We have, by Jensen's inequality,

$$\begin{aligned}&\frac{1}{p-\varepsilon}\frac{1}{|2m|^\varepsilon}\iint\limits_{x\in B_R;\ |x-y|<r} \frac{\big|u(x)-u(y)\big|^{p}}{|x-y|^{p-\varepsilon}} \frac{\varepsilon}{|x-y|^{N-\frac{\varepsilon\gamma}{p}-\varepsilon}}\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad \ge \frac{1}{p-\varepsilon}\frac{1}{|2m|^\varepsilon}\iint\limits_{x\in B_{R-1};\ |x-y|<r} \frac{\big|u_k(x)-u_k(y)\big|^{p}}{|x-y|^{p}} \frac{\varepsilon}{|x-y|^{N-\frac{\varepsilon\gamma}{p}-\varepsilon}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{46}$$

Since, thanks to fact that $\gamma > 0$,

$$\lim_{\varepsilon \to 0_+} \frac{\varepsilon a^{p-\varepsilon}}{p-\varepsilon} \iint\limits_{x \in B_R;\ |x-y|<r} |x-y|^{-N+\gamma} \, dx \, dy = 0, \tag{47}$$

we derive from (44), (45), and (46) that

$$\limsup_{\varepsilon \to 0_+} \int_a^\infty \frac{\varepsilon}{\lambda^{1-\varepsilon}} \Phi_\lambda(u) \, d\lambda \geq \liminf_{\varepsilon \to 0_+} \frac{1}{p-\varepsilon} \frac{1}{|2m|^\varepsilon} \iint\limits_{x \in B_{R-1};\ |x-y|<1} \frac{\big|u_k(x) - u_k(y)\big|^p}{|x-y|^p} \frac{\varepsilon}{|x-y|^{N - \frac{\varepsilon\gamma}{p} - \varepsilon}} \, dx \, dy.$$

Using (43), we obtain

$$\limsup_{\lambda \to +\infty} \Phi_\lambda(u) \geq \frac{K_{N,p}}{\gamma + p} \int_{B_{R-1}} |\nabla u_k|^p \, dx \tag{48}$$

since, for $-1 < \gamma < 0$,

$$\lim_{\varepsilon \to 0_+} \int_0^1 \frac{\varepsilon}{s^{1 - \frac{\gamma\varepsilon}{p} - \varepsilon}} \, ds = \frac{p}{\gamma + p}.$$

Since $R > 2$ is arbitrary, we reach

$$\limsup_{\lambda \to +\infty} \Phi_\lambda(u) \geq \frac{K_{N,p}}{\gamma + p} \int_{\mathbb{R}^N} |\nabla u_k|^p \, dx. \tag{49}$$

Letting $k \to +\infty$, we reach the conclusion when $u \in L^\infty(\mathbb{R}^N)$.

The proof in the general case follows from the case where $u \in L^\infty(\mathbb{R}^N)$ as in the proof of Theorem 4 after noting that

$$\Phi_\lambda(u) \geq \Phi_\lambda(u_A),$$

where

$$u_A = \min\big\{\max\{u, -A\}, A\big\}.$$

The details are omitted.

5. Proof of Theorem 18

The proof is again based on the ideas of the convexity and the BBM formula as mentioned in the proof of (18) related to Theorem 4. Without loss of generality, one might assume that

$$\limsup_{\lambda \to 0_+} \Phi_\lambda u < +\infty. \tag{50}$$

This implies, for all $1 < \Lambda < +\infty$,

$$\Phi_\lambda u < C_\Lambda \quad \text{for } \Lambda^{-1} \leq \lambda \leq \Lambda. \tag{51}$$

We first assume that

$$\|u\|_{L^\infty} \leq m \quad \text{and} \quad m \geq 1.$$

Then, by (51), it holds

$$\limsup_{\lambda \to +\infty} \Phi_\lambda(u) \geq \limsup_{\varepsilon \to 0_+} \int_1^\infty \frac{\varepsilon}{\lambda^{1+\varepsilon}} \Phi_\lambda(u) \, d\lambda. \tag{52}$$

We have

$$\int_1^\infty \frac{\varepsilon}{\lambda^{1+\varepsilon}} \Phi_\lambda(u) \, d\lambda = \int_1^\infty \frac{\varepsilon}{\lambda^{1+\varepsilon}} \lambda^p \iint\limits_{\frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}} > \lambda} |x-y|^{-N+\gamma} \, dx \, dy$$

$$= \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^{-N+\gamma} \, dx \, dy \int_1^\infty \varepsilon \lambda^{p-1-\varepsilon} \mathbb{1}_{\lambda < \frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}}} \, d\lambda.$$

This implies, for every $R > 2$ and $0 < r < 1$,

$$\int_1^\infty \frac{\varepsilon}{\lambda^{1+\varepsilon}} \Phi_\lambda(u)\, d\lambda \ge \frac{1}{p-\varepsilon} \iint\limits_{\substack{x \in B_R;\ |x-y|<r \\ \frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}} > 1}} \frac{|u(x)-u(y)|^{p-\varepsilon}}{|x-y|^{p-\varepsilon}} \frac{\varepsilon}{|x-y|^{N-\frac{\varepsilon\gamma}{p}}}\, dx\, dy \\ - \frac{\varepsilon}{p-\varepsilon} \iint\limits_{\substack{x \in B_R;\ |x-y|<r \\ \frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}} > 1}} |x-y|^{-N+\gamma}\, dx\, dy. \quad (53)$$

On the other hand,

$$\frac{1}{p-\varepsilon} \iint\limits_{\substack{x \in B_R;\ |x-y|<r \\ \frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}} > 1}} \frac{|u(x)-u(y)|^{p-\varepsilon}}{|x-y|^{p-\varepsilon}} \frac{\varepsilon}{|x-y|^{N-\frac{\varepsilon\gamma}{p}}}\, dx\, dy \\ \ge \frac{1}{p-\varepsilon} \frac{1}{|2m|^{2\varepsilon}} \iint\limits_{\substack{x \in B_R;\ |x-y|<r \\ \frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}} > 1}} \frac{|u(x)-u(y)|^{p+\varepsilon}}{|x-y|^{p+\varepsilon}} \frac{\varepsilon}{|x-y|^{N-\frac{\varepsilon\gamma}{p}-2\varepsilon}}\, dx\, dy. \quad (54)$$

Combining (53), and (54) yields

$$\int_1^\infty \frac{\varepsilon}{\lambda^{1+\varepsilon}} \Phi_\lambda(u)\, d\lambda \ge \frac{1}{p-\varepsilon} \frac{1}{|2m|^{2\varepsilon}} \iint\limits_{\substack{x \in B_R;\ |x-y|<r \\ \frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}} > 1}} \frac{|u(x)-u(y)|^{p+\varepsilon}}{|x-y|^{p+\varepsilon}} \frac{\varepsilon}{|x-y|^{N-\frac{\varepsilon\gamma}{p}-2\varepsilon}}\, dx\, dy \\ - \frac{\varepsilon}{p-\varepsilon} \iint\limits_{\substack{x \in B_R;\ |x-y|<r \\ \frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}} > 1}} |x-y|^{-N+\gamma}\, dx\, dy. \quad (55)$$

Similarly, by (51), it holds

$$\limsup_{\lambda \to 0_+} \Phi_\lambda(u) \ge \limsup_{\varepsilon \to 0_+} \int_0^1 \frac{\varepsilon}{\lambda^{1-\varepsilon}} \Phi_\lambda(u)\, d\lambda. \quad (56)$$

We have

$$\int_0^1 \frac{\varepsilon}{\lambda^{1-\varepsilon}} \Phi_\lambda(u)\, d\lambda = \int_0^1 \frac{\varepsilon}{\lambda^{1-\varepsilon}} \lambda^p \iint\limits_{\frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}} > \lambda} |x-y|^{-N+\gamma}\, dx\, dy \\ = \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^{-N+\gamma}\, dx\, dy \int_0^1 \varepsilon \lambda^{p-1+\varepsilon} \mathbb{1}_{\lambda < \frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}}}\, d\lambda.$$

This implies, for every $R > 2$ and $0 < r < 1$,

$$\int_0^1 \frac{\varepsilon}{\lambda^{1-\varepsilon}} \Phi_\lambda(u)\, d\lambda \ge \frac{1}{p+\varepsilon} \iint\limits_{\substack{x \in B_R;\ |x-y|<r \\ \frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}} \le 1}} \frac{|u(x)-u(y)|^{p+\varepsilon}}{|x-y|^{p+\varepsilon}} \frac{\varepsilon}{|x-y|^{N+\frac{\varepsilon\gamma}{p}}}\, dx\, dy \\ + \frac{\varepsilon}{p+\varepsilon} \iint\limits_{\substack{x \in B_R;\ |x-y|<r \\ \frac{|u(x)-u(y)|}{|x-y|^{1+\frac{\gamma}{p}}} > 1}} |x-y|^{-N+\gamma}\, dx\, dy. \quad (57)$$

Combining (55) and (57) yields, since $\frac{\varepsilon\gamma}{p} > -\frac{\varepsilon\gamma}{p} - 2\varepsilon$ and $r < 1$,

$$\int_1^\infty \frac{\varepsilon}{\lambda^{1+\varepsilon}} \Phi_\lambda(u)\,\mathrm{d}\lambda + \frac{p+\varepsilon}{p-\varepsilon} \int_0^1 \frac{\varepsilon}{\lambda^{1-\varepsilon}} \Phi_\lambda(u)\,\mathrm{d}\lambda \\ \geq \frac{1}{p-\varepsilon} \frac{1}{|2m|^{2\varepsilon}} \iint\limits_{x\in B_R;\ |x-y|<r} \frac{\left|u(x)-u(y)\right|^{p+\varepsilon}}{|x-y|^{p+\varepsilon}} \frac{\varepsilon}{|x-y|^{N-\frac{\varepsilon\gamma}{p}-2\varepsilon}}\,\mathrm{d}x\,\mathrm{d}y. \tag{58}$$

Let $(\varphi_k)_{k\geq 1}$ be a smooth non-negative sequence of approximations to the identity such that $\operatorname{supp}\varphi_k \subset B_{1/k}$. Set

$$u_k = \varphi_k * u \quad \text{for } k \geq 1.$$

We have, by Jensen's inequality,

$$\frac{1}{p-\varepsilon} \frac{1}{|2m|^{2\varepsilon}} \iint\limits_{x\in B_R;\ |x-y|<r} \frac{\left|u(x)-u(y)\right|^{p+\varepsilon}}{|x-y|^{p+\varepsilon}} \frac{\varepsilon}{|x-y|^{N-\frac{\varepsilon\gamma}{p}-2\varepsilon}}\,\mathrm{d}x\,\mathrm{d}y \\ \geq \frac{1}{p-\varepsilon} \frac{1}{|2m|^{2\varepsilon}} \iint\limits_{x\in B_{R-1};\ |x-y|<r} \frac{\left|u_k(x)-u_k(y)\right|^{p+\varepsilon}}{|x-y|^{p+\varepsilon}} \frac{\varepsilon}{|x-y|^{N-\frac{\varepsilon\gamma}{p}-2\varepsilon}}\,\mathrm{d}x\,\mathrm{d}y. \tag{59}$$

Since

$$\liminf_{\varepsilon\to 0_+} \iint\limits_{x\in B_{R-1};\ |x-y|<r} \frac{\left|u_k(x)-u_k(y)\right|^{p+\varepsilon}}{|x-y|^{p+\varepsilon}} \frac{\varepsilon}{|x-y|^{N-\frac{\varepsilon\gamma}{p}-2\varepsilon}}\,\mathrm{d}x\,\mathrm{d}y \geq \frac{K_{N,p}}{\gamma+2p} \int_{B_{R-1}} |\nabla u_k|^p\,\mathrm{d}x, \tag{60}$$

it follows from (58) and (59) that

$$\liminf_{\varepsilon\to 0_+} \left(\int_1^\infty \frac{\varepsilon}{\lambda^{1+\varepsilon}} \Phi_\lambda(u)\,\mathrm{d}\lambda + \int_0^1 \frac{\varepsilon}{\lambda^{1-\varepsilon}} \Phi_\lambda(u)\,\mathrm{d}\lambda \right) \geq \frac{K_{N,p}}{\gamma+2p} \int_{B_{R-1}} |\nabla u_k|^p\,\mathrm{d}x. \tag{61}$$

Since $R > 2$ is arbitrary, we reach

$$\limsup_{\lambda\to 0_+} \Phi_\lambda(u) + \limsup_{\lambda\to+\infty} \Phi_\lambda(u) \geq \frac{K_{N,p}}{\gamma+2p} \int_{\mathbb{R}^N} |\nabla u_k|^p\,\mathrm{d}x. \tag{62}$$

Letting $k \to +\infty$, we reach the conclusion when $u \in L^\infty(\mathbb{R}^N)$.

The proof in the general case follows from the case where $u \in L^\infty(\mathbb{R}^N)$ as in the proof of Theorem 4 after noting that

$$\Phi_\lambda(u) \geq \Phi_\lambda(u_A),$$

where

$$u_A = \min\big\{\max\{u, -A\}, A\big\}.$$

The details are omitted.

6. Proof of Proposition 17

This section consisting of two subsections is devoted to the proof of Proposition 17. In the first subsection, we state and prove a lemma which is used in the proof of Proposition 17. The proof of Proposition 17 is given in the second subsection.

6.1. *A useful lemma*

Here is the result of this section.

Lemma 20. *Let $1 \le p < +\infty$, $\beta \in \mathbb{R}$, and let g be a measurable function on $\mathbb{R}^N$. There exist positive constants c_1, c_2 independent of g such that, for $\gamma \in \mathbb{R} \setminus \{0\}$,*

$$\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \int_{\mathbb{R}}_{\frac{|g(x_i', x_i) - g(x_i', y_i)|}{|x_i - y_i|^\beta} > c_1 \lambda} |x_i - y_i|^{-1+\gamma} \, dx_i \, dy_i \, dx_i' \le c_2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N}_{\frac{|g(x) - g(y)|}{|x-y|^\beta} > \lambda} |x-y|^{-N+\gamma} \, dx \, dy \quad \forall\, \lambda > 0, \quad \text{for } 1 \le i \le N, \tag{63}$$

and, for $\gamma < 1$ and $\beta \ge 0$,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N}_{\frac{|g(x) - g(y)|}{|x-y|^\beta} > c_1 \lambda} |x-y|^{-N+\gamma} \, dx \, dy \le c_2 \sum_{i=1}^N \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \int_{\mathbb{R}}_{\frac{|g(x_i', x_i) - g(x_i', y_i)|}{|x_i - y_i|^\beta} > \lambda} |x_i - y_i|^{-1+\gamma} \, dx_i \, dy_i \, dx_i', \quad \forall\, \lambda > 0. \tag{64}$$

Here and in what follows, for $1 \le i \le N$ and for a mesurable function f defined in $\mathbb{R}^N$, we denote

$$f(x_i', x_i) = f(x) \quad \text{for } x = (x_1, \ldots, x_N) \in \mathbb{R}^N$$

where

$$x_i' = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in \mathbb{R}^{N-1}.$$

Proof. We begin with the proof of (63). For notational ease, we only consider the case $i = N$ and denote $x_N' = x'$ in this case. We have

$$\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \int_{\mathbb{R}}_{\frac{|g(x', x_N) - g(x', y_N)|}{|x_N - y_N|^\beta} > c_1 \lambda} |x_N - y_N|^{-1+\gamma} \, dx_N \, dy_N \, dx' = \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \int_{\mathbb{R}}_{\frac{|g(x', x_N) - g(x', y_N)|}{|x_N - y_N|^\beta} > c_1 \lambda} \fint_{B_{\mathbb{R}^{N-1}}(x', |y_N - x_N|/2)} |x_N - y_N|^{-1+\gamma} \, dz' \, dx_N \, dy_N \, dx'. \tag{65}$$

Here, for $z' \in \mathbb{R}^{N-1}$ and $r > 0$, we denote $B_{\mathbb{R}^{N-1}}(z', r)$ the open ball in $\mathbb{R}^{N-1}$ centered at $z' \in \mathbb{R}^{N-1}$ and of radius $r > 0$, and $\fint_{B_{\mathbb{R}^{N-1}}(z',r)} = \frac{1}{|B_{\mathbb{R}^{N-1}(z',r)}|} \int_{B_{\mathbb{R}^{N-1}}(z',r)}$.

Since $\left|g(x', x_N) - g(x', y_N)\right| \le \left|g(x', x_N) - g\big(z, \frac{x_N + y_N}{2}\big)\right| + \left|g\big(z, \frac{x_N + y_N}{2}\big) - g(x', y_N)\right|$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \int_{\mathbb{R}}_{\frac{|g(x', x_N) - g(x', y_N)|}{|x_N - y_N|^\beta} > c_1 \lambda} \fint_{B_{\mathbb{R}^{N-1}}(x', |y_N - x_N|/2)} |x_N - y_N|^{-1+\gamma} \, dz' \, dx_N \, dy_N \, dx' \\
&\le \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \int_{\mathbb{R}}_{\frac{\left|g(x', y_N) - g\left(z', \frac{x_N + y_N}{2}\right)\right|}{|x_N - y_N|^\beta} > c_1 \lambda/2} \fint_{B_{\mathbb{R}^{N-1}}(x', |y_N - x_N|/2)} |x_N - y_N|^{-1+\gamma} \, dz' \, dx_N \, dy_N \, dx' \\
&\quad + \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \int_{\mathbb{R}}_{\frac{\left|g(x', x_N) - g\left(z', \frac{x_N + y_N}{2}\right)\right|}{|x_N - y_N|^\beta} > c_1 \lambda/2} \fint_{B_{\mathbb{R}^{N-1}}(x', |y_N - x_N|/2)} |x_N - y_N|^{-1+\gamma} \, dz' \, dx_N \, dy_N \, dx'.
\end{aligned} \tag{66}$$

We have, by a change of variables, for c_1 sufficiently large,

$$\int_{\mathbb{R}^{N-1}} \iint_{\substack{\mathbb{R}\ \mathbb{R} \\ \frac{\left|g(x',x_N)-g\left(z',\frac{x_N+y_N}{2}\right)\right|}{|x_N-y_N|^\beta}>c_1\lambda/2}} \fint_{B_{\mathbb{R}^{N-1}}(x',|y_N-x_N|/2)} |x_N-y_N|^{-1+\gamma}\,\mathrm{d}z'\,\mathrm{d}x_N\,\mathrm{d}y_N\,\mathrm{d}x' \le C \iint_{\substack{\mathbb{R}^N\times\mathbb{R}^N \\ \frac{|g(x)-g(y)|}{|x-y|^\beta}>\lambda}} |x-y|^{-N+\gamma}\,\mathrm{d}x\,\mathrm{d}y, \tag{67}$$

and

$$\int_{\mathbb{R}^{N-1}} \iint_{\substack{\mathbb{R}\ \mathbb{R} \\ \frac{\left|g(x',y_N)-g\left(z',\frac{x_N+y_N}{2}\right)\right|}{|x_N-y_N|^\beta}>c_1\lambda/2}} \fint_{B_{\mathbb{R}^{N-1}}(x',|y_N-x_N|/2)} |x_N-y_N|^{-1+\gamma}\,\mathrm{d}z'\,\mathrm{d}x_N\,\mathrm{d}y_N\,\mathrm{d}x' \le C \iint_{\substack{\mathbb{R}^N\times\mathbb{R}^N \\ \frac{|g(x)-g(y)|}{|x-y|^\beta}>\lambda}} |x-y|^{-N+\gamma}\,\mathrm{d}x\,\mathrm{d}y. \tag{68}$$

Combining (65), (66), (67), and (68) yields (63).

We next give the proof of (64). For notational ease, we only consider the case $N=2$. The general case follows similarly. Since $\big|g(x_1,x_2)-g(y_1,y_2)\big| \le \big|g(x_1,x_2)-g(x_1,y_2)\big| + \big|g(x_1,y_2)-g(x_2,y_2)\big|$, and $\beta\ge 0$, we derive that, for c_1 sufficiently large,

$$\iint_{\substack{\mathbb{R}^2\times\mathbb{R}^2 \\ \frac{|g(x_1,x_2)-g(y_1,y_2)|}{|x-y|^\beta}>c_1\lambda}} |x-y|^{-2+\gamma}\,\mathrm{d}x\,\mathrm{d}y \le \iint_{\substack{\mathbb{R}^2\times\mathbb{R}^2 \\ \frac{|g(x_1,x_2)-g(x_1,y_2)|}{|x_2-y_2|^\beta}>\lambda}} |x-y|^{-2+\gamma}\,\mathrm{d}x\,\mathrm{d}y + \iint_{\substack{\mathbb{R}^2\times\mathbb{R}^2 \\ \frac{|g(x_1,y_2)-g(y_1,y_2)|}{|x_1-y_1|^\beta}>\lambda}} |x-y|^{-2+\gamma}\,\mathrm{d}x\,\mathrm{d}y. \tag{69}$$

Since $\gamma<1$, it follows that

$$\iint_{\substack{\mathbb{R}^2\times\mathbb{R}^2 \\ \frac{|g(x_1,x_2)-g(x_1,y_2)|}{|x_2-y_2|^\beta}>\lambda}} |x-y|^{-2+\gamma}\,\mathrm{d}x\,\mathrm{d}y \le C\int_{\mathbb{R}} \iint_{\substack{\mathbb{R}\times\mathbb{R} \\ \frac{|g(x_1,x_2)-g(x_1,y_2)|}{|x_2-y_2|^\beta}>\lambda}} |x_2-y_2|^{-1+\gamma}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}y_2, \tag{70}$$

and

$$\iint_{\substack{\mathbb{R}^2\times\mathbb{R}^2 \\ \frac{|g(x_1,y_2)-g(y_1,y_2)|}{|x_1-y_1|^\beta}>\lambda}} |x-y|^{-2+\gamma}\,\mathrm{d}x\,\mathrm{d}y \le C\int_{\mathbb{R}} \iint_{\substack{\mathbb{R}^2\times\mathbb{R}^2 \\ \frac{|g(x_1,y_2)-g(y_1,y_2)|}{|x_1-y_1|^\beta}>\lambda}} |x_1-y_1|^{-1+\gamma}\,\mathrm{d}x_1\,\mathrm{d}y_1\,\mathrm{d}y_2. \tag{71}$$

Combining (69), (70), and (71) yields (64).

The proof is complete. □

Remark 21. The method used to prove Lemma 20 appeared in the theory of fractional Sobolev spaces (see, e.g., [1, Chapter 7]). In this context, it is due to Besov.

6.2. *Proof of Proposition 17*

Set

$$\beta := 1+\gamma/p.$$

Then $\beta>0$ since $-1<\gamma<0$. Consider

$$u = \mathbb{1}_Q \quad \text{where } Q=(0,1)^N \subset \mathbb{R}^N.$$

By Lemma 20, there exist two positive constants c_1, c_2 such that, for all $\lambda > 0$,

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}_{\frac{|u(x)-u(y)|}{|x-y|^\beta}>c_1\lambda} |x-y|^{-N+\gamma}\,\mathrm{d}x\,\mathrm{d}y &\le c_2 \sum_{i=1}^N \int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}}\int_{\mathbb{R}}_{\frac{|u(x_i',x_i)-u(x_i',y_i)|}{|x_i-y_i|^\beta}>\lambda} |x_i-y_i|^{-1+\gamma}\,\mathrm{d}x_i\,\mathrm{d}y_i\,\mathrm{d}x_i' \\ &= c_2 N \int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}}\int_{\mathbb{R}}_{\frac{|u(x',x_N)-u(x',y_N)|}{|x_N-y_N|^\beta}>\lambda} |x_N-y_N|^{-1+\gamma}\,\mathrm{d}x_N\,\mathrm{d}y_N\,\mathrm{d}x',\end{aligned} \tag{72}$$

where we denote x_N' by x' for notational ease. Since

$$\begin{aligned}&\left\{(x', x_N, y_N) \in \mathbb{R}^{N-1}\times\mathbb{R}\times\mathbb{R};\ \frac{|u(x',x_N)-u(x',y_N)|}{|x_N-y_N|^\beta} > \lambda\right\} \\ &\quad = \left\{(x', x_N, y_N) \in (0,1)^{N-1} \times \Big(\big((0,1)\times(\mathbb{R}\setminus(0,1))\big)\cup\big((\mathbb{R}\setminus(0,1))\times(0,1)\big)\Big);\ |x_N-y_N|^\beta < 1/\lambda\right\},\end{aligned}$$

it follows that, for $\lambda^{1/\beta} < 1/4$,

$$\lambda^p \int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}}\int_{\mathbb{R}}_{\frac{|u(x',x_N)-u(x',y_N)|}{|x_N-y_N|^\beta}>\lambda} |x_N-y_N|^{-1+\gamma}\,\mathrm{d}x_N\,\mathrm{d}y_N\,\mathrm{d}x' \le C\lambda^p \int_0^1 x_N^\gamma\,\mathrm{d}x_N \le C_\gamma \lambda^p. \tag{73}$$

The conclusion now follows from (72) and (73).

7. Proof of Theorem 16

7.1. *A fundamental lemma*

We follow the approach of Bourgain and Nguyen [9]. The following lemma is the key ingredient.

Lemma 22. *Let $1 \le p < +\infty$, $-p \le \gamma \le -1$, and let f be a measurable function on a bounded open nonempty interval I. Then*

$$\liminf_{\lambda\to 0_+} \Phi_{\lambda,I}(f) \ge c\frac{1}{|I|^{p-1}}\left(\operatorname*{ess\,sup}_I f - \operatorname*{ess\,inf}_I f\right)^p, \tag{74}$$

where $c = c_p$ is a positive constant depending only on p.

Proof. Set

$$\beta = 1 + \gamma/p.$$

Then

$$0 \le \beta < 1. \tag{75}$$

In what follows, we assume that

$$\liminf_{\lambda\to 0_+} \Phi_{\lambda,I}(u) < +\infty \tag{76}$$

since there is nothing to prove otherwise.

The proof is now divided into two steps.

Step 1: Proof of (74) for $f \in L^\infty(I)$. By rescaling, we may assume $I = (0,1)$. Denote $s_+ = \operatorname{ess\,sup}_I f$ and $s_- = \operatorname{ess\,inf}_I f$. Rescaling f, one may also assume

$$s_+ - s_- = 1 \tag{77}$$

(unless f is constant on I and there is nothing to prove in this case).

Take $0 < \delta < 1$ small enough to ensure that there are (density) points $t_+,\ t_- \in [40\delta, 1-40\delta] \subset [0,1]$ with

$$\begin{cases} \left|\left[t_+ - \tau, t_+ + \tau\right] \cap \left[f > \frac{3}{4}s_+ + \frac{1}{4}s_-\right]\right| > \frac{9}{5}\tau, \\ \left|\left[t_- - \tau, t_- + \tau\right] \cap \left[f < \frac{3}{4}s_- + \frac{1}{4}s_+\right]\right| > \frac{9}{5}\tau, \end{cases} \qquad \forall\, 0 < \tau < 40\delta. \tag{78}$$

Take $K \in \mathbb{Z}_+$ such that $\delta < 2^{-K} \le 5\delta/4$ and denote

$$J = \left\{ j \in \mathbb{Z}_+;\ \frac{3}{4}s_- + \frac{1}{4}s_+ < j2^{-K} < \frac{3}{4}s_+ + \frac{1}{4}s_- \right\}.$$

Then

$$|J| \ge 2^{K-1} - 2 \approx \frac{1}{\delta}. \tag{79}$$

For each j, define the following sets

$$A_j = \left\{ x \in [0,1];\ (j-1)2^{-K} \le f(x) < j2^{-K} \right\}, \quad B_j = \bigcup_{j'<j} A_{j'} \quad \text{and} \quad C_j = \bigcup_{j'>j} A_{j'},$$

so that $B_j \times C_j \subset \left[\left|f(x) - f(y)\right| \ge 2^{-K}\right] \subset \left[\left|f(x) - f(y)\right| > \delta\right]$.

Since the sets A_j are disjoint, it follows from (79) that

$$\operatorname{card}(G) \ge 2^{K-2} - 3 \approx \frac{1}{\delta}, \tag{80}$$

where G is defined by

$$G = \left\{ j \in J;\ |A_j| < 2^{-K+2} \right\}.$$

For each $j \in G$, set $\lambda_{1,j} = |A_j|$ and consider the function $\psi_1(t)$ defined as follows:

$$\psi_1(t) = \left| \left[t - 4\lambda_{1,j}, t + 4\lambda_{1,j}\right] \cap B_j \right|, \quad \forall\ t \in [40\delta, 1-40\delta].$$

Then, from (78), $\psi_1(t_+) < 4\lambda_{1,j}$ and $\psi_1(t_-) > 4\lambda_{1,j}$. Hence, since ψ_1 is a continuous function on the interval $[40\delta, 1-40\delta]$ containing the two points t_+ and t_-, there exists $t_{1,j} \in [40\delta, 1-40\delta]$ such that

$$\psi_1(t_{1,j}) = 4\lambda_{1,j}. \tag{81}$$

We have

$$\iint\limits_{\substack{I\times I \\ |f(x)-f(y)|>\delta}} |x-y|^{-2}\,\mathrm{d}x\,\mathrm{d}y < +\infty$$

by (76) and the fact that $\beta \ge 0$ and $\gamma \le -1$. It follows that $\left|\left[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}\right] \cap A_j\right| > 0$. Indeed, suppose $\left|\left[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}\right] \cap A_j\right| = 0$. Then

$$\iint\limits_{\substack{x \in \left[t_{1,j}-4\lambda_{1,j}, t_{1,j}+4\lambda_{1,j}\right] \cap B_j \\ y \in \left[t_{1,j}-4\lambda_{1,j}, t_{1,j}+4\lambda_{1,j}\right] \setminus B_j}} \frac{1}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y \le \iint\limits_{\substack{I\times I \\ |f(x)-f(y)|>\delta}} \frac{1}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y < +\infty.$$

Hence $\left|\left[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}\right] \cap B_j\right| = 0$ or $\left|\left[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}\right] \setminus B_j\right| = 0$. This is a contradiction since $\psi_1(t_{1,j}) = \left|\left[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}\right] \cap B_j\right| = 4\lambda_{1,j}$ (see (81)).

If $\left|\left[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}\right] \cap A_j\right| < \lambda_{1,j}/4$, then take $\lambda_{2,j} > 0$ such that $\lambda_{1,j}/\lambda_{2,j} \in \mathbb{Z}_+$ and

$$\frac{\lambda_{2,j}}{2} < \left|\left[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}\right] \cap A_j\right| \le \lambda_{2,j}.$$

Since $\left|\left[t_{1,j} - 4\lambda_{1,j}, t_{1,j} + 4\lambda_{1,j}\right] \cap A_j\right| < \lambda_{1,j}/4$, we infer that $\lambda_{2,j} \le \lambda_{1,j}/2$.

Set $E_{2,j} = \left[t_{1,j} - 4\lambda_{1,j} + 4\lambda_{2,j}, t_{1,j} + 4\lambda_{1,j} - 4\lambda_{2,j}\right]$ and consider the function $\psi_2(t)$ defined as follows

$$\psi_2(t) = \left|\left[t - 4\lambda_{2,j}, t + 4\lambda_{2,j}\right] \cap B_j\right|, \quad \forall\ t \in E_{2,j}.$$

We claim that there exists $t_{2,j} \in E_{2,j}$ such that $\psi_2(t_{2,j}) = 4\lambda_{2,j}$.

To see this, we argue by contradiction. Suppose that $\psi_2(t) \neq 4\lambda_{2,j}$, for all $t \in E_{2,j}$. Since ψ_2 is a continuous function on $E_{2,j}$, we assume as well that $\psi_2(t) < 4\lambda_{2,j}$, for all $t \in E_{2,j}$. Since $\lambda_{1,j}/\lambda_{2,j} \in \mathbb{Z}_+$, it follows that $\psi_1(t_{1,j}) < 4\lambda_{1,j}$, hence we have a contradiction to (81). Thus the claim is proved.

It is clear that

$$\iint\limits_{\substack{[t_{2,j}-4\lambda_{2,j},t_{2,j}+4\lambda_{2,j}]^2\\ |f(x)-f(y)|>\delta}} \frac{1}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y \geq \iint\limits_{\substack{[t_{2,j}-4\lambda_{2,j},t_{2,j}+4\lambda_{2,j}]^2\\ x\in B_j;\ y\in C_j}} \frac{1}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y \gtrsim 1.$$

If $\left|\left[t_{2,j}-4\lambda_{2,j}, t_{2,j}+4\lambda_{2,j}\right]\cap A_j\right| < \lambda_{2,j}/4$, then take $\lambda_{3,j}$ ($\lambda_{3,j} \leq \lambda_{2,j}/2$) and $t_{3,j}$, etc. On the other hand, since

$$\iint\limits_{\substack{I\times I\\ |f(x)-f(y)|>\delta}} \frac{1}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y < +\infty,$$

we have

$$\limsup_{\substack{\mathrm{diam}(Q)\to 0\\ Q:\ \text{an interval of } I}} \iint\limits_{\substack{Q\times Q\\ |f(x)-f(y)|>\delta}} \frac{1}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{82}$$

Thus, from (82) and the construction of $t_{k,j}$ and $\lambda_{k,j}$, there exist $t_j \in [40\delta, 1-40\delta]$ and $\lambda_j > 0$ ($t_j = t_{k,j}$ and $\lambda_j = \lambda_{k,j}$ for some k) such that

$$\left|\left[t_j-4\lambda_j, t_j+4\lambda_j\right]\cap B_j\right| = 4\lambda_j$$

and

$$\frac{\lambda_j}{4} \leq \left|\left[t_j-4\lambda_j, t_j+4\lambda_j\right]\cap A_j\right| \leq \lambda_j. \tag{83}$$

Set $\lambda = \inf_{j\in G}\lambda_j$ ($\lambda > 0$ since G is finite). Suppose $G = \bigcup_{i=1}^n I_m$, where I_m is defined as follows

$$I_m = \left\{j \in G;\ 2^{m-1}\lambda \leq \lambda_j < 2^m\lambda\right\}, \quad \forall\, m \geq 1.$$

Then it follows from (80) that

$$\sum_{m=1}^{n} \mathrm{card}(I_m) \gtrsim \frac{1}{\delta}. \tag{84}$$

For each m ($1 \leq m \leq n$), since $A_j \cap A_k = \emptyset$ for $j \neq k$, it follows from (83) that there exists $J_m \subset I_m$ such that

$$\mathrm{card}(J_m) \gtrsim \mathrm{card}(I_m) \tag{85}$$

and

$$|t_i - t_j| > 2^{m+3}\lambda, \quad \forall\, i,j \in J_m. \tag{86}$$

Then, from (86) and the definition of I_m,

$$\left[t_i-4\lambda_i, t_i+4\lambda_i\right]\cap\left[t_j-4\lambda_j, t_j+4\lambda_j\right] = \emptyset, \quad \forall\, i,j \in J_m. \tag{87}$$

Set $U_0 := \emptyset$ and

$$\begin{cases} L_m = \left\{j \in J_m;\ \left|\left[t_j-4\lambda_j, t_j+4\lambda_j\right]\setminus U_{m-1}\right| \geq 6\lambda_j\right\}, \\ U_m = \left(\bigcup_{j\in L_m}\left[t_j-4\lambda_j, t_j+4\lambda_j\right]\right)\cup U_{m-1}, \\ a_m = \mathrm{card}(J_m) \quad \text{and} \quad b_m = \mathrm{card}(L_m), \end{cases} \qquad \text{for } m = 1,2,\ldots,n.$$

From (87) and the definitions of J_m and L_m,

$$\frac{1}{4}2^{m-1}(a_m - b_m) \leq \sum_{i=1}^{m-1} 2^i b_i$$

which shows that

$$a_m \le b_m + 8 \sum_{i=1}^{m-1} 2^{(i-m)} b_i.$$

Consequently,

$$\sum_{m=1}^{n} a_m \le \sum_{m=1}^{n} b_m + 8 \sum_{m=1}^{n} \sum_{i=1}^{m-1} 2^{(i-m)} b_i = \sum_{m=1}^{n} b_m + 8 \sum_{i=1}^{n} b_i \sum_{m=i+1}^{n} 2^{(i-m)}.$$

Since $\sum_{i=1}^{\infty} 2^{-i} = 1$, it follows from (84) and (85) that

$$\sum_{m=1}^{n} b_m \ge \frac{1}{9} \sum_{m=1}^{n} a_m \gtrsim \frac{1}{\delta}.$$

Take λ such that

$$\delta \ge \lambda (8\delta)^{\beta} \ge \delta/2.$$

We then have, since $\gamma \le -1$,

$$\iint\limits_{\substack{I\times I \\ \frac{|f(x)-f(y)|}{|x-y|^{\beta}} > \lambda}} \lambda^p |x-y|^{-1+\gamma}\, \mathrm{d}x\, \mathrm{d}y \ge \sum_{m=1}^{n} \sum_{j \in L_m} \iint\limits_{\substack{([t_j - 4\lambda_j, t_j + 4\lambda_j] \setminus U_{m-1})^2 \\ x \in B_j,\ y \in C_j}} \lambda^p |x-y|^{-1+\gamma}\, \mathrm{d}x\, \mathrm{d}y$$

$$\gtrsim \sum_{m=1}^{n} b_m \lambda^p \delta^{1+\gamma} \gtrsim \lambda^p \delta^{\gamma} \gtrsim 1.$$

which yields (74) by (75).

Step 2: Proof of (74) in the general case. For $A > 0$, denote

$$f_A = \big(f \vee (-A)\big) \wedge A,$$

then

$$\big|f_A(x) - f_A(y)\big| \le \big|f(x) - f(y)\big| \quad \text{for } (x, y) \in I^2.$$

Applying the result of Step 1 to the sequence f_A and letting A go to infinity, we deduce that (74) holds for any measurable function f on I.

The proof is complete. □

7.2. *Proof of Theorem 16*

Step 1: Proof of Theorem 16 when $N = 1$. Set

$$\tau_h(u)(x) = \frac{u(x+h) - u(x)}{h}, \quad \forall\, x \in \mathbb{R},\ 0 < h < 1.$$

For each $m \ge 2$, take $K \in \mathbb{R}_+$ such that $Kh \ge m$, then

$$\int_{-m}^{m} \big|\tau_h(u)(x)\big|^p \,\mathrm{d}x \le \sum_{k=-K}^{K} \int_{kh}^{(k+1)h} \big|\tau_h(u)(x)\big|^p \,\mathrm{d}x.$$

Thus, since

$$\int_a^{a+h} \big|\tau_h(u)(x)\big|^p \,\mathrm{d}x \le \int_a^{a+h} \frac{1}{h^p} \Big| \operatorname*{ess\,sup}_{x \in (a, a+2h)} u - \operatorname*{ess\,inf}_{x \in (a, a+2h)} u \Big|^p \,\mathrm{d}x,$$

it follows from Lemma 22 that, for some constant $c = c_p > 0$,

$$\int_{-m}^{m} \big|\tau_h(u)(x)\big|^p \,\mathrm{d}x \le c \liminf_{\lambda \to 0} \Phi_\lambda(u). \tag{88}$$

Since $m \ge 2$ is arbitrary, (88) shows that

$$\int_{\mathbb{R}} \big|\tau_h(u)(x)\big|^p \,\mathrm{d}x \le c \liminf_{\lambda \to 0} \Phi_\lambda(u). \tag{89}$$

Since (89) holds for all $0 < h < 1$, it follows that $\nabla \in L^p(\mathbb{R})$ for $p > 1$ (see, e.g., [11, Chapter 8]) and $\nabla u \in \mathscr{M}$ (see, e.g., [24]). Moreover, (33) holds.

Step 2: Proof of Theorem 16 when $N \geq 2$. The result in this case is a consequence of Lemma 22, Lemma 20, and [35, Proposition 3]. As in the proof of the case $N = 1$, one can show that $\operatorname{ess V}(u, j)$ is bounded by $C \liminf_{\lambda \to 0} \Phi_\lambda(u)$ for $1 \leq j \leq N$ where $\operatorname{ess V}(u, j)$ is the essential variation of u in the j-th direction whose definition is given in [35, Definition 5]. The details are omitted.

8. Gamma-convergence

In this section, we discuss the Γ-convergence of the families of nonlocal functionals mentioned in the introduction. Concerning the BBM formula, the following result was established by Ponce [45].

Theorem 23 (Ponce). *Let $N \geq 1$, $1 \leq p < +\infty$, and let $(\rho_n)_{n \geq 1}$ be a sequence of non-negative mollifiers. Denote*

$$J_n(u) := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\left|u(x) - u(y)\right|^p}{|x - y|^p} \rho_n\big(|x - y|\big) \, \mathrm{d}x \, \mathrm{d}y.$$

Then, J_n Γ-converges to $K_{N,p}\Phi$ in $L^p(\mathbb{R}^N)$ as $n \to +\infty$.

To establish Theorem 23, one needs to show that, given $u \in L^p(\mathbb{R}^N)$:

(i) there exists $(u_n) \to u$ in $L^p(\mathbb{R}^N)$ such that

$$\limsup_{n \to +\infty} J_n(u_n) \leq K_{N,p}\Phi(u);$$

(ii) for all $(u_n) \to u$ in $L^p(\mathbb{R}^n)$, we have

$$\liminf_{n \to +\infty} J_n(u_n) \geq K_{N,p}\Phi(u).$$

Assertion (i) follows from Theorem 1 by taking $u_n = u$ for $n \geq 1$. The proof of assertion (ii), following the arguments in the work of Brezis and Nguyen [14], where related results were also considered, can be carried out as follows. Let $(\varphi_k)_{k \geq 1}$ be a smooth non-negative sequence of approximations to the identity such that $\operatorname{supp} \varphi_k \subset B_{1/k}$. Since

$$|a|^p - |b|^p \leq C_p |a - b|^p,$$

it follows that, for $u, v \in L^p(\mathbb{R}^N)$,

$$\big|J_n(u) - J_n(v)\big| \leq C_{N,p} J_n(u - v). \tag{90}$$

We derive that

$$\begin{aligned} \big|J_n(\phi_k * u) - J_n(\phi_k * u_n)\big| &\leq C_{N,p} J_n\big(\phi_k * u - \phi_k * u_n\big) \\ &\overset{\text{Theorem 1}}{\leq} C_{N,p} \big\|\nabla \phi_k * u - \nabla \phi_k * u_n\big\|_{L^p(\mathbb{R}^N)} \\ &\leq C_{N,p,k} \| u_n - u\|_{L^p(\mathbb{R}^N)}. \end{aligned} \tag{91}$$

This implies

$$J_n(\phi_k * u) \leq J_n(\phi_k * u_n) + C_{N,p,k} \| u_n - u\|_{L^p(\mathbb{R}^N)} \leq J_n(u_n) + C_{N,p,k} \| u_n - u\|_{L^p(\mathbb{R}^N)} \tag{92}$$

By letting $n \to +\infty$, we obtain

$$K_{N,p}\Phi(\phi_k * u) \leq \liminf_{n \to +\infty} J_n(u_n). \tag{93}$$

By letting $k \to +\infty$, we obtain assertion (ii).

We next discuss the setting given in Theorem 4 and 7. Set

$$\kappa_{N,p,\gamma} := \begin{cases} \inf \liminf_{\lambda \to 0_+} \Phi_\lambda(h_\lambda) & \text{if } \gamma < 0, \\ \inf \liminf_{\lambda \to +\infty} \Phi_\lambda(h_\lambda) & \text{if } \gamma > 0, \end{cases} \tag{94}$$

where the infimum is taken over all families of measurable functions $(h_\lambda)_{\lambda \in \mathbb{R}}$ defined on the open unit cube $Q = (0,1)^N$ such that $h_\lambda \to h(x) \equiv \frac{x_1 + \dots + x_N}{\sqrt{N}}$ in Lebesgue measure on Q as $\lambda \to 0$ for $\gamma < 0$ and as $\lambda \to +\infty$ for $\gamma > 0$. It is clear that $\kappa_{N,p,\gamma} \geq 0$.

The following result was obtained in [32,35].

Theorem 24 (Nguyen). *Let $N \geq 1$, $p \geq 1$, and $\gamma = -p$. Then, $0 < \kappa_{N,p,\gamma} < \frac{K_{N,p}}{p}$ and Φ_λ Γ-converges to $\kappa_{N,\gamma,p}\Phi$ in $L^p(\mathbb{R}^N)$ as $\lambda \to 0$.*

Remark 25. As a consequence of Theorem 24, the Γ-limit is strictly smaller than the pointwise limit viewing Theorem 4. This fact is quite surprising. The proof of the fact $\kappa_{N,p,-p} > 0$ is based on the work of Bourgain and Nguyen [9], see also the proof of Theorem 16 in Section 7. The proof of the fact $\kappa_{N,p,-p} < \frac{K_{N,p}}{p}$ is based on the construction of an example. The proof of the Γ-convergence is based on essentially the monotonicity property of Φ_λ in the case $\gamma = -p$. The analysis is delicate but quite robust and can be extended to more general functionals, which are different from Φ_δ, by Brezis and Nguyen [15]. In the one dimensional case, we can obtain even a more general result [16].

Remark 26. The explicit value of $\kappa_{N,p,-p}$ was conjectured in [32] for $N = 1$. This value was later confirmed by Antonucci, Gobbino, Migliorini, and Picenni [3] where the case $N \geq 2$ was also established.

Remark 27. Γ-convergence results for functionals reminiscent of Φ_λ in the case $\gamma = -p$ were studied by Ambrosio, De Philippis, and Martinazzi [2].

We conjecture that

Conjecture 28. *Let $N \geq 1$, $p \geq 1$, and $\gamma \in \mathbb{R} \setminus \{0\}$. Then, Φ_λ Γ-converges to $\kappa_{N,p,\gamma}\Phi$ in $L^p(\mathbb{R}^N)$ for $\lambda \to +\infty$ if $\gamma > 0$ and for $\lambda \to 0_+$ if $\gamma < 0$.*

Concerning Conjecture 28, we have the following results.

Theorem 29. *Let $N \geq 1$, $p \geq 1$, and $-1 < \gamma < 0$. Then Φ_λ Γ-converges to 0 in $L^p(\mathbb{R}^N)$.*

Proof. It suffices to prove that for each $u \in L^p(\mathbb{R}^N)$, there exists $(u_\lambda)_{\lambda \in (0,1)}$ such that $u_\lambda \to u$ in $L^p(\mathbb{R}^N)$ and $\Phi_\lambda(u_\lambda) \to 0$ as $\lambda \to 0_+$. For notational ease, we only consider the case $N = 2$. The general case follows similarly.

To this end, we first show this for each $u \in C_c^\infty(\mathbb{R}^2)$ with compact support. Let $m \in \mathbb{N}$ be such that

$$\operatorname{supp} u \subset [-m, m]^2.$$

For $k \in \mathbb{N}$, denote

$$\mathcal{Q}_k = \Big\{\text{open cubes } Q_{i,j,k} := \big(i2^{-k}, (i+1)2^{-k}\big) \times \big(j2^{-k}, (j+1)2^{-k}\big) \subset \mathbb{R}^2;\ i, j \in \mathbb{Z}\Big\}.$$

Set

$$u_k(x) = \sum_{-m2^k \leq i,j \leq m2^k} v_{i,j,k}(x),$$

where

$$v_{i,j,k} = u(x_{i,j,k}) \mathbb{1}_{Q_{i,j,k}} \quad \text{with} \quad x_{i,j,k} = \big((i+1/2)2^{-k}, (j+1/2)2^{-k}\big).$$

We have, for some positive constants c_k independent of λ,

$$\Phi_{c_k\lambda}(u_k) \leq \sum_{-m2^k \leq i,j \leq m2^k} \Phi_\lambda(v_{i,j,k}).$$

Similar to the proof of Proposition 17, we have

$$\lim_{\lambda\to 0_+} \Phi_\lambda(v_{i,j,k}) = 0.$$

We then derive that there exists a family $(u_\lambda)_{\lambda\in(0,1)}$ such that $u_\lambda \to u$ in $L^p(\mathbb{R})$ and

$$\lim_{\lambda\to 0_+} \Phi_\lambda(u_\lambda) = 0.$$

Since for each $u \in L^p(\mathbb{R}^N)$, there exists a sequence $(U_n) \subset C_c^\infty(\mathbb{R}^N)$ such that $U_n \to u$ in $L^p(\mathbb{R}^N)$. The conclusion then follows from the case $u \in C_c^\infty(\mathbb{R}^N)$ by a standard approximation. □

Remark 30. Brezis, Seeger, Van Schaftingen, and Yung [18] (see also [12]) asked whether the Γ-convergence holds in L^1_{loc} and the limit is Φ up to a positive constant. Theorem 29 gives a negative answer to this question in the case $-1 < \gamma < 0$.

Remark 31. It would be interesting to revisit the arguments in [15,35] to establish Conjecture 28.

9. Inequalities related to the nonlocal functionals

Since Φ_λ characterize Sobolev norms and the total variations, it is natural to ask whether or not one can obtain properties of Sobolev spaces using the information of Φ_λ instead of the one of Φ. We addressed this question in the case $\gamma = -p$ [36]. In particular, it was shown [36] that a variant of Poincaré's inequality holds in this case.

Theorem 32 (Nguyen). *Let $N \geq 1$, $p \geq 1$, and u be a real measurable function defined in a ball $B \subset \mathbb{R}^N$. We have*

$$\int_B\int_B \left|u(x) - u(y)\right|^p \,\mathrm{d}x\,\mathrm{d}y \leq C_{N,p}\left(|B|^{\frac{N+p}{N}} \underset{|u(x)-u(y)|>\delta}{\int_B\int_B} \frac{\delta^p}{|x-y|^{N+p}}\,\mathrm{d}x\,\mathrm{d}y + \delta^p|B|^2\right). \tag{95}$$

The proof of Theorem 32 is based on the arguments of Bourgain and Nguyen [9] used in the proof of Lemma 22, and the inequalities associated with BMO-functions due to John and Nirenberg [26].

Applying Theorem 32, one can derive that $u \in \mathrm{BMO}(\mathbb{R}^N)$, the space of all functions of bounded mean oscillation defined in $\mathbb{R}^N$ if $u \in L^1(\mathbb{R}^N)$ and $\Phi_\delta(u) < +\infty$ for $\gamma = -p$ and $p = N$, and for some $\lambda > 0$. Moreover, there exists a positive constant C, depending only on N, such that, for $\gamma = -p$ and $p = N$,

$$|g|_{\mathrm{BMO}} := \sup_B \fint_B\fint_B \left|g(x) - g(y)\right| \mathrm{d}x\,\mathrm{d}y \leq C\left(\Phi_\lambda^{\frac{1}{N}}(u) + \lambda\right),$$

where the supremum is taken over all balls of $\mathbb{R}^N$. In a joint work with Brezis [13], we also show that if $u \in L^1(\mathbb{R}^N)$ and $\Phi_\lambda(u) < +\infty$ with $\gamma = -p$ and $p = N$ for all $\lambda > 0$, then $g \in \mathrm{VMO}(\mathbb{R}^N)$, the space of all functions of vanishing mean oscillation.

Using Theorem 32, we can establish variants of Sobolev's inequalities and Rellich–Kondrachov's compactness criterion [36]. The proof of Rellich–Kondrachov's compactness criterion using Theorem 32 is quite standard. The idea of the proof of the Sobolev inequalities using the Poincaré inequalities is as follows. We first establish the Sobolev inequalities for the weak type from the Poincaré inequalities using covering lemmas. We then use the truncation arguments due to Maz'ya, see, e.g., [29], and an inequality related to sharp maximal functions due to Fefferman and Stein, see, e.g., [47], to obtain the desired estimates from the weak-type ones. This kind of arguments have been extended to obtain the full range of Gagliardo & Nirenberg and Caffarelli & Kohn & Nirenberg interpolation inequalities associated with Coulomb–Sobolev spaces [28], a result obtained in a collaboration with Mallick.

In another direction, one can also derive variants of the Hardy inequalities and the Caffarelli & Kohn & Nirenberg inequalities using the information of Φ_λ instead of Φ in the case $\gamma = -p$. This is given in a joint work with Squassina [41]. Interestingly, our proofs are quite elementary and mainly based on the Poincaré and Sobolev inequalities for an annulus; the integration-by-part arguments are not required. Our analysis is inspired from the harmonic one, nevertheless, instead of using dyadic decomposition for the frequency, we do it for the space variables. These arguments have also been used by us to obtain the full range of the Caffarelli & Kohn & Nirenberg inequalities for fractional Sobolev's spaces [39], which generalizes the Caffarelli & Kohn & Nirenberg inequalities in [21].

In this direction, concerning Φ_λ, we ask the following question viewing Proposition 17.

Open question 33. *Let Q be a unit cube of $\mathbb{R}^N$, and let $p \geq 1$, $\gamma \in \mathbb{R} \setminus (-1, 0]$ be such that $\kappa_{N,p,\gamma} > 0$. Is it true that*

$$\iint_{Q\times Q} \left|u(x) - u(y)\right|^p \,\mathrm{d}x\,\mathrm{d}y \leq C_{N,p}\big(\Phi_\lambda(u) + \lambda^p\big) \quad \textit{for } u \in L^p(Q)\,?$$

Acknowledgments

The author is grateful to the referee for a careful reading of the manuscript and for valuable suggestions.

Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

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