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Remark on the energy channel property for the radial linear wave equation

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In the memory of Haïm Brezis

Abstract. This note concerns an exterior energy bound (channel of energy property) for the linear radial wave equations, which is crucial in the proof of the soliton resolution for the energy-critical wave equations. We give a short and synthetic proof that this property in general odd space dimension follows from the case of space dimension 3, which is elementary. This gives a simple proof of the channel of energy property in all odd space dimensions. We also show by the same proof that the analogous bound in even space dimensions follows from the case of space dimension 4.

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1. Introduction and statement of the result

Let $N \geq 3$, and denote by $\dot{H}^1_N(R)$ and $L^2_N(R)$ the vector space of radial functions in $\{x \in \mathbb{R}^N, |x| > R\}$, with the norms:

$$\|f\|^2_{\dot{H}^1_N} = \int_R^\infty |f'(r)|^2 r^{N-1}\,\mathrm{d}r, \qquad \|g\|^2_{L^2_N} = \int_R^\infty |g(r)|^2 r^{N-1}\,\mathrm{d}r.$$

When there is no ambiguity on the dimension, we will drop the index N. We will consider the radial wave equation in space dimension N:

$$\partial_t^2 u - \partial_r^2 u - \frac{N-1}{r}\partial_r u = 0, \quad r > R + |t|, \tag{1}$$

with initial data

$$(u, \partial_t u)_{\restriction t=0} = (u_0, u_1) \in \dot{H}^1(R) \times L^2(R). \tag{2}$$

Extending the initial data (u_0, u_1) to an element of $\dot{H}^1 \times L^2(\mathbb{R}^N)$, we see that the solution u of (1), (2) is defined without ambiguity on $\{r > R+|t|\}$. Indeed, by finite speed of propagation the restriction of u to this set is independent of the choice of the extension of (u_0, u_1).

We define the following subspace of $\dot{H}^1_N(R)$ (respectively $L^2_N(R)$):

$$P_N(R) = \operatorname{span}\left\{ \frac{1}{r^{N-2\ell}} \,\middle|\, \ell = 1, \ldots, \left\lfloor \frac{N+1}{4} \right\rfloor \right\},$$

$$Q_N(R) = \operatorname{span}\left\{ \frac{1}{r^{N-2\ell}} \,\middle|\, \ell = 1, \ldots, \left\lfloor \frac{N-1}{4} \right\rfloor \right\},$$

and note that $P_N(R)$ (respectively $Q_N(R)$) is exactly the intersection of the generalized kernel of Δ with $\dot{H}^1_N(R)$ (respectively $L^2_N(R)$).

Theorem 1. *There exists a constant $C > 0$, independent of N, with the following property. Assume that $N \geq 3$ is an integer, and $R > 0$. Let u be a solution of* (1), (2).

- *Assume that N is odd, and that (u_0, u_1) is in the orthogonal of $P_N(R) \times Q_N(R)$ in $\dot{H}^1(R) \times L^2(R)$. Then:*

$$\sum_{\pm} \lim_{t \to \pm\infty} \int_{R+|t|}^{\infty} \left(\partial_{t,r} u(t,r)\right)^2 r^{N-1}\, \mathrm{d}r \geq \left\| (u_0, u_1) \right\|^2_{\dot{H}^1(R) \times L^2(R)}. \tag{3}$$

- *Assume that $N = 4k$, $k \in \mathbb{N}$, and that u_0 is orthogonal to $P_N(R)$ in $\dot{H}^1(R)$. Then:*

$$\sum_{\pm} \lim_{t \to \pm\infty} \int_{R+|t|}^{\infty} \left(\partial_{t,r} u(t,r)\right)^2 r^{N-1}\, \mathrm{d}r \geq \frac{1}{C} \| u_0 \|^2_{\dot{H}^1(R)}. \tag{4}$$

- *Assume that $N = 4k+2$, $k \in \mathbb{N} \setminus \{0\}$, and that u_1 is orthogonal to $Q_N(R)$ in $L^2(R)$. Then:*

$$\sum_{\pm} \lim_{t \to \pm\infty} \int_{R+|t|}^{\infty} \left(\partial_{t,r} u(t,r)\right)^2 r^{N-1}\, \mathrm{d}r \geq \frac{1}{C} \| u_1 \|^2_{L^2(R)}. \tag{5}$$

The estimates of Theorem 1 are crucial for the proof of the soliton resolution for the energy critical semi-linear wave equation: see [1,5,7,8].

Theorem 1 was proved in space dimension 3 in [6] and generalized to higher odd space dimensions in [9]. The difficulty of the proof in [9] increases with the dimensions.

Channels of energy in even space dimensions were first studied in [2], where the limit of the outer energy (the left-hand side of (4) and (5)) is computed in the case $R = 0$. Theorem 1 was proved for $N = 4$ and $N = 6$ in [5], using the formula in [2] and explicit computation. The general case of Theorem 1 in even space dimensions was obtained in [10]. The proof in [10] relies on explicit formulas for the solution of the wave equation whose complexity increases with the dimension. We refer to [3,4] for other recent works on channels of energy.

In this note, we remark, using a short argument, that it is sufficient to prove Theorem 1 in space dimensions 3 and 4. More precisely, the estimate (3) in all odd space dimensions follows from the same estimate in space dimension 3, and the estimates (4) and (5) in all even space dimensions follow from the estimate (4) in space dimension 4.

Theorem 1 in the case $N = 3$ (proved in [6]) is elementary: it follows immediately from the explicit formula for radial solutions of (1), (2) for $N = 3$:

$$u(t,r) = \frac{1}{r}\left(\varphi(r+t) - \varphi(t-r)\right), \tag{6}$$

where

$$\varphi(\eta) = \frac{1}{2}\eta u_0(|\eta|) + \frac{1}{2}\int_0^{\eta} \sigma u_1(|\sigma|)\, \mathrm{d}\sigma. \tag{7}$$

We thus have a very simple proof of the odd dimension channels of energy estimate (3) in every odd dimension.

The fourth author is particularly pleased to honour Haïm with a short and elegant proof, which was one of the pleasures he has cherished in mathematics.

2. Proof

The proof, by induction, is based on a transformation of the radial linear wave equation, that was introduced in [9], and also used in [5] to deduce the case $N=6$ from the case $N=4$. More precisely, we will introduce a transformation that maps a solution u of (1) to a solution v of:

$$\partial_t^2 v - \partial_r^2 v - \frac{N-3}{r}\partial_r v = 0, \quad r > R+|t|. \tag{8}$$

We prove Theorem 1 when N is even. The proof when N is odd is quasi identical.

We thus fix $N \geq 6$ even and assume that the conclusion of the theorem holds with $N-2$ instead of N. We consider a solution u of (1). We let

$$v(t,r) = \int_r^\infty \rho \partial_t u(t,\rho)\,\mathrm{d}\rho, \qquad v_0(r) = v(0,r), \qquad v_1(r) = \partial_t v(0,r). \tag{9}$$

Then by (1),

$$\partial_r v = -r\partial_t u, \qquad \partial_t v = -(N-2)u(t,r) - r\partial_r u(t,r), \tag{10}$$

so that v satisfies (8). Also,

$$\int_{R+|t|}^\infty (\partial_r v)^2 r^{N-3}\,\mathrm{d}r = \int_{R+|t|}^\infty (\partial_t u)^2 r^{N-1}\,\mathrm{d}r \tag{11}$$

and (by a straightforward integration by parts)

$$\int_{R+|t|}^\infty (\partial_t v)^2 r^{N-3}\,\mathrm{d}r = \int_{R+|t|}^\infty (\partial_r u)^2 r^{N-1}\,\mathrm{d}r - (N-2)\big(R+|t|\big)^{N-2} u^2\big(t, R+|t|\big), \tag{12}$$

and thus

$$\lim_{t\to\pm\infty} \int_{R+|t|}^\infty (\partial_t v)^2 r^{N-3}\,\mathrm{d}r \leq \lim_{t\to\pm\infty} \int_{R+|t|}^\infty (\partial_r u)^2 r^{N-1}\,\mathrm{d}r \tag{13}$$

(indeed it easy to show the equality between these two limits but the inequality (13) is sufficient for our purpose).

We will denote by $P_N^\perp(R)$ the orthogonal of $P_N(R)$ in $\dot{H}^1(R)$, and $Q_N^\perp(R)$ the orthogonal of $Q_N(R)$ in $L^2(R)$.

Case 1. We assume $N = 4k$, $k \in \mathbb{N}$, $u_1 = 0$ and $u_0 \in P_N^\perp(R)$. By (10), $v_0 = 0$. We claim

$$v_1 \in Q_{N-2}^\perp(R), \tag{14}$$

i.e. that v_1 is orthogonal to $1/r^{2j}$, $j \in \{k,\dots,2k-2\}$ in $L^2_{N-2}(R)$. For such j, we compute, using (10),

$$\int_R^\infty v_1(r)\frac{1}{r^{2j}} r^{N-3}\,\mathrm{d}r = -\int_R^\infty \big((N-2)u_0(r) + r\partial_r u_0(r)\big)\frac{1}{r^{2j}} r^{N-3}\,\mathrm{d}r.$$

We have

$$\int_R^\infty r\frac{\mathrm{d}u_0}{\mathrm{d}r}\frac{1}{r^{2j}} r^{N-3}\,\mathrm{d}r = \int_R^\infty \frac{\mathrm{d}u_0}{\mathrm{d}r}\frac{1}{r^{2j+1}} r^{N-1}\,\mathrm{d}r = 0,$$

since $u_0 \in P_N^\perp(R)$. On the other hand, integrating by parts, we obtain

$$\int_R^\infty u_0(r) r^{N-3-2j}\,\mathrm{d}r = -\frac{1}{N-2-2j}\int_R^\infty \frac{\mathrm{d}u_0}{\mathrm{d}r}\frac{1}{r^{1+2j}} r^{N-1}\,\mathrm{d}r - \frac{1}{N-2-2j} R^{N-2-2j} u_0(R).$$

Furthermore

$$u_0(R) = -\int_R^\infty \frac{\mathrm{d}u_0}{\mathrm{d}r}\,\mathrm{d}r = -\int_R^\infty \frac{\mathrm{d}u_0}{\mathrm{d}r}\frac{1}{r^{4k-1}} r^{N-1}\,\mathrm{d}r = 0,$$

where the last equality follows again from the fact that $u_0 \in P_N^\perp(R)$, and thus that u_0 is orthogonal to $1/r^{4k-2}$ in $\dot{H}^1_N(R)$. This concludes the proof of (14).

By (13), (14) and the induction hypothesis,

$$\sum_{\pm}\lim_{t\to\pm\infty}\int_R^{\infty}\left|\partial_{t,r}u(t,r)\right|^2 r^{N-1}\,\mathrm{d}r \geq \frac{1}{C}\|v_1\|^2_{L^2_{N-2}(R)}. \tag{15}$$

By (12)

$$\|v_1\|^2_{L^2_{N-2}(R)} = \|u_0\|^2_{\dot{H}^1_N(R)} - (N-2)R^{N-2}u_0^2(R).$$

Since we have proved that $u_0(R) = 0$, this implies the desired inequality (4).

Case 2. We assume $N = 4k+2$, $u_0 = 0$ and $u_1 \in Q_N^{\perp}(R)$. We claim

$$v_0 \in P^{\perp}_{N-2}(R), \tag{16}$$

that is v_0 is orthogonal to $\frac{1}{r^{2j}}$, $k \leq j \leq 2k-1$. Indeed, by (10), if $k \leq j \leq 2k-1$,

$$\int_R^{\infty}\frac{\mathrm{d}v_0}{\mathrm{d}r}\frac{1}{r^{2j+1}}r^{N-3}\,\mathrm{d}r = -\int_R^{\infty}u_1\frac{1}{r^{2j+2}}r^{N-1}\,\mathrm{d}r = 0,$$

by the assumption that $u_0 \in Q_N^{\perp}(R)$. This proves (16). By (13), (16), and the induction hypothesis,

$$\sum_{\pm}\lim_{t\to\pm\infty}\int_{R+|t|}^{\infty}\left(\partial_{t,r}u(t,r)\right)^2 r^{N-1}\,\mathrm{d}r \geq \frac{1}{C}\|v_0\|^2_{\dot{H}^1_{N-2}(R)}. \tag{17}$$

By (11), $\|v_0\|^2_{\dot{H}^1_{N-2}(R)} = \|u_1\|^2_{L^2_N(R)}$, which yields (5) and concludes the proof.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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