Comptes Rendus

Mathématique

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Regional and partial observability and control of waves

Research article
Control theory

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Abstract. We establish sharp regional observability results for solutions of the wave equation in a bounded domain $\Omega \subset \mathbb{R}^n$, in arbitrary spatial dimension. Assuming the waves are observed on a non-empty open subset $\omega \subset \Omega$ and that the initial data are supported in another open subset $\mathcal{O} \subset \Omega$, we derive estimates for the energy of initial data localized in $\mathcal{O}$, in terms of the energy measured on the observation set $(0, T) \times \omega$. This holds under a suitable geometric condition relating the time horizon T and the pair of subdomains $(\omega, \mathcal{O})$.

Roughly speaking, this geometric condition requires that all rays of geometric optics in $\overline{\Omega}$, emanating from $\overline{\mathcal{O}}$, must reach the observation region $(0, T) \times \omega$. Our result generalizes classical observability results, which recover the total energy of all solutions when the observation set ω satisfies the so-called Geometric Control Condition (GCC) — a particular case corresponding to $\mathcal{O} = \Omega$.

A notable feature of our approach is that it remains effective in settings where Holmgren's uniqueness does not guarantee unique continuation. As a consequence of our analysis, unique continuation is nonetheless recovered for wave solutions observed on $(0, T) \times \omega$ with initial data supported in $\mathcal{O}$.

The proof of our result combines a high-frequency observability estimate — based on the propagation of singularities — with a compactness-uniqueness argument that exploits the unique continuation properties of elliptic operators.

By duality, this observability result leads to controllability results for the wave equation, ensuring that the projection of the solution onto $\mathcal{O}$ can be controlled by means of controls supported in ω, with optimal spatial support.

We also present several extensions of the main result, including the case of boundary observations, as well as a characterization of the observable fraction of the energy of the initial data from partial measurements on $(0, T) \times \omega$. Applications to wave control are discussed accordingly.

Keywords. Observability, wave equations, geometric conditions.

2020 Mathematics Subject Classification. 35A18, 35L05, 35Q93, 93B07, 93C20.

Funding. The first author (B.D.) is partially supported by the Tunisian Ministry for Higher Education and Scientific Research within the LR-99-ES20 program.

1. Introduction and first results

1.1. *Problem formulation*

Let Ω be a bounded domain of $\mathbb{R}^n$, with boundary $\partial\Omega$ of class $\mathscr{C}^\infty$. We set

$$\mathcal{L} = \mathbb{R} \times \Omega \qquad \text{and} \qquad \partial\mathcal{L} = \mathbb{R} \times \partial\Omega.$$

We also introduce $A = \big(a_{ij}(x)\big)$, a $n \times n$ matrix of $\mathscr{C}^\infty$ coefficients, symmetric, uniformly definite positive on a neighborhood of $\overline{\Omega}$, and we denote $\Delta_A = \sum_{i,j=1}^n \partial_{x_j}\big(a_{ij}(x)\partial_{x_i}\cdot\big)$ the corresponding Laplacian operator.

In addition, we will assume that the geodesics of $\overline{\Omega}$ with respect to the metric $\big(a^{ij}(x)\big)_{ij}$ have no contacts of infinite order with $\partial\Omega$. This is a standing assumption used to define the global Melrose–Sjöstrand flow, see Section 3.

We consider then the following wave equation:

$$\begin{cases} \partial_t^2 u - \Delta_A u = 0 & \text{in } \mathcal{L}, \\ u(t,\cdot) = 0 & \text{on } \partial\mathcal{L}, \\ \big(u(0,\cdot), \partial_t u(0,\cdot)\big) = (u_0, u_1). \end{cases} \tag{1}$$

We recall that, for $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the equation (1) is well posed and admits a unique solution in the space $\mathscr{C}^0\big(\mathbb{R}, H_0^1(\Omega)\big) \cap \mathscr{C}^1\big(\mathbb{R}, L^2(\Omega)\big)$. It is also well posed in $L^2(\Omega) \times H^{-1}(\Omega)$ with the unique solution lying in $\mathscr{C}^0\big(\mathbb{R}, L^2(\Omega)\big) \cap \mathscr{C}^1\big(\mathbb{R}, H^{-1}(\Omega)\big)$.

The observation subdomain is denoted by ω, a non-empty open subset of Ω where waves will be observed, and, for a given time-horizon $T > 0$, we set $\omega_T = (0, T) \times \omega$ and $\mathcal{L}_T = (0, T) \times \Omega$.

Thus, in the following, ω_T corresponds to the space-time subset on which u is measured, observed and known, and our goal is to recover the initial data (u_0, u_1) out of this partial measurement.

In other words, our aim is to analyze the inverse of the map

$$\begin{aligned} \mathrm{Obs}_T \colon L^2(\Omega) \times H^{-1}(\Omega) &\longrightarrow L^2(\omega_T) \\ (u_0, u_1) &\longmapsto u\big|_{\omega_T}, \quad \text{where } u \text{ solves (1)}. \end{aligned} \tag{2}$$

This is a classical question, motivated, in particular, by control theory, and commonly addressed in the context of the observability inequality

$$\big\|(u_0, u_1)\big\|_{L^2(\Omega) \times H^{-1}(\Omega)} \leq C \|u\|_{L^2(\omega_T)}, \tag{3}$$

which has been the focus of extensive research. This inequality refers to the possibility of inverting continuously the observation operator Obs_T.

Traditionally it has been addressed for all possible solutions with initial data in $L^2(\Omega) \times H^{-1}(\Omega)$ and it is equivalent to the property of exact controllability of system (1) in the space $H_0^1(\Omega) \times L^2(\Omega)$ with controls in $L^2(\omega_T)$, see [13].

Problem (3) is well understood, and there is an almost necessary and sufficient condition for the validity of (3), the so-called Geometric Control Condition (in short GCC, see [2,3,19]) which, roughly, asserts that all rays of geometric optics starting at time $t = 0$ from any point in $\overline{\Omega}$ meet the observation set ω_T. As we shall see below (see Section 3) in more detail, these rays are the space-time projections of the generalized bicharacteristics of Melrose–Sjöstrand, [16], for the operator $P_A = \partial_t^2 - \Delta_A$, which bounce on the boundary according to the Descartes–Snell law (see Section 3.2).

The GCC imposes a significant restriction on the class of observation domains ω for which the observability estimate (3) holds. In this work, we adopt a complementary viewpoint: rather than fixing the solution space and seeking suitable observation sets, we aim to consider all possible subdomains ω, which is particularly relevant from an applied perspective due to practical constraints on the placement and availability of sensors or actuators. This naturally leads to the following question: can we characterize the subclass of solutions to (1) for which the observability estimate (3) holds, given an arbitrary observation region ω?

The main novelty of this paper lies in the sharp characterization of a class of solutions for which the observability estimate (3) holds without imposing any geometric conditions on the observation set ω. Specifically, we show that (3) remains valid when the initial data of the solutions are supported in another subset $\mathcal{O} \subset \Omega$, provided a suitable microlocal geometric condition is satisfied. This condition involves the time horizon T and the pair $(\omega, \mathcal{O})$, and can be interpreted as a localized version of the Geometric Control Condition (GCC): it requires that all rays of geometric optics emanating from $\mathcal{O}$ reach the observation region ω within time T.

1.2. *Main results*

Our first main result is as follows.

Theorem 1. *Given the domain Ω, the observation subdomain $\omega \subset \Omega$, and the time-horizon $T > 0$, let the subdomain $\mathcal{O}$ be such that every generalized bicharacteristic ray (see Definition 12 for their precise definition) starting from $\{t = 0\} \times \overline{\mathcal{O}}$ intersects the set $(0, T) \times \omega$.*

Then, there exists $C > 0$ such that the solution of (1) satisfies the observability estimate

$$\left\| (u_0, u_1) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)} \leqslant C \|u\|_{L^2(\omega_T)}, \tag{4}$$

for any initial data $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ with support in $\overline{\mathcal{O}}$, i.e., satisfying

$$\mathrm{Supp}(u_0, u_1) \subset \overline{\mathcal{O}}. \tag{5}$$

Similarly, the observability estimate

$$\left\| (u_0, u_1) \right\|_{H_0^1(\Omega) \times L^2(\Omega)} \leqslant C \|\partial_t u\|_{L^2(\omega_T)} \tag{6}$$

holds for any initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ with support in $\overline{\mathcal{O}}$.

Remark 2. This result holds for all subdomains ω and initial data with support in $\overline{\mathcal{O}}$, under the condition that the pair $(\omega, \mathcal{O})$ fulfills a mutual or joint microlocal relation, so that all rays emanating from $\overline{\mathcal{O}}$ reach $(0, T) \times \omega$. It is a natural extension of the classical one guaranteeing the observability of all solutions, namely the GCC, which corresponds to the case $\mathcal{O} = \Omega$ in our setting. Indeed, it is natural that, when the support of the initial data lies in $\overline{\mathcal{O}}$, its observation only depends on the dynamics of the rays emanating from $\overline{\mathcal{O}}$, independently of what other rays (the ones departing away from $\overline{\mathcal{O}}$) do.

Our result extends the classical ones, allowing to consider all possible subdomains ω, not only those fulfilling the highly demanding GCC, and identifying a class of observable initial data. This is particularly relevant in applications where the available observations are limited either because of the lack of accessibility to some regions of the domain where waves propagate or due to the lack of sufficient sensing devices.

After this work had been submitted, the referee kindly pointed out to us the work [1, Section 2], which deals with a similar problem in a particular setting in which $\mathscr{O}$ is a ball $B(r)$ strictly included in Ω and the metric is flat, under a slightly more demanding condition. Namely, the article [1] requires that there is a neighborhood $B(r + \delta)$ ($\delta > 0$) of the ball for which every generalized bicharacteristic ray passing through $[0, \delta] \times B(r + \delta)$ intersects $(0, T) \times \omega$ (or $(0, T) \times \Gamma$ at a non-diffractive point if Γ is a part of the boundary on which the observation is done, see Theorem 4 for the counterpart of Theorem 1 when the observation is performed on the boundary). Using partition of unity, we could derive from [1] cases in which $\mathscr{O}$ is strictly included in Ω, but it will not be sufficient to handle cases in which $\overline{\mathscr{O}}$ touches the boundary $\partial\Omega$ as in the example presented in Figure 3 afterwards.

Remark 3. These results enter in the context of "enlarged observability/controllability" introduced in [13], according to which when limiting the class of solutions under consideration the requirements for observability can be weakened. However, in [13], because of the use of the multiplier method, improvements were only achieved at the level of the needed observability time. The results of the present paper constitute a more precise answer to these questions.

A similar result holds when the observation is done along the boundary. For this, we need the notion of nondiffractive points of the boundary that will be detailed in Definition 11, see also [2, Definition, p. 1037].

Theorem 4. *Let Γ be a non-empty open subset of the boundary $\partial\Omega$ and $\mathscr{O}$ be a non-empty open set of Ω such that the pair $(\Gamma, \mathscr{O})$ satisfies the following microlocal condition for some $T > 0$: every generalized bicharacteristic ray starting from $\{t = 0\} \times \overline{\mathscr{O}}$ intersects the set $(0, T) \times \Gamma$ at a nondiffractive point. Then there exists $C > 0$ such that for any initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ supported in $\overline{\mathscr{O}}$, the solution of* (1) *satisfies the observability estimate*

$$\left\| (u_0, u_1) \right\|_{H_0^1(\Omega) \times L^2(\Omega)} \leqslant C \left\| \partial_n u \right|_{\partial\Omega} \right\|_{L^2(\Gamma_T)}. \tag{7}$$

1.3. *Examples*

A 1-d example. The microlocal assumption on the pair $(\omega, \mathscr{O})$ in Theorem 1 is, in general, weaker than the GCC since it only concerns the rays emanating from $\overline{\mathscr{O}}$. It also requires a shorter observation time. This is even the case in 1-d.

Indeed, consider the simple example of the 1-d wave equation set on $\Omega = (-1, 1)$, with control in $\omega = (-1, -3/4) \cup (3/4, 1)$, and initial data localized in $\mathscr{O} = (-1/4, 1/4)$, as in Figure 1. In this case, the geodesic rays are simply the characteristics $t \mapsto x_0 \pm t$, bouncing when meeting the boundaries $\{-1, 1\}$.

By symmetry considerations, it is then easy to check that Theorem 1 holds for any $T > 1$. This minimal time corresponds to the arrival in ω of a characteristic starting from $x = -1/4$, propagating to the right.

However, the classical, sharp condition for unique continuation in (8) (in the following, T_{UC} stands for the critical time for unique continuation; it is given by $2 \sup_{x \in \Omega} d(x, \omega)$ from the Hörmander–Tataru–Robbiano–Zuily uniqueness theorem [8,21,24]) or the GCC require $T > T_{\text{UC}} = 3/2$. This condition is indeed optimal when aiming to observe all solutions since one can build initial data localized in $(-3/4, -3/4 + \varepsilon)$ ($\varepsilon > 0$ small) leading to waves propagating towards the right at speed one, and vanishing in ω during the time interval $(0, 3/2 - \varepsilon)$.

Therefore, the global observability estimates (4) or (6) do not hold for *all* initial data for the intermediate times $1 < T < 3/2$, but, according to our result, they do hold for initial data localized in $\overline{\mathscr{O}}$.
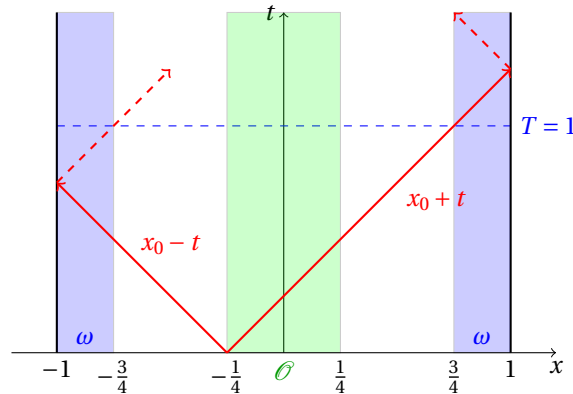

Figure 1. Illustration of the 1-d example: $\Omega = (-1, 1)$, $\omega = (-1, -3/4) \cup (3/4, 1)$, $\mathscr{O} = (-1/4, 1/4)$. The critical time given by Theorem 1 is $T_{0,\mathrm{crit}} = 1$, while the times for unique continuation and the GCC coincide and are equal to $T_{\mathrm{UC}} = 3/2$.

Multi-d examples. We present below two additional multi-d examples.

(1) Let us consider the case illustrated in Figure 2, in which Ω is the unit ball, ω is the ε-neighborhood of its boundary and $\mathscr{O}$ is the interior ball centered at the origin, of radius α, with $0 < \alpha < 1 - \varepsilon$. Then the critical time given by Theorem 1 is $T_{0,\mathrm{crit}} = 1 + \alpha - \varepsilon$. But, in this case, the time for unique continuation and GCC is larger, namely, $2(1 - \varepsilon)$.

(2) Another example, illustrated in Figure 3, still when Ω is the unit ball of $\mathbb{R}^2$, corresponds to an observation subdomain ω which is an ε neighborhood of one third of the boundary of Ω, with angles in $(-\pi/3, \pi/3)$, and $\mathscr{O}$ being the vertical strip $\{x = (x_1, x_2) \in \Omega \text{ with } x_1 < \cos(\alpha)\}$ for some $\alpha \in (\pi, 4\pi/3)$. The critical time T given by Theorem 1 is finite but GCC fails, whatever the time-horizon is, due to the vertical diameter, corresponding to a geodesic ray which bounces back and forth endless, without ever entering the observation set ω.

  The longest geodesic that starts from $\mathscr{O}$ and stays away from ω is the one that starts from $(\cos(\alpha), \sin(\alpha))$ and goes to $(\cos(\pi/3), \sin(\pi/3))$ (or rather an ε-neighborhood of it): it is not difficult to check that this geodesic has length $4k_c \sin((\alpha - \pi/3)/2)$ where k_c is the first integer such that $2k_c(\pi - (\alpha - \pi/3)) > 5\pi/3 - \alpha$.

Remark 5. The examples above can be easily adapted to the boundary-observation setting by "smashing" the set ω to the boundary of Ω.

1.4. *Some relevant consequences*

As we will see in Section 5 below, the observability results in Theorems 1 and 4 have their dual controllability counterparts: one can exactly control the projection of wave solutions over $\mathscr{O}$ at time T, by means of controls in $L^2(\omega_T)$, see Theorem 28.

  Such result lays in between the classical properties of approximate and exact controllability, since it assures the exact control over the projection onto $\mathscr{O}$, but without providing any information on what happens outside $\mathscr{O}$.
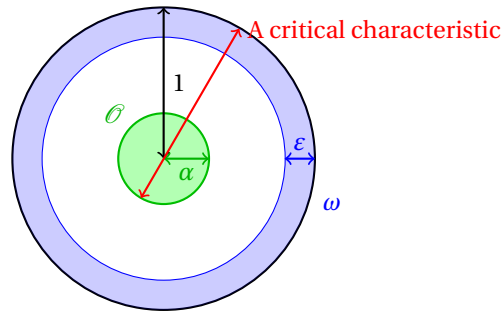
Figure 2. Illustration of the multi-d example (1): $\Omega = B(1)$, $\omega = B(1) \setminus B(1-\varepsilon)$, $\mathscr{O} = B(\alpha)$. The critical time given by Theorem 1 is $T_{0,\text{crit}} = 1 + \alpha - \varepsilon$, while the times for unique continuation and GCC coincide: $T_{\text{UC}} = 2(1-\varepsilon)$.
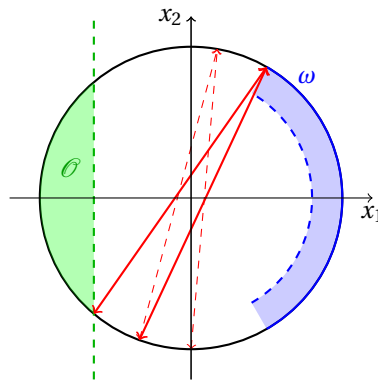

Figure 3. Illustration of the multi-d example (2), with $\alpha = 230°$: in this case, it is clear that the critical time T given by Theorem 1 is finite, while the geometric control condition is never satisfied. In red, we have plotted the first few reflections of the longest geodesic starting from $\mathscr{O}$ and staying outside ω.

Note that, if we further assume that the time horizon T is sufficiently large to guarantee the injectivity of the operator Obs_T in (2) (in fact, this corresponds to the condition $T > 2 \sup d(x, \omega)$, see (8) afterwards), we can find controls that simultaneously assure the exact control of the projection over $\mathscr{O}$ and the approximate controllability property everywhere in the domain Ω, see Remark 30.

The problem of controllability in the absence of GCC has been the object of extensive study, see, for instance, [10,11,20,22]. In those papers, one aims to quantify the property of approximate controllability, by identifying subspaces of initial data that can actually be controlled. These spaces are typically very small, imposing suitable analyticity restrictions. Surprisingly, analyzing what can be controlled for solutions of the wave equation in the usual energy space in the absence of GCC has not received much attention, and, besides our work, we are only aware of the work [1] focusing on a very particular case.

Our result, valid for arbitrary observation sets ω, encompasses as a particular case the classical setting in which $\mathscr{O} = \Omega$, that is, when observability is required for all solutions of the wave equation. In this case, our condition naturally reduces to the well-known Geometric Control Condition (GCC).

Our main result, as we shall see, is even sharper, since it allows to identify the observable

microlocal projections of solutions for any subdomain ω, something that might be also relevant in applications.

By duality, these observation type properties yield control results for the wave equation in which the control objective is to control the projection of the solutions over $\mathscr{O}$, or its microlocal projection up to some smoother error terms (see Section 5 for the precise statement).

Throughout the paper, we also present several illustrative examples involving different domains Ω, observation subsets ω, and time horizons T, which do not satisfy the classical GCC. Nonetheless, our results apply in those cases, as our goal is to discuss all the information on the solutions of the wave equation that can be derived from a given observation set $(0, T) \times \omega$.

Our analysis has also important consequences in what concerns the classical property of unique continuation. Indeed, the injectivity of the operator Obs_T is equivalent to the unique continuation property:

$$\text{"for } u \text{ solution of (1), } u\big|_{\omega_T} = 0 \ \Rightarrow \ (u_0, u_1) \equiv (0, 0)\text{"}.$$

Such property is known to hold for all possible solutions of the wave equation when A is analytic, thanks to Holmgren's uniqueness theorem [9], or for smooth time-independent coefficients, by [8,21,24], provided the time T satisfies

$$T > 2 \sup_{x \in \Omega} d(x, \omega), \tag{8}$$

where $d(\cdot, \omega)$ stands for the geodesic distance to ω. Accordingly, unique continuation holds for all non-empty open subset ω provided T is large enough as in (8). In this setting, a quantitative logarithmic stability estimate was recently proved in [10]. Thus, under the only condition (8), the operator Obs_T is one-to-one, but in general, its inverse is ill-posed and it is not a bounded operator, unless the additional GCC is satisfied.

As an interesting corollary of our observability result, we derive unique continuation properties for specific classes of solutions — such as those with initial data supported in $\mathscr{O}$ — even in settings where the classical condition (8) fails. Consequently, these results apply in situations where existing uniqueness theorems of Holmgren do not suffice.

Our contribution may also be seen as a complement or alternative to the results developed in [11] and the subsequent literature. In that context, for general subdomains ω, and under the sole assumption of the unique continuation condition (8) — in the absence of the GCC — observability for general solutions was established in a weaker, generalized framework, where the observability constant depends, roughly, exponentially on the frequency of the solutions.

In contrast, our approach aims to recover the classical observability inequality in the natural energy spaces, even without the GCC, by identifying — microlocally — specific classes of initial data for which the inequality still holds. Rather than relaxing the inequality to include all solutions at the cost of weakening the estimate, we preserve its sharp form within a suitably restricted solution space. Throughout the paper, we clarify the relationship between both approaches and highlight the connections between their respective results.

1.5. *Methodology of proof*

The proof strategy for Theorems 1 and 4 relies on microlocal analysis techniques, as is customary in the study of wave propagation phenomena. More precisely, it combines the following key ingredients:

- the microlocal geometric property satisfied by the pair $(\omega, \mathscr{O})$ (or its boundary counterpart $(\Gamma, \mathscr{O})$) allows to propagate the energy of solutions at high-frequencies from $(0, T) \times \omega$ towards $\{t = 0\} \times \mathscr{O}$, and this leads to a weak version of the observability estimate with a compact remainder term;

- removing the compact remainder requires a unique continuation property; this property needs to be proved in an ad-hoc manner since the assumptions made on the time-horizon do not assure that (8) is fulfilled;
- this is achieved by means of an added compactness-uniqueness argument, which reduces the unique continuation property to an elliptic context in which it holds by classical Carleman inequalities.

1.6. *Outline*

Section 2 is devoted to the proofs of the main results — Theorems 1 and 4 — following the methodology outlined in Section 1.5. In Section 3, we recall the Melrose–Sjöstrand cotangent bundle framework and introduce several technical microlocal analysis tools that are used throughout the paper. Section 4 presents extensions of the observability results, where the assumptions on the support of the initial data are relaxed. In Section 5, we establish the control counterparts of the observability results. The article concludes with a final section discussing open problems and future perspectives.

2. Proofs of the main results

We essentially focus on the proof of Theorem 1, i.e., of the estimate (6), since the proof of estimate (4) is similar, and is thus left to the reader, with the indication of the additional steps needed, see Section 2.2 afterwards. The general strategy of the proof follows the program described in Section 1.5.

2.1. *Proof of Theorem 1*

We start with the following lemma that describes the propagation of regularity for solutions to system (1). The proof of this lemma requires the use of microlocal tools and is a consequence of [7,16], and we refer the unfamiliar reader to Section 3 for a precise description of these notions. In particular, the proof of Lemma 6 can be found in Section 3.6.

Lemma 6. *Under the assumptions of Theorem 1, consider a solution u to system* (1) *with initial data $(u_0, u_1) \in H^{-1}(\Omega) \times (H^2 \cap H_0^1)'(\Omega)$ supported in $\overline{\mathcal{O}}$, and satisfying $u \in L^2(\omega_T)$. Then $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $u \in \mathscr{C}^0(\mathbb{R}, L^2(\Omega)) \cap \mathscr{C}^1(\mathbb{R}, H^{-1}(\Omega))$.*

Similarly, if $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ is supported in $\overline{\mathcal{O}}$, and the solution u satisfies $\partial_t u \in L^2(\omega_T)$, one has $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $u \in \mathscr{C}^0(\mathbb{R}, H_0^1(\Omega)) \cap \mathscr{C}^1(\mathbb{R}, L^2(\Omega))$.

We then deduce the following corollary, which fulfills the first step of the proof, providing a first observability estimate with a compact remainder.

Corollary 7. *Under assumptions of Theorem 1, there exists $C > 0$ such that the solution of* (1) *satisfies*

$$\left\| (u_0, u_1) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)} \leq C \| u \|_{L^2(\omega_T)} + C \left\| (u_0, u_1) \right\|_{H^{-1}(\Omega) \times (H^2 \cap H_0^1)'(\Omega)}, \tag{9}$$

for any initial data $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ supported in $\overline{\mathcal{O}}$ (i.e., satisfying (5)*).*

Similarly,

$$\left\| (u_0, u_1) \right\|_{H_0^1(\Omega) \times L^2(\Omega)} \leq C \| \partial_t u \|_{L^2(\omega_T)} + C \left\| (u_0, u_1) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)} \tag{10}$$

holds for any initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ supported in $\overline{\mathcal{O}}$ (i.e., satisfying (5)*).*

Proof. We focus on the proof of (10), (9) being similar. Consider the following Hilbert space

$$E = \left\{ (u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega), \ \mathrm{supp}(u_0, u_1) \subset \overline{\mathcal{O}}, \ \text{and} \ \partial_t u \in L^2(\omega_T) \right\}$$

equipped with the norm

$$\left\| (u_0, u_1) \right\|_E^2 = \left\| (u_0, u_1) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)}^2 + \left\| \partial_t u \right\|_{L^2(\omega_T)}^2,$$

and the energy space $F = H_0^1(\Omega) \times L^2(\Omega)$ equipped with its natural norm.

Thanks to Lemma 6, the identity map

$$E \longrightarrow F = H_0^1(\Omega) \times L^2(\Omega), \qquad (u_0, u_1) \longmapsto (u_0, u_1)$$

is well defined. Consequently, the closed graph theorem yields its continuity and estimate (10).

The proof of estimate (9) is similar, except that it relies on the propagation of the L^2-wave front set along the bicharacteristic flow, that is the first item of Lemma 6. $\qquad\square$

As a second step in the proof, and as a consequence of the previous estimate, the following unique continuation property holds.

Lemma 8. *Under the assumptions of Theorem 1, any solution u of system* (1) *with initial data in $H^{-1}(\Omega) \times \left(H^2 \cap H_0^1(\Omega) \right)'$ with support in $\overline{\mathcal{O}}$, and satisfying $u = 0$ in ω_T, vanishes everywhere, i.e., $u \equiv 0$.*

Similarly, any solution u of system (1) *with initial data in $L^2(\Omega) \times H^{-1}(\Omega)$ supported in $\overline{\mathcal{O}}$, and satisfying $\partial_t u = 0$ in ω_T, vanishes everywhere, i.e., $u \equiv 0$.*

Remark 9. At this point it is worth noticing that this uniqueness result is not standard since it only applies to the solutions with initial data supported in $\overline{\mathcal{O}}$. It is not a consequence of Holmgren's uniqueness theorem nor any of its generalizations, but rather a corollary of the Corollary 7, which establishes a relaxed version of the observability inequalities we aim, with an added compact additive remainder term.

Proof. Similarly as in the proof of Corollary 7, we focus on finite energy solutions, the case of weaker solutions being similar.

Our goal is to prove that the closed linear subspace of $L^2(\Omega) \times H^{-1}(\Omega)$ defined by

$$\mathcal{N} = \left\{ (u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega), \ \mathrm{Supp}(u_0, u_1) \subset \overline{\mathcal{O}}, \ \partial_t u \big|_{\omega_T} = 0 \right\}$$

is reduced to $\mathcal{N} = \{(0,0)\}$.

Thanks to the estimate (10) in Corollary 7, it is clear that $\mathcal{N} \subset H_0^1(\Omega) \times L^2(\Omega)$ and

$$\left\| (u_0, u_1) \right\|_{H_0^1(\Omega) \times L^2(\Omega)} \leqslant C \left\| (u_0, u_1) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)}$$

for every $(u_0, u_1) \in \mathcal{N}$.

Using then the compact embedding[1] $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ we deduce that $\mathcal{N}$ has a finite dimension. In addition, the matrix operator $\mathcal{A} = \left(\begin{smallmatrix} 0 & 1 \\ \Delta_A & 0 \end{smallmatrix} \right)$ defines a linear bounded operator in $\mathcal{N}$. This is so since the wave equation, written on the column vector unknown $U = \left(\begin{smallmatrix} u \\ u_t \end{smallmatrix} \right)$ takes the form $U_t = \mathcal{A}U$. Therefore, $\mathcal{A}$ operates continuously in $\mathcal{N}$ and corresponds to the application of the time derivative on the solutions of the wave equation (1), transferring the initial data (u_0, u_1) into $(u_1, \Delta_A u_0)$.

If $\mathcal{N}$ were non-trivial, the operator $\mathcal{A}$ would have an eigenvalue corresponding to an eigenvector in $\mathcal{N}$. But, as we shall see, this is impossible, concluding, by contradiction, that $\mathcal{N} = \{(0,0)\}$.

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[1]In fact, since we are considering data which are supported in $\overline{\mathcal{O}}$, we can also consider the compact embedding of $H_0^1(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$. This remark allows to deduce the same results even in cases in which Ω is unbounded, provided $\mathcal{O}$ is bounded.

Indeed, if $\lambda \in \mathbb{C}$ is an eigenvalue and $(u_0, u_1) \neq (0,0)$ in $H_0^1(\Omega) \times L^2(\Omega)$ is an eigenvector of $\mathcal{A}$ in $\mathcal{N}$, we have

$$\Delta_A u_0 - \lambda^2 u_0 = 0 \text{ in } \Omega, \qquad u_0 = 0 \text{ on } \partial\Omega, \qquad \text{and} \qquad \lambda u_0 = 0 \text{ in } \omega.$$

It is easy to check that this cannot happen unless $u_0 \equiv 0$, which also implies that $u_1 \equiv 0$. Indeed, if $\lambda = 0$, given that $\Delta_A u_0 = 0$ and u_0 vanishes on the boundary, we conclude that $u_0 \equiv 0$. On the other hand, when $\lambda \neq 0$, u_0 vanishes in ω and by elliptic unique continuation applied to the equation $\Delta_A u_0 - \lambda^2 u_0 = 0$ we deduce that $u_0 \equiv 0$ everywhere.

This concludes the proof of Lemma 8. $\qquad\square$

We are now in conditions to conclude the proof of Theorem 1, i.e., of estimate (6).

We use a contradiction argument and we assume that estimate (6) is false. Consider a sequence of initial data $(u_{0,k}, u_{1,k}) \in H_0^1(\Omega) \times L^2(\Omega)$ with support in $\overline{\mathcal{O}}$, and (u_k) the corresponding solutions, with

$$\left\|(u_{0,k}, u_{1,k})\right\|_{H_0^1(\Omega) \times L^2(\Omega)} = 1 \qquad \text{and} \qquad \lim_{k \to \infty} \|\partial_t u_k\|_{L^2(\omega_T)} = 0. \tag{11}$$

The sequence (u_k) is bounded in the energy space $\mathscr{C}^0\big([0,T]; H_0^1(\Omega)\big) \cap \mathscr{C}^1\big([0,T]; L^2(\Omega)\big)$. Thus, after extracting a subsequence, we may assume that it converges weakly in $H^1(\mathcal{L}_T)$ to another solution $u \in H^1(\mathcal{L}_T)$ of (1), corresponding to an initial datum $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ with support in $\overline{\mathcal{O}}$, weak limit of $(u_{0,k}, u_{1,k})$ in the energy space.

Passing to the limit $k \to \infty$ in (11), we obtain

$$\partial_t u\big|_{\omega_T} = 0. \tag{12}$$

Then, the unique continuation result of Lemma 8 assures that $u \equiv 0$. This implies that $(u_{0,k}, u_{1,k})$ strongly converges to $(0,0)$ in $L^2(\Omega) \times H^{-1}(\Omega)$.

On the other hand, in view of the relaxed observability estimate (10) applied to u_k, we have

$$1 = \left\|(u_{0,k}, u_{1,k})\right\|_{H_0^1(\Omega) \times L^2(\Omega)} \leqslant C \|\partial_t u_k\|_{L^2(\omega_T)} + C \left\|(u_{0,k}, u_{1,k})\right\|_{L^2(\Omega) \times H^{-1}(\Omega)}.$$

But the right-hand side tends to 0 as $k \to \infty$ thanks to (11) and the fact that $(u_{0,k}, u_{1,k}) \to (0,0)$ in $L^2(\Omega) \times H^{-1}(\Omega)$.[2] This yields a contradiction.

2.2. *Proof of Theorem 4*

To conclude this section, we outline the proof of Theorem 4, which closely follows that of Theorem 1. The only difference lies in the replacement of Lemma 6 with the following lemma, whose proof is sketched in Remark 16 in Section 3.6.

Lemma 10. *Under the assumptions of Theorem 4, consider a solution u to system* (1) *with initial data $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ supported in $\overline{\mathcal{O}}$, and satisfying $\partial_n u \in L^2(\Gamma_T)$. Then $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $u \in \mathscr{C}^0\big(\mathbb{R}, H_0^1(\Omega)\big) \cap \mathscr{C}^1\big(\mathbb{R}, L^2(\Omega)\big)$.*

3. Some geometric facts, operators and wave fronts

In this section, we analyze the geometry of the domain Ω near the boundary and we provide the microlocal material used in this paper. More precisely, we present the generalized bicharacteristic flow of Melrose–Sjöstrand, the notion of wave front set up to the boundary and the theorem of propagation of singularities. All these notions are borrowed to Melrose–Sjöstrand [16] and Hörmander [7].

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[2]Note that, here again, the same argument applies if Ω is unbounded but $\mathcal{O}$ is bounded, since the compact embedding of $H_0^1(\mathcal{O}) \times L^2(\mathcal{O})$ into $L^2(\mathcal{O}) \times H^{-1}(\mathcal{O})$ can be employed.

Recall that the compressed cotangent bundle of Melrose–Sjöstrand is given by

$$T_b^* \mathcal{L} = T^* \mathcal{L} \cup T^* \partial \mathcal{L},$$

and that we have a natural projection

$$\pi\colon T^* \mathbb{R}^{n+1}\big|_{\overline{\Omega}} \longrightarrow T_b^* \mathcal{L}. \tag{13}$$

We equip $T_b^* \mathcal{L}$ with the induced topology.

Given the matrix $A(x) = \big(a_{ij}(x)\big)$ and $\xi \in \mathbb{R}^n$, we set $|\xi|_x^2 = {}^t\xi A(x)\xi$. We also denote by $p_A(t,x;\tau,\xi) = -\tau^2 + |\xi|_x^2$, the principal symbol of $P_A = \partial_t^2 - \sum_{i,j=1}^n \partial_{x_j}(a_{ij}(x)\partial_{x_i} \cdot)$. Finally, we define the characteristic set

$$\mathrm{Char}(P_A) = \Big\{(t,x;\tau,\xi) \in T^* \mathbb{R}^{n+1} \backslash 0, \ p_A(t,x;\tau,\xi) = 0\Big\},$$

and $\Sigma_A = \pi\big(\mathrm{Char}(P_A)\big)$.

3.1. *Local geodesic coordinates*

Near a point m_0 of the boundary $\partial\Omega$, taking advantage of the regularity of Ω which is of class $\mathscr{C}^\infty$, we can define a system of geodesic coordinates $x = (x_1, x_2, \ldots, x_n) \mapsto y = (y_1, y_2, \ldots, y_n)$ such that we have locally

$$\Omega = \big\{(y_1, y_2, \ldots, y_n), \ y_n > 0\big\}, \qquad \partial\Omega = \big\{(y_1, y_2, \ldots, y_{n-1}, 0)\big\} = \big\{(y', 0)\big\}, \tag{14}$$

and the corresponding wave operator is given by

$$P_A = \partial_t^2 - \left(\partial_{y_n}^2 + \sum_{1 \leqslant i,j \leqslant n-1} \partial_{y_j}\big(b_{ij}(y)\partial_{y_i}\big) \right) + M_0(y)\partial_{y_n} + M_1(y,\partial_{y'}).$$

Here, the matrix $\big(b_{ij}(y)\big)_{ij}$ is of class $\mathscr{C}^\infty$, symmetric, uniformly definite positive on a neighborhood of m_0, $M_0(y)$ is a real valued function of class $\mathscr{C}^\infty$, and $M_1(y,\partial_{y'})$ is a tangential differential operator of order 1 with $\mathscr{C}^\infty$ coefficients.

In the sequel, for convenience, we will use the same notation $(t,x) = (t,x',x_n)$ to denote (t,y',y_n), and we shall write

$$P_A = -\partial_n^2 - R(x_n, x', \partial_{x',t}) + M_0(x)\partial_n + M_1(x, \partial_{x'}). \tag{15}$$

Notice that, in this system of coordinates, the principal symbol of the wave operator P_A is given by

$$p_A = \xi_n^2 - r(x,\tau,\xi') = \xi_n^2 - \left(\tau^2 - \sum_{1 \leqslant i,j \leqslant n-1} a_{ij}(x)\xi_i\xi_j \right).$$

We shall set $r_0(x',\tau,\xi') = r(x',0,\tau,\xi')$.

3.2. *Generalized bicharacteristic rays*

First, let us recall that the hamiltonian field associated to p_A is given by

$$H_{p_A} = -2\tau\partial_t + 2\,{}^t\xi A(x)\nabla_x - \sum_{k=1}^n {}^t\xi\big(\partial_{x_k} A(x)\big)\xi\partial_{\xi_k}.$$

We also recall the following partition of $T^*(\partial\mathcal{L})$ into elliptic, hyperbolic and glancing sets:

$$\#\Big\{\pi^{-1}(\rho) \cap \mathrm{Char}(P_A)\Big\} = \begin{cases} 0 & \text{if } \rho \in \mathcal{E}, \\ 1 & \text{if } \rho \in \mathcal{G}, \\ 2 & \text{if } \rho \in \mathcal{H}. \end{cases} \tag{16}$$

For the sake of simplicity, we develop the rest of this section in a system of local geodesic coordinates as introduced in Section 3.1. Therefore we have locally

$$\mathcal{E} = \{r_0 < 0\}, \qquad \mathcal{H} = \{r_0 > 0\}, \qquad \mathcal{G} = \{r_0 = 0\}.$$

In addition, using the projection π, one can identify the glancing set $\mathcal{G}$ with a subset of $T^*\mathbb{R}^{n+1}$.

Following [2,7], we have the precise description of the glancing set $\mathcal{G}$.

Definition 11. *Let ρ be a point of $T^*\partial\mathcal{L}\backslash 0$. We say that:*

(1) *ρ is diffractive if $\rho \in \mathcal{G}$ and $H^2_{p_A}(x_n)(\rho) > 0$; this means that the free bicharacteristic ray γ issued from ρ belongs to $T^*\mathcal{L}$ in a neighborhood of 0, except at $s = 0$, i.e., there exists $\varepsilon > 0$ such that $(\exp s H_{p_A})\tilde{\rho} \in T^*\mathcal{L}$ for $0 < |s| < \varepsilon$, with $\tilde{\rho} = \pi^{-1}(\rho)$;*

(2) *ρ is nondiffractive if (a) $\rho \in \mathcal{H}$ or (b) $\rho \in \mathcal{G}$ and the free bicharacteristic ray $(\exp s H_{p_A})\tilde{\rho}$ passes over the complement of $\mathcal{L}$ for arbitrarily small values of s.*

We shall denote by $\mathcal{G}_d$ the set of diffractive points. Notice that in local geodesic coordinates, the set $\mathcal{G}_d$ is given by

$$\mathcal{G}_d = \left\{ \xi_n = r_0 = 0, \ \partial_n r\big|_{x_n=0} > 0 \right\}. \tag{17}$$

Definition 12. *A generalized bicharacteristic ray is a continuous map*

$$\mathbb{R} \supset I \backslash B \ni s \longmapsto \gamma(s) \in T^*\mathcal{L} \cup \mathcal{G} \subset T^*\mathbb{R}^{n+1}$$

where I is an interval of $\mathbb{R}$, B is a set of isolated points, for every $s \in I \backslash B$, $\gamma(s) \in \Sigma_A$ and γ is differentiable as a map with values in $T^*\mathbb{R}^{n+1}$, and:

(1) *if $\gamma(s_0) \in T^*\mathcal{L} \cup \mathcal{G}_d$ then $\dot{\gamma}(s_0) = H_{p_A}(\gamma(s_0))$;*

(2) *if $\gamma(s_0) \in \mathcal{G} \backslash \mathcal{G}_d$ then $\dot{\gamma}(s_0) = H^G_{p_A}(\gamma(s_0))$, where $H^G_{p_A} = H_{p_A} + (H^2_{p_A} x_n / H^2_{x_n} p_A) H_{x_n}$;*

(3) *for every $s_0 \in B$, the two limits $\gamma(s_0 \pm 0)$ exist and are the two different points of the same hyperbolic fiber of the projection π.*

Several remarks are in order:

(1) if Ω has no contact of infinite order with its tangents, the Melrose–Sjöstrand flow is globally well defined, see [16];

(2) in the interior, i.e., in $T^*\mathcal{L}$, a generalized bicharacteristic is simply a classical bicharacteristic ray of the wave operator whose projection on the space variable is a geodesic of Ω equipped with the metric $(a^{ij}) = (a_{ij})^{-1}$;

(3) finally, any generalized bicharacteristic ray γ can be considered as a continuous map on the interval I with values in $T^*_b\mathcal{L}$.

3.3. *Sets of interest*

In what follows, we introduce several geometric sets associated with the hamiltonian flow and linked to the observation region ω_T.

A microlocal open subset of $T^*_b\mathcal{L}$. We first introduce the set $\mathcal{R}(\omega_T)$ defined by

$$\mathcal{R}(\omega_T) = \left\{ \rho = (t, x, \tau, \xi) \in T^*_b\mathcal{L}\backslash 0, \ \text{s.t.} \ \rho \notin \Sigma_A \ \text{or} \ \gamma_\rho(\mathbb{R}) \cap T^*_b\omega_T \neq \varnothing \right\}, \tag{18}$$

which is the union of the set in which P_A is elliptic, and of the set corresponding to the collection of bicharacteristic rays that meet the observation set $T^*_b\omega_T$. As we will see next, this is the set on which we can recover regularity properties on solutions of the wave equation from the regularity of the solution on ω_T.

A microlocal open subset of $T_b^* \Omega$. Another set, which will be of interest when discussing the recovery of microlocal information at $t = 0$, is the set $\mathcal{R}_0(\omega_T)$ defined by

$$\mathcal{R}_0(\omega_T) = \left\{ (x, \xi) \in T_b^* \Omega \backslash 0 \; \middle| \; \begin{array}{l} \text{any } \gamma_\rho \text{ emanating from } (x, \xi) \text{ at } t = 0 \\ \text{satisfies } \gamma_\rho \cap T_b^*(\omega_T) \neq \emptyset \end{array} \right\}. \tag{19}$$

In other words, $(x, \xi) \in \mathcal{R}_0(\omega_T)$ if any bicharacteristic ray emanating from (x, ξ) at $t = 0$ enters in ω before the time T. Let us emphasize immediately that for any $(x, \xi) \in \mathcal{R}_0(\omega_T)$, there are at least two bicharacteristics emanating from (x, ξ) at $t = 0$.

To be more precise, we introduce the map $j \colon T_b^* \mathcal{L} \big|_{t=0} \to T_b^* \Omega$ defined by

$$\begin{cases} j(0, x; \tau, \xi) = (x, \xi) & \text{if } (x, \xi) \in T^* \Omega, \\ j(0, x; \tau, \xi') = (x, \xi') & \text{if } (x, \xi') \in T^* \partial \Omega. \end{cases} \tag{20}$$

In the sequel we will denote by (x, ξ) the current point of $T_b^* \Omega$. If x is a boundary point, (x, ξ) has to be understood as $(x, \xi') \in T^* \partial \Omega$, that is $\xi' \in \mathbb{R}^{n-1}$.

Recalling that $\Sigma_A = \pi(\text{Char}(P_A))$, we note that for $\widetilde{\rho} = (x, \xi) \in T_b^* \Omega$, the set $j^{-1}\{\widetilde{\rho}\} \cap \Sigma_A$ is not empty.

Now, we make precise the notion of bicharacteristic curves of P_A, denoted by γ, emanating from (x, ξ) at $\{t = 0\}$. Consider $\widetilde{\rho} = (x, \xi) \in T_b^* \Omega \backslash 0$.

- If x is an interior point, that is $x \in \Omega$, we have $j^{-1}\{\widetilde{\rho}\} \cap \Sigma_A = \{(0, x; \tau = \pm |\xi|_x, \xi)\}$. Therefore, we have two bicharacteristic curves issued from $\widetilde{\rho}$, namely the curve γ^+ issued from the point $\rho_+ = (0, x; \tau = |\xi|_x, \xi)$, and the curve γ^- issued from the point $\rho_- = (0, x; \tau = -|\xi|_x, \xi)$.

- If x is a boundary point, that is $x \in \partial \Omega$, working in local geodesic coordinates, we have in this case $\rho = (0, x; \tau, \xi') \in \Sigma_A \Leftrightarrow \tau^2 \geq |\xi'|_x^2$.

  (a) If $|\tau| = |\xi'|_x$, we are dealing with a glancing point, and we know that for each $\tau = \pm |\xi'|_x$, there exists a unique ray γ_ρ passing through $\rho = (t = 0, x; \tau, \xi')$. More precisely, if $\rho \in \mathcal{G}_d$, we then identify ρ to the point $(t = 0, x; \tau = \pm |\xi'|_x, \xi', \xi_n = 0) \in T^* \mathbb{R}^{d+1}$, and γ_ρ is an integral curve of the (free) hamiltonian field H_p. And if $\rho \in \mathcal{G} \backslash \mathcal{G}_d$, then γ_ρ is an integral curve of the gliding field H_p^G, see Definition 12.

  (b) If $|\tau| > |\xi'|_x$, we are dealing with a hyperbolic point. γ is then one of the two hyperbolic fibers of P_A at ρ. According to the hamiltonian equations, on sees that the bicharacteristic curve γ corresponds to the integral curve of the (free) hamiltonian field H_p issued from the point

$$\rho_- = \left(0, x; -|\tau|, \xi', \xi_n = +\sqrt{\tau^2 - |\xi'|_x^2}\right) \quad \text{or} \quad \rho_+ = \left(0, x; +|\tau|, \xi', \xi_n = +\sqrt{\tau^2 - |\xi'|_x^2}\right).$$

Let us emphasize that

$$(x, \xi) \in \mathcal{R}_0(\omega_T) \iff j^{-1}(x, \xi) \subset \mathcal{R}(\omega_T).$$

An open subset of $\overline{\Omega}$. The last set of interest is the set

$$\mathcal{O}(\omega_T) = \left\{ x \in \overline{\Omega}, \text{ s.t. } T_b^* \Omega \big|_{\{x\}} \backslash 0 \subset \mathcal{R}_0(\omega_T) \right\},$$

which corresponds to the set of $x \in \overline{\Omega}$, from which all bicharacteristics emanating at $t = 0$ meet the observation set ω_T.

Let us finally point out some basic properties of the sets $\mathcal{R}(\omega_T)$, $\mathcal{R}_0(\omega_T)$, and $\mathcal{O}(\omega_T)$:

- all these sets are non-empty since obviously $T_b^* \mathcal{L} \big|_{\omega_T} \backslash 0 \subset \mathcal{R}(\omega_T)$, $T_b^* \Omega \big|_\omega \backslash 0 \subset \mathcal{R}_0(\omega_T)$, and $\omega \subset \mathcal{O}(\omega_T)$;

- $\mathcal{R}(\omega_T)$, $\mathcal{R}_0(\omega_T)$, and $\mathcal{O}(\omega_T)$ respectively are open subsets of $T_b^* \mathcal{L}$, $T_b^* \Omega$ and $\overline{\Omega}$, according to the continuity of the Melrose–Sjöstrand flow.

Note that the classical GCC for ω_T can be simply expressed as one of the following equivalent formulations: $\mathscr{R}(\omega_T) = T_b^*\mathcal{L}\backslash 0$, $\mathcal{R}_0(\omega_T) = T_b^*\Omega\backslash 0$, or $\mathcal{O}(\omega_T) = \overline{\Omega}$.

The geometric condition of Theorem 1 can in fact be simply stated as $\overline{\mathscr{O}} \subset \mathcal{O}(\omega_T)$. In other words, Theorem 1 applies for any open set $\mathscr{O}$ strictly included in $\mathcal{O}(\omega_T)$.

3.4. *Pseudo-differential operators*

Following [12], we define the set $\mathcal{A}$ of pseudo-differential operators on $\mathbb{R} \times \mathbb{R}^n$ of the form $Q = Q_i + Q_\partial$ where Q_i is a classical pseudo-differential operator, compactly supported in $\mathcal{L}$ and Q_∂ is a classical pseudo-differential operator tangential to the boundary $\partial\mathcal{L}$, compactly supported near $\partial\mathcal{L}$. More precisely, $Q_i = \varphi Q_i \varphi$ for some $\varphi \in \mathscr{C}_0^\infty(\mathcal{L})$, and $Q_\partial = \psi Q_\partial \psi$ for some $\psi \in \mathscr{C}_0^\infty(U_{\partial\mathcal{L}})$, where $U_{\partial\mathcal{L}}$ is a small neighborhood of $\partial\mathcal{L}$ in $\mathbb{R} \times \mathbb{R}^n$. For $s \in \mathbb{R}$, $\mathcal{A}^s$ denotes the set of elements of order s of $\mathcal{A}$.

In a similar way, we also define the set $\mathcal{B}$ of pseudo-differential operators on $\mathbb{R}^n$, i.e., on the space variable, of the form $\psi = \psi_i + \psi_\partial$ where ψ_i is a classical pseudo-differential operator, compactly supported in Ω, and ψ_∂ is a classical pseudo-differential operator tangential to the boundary $\partial\Omega$, compactly supported near $\partial\Omega$. Similarly as above, for $s \in \mathbb{R}$, $\mathcal{B}^s$ denotes the set of the elements of $\mathcal{B}$ of order s.

3.5. *Wave front sets and propagation results*

In this section, we recall the notion of wave front set up to the boundary and the classical propagation results of Melrose–Sjöstrand [16] and Hörmander [7].

For a distribution u defined on the cylinder $\mathcal{L} = \mathbb{R} \times \Omega$, we define the H^s-wave front set up to the boundary, denoted $\mathrm{WF}_b^s(u)$, as a subset of the compressed cotangent bundle in the sense of Melrose–Sjöstrand, $T_b^*\mathcal{L} = T^*\mathcal{L} \cup T^*\partial\mathcal{L}$. This set coincides with the classical wave front set $\mathrm{WF}^s(u)$ in the interior of $\mathcal{L}$, i.e., in $T^*\mathcal{L}$, and extends the notion to describe the H^s-microlocal regularity of u up to the boundary. We follow here the definition of Chazarain (see [4]) that is used by Melrose and Sjöstrand (see [16]). In addition, for solutions of $P_A u \in \mathscr{C}^\infty$, it agrees with the intrinsic notion of Melrose (see Hörmander [7, Corollary 18.3.33]), which does not depend on P_A. In the following, we use the spaces of pseudodifferential operators $\mathcal{A}^0$ and $\mathcal{B}^0$, introduced in Section 3.4.

Consider $s \in \mathbb{R}$ and $u \in \mathscr{D}'(\mathcal{L})$ (later, u will be a solution of $P_A u = 0$ in $\mathcal{L}$). Also, for $q \in \mathbb{R}^{n+1}$ and $r > 0$, denote by $B_r(q)$ the euclidean ball of center q and radius r.

Definition 13. *For $\rho = (q, \eta) \in T_b^*\mathcal{L}$, we say that $\rho \notin \mathrm{WF}_b^s u$ if there exists a pseudodifferential operator $Q \in \mathcal{A}^0$ such that Q is elliptic at ρ, and $Qu \in H^s\big(B_r(q) \cap \mathcal{L}\big)$ for some $r > 0$.*
   More precisely;
- *if $\rho = (q, \eta) \in T^*\mathcal{L}$, i.e., q is an interior point, there exists a pseudodifferential operator $Q = Q_i \in \mathcal{A}^0$, elliptic at ρ, such that $Qu \in H^s\big(B_r(q)\big)$ for some $r > 0$, $B_r(q) \subset \mathcal{L}$;*
- *if $\rho = (q, \eta) \in T^*\partial\mathcal{L}$, i.e., q is a boundary point, there exists a tangential pseudodifferential operator $Q = Q_\partial \in \mathcal{A}^0$, elliptic at ρ, such that $Qu \in H^s\big(B_r(q) \cap \mathcal{L}\big)$ for some $r > 0$.*

Remark 14. *For $v \in \mathscr{D}'(\Omega)$ and $\rho' \in T_b^*\Omega$, we have a similar definition for $\rho' \notin \mathrm{WF}_b^s v$.*

Here we recall the Melrose–Sjöstrand theorem for propagation of regularity. For the convenience of the reader, we give a statement adapted to the framework of system (1).

Remind that for $\rho \in \Sigma_A \subset T_b^*\mathcal{L}$, we denote by γ_ρ the generalized bicharacteristic curve of P_A, issued from ρ as described in Section 3.3 above.

Theorem 15 (Melrose–Sjöstrand [16]). *Let u be a solution of system* (1) *with $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, and assume that a point $\rho \in T_b^*\mathcal{L}$ is such that $\rho \notin \mathrm{WF}_b^1 u$. Then $\gamma_\rho \cap \mathrm{WF}_b^1 u = \emptyset$.*

3.6. *Proof of Lemma 6*

First, we notice that if $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, and the corresponding solution satisfies $\partial_t u \in L^2(\omega_T)$, then u lies in $H^1_{\text{loc}}(\omega_T)$ by microlocal elliptic regularity.

We will deduce that u actually belongs to $H^1\big((0, T) \times \Omega\big)$. Indeed, let us consider $\rho_0 \in T^*_b\big((0, T) \times \Omega\big)$. If ρ_0 is an elliptic point (independently if it is an interior or a boundary point), it is classical that $\rho_0 \notin \text{WF}^2_b u$, i.e., u is in H^2 microlocally near ρ_0. Here, a special care must be taken at the boundary, see Hörmander [7, Theorem 20.1.14].

If ρ_0 is not an elliptic point, denote by γ_{ρ_0} the generalized bicharacteristic ray issued from ρ_0. We have two possibilities:

(a) γ_{ρ_0} intersects $T^*(\omega_T)$;
(b) $\gamma_{\rho_0} \cap T^*(\omega_T) = \emptyset$.

In case (a), since $u \in H^1_{\text{loc}}(\omega_T)$, $\rho_0 \notin \text{WF}^1_b u$ by propagation of the H^1-wave front, see Theorem 15 above.

In case (b), following γ_{ρ_0} backward in time, let us set $\rho_1 = \gamma_{\rho_0} \cap \{t = 0\}$. According to the microlocal assumption on the pair $(\omega, \mathcal{O})$, we have $x(\rho_1) \notin \overline{\mathcal{O}}$. Therefore, the initial data (u_0, u_1) is vanishing in a neighborhood of $x(\rho_1)$ and so does the solution u in some space-time cylinder $(-\alpha, \alpha) \times B\big(x(\rho_1), r\big)$, $\alpha > 0$, $r > 0$ small. Consequently, $\rho_1 \notin \text{WF}^1_b u$, and again, we obtain $\rho_0 \notin \text{WF}^1_b u$, by propagation of the H^1-wave front up to the boundary.

Accordingly, the solution u of the wave system (1) lies in $H^1\big((0, T) \times \Omega\big)$. Thus, by conservation of energy in time, we also conclude that the initial data has finite energy, i.e., (u_0, u_1) belongs to $H^1_0(\Omega) \times L^2(\Omega)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 16. The proof of the propagation of regularity from the boundary stated in Lemma 10 follows the same strategy as the one of Lemma 6. The only difference is that one has to use the propagation of the H^1-wave front set from an observation on the boundary, namely out of the information that $\partial_n u\big|_{\partial\Omega} \in L^2(\Gamma_T)$. This can be done on nondiffractive points. Indeed, under the assumption $\partial_n u\big|_{\partial\Omega} \in L^2(\Gamma_T)$, by the lifting lemma in [2, Theorem 2.2], we deduce that every nondiffractive point $\rho_0 \in T^*(\Gamma_T)$ satisfies $\rho_0 \notin \text{WF}^1_b(u)$. In other words, the solution u is H^1 microlocally near this point, up to the boundary. This suffices to conclude the proof of Lemma 10 in view of the imposed microlocal geometric condition on the pair $(\Gamma, \mathcal{O})$.

3.7. *Further technical results*

The goal of this section is to prove the following generalization of Lemma 6, which underlines the role played by the various sets $\mathcal{R}(\omega_T)$, $\mathcal{R}_0(\omega_T)$, and $\mathcal{O}(\omega_T)$ introduced in Section 3.3.

Lemma 17. *Let u be a solution u of* (1) *with initial data* $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, *and satisfying* $\partial_t u \in L^2(\omega_T)$. *Then:*

(1a) $\text{WF}^1_b(u) \cap \mathcal{R}(\omega_T) = \emptyset$;
(2a) $\big(\text{WF}^1_b(u_0) \cup \text{WF}^0_b(u_1)\big) \cap \mathcal{R}_0(\omega_T) = \emptyset$;
(3a) $(u_0, u_1) \in H^1_{\text{loc}}\big(\mathcal{O}(\omega_T)\big) \times L^2_{\text{loc}}\big(\mathcal{O}(\omega_T)\big)$.

Similarly, if u solution of (1), *with initial data* $(u_0, u_1) \in H^{-1}(\Omega) \times (H^2 \cap H^1_0)'(\Omega)$, *satisfies* $u \in L^2(\omega_T)$, *we have:*

(1b) $\text{WF}^0_b(u) \cap \mathcal{R}(\omega_T) = \emptyset$;
(2b) $\big(\text{WF}^0_b(u_0) \cup \text{WF}^{-1}_b(u_1)\big) \cap \mathcal{R}_0(\omega_T) = \emptyset$;
(3b) $(u_0, u_1) \in L^2_{\text{loc}}\big(\mathcal{O}(\omega_T)\big) \times H^{-1}_{\text{loc}}\big(\mathcal{O}(\omega_T)\big)$.

Proof of Lemma 17. (1). The proof of this item is in fact included in the proof of Lemma 6. Let us consider u a solution of (1) with initial data $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, and satisfying $\partial_t u \in L^2(\omega_T)$ (the other case, being completely similar, is left to the reader).

For $\rho_0 \in T_b^*\mathcal{L} \cap \mathscr{R}(\omega_T)$, there are two possibilities: either $\rho_0 \notin \Sigma_A$, or $\rho_0 \in \Sigma_A$ and $\gamma_{\rho_0} \cap T_b^*(\omega_T) \neq \emptyset$: when $\rho_0 \notin \Sigma_A$, the operator is elliptic, so u is in H^2 microlocally near ρ_0; when $\rho_0 \in \Sigma_A$ and $\gamma_{\rho_0} \cap T_b^*(\omega_T) \neq \emptyset$, Theorem 15 guarantees that u is in H^1 microlocally near ρ_0.

(3). Here again, we consider a solution u of (1) with initial data $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, and satisfying $\partial_t u \in L^2(\omega_T)$ (the other case being, completely similar, is left to the reader).

Let $x_0 \in \mathcal{O}(\omega_T)$. Since $\mathcal{O}(\omega_T)$ is open in $\overline{\Omega}$, by continuity of the flow of Melrose–Sjöstrand, there exists $\varepsilon > 0$ and $\varphi \in \mathscr{C}_c^\infty\big((-\varepsilon, \varepsilon) \times \overline{\Omega}\big)$ such that $\varphi \equiv 1$ in a neighborhood of $(0, x_0)$, such that all rays emanating from the support of φ meet ω_T.

By item (1), we know that u is locally H^1 near any point of the support of φ. To conclude, we simply note that $v = \varphi u$ solves

$$\begin{cases} P_A v = [P_A, \varphi] u \in L^2(\mathcal{L}), \\ v|_{\partial\Omega} = 0, \\ v(\varepsilon, \cdot) = \partial_t v(\varepsilon, \cdot) = 0, \end{cases}$$

thus $(v, \partial_t v)\big|_{t=0} \in H_0^1(\Omega) \times L^2(\Omega)$. This concludes the proof since $u = v$ in a neighborhood of x_0.

(2). The proof of item (2) is more involved and requires several additional technical results, which are presented in detail below.

Using the continuity of Melrose–Sjöstrand flow, there exists $\varepsilon > 0$ such that the set

$$E_\varepsilon = \Big\{ \rho = (t, x; \tau, \xi) \in T_b^*\mathcal{L}, \ (x, \xi) \in \mathscr{R}_0(\omega_T), \ t \in (0, \varepsilon) \Big\}$$

satisfies $E_\varepsilon \subset \mathscr{R}(\omega_T)$ and thus $E_\varepsilon \cap \mathrm{WF}_b^1 u = \emptyset$ by item (1).

Our first goal is to prove that for any $\psi(x, D_x) \in \mathcal{B}^0$ supported in $\mathscr{R}_0(\omega_T)$, we have $\psi(x, D_x) u \in H^1\big((0, \varepsilon) \times \Omega\big)$. And for this end, we will prove that any point $\rho_0 = (t_0, x_0; \tau_0, \xi_0) \in T_b^*\mathcal{L}, t_0 \in (0, \varepsilon)$ satisfies $\rho_0 \notin \mathrm{WF}_b^1\big(\psi(x, D_x) u\big)$.

Case 1. $\rho_0 \in T^*\mathcal{L}$, i.e., it is an interior point and $x_0 \in \Omega$.

Take $\varphi = \varphi(t, x) \in \mathscr{C}_0^\infty\big((0, \varepsilon) \times \Omega\big)$, supported near (T, x_0). In the operators space $\mathcal{A}^0$ consider the local identity partition

$$\varphi(t, x) = q_1(t, x; D_t, D_x) + q_2(t, x; D_t, D_x) + \mathcal{R},$$

with

$$\mathrm{Supp}(q_1) \subset \Big\{ \tfrac{1}{2}|\xi| \leqslant |\tau| \leqslant 2|\xi| \Big\},$$
$$\mathrm{Supp}(q_2) \subset \Big\{ |\tau| \leqslant \tfrac{3}{4}|\xi| \Big\} \cup \Big\{ |\tau| \geqslant \tfrac{3}{2}|\xi| \Big\},$$

and $\mathrm{Supp}_{(t,x)}(q_j)$, $j = 1, 2$ is a compact of $(0, \varepsilon) \times \Omega$, and $\mathcal{R}$ is infinitely smoothing.

Clearly, $\mathrm{Supp}(q_2)$ is contained in the elliptic set of $T^*\mathcal{L}$, so $q_2(t, x; D_t, D_x) u \in H^1(\mathcal{L}_\varepsilon)$, and $\psi(x, D_x) q_2 u \in H^1(\mathcal{L}_\varepsilon)$.

Let us now examine the first term $q_1(t, x; D_t, D_x) u$. Here we notice that the composition $\psi(x, D_x) q_1(t, x; D_t, D_x)$ provides a well defined global pseudodifferential operator, according to [7, Theorem 18.1.35]. In addition, if $\rho = (t, x; \tau, \xi) \in \mathrm{Supp}\big(\sigma(\psi q_1)\big)$, where σ denotes the symbol of the operator ψq_1, ρ is either an elliptic or hyperbolic point lying in the set E_ε, which does not intersect $\mathrm{WF}^1 u$. Therefore, $\psi(x, D_x) q_1(t, x; D_t, D_x) u \in H^1(\mathcal{L}_\varepsilon)$.

Hence we deduce $\varphi\psi(x, D_x) u = \psi(x, D_x)(\varphi u) - \big[\varphi, \psi(x, D_x)\big] u \in H^1(\mathcal{L}_\varepsilon)$.

Case 2. $\rho_0 \in T^*\partial\mathcal{L}$, i.e., it is a boundary point and $x_0 \in \partial\Omega$.

Here we shall work in a system of local geodesic coordinates $(t, x', x_n; \tau, \xi', \xi_n)$ with $\partial\Omega = \{x_n = 0\}$ and $\Omega = \{x_n > 0\}$, see Section 3.1. Hence we will set $\rho_0 = (T, x_0', \tau_0, \xi_0')$. Recall that the operators of $\mathcal{A}^0$ (resp. of $\mathcal{B}^0$) take the form $q(x_n, t, x', D_t, D_{x'})$ (resp. $\psi(x_n, x', D_{x'})$).

As in Case 1 above, we consider $\varphi = \varphi(t,x) \in \mathscr{C}_0^\infty\big((0,\varepsilon) \times \mathbb{R}^n\big)$, supported near (T, x_0), and a local partition of the identity with tangential pseudodifferential operators, of the form

$$\varphi(t,x) = q_1(x_n, t, x'; D_t, D_{x'}) + q_2(x_n, t, x'; D_t, D_{x'}) + \mathcal{R},$$

with, this time,

$$\text{Supp}(q_1) \subset \big\{ |\tau| \leqslant 3|\xi'| \big\} \qquad \text{and} \qquad \text{Supp}(q_2) \subset \big\{ |\tau| \geqslant 2|\xi'| \big\}.$$

With the notations of Section 3.2, $\text{Supp}(q_2) \subset \mathcal{H}$, the hyperbolic subset of $T^*\partial\mathcal{L}$. Therefore since $\text{Supp}(\psi) \subset \mathscr{R}_0(\omega_T)$, we get $q_2(x_n, t, x'; D_t, D_{x'})\psi(x_n, x', D_{x'})u \in H^1(\mathcal{L}_T)$.

Finally, as in Case 1, we notice that the composition $q_1(x_n, t, x'; D_t, D_{x'})\psi(x_n, x', D_{x'})$ provides a well defined global tangential pseudodifferential operator, see [7, Theorem 18.1.35], whose support is contained in $\mathcal{E} \cup \mathscr{R}(\omega_T)$. And this yields $q_1(x_n, t, x'; D_t, D_{x'})\psi(x_n, x', D_{x'})u \in H^1(\mathcal{L}_T)$ according to item (1).

Let us now examine the regularity of the trace (u_0, u_1). For this, consider a function $h(t) \in \mathscr{C}_0^\infty(\mathbb{R})$, $h(t) = 1$ for $|t| \leqslant \varepsilon/2$ and $h(t) = 0$ for $|t| \geqslant 3/4\varepsilon$. For $u(t,x)$ solution to (1), the function $v = h(t)\psi(x, D_x)u$ satisfies the wave system

$$\begin{cases} P_A v = \big[\partial_t^2, h(t)\big]\psi(x, D_x)u - h(t)\big[\Delta_A, \psi(x, D_x)\big]u, \\ v\big|_{\partial\Omega} = 0, \\ v(\varepsilon, \cdot) = \partial_t v(\varepsilon, \cdot) = 0. \end{cases} \tag{21}$$

The right-hand side of this equation lies in $L^2\big((0,\varepsilon) \times \Omega\big)$ according to the argument above. Therefore $\big(v(0,\cdot), \partial_t v(0,\cdot)\big) = \psi(x, D_x)(u_0, u_1)$ belongs to $H_0^1(\Omega) \times L^2(\Omega)$. This ends the proof of item (2), since $\psi(x, D)$ is any operator in $\mathcal{B}^0$ supported in $\mathscr{R}_0(\omega_T)$. $\qquad\square$

3.8. *A 1-d example*

To better understand the geometric statements given by Lemma 17, we briefly study the 1-d case when $\Omega = (-10, 10)$.

In this case, the wave equation

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & \text{in } (0, T) \times (-10, 10), \\ u(t, -10) = u(t, 10) = 0 & \text{on } (0, T), \\ \big(u(0, \cdot), \partial_t u(0, \cdot)\big) = (u_0, u_1), \end{cases} \tag{22}$$

can be solved explicitly using the characteristics. Indeed, setting

$$w_+(t,x) = (\partial_t u - \partial_x u)(t,x), \quad w_-(t,x) = (\partial_t u + \partial_x u)(t,x), \quad \text{for } (t,x) \in (0, T) \times (-10, 10),$$

the 1-d wave equation can be recast into a system of transport equation coupled from the boundary

$$\begin{cases} \partial_t w_+ + \partial_x w_+ = 0 & \text{in } (0, T) \times (-10, 10), \\ \partial_t w_- - \partial_x w_- = 0 & \text{in } (0, T) \times (-10, 10), \\ (w_+ + w_-)(t, -10) = (w_+ + w_-)(t, 10) = 0 & \text{on } (0, T), \\ \big(w_+(0, \cdot), w_-(0, \cdot)\big) = (u_1 - \partial_x u_0, u_1 + \partial_x u_0). \end{cases} \tag{23}$$

In this case, the bicharacteristic rays are particularly simple: they are the curves $t \mapsto x_0 + t$ and $t \mapsto x_0 - t$ for $x_0 \in (-10, 10)$ while these curves stay in the domain, bouncing back when meeting the boundary. Accordingly, in 1-d, we can identify the characteristic manifold $\text{Char}(P_A)$ with $\mathbb{R} \times \Omega \times \{-1, 1\}$, depending if $\tau = |\xi|$, corresponding to $\epsilon = 1$, or $\tau = -|\xi|$ corresponding to $\epsilon = -1$.

Let us now fix $\omega = (-2, -1) \cup (1, 2)$, and $T = 3$. The sets $\mathscr{R}(\omega_T)$, $\mathscr{R}_0(\omega_T)$ and $\mathcal{O}(\omega_T)$ can then be computed explicitly:

$$\mathscr{R}(\omega_T)\big|_{t \in [0, T]} = \mathscr{R}(\omega_T)^+ \cup \mathscr{R}(\omega_T)^- \tag{24}$$

with

$$\mathscr{R}(\omega_T)^+ = \left\{(t, x, +1) \text{ s.t. } t \in [0, T] \text{ and } -5 + t < x < 2 + t\right\},$$
$$\mathscr{R}(\omega_T)^- = \left\{(t, x, -1) \text{ s.t. } t \in [0, T] \text{ and } -2 - t < x < 5 - t\right\},$$

and

$$\mathscr{R}_0(\omega_T) = (-2, 2) \times \mathbb{R}^*,$$
$$\mathcal{O}(\omega_T) = (-2, 2).$$

To illustrate Lemma 17, due to the structure of the solutions of the wave given by (23), it is clear that if $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ with $u \in H^1(\omega_T)$, which of course implies w_+ and w_- belong to $L^2(\omega_T)$:

(1) $w_+ \in L^2(\mathscr{R}(\omega_T)^+)$ and $w_- \in L^2(\mathscr{R}(\omega_T)^+)$;
(2) $w_+|_{t=0} \in L^2(-5, 2)$ and $w_-|_{t=0} \in L^2(-2, 5)$;
(3) $(u_0, u_1) \in H^1(-2, 2) \times L^2(-2, 2)$.

It is also clear due to the explicit character of the solutions of (23) that we cannot improve these sets of regularity for general data.

4. Further observability results

The aim of this section is to refine Theorem 1 by analyzing which microlocal components of the initial data of general wave solutions can be effectively observed from measurements taken on ω_T.

Up to this point, our focus has been on initial data supported in a set $\overline{\mathcal{O}}$, such that the pair $(\omega, \mathcal{O})$ satisfies the required microlocal geometric condition. We now adopt a complementary viewpoint: given a fixed observation region ω_T, we seek to extract the maximum amount of information possible from the available measurements. As we shall see, we can recover, in a precise sense, the energy associated with the microlocal projection of the initial data that propagates along rays entering the observation region ω.

The proofs of these refined results follow the same general strategy as before, relying in particular on Lemma 17, which ensures propagation of microlocal regularity. The final observability estimates depend on whether a suitable unique continuation property is available, which determines our ability to eliminate the compact remainder term.

4.1. *Statement of the results*

We start with the following microlocal observability estimates.

Theorem 18 (Relaxed microlocal observability estimates). *Let ω be a non-empty open set of Ω and $T > 0$.*

(1a) *For every operator $\psi(t, x, D_t, D_x) \in \mathcal{A}^0$ such that $\mathrm{Supp}(\psi) \cap T_b^* \mathcal{L} \subset \mathcal{R}(\omega_T)$, there exists $C > 0$ such that for every initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the solution of* (1) *satisfies*

$$\left\| \psi(t, x, D_t, D_x) u \right\|_{H^1(\mathcal{L}_T)} \leqslant C \|\partial_t u\|_{L^2(\omega_T)} + C \|u\|_{L^2(\mathcal{L}_T)}. \tag{25}$$

(2a) *For every operator $\psi(x, D_x) \in \mathcal{B}^0$ such that $\mathrm{Supp}(\psi) \cap T_b^* \Omega \subset \mathcal{R}_0(\omega_T)$, there exists $C > 0$ such that for every initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the solution of* (1) *satisfies*

$$\left\| \psi(x, D_x)(u_0, u_1) \right\|_{H_0^1(\Omega) \times L^2(\Omega)} \leqslant C \|\partial_t u\|_{L^2(\omega_T)} + C \left\| (u_0, u_1) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)}. \tag{26}$$

(3a) *For every function $\psi = \psi(x) \in \mathscr{C}_c^\infty(\mathcal{O}(\omega_T))$, there exists $C > 0$ such that for every initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the solution of (1) satisfies*

$$\left\| \psi(x)(u_0, u_1) \right\|_{H_0^1(\Omega) \times L^2(\Omega)} \leq C \|\partial_t u\|_{L^2(\omega_T)} + C \left\| (u_0, u_1) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)}. \tag{27}$$

Similarly:

(1b) *for every operator $\psi(t, x, D_t, D_x) \in \mathcal{A}^0$ such that $\mathrm{Supp}(\psi) \cap T_b^* \mathcal{L} \subset \mathcal{R}(\omega_T)$, there exists $C > 0$ such that for every initial data $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, the solution of (1) satisfies*

$$\left\| \psi(t, x, D_t, D_x) u \right\|_{L^2(\mathcal{L}_T)} \leq C \|u\|_{L^2(\omega_T)} + C \|u\|_{H^{-1}(\mathcal{L}_T)}; \tag{28}$$

(2b) *for every operator $\psi(x, D_x) \in \mathcal{B}^0$ such that $\mathrm{Supp}(\psi) \cap T_b^* \Omega \subset \mathcal{R}_0(\omega_T)$, there exists $C > 0$ such that for every initial data $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, the solution of (1) satisfies*

$$\left\| \psi(x, D_x)(u_0, u_1) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)} \leq C \|u\|_{L^2(\omega_T)} + C \left\| (u_0, u_1) \right\|_{H^{-1}(\Omega) \times (H^2 \cap H_0^1)'(\Omega)}; \tag{29}$$

(3b) *for every function $\psi = \psi(x) \in \mathscr{C}_c^\infty(\mathcal{O}(\omega_T))$, there exists $C > 0$ such that for every initial data $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, the solution of (1) satisfies*

$$\left\| \psi(x)(u_0, u_1) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)} \leq C \|u\|_{L^2(\omega_T)} + C \left\| (u_0, u_1) \right\|_{H^{-1}(\Omega) \times (H^2 \cap H_0^1)'(\Omega)}. \tag{30}$$

Theorem 18, items (3a) and (3b), and Lemma 8 allow to generalize the result of Theorem 1 as follows.

Corollary 19. *For every function $\psi = \psi(x) \in \mathscr{C}_c^\infty(\mathcal{O}(\omega_T))$, there exists $C > 0$ such that for any $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the solution u of (1) satisfies the observability estimate*

$$\left\| \psi(x)(u_0, u_1) \right\|_{H_0^1(\Omega) \times L^2(\Omega)} \leq C \|\partial_t u\|_{L^2(\omega_T)} + C \left\| (1 - \psi(x))(u_0, u_1) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)}. \tag{31}$$

Similarly, for any initial data $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$,

$$\left\| \psi(x)(u_0, u_1) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)} \leq C \|u\|_{L^2(\omega_T)} + C \left\| (1 - \psi(x))(u_0, u_1) \right\|_{H^{-1}(\Omega) \times (H^2 \cap H_0^1)'(\Omega)}. \tag{32}$$

The proof can be done similarly as the one of Theorem 1, passing from the estimates (27) and (30) to the estimates (31)–(32) by contradiction, and using the unique continuation result given by Lemma 8 for solutions of the wave equation with supported in $\mathrm{Supp}\,\varphi$. Details are left to the reader.

Remark 20. Several comments are in order.

- The estimates in Theorem 18 hold true independently of any unique continuation consideration. They only rely on the propagation of regularity for solutions to the wave operator from ω_T to capture the energy of the projections of the data determined in the sets $\mathcal{R}(\omega_T)$, $\mathcal{R}_0(\omega_T)$ and $\mathcal{O}(\omega_T)$ introduced in Section 3.3 according to Lemma 17.
- The constants appearing in Theorem 18 depend on both the observation time T and the underlying metric $(a^{ij}(x))$. The dependence on the metric is very subtle and difficult to quantify explicitly, as it involves a large number of derivatives of the coefficients a^{ij}. This dependence can, in principle, be traced through a careful analysis of the proof of Theorem 15 in [16]. However, it is not readily expressible in closed form, since the operator norm of a pseudo-differential operator typically involves multiple derivatives of its symbol, making the exact dependence implicit and technically intricate.
- Due to the potential failure of the unique continuation property, an additive remainder term is required to ensure the validity of the inequalities. In the following, we will discuss how this remainder can be weakened or even eliminated under additional geometric assumptions.

Remark 21. The remainder terms $\|u\|_{L^2(\mathcal{L}_T)}$ in (25), and $\left\| (u_0, u_1) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)}$ in (26) and (27), can be weakened to $\|u\|_{H^{-1}(\mathcal{L}_T)}$ in (25) and $\left\| (u_0, u_1) \right\|_{H^{-1}(\Omega) \times (H^2 \cap H_0^1(\Omega))'}$ respectively, using the same proof as in Theorem 18. More generally, it is clear that these remainder terms can be

replaced by norms in any Hilbert spaces of negative order, as long as they are appropriately adapted to the boundary conditions of the problem.

We also emphasize that this observation applies to the remainder term in the observability inequality (31).

Remark 22. In the proof of Theorem 1, we showed that the remainder terms in (27) and (30) can be removed through a simple analysis of the invisible set (see Lemma 8), even in cases where the unique continuation property does not hold for all solutions of the wave equation (1). Further improvements along these lines are discussed in Corollary 19.

It is natural to ask whether a similar strategy, as used in Lemma 8, can be applied to remove the remainder terms in the microlocal estimates (25), (26), (28), and (29), at least for initial data microlocally supported in suitable regions. Unfortunately, this approach appears not to be effective in this context.

To illustrate the difficulty, consider the inequality (25). Let $\psi = \psi(t, x, D_t, D_x) \in \mathcal{A}^0$ be a pseudodifferential operator with $\operatorname{supp} \psi \cap T_b^* \mathcal{L} \subset \mathscr{R}(\omega_T)$, and define the set

$$\mathcal{N}_\psi = \left\{ u \in L^2_{\mathrm{loc}}(\mathcal{L}) \ \middle| \ \begin{array}{l} u \text{ solves (1)}, \\ (I - \psi) u = 0 \text{ in } \mathcal{L}, \\ \partial_t u = 0 \text{ in } \omega_T \end{array} \right\}.$$

By (26), any $u \in \mathcal{N}_\psi$ satisfies $u = \psi u \in H^1_{\mathrm{loc}}(\mathcal{L})$, implying that $\mathcal{N}_\psi$ is compact and hence finite-dimensional.

Now consider $v = \partial_t u$ for $u \in \mathcal{N}_\psi$. Clearly, v also solves the wave equation and satisfies $\partial_t v = 0$ in ω_T. However, there is no guarantee that v belongs to $\mathcal{N}_\psi$, as we do not have $(I - \psi) v = 0$. In fact, since $\psi v = \psi(\partial_t u)$ and $v = \partial_t u$, their difference involves the commutator $[\psi, \partial_t] u$, which does not vanish in general. Therefore, the naive use of the operator ∂_t does not yield an operator acting invariantly on $\mathcal{N}_\psi$.

We have not been able to further analyze the structure of the sets $\mathcal{N}_\psi$. Whether or not $\mathcal{N}_\psi$ is non-trivial remains an open problem.

When, in addition, the unique continuation property holds for (1), i.e., when the uniqueness condition (8) is satisfied, we can get the following result.

Theorem 23 (A second relaxed microlocal observability estimate). *Let ω be a non-empty open set of Ω and $T > 0$ such that*

$$T > 2 \sup_{\Omega} d(x, \omega). \tag{33}$$

(1a) *For every operator $\psi(t, x, D_t, D_x) \in \mathcal{A}^0$ with $\operatorname{Supp}(\psi) \cap T_b^* \mathcal{L} \subset \mathscr{R}(\omega_T)$, there exists $C > 0$ such that for every initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the solution u of (1) satisfies the observability estimate*

$$\left\| \psi(t, x, D_t, D_x) u \right\|_{H^1(\mathcal{L}_T)} \leq C \| \partial_t u \|_{L^2(\omega_T)} + C \left\| \left(I - \psi(t, x, D_t, D_x) \right) u \right\|_{L^2(\mathcal{L}_T)}. \tag{34}$$

(2a) *For every operator $\psi(x, D_x) \in \mathcal{B}^0$ with $\operatorname{Supp}(\psi) \cap T_b^* \Omega \subset \mathscr{R}_0(\omega_T)$, there exists $C > 0$ such that for every initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the solution u of (1) satisfies the observability estimate*

$$\left\| \psi(x, D_x)(u_0, u_1) \right\|_{H_0^1(\Omega) \times L^2(\Omega)} \leq C \| \partial_t u \|_{L^2(\omega_T)} + C \left\| \left(I - \psi(x, D_x) \right)(u_0, u_1) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)}. \tag{35}$$

Similarly

(1b) *For every operator $\psi(t, x, D_t, D_x) \in \mathcal{A}^0$ with $\operatorname{Supp}(\psi) \cap T_b^* \mathcal{L} \subset \mathscr{R}(\omega_T)$, there exists $C > 0$ such that for every initial data $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, the solution u of (1) satisfies the observability estimate*

$$\left\| \psi(t, x, D_t, D_x) u \right\|_{L^2(\mathcal{L}_T)} \leq C \| u \|_{L^2(\omega_T)} + C \left\| \left(I - \psi(t, x, D_t, D_x) \right) u \right\|_{H^{-1}(\mathcal{L}_T)}. \tag{36}$$

(2b) *For every operator $\psi(x, D_x) \in \mathcal{B}^0$ with $\mathrm{Supp}(\psi) \cap T_b^* \Omega \subset \mathscr{R}_0(\omega_T)$, there exists $C > 0$ such that for every initial data $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, the solution u of (1) satisfies the observability estimate*

$$\left\| \psi(x, D_x)(u_0, u_1) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)} \leq C \|u\|_{L^2(\omega_T)} + C \left\| (I - \psi(x, D_x))(u_0, u_1) \right\|_{H^{-1}(\Omega) \times (H^2 \cap H_0^1)'(\Omega)}. \quad (37)$$

Remark 24. Note that the main difference between Theorem 18 and Theorem 23 is that, in the latter, the compact remainder term is localized through the projection realized by the pseudodifferential operator $\left(I - \psi(t, x, D_t, D_x)\right)$ or $\left(I - \psi(x, D_x)\right)$, while, in the first theorem, the remainder involves the whole initial data. But for this to be done, we have assumed the condition (33) guaranteeing the time-horizon is large enough to ensure that unique continuation holds. Whether the results of Theorem 23 can be achieved from those in Theorem 18 without any additional further unique continuation assumption by means of a compactness-uniqueness argument is an interesting open problem, as we have discussed above in Remark 22.

Remark 25. Let us point out that, using Remark 21, we can weaken the remainder terms in the estimates of Theorem 23, replacing the terms $\left\| (I - \psi(t, x, D_t, D_x))u \right\|_{L^2(\mathcal{L}_T)}$ in (34) and $\left\| (I - \psi(x, D_x))(u_0, u_1) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)}$ in (35) by the weaker terms $\left\| (I - \psi(t, x, D_t, D_x))u \right\|_{H^{-1}(\mathcal{L}_T)}$ in (34) and $\left\| (I - \psi(x, D_x))(u_0, u_1) \right\|_{H^{-1}(\Omega) \times (H^2 \cap H_0^1(\Omega))'}$ in (35).

Remark 26 (Examples). The typical example in which Theorem 23 applies is for instance when Ω is the unit ball, $A = \mathrm{Id}$ (i.e., the standard constant coefficients wave equation), and ω is the ball of radius $1/2$. In such case, the GCC is not satisfied in any time, due to the whispering gallery phenomenon, i.e., the existence of rays localized in a neighborhood of the boundary. However, as soon as $T > 1$, unique continuation holds and the above theorem applies.

Another example corresponds to the case in which Ω is the unit square, $A = \mathrm{Id}$, and ω is an $\varepsilon(> 0)$-neighborhood of the whole boundary. Note that, even if the square is not a smooth bounded domain and its boundary has tangencies of infinite order, this is not an impediment for our results to apply since the observation is made on a neighborhood of the whole boundary and, therefore, boundary phenomena are irrelevant. In this case, the GCC holds as soon as $T > \sqrt{2}(1 - 2\varepsilon)$ while unique continuation holds as soon as $T > (1 - 2\varepsilon)$. Therefore, when T belongs to the intermediate interval $\left((1 - 2\varepsilon), \sqrt{2}(1 - 2\varepsilon)\right)$, Theorem 23 applies, but the classical observability inequality based on GCC does not hold.

The same occurs in most domains Ω since an ε-neighborhood of the boundary always guarantees GCC when the time-horizon is long enough, but, normally, the unique continuation property holds in shorter times.

We conclude this section with the following result, which goes a step further by addressing the case where global unique continuation fails. In such situations, it becomes necessary to assume that either the initial position u_0 or the initial velocity u_1 vanishes.

Theorem 27 (When unique continuation does not hold and u_0 or u_1 vanishes). *Let Ω be a smooth (possibly unbounded) domain, ω a non-empty bounded open set of Ω and $T > 0$.*

Then all the items (1a), (2a), (1b) *and* (2b) *in Theorem 23 hold for solutions u of the wave equation* (1) *corresponding to initial data (u_0, u_1) with either $u_0 = 0$ or $u_1 = 0$.*

The typical example in which Theorem 27 applies is when $\Omega = \mathbb{R}^d$ and ω is the unit ball, see Figure 4.

4.2. *Proofs*

Proof of Theorem 18. All the estimates of Theorem 18 can be proved through a direct application of the closed graph theorem and Lemma 17. Below, we present the proof of the estimate (26), the other proofs being completely similar and left to the reader.
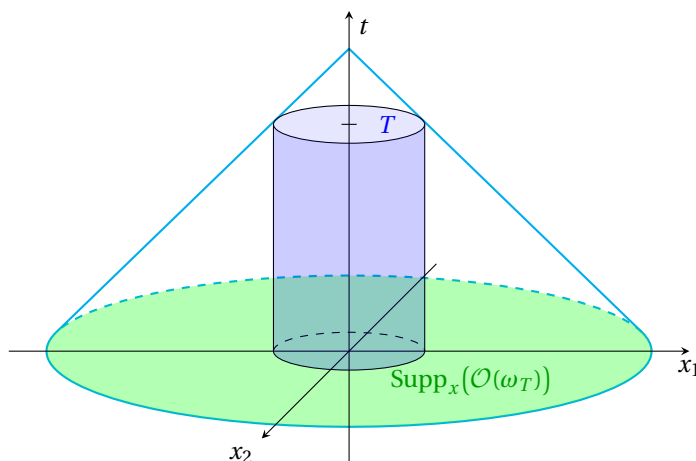
Figure 4. When $\Omega = \mathbb{R}^d$, ω is the unit ball and Δ_A is the flat Laplacian, the set $\mathcal{O}(\omega_T)$, when projected in the physical space is supported in the ball of size T. In fact, in this case, as the bicharacteristics in this setting correspond to straight lines,

$$\mathcal{O}(\omega_T) = \Big\{ (x, \xi) \in T_b^* \Omega \backslash 0 \text{ s.t. } \exists\, t \in [0, T] \text{ satisfying } x + \xi t \in \omega \Big\}.$$

We will closely follow the proof of estimate (10) of Corollary 7. Consider the following Hilbert space

$$E' = \Big\{ (u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega), \text{ and } \partial_t u \in L^2(\omega_T) \Big\}$$

equipped with the norm

$$\big\| (u_0, u_1) \big\|_{E'}^2 = \big\| (u_0, u_1) \big\|_{L^2(\Omega) \times H^{-1}(\Omega)}^2 + \| \partial_t u \|_{L^2(\omega_T)}^2,$$

and the energy space $F = H_0^1(\Omega) \times L^2(\Omega)$ equipped with its natural norm. According to Lemma 17, item (2), for every operator $\psi(x, D_x) \in \mathcal{B}^0$ such that $\mathrm{Supp}(\psi) \cap T_b^* \Omega \subset \mathcal{R}_0(\omega_T)$, the map

$$E' \longrightarrow F = H_0^1(\Omega) \times L^2(\Omega), \qquad (u_0, u_1) \longmapsto \psi(x, D_x)(u_0, u_1)$$

is well defined.

Consequently, the closed graph theorem yields its continuity and estimate (26).   □

Proof of Theorem 23. Here again, all the items of Theorem 23 can be proved similarly using a classical compactness-uniqueness argument. Below, we only present the proof of the estimate (35), as the other ones follow exactly the same path.

Let $\psi(x, D_x) \in \mathcal{B}^0$ with $\mathrm{Supp}(\psi) \cap T_b^* \Omega \subset \mathcal{R}_0(\omega_T)$. In view of (26) it is sufficient to show the existence of a constant $C > 0$ such that

$$\big\| \psi(x, D_x)(u_0, u_1) \big\|_{L^2(\Omega) \times H^{-1}(\Omega)} \leq C \Big(\| \partial_t u \|_{L^2(\omega_T)} + \big\| (I - \psi(x, D_x))(u_0, u_1) \big\|_{L^2(\Omega) \times H^{-1}(\Omega)} \Big), \qquad (38)$$

for all solution $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

We argue by contradiction. If that were not the case there would exist a sequence $(u_{0,k}, u_{1,k})_{k \in \mathbb{N}} \subset H_0^1(\Omega) \times L^2(\Omega)$ such that

$$\big\| \psi(x, D_x)(u_{0,k}, u_{1,k}) \big\|_{L^2(\Omega) \times H^{-1}(\Omega)} = 1, \qquad (39)$$

$$\lim_{k \to \infty} \Big(\| \partial_t u_k \|_{L^2(\omega_T)} + \big\| (I - \psi(x, D_x))(u_{0,k}, u_{1,k}) \big\|_{L^2(\Omega) \times H^{-1}(\Omega)} \Big) = 0. \qquad (40)$$

Accordingly, $(u_{0,k}, u_{1,k})_{k \in \mathbb{N}}$ is bounded in $L^2(\Omega) \times H^{-1}(\Omega)$, and up to the extraction of a subsequence still denoted the same, weakly converges to some (u_0, u_1), and from the above condition, we also have that $(I - \psi(x, D_x))(u_0, u_1) = 0$, that is $(u_0, u_1) = \psi(x, D_x)(u_0, u_1)$.

Then, in view of (26), $\psi(x, D_x)(u_{0,k}, u_{1,k})$ is bounded in $H_0^1(\Omega) \times L^2(\Omega)$, so it weakly converges to $\psi(x, D_x)(u_0, u_1) = (u_0, u_1)$ in $H_0^1(\Omega) \times L^2(\Omega)$, entailing in particular that $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, and corresponds to a solution u of (1) such that $\partial_t u = 0$ in ω_T.

By unique continuation we deduce that the limit $u \equiv 0$ and therefore $(u_0, u_1) \equiv (0, 0)$. But then the sequence $(\psi(x, D)(u_{0,k}, u_{1,k}))_{k \in \mathbb{N}}$ strongly converges to $(0, 0)$ in $L^2(\Omega) \times H^{-1}(\Omega)$. This contradicts (39) and concludes the proof. $\qquad \square$

Proof of Theorem 27. The proof of Theorem 27 follows the one of Theorem 23, the only difference being the unique continuation property we shall rely on, which is the following one: if u is a solution of the wave equation (1) corresponding to an initial datum satisfying $u_0 = 0$ (respectively $u_1 = 0$) such that $\partial_t u = 0$ in $(0, T) \times \omega$, then u vanishes identically in $\{(t, x) \in (-T, T) \times \Omega, \, d(x, \omega) + |t| \leq T\}$, and thus u_0 (respectively u_1) vanishes in the set $O^T = \{x \in \overline{\Omega}, \, d(x, \omega) < T\}$.

Indeed, if u_0 (respectively u_1) vanishes, the function u extended in an odd (respectively even) manner is a solution of the wave equation (1) on $(-T, T) \times \Omega$. We can then use the classical unique continuation result for the wave equation [24] which asserts that, if $\partial_t u = 0$ in $(-T, T) \times \omega$ for a solution u of (1) on $(-T, T) \times \Omega$, then u vanishes in the set $\{(t, x) \in (-T, T) \times \Omega, \, d(x, \omega) + |t| \leq T\}$.

Let us now explain how it can be used to prove for instance (again, all the other statements in Theorem 27 can be proved similarly) that for every operator $\psi(x, D_x) \in \mathcal{B}^0$ with $\text{Supp}(\psi) \cap T_b^* \Omega \subset \mathcal{R}_0(\omega_T)$, there exists $C > 0$ such that for any initial data (u_0, u_1) with $u_0 = 0$ and $u_1 \in L^2(\Omega)$, the solution of (1) satisfies the observability estimate

$$\left\| \psi(x, D_x) u_1 \right\|_{H^{-1}(\Omega)} \leq C \| \partial_t u \|_{L^2(\omega_T)} + C \left\| (I - \psi(x, D_x)) u_1 \right\|_{H^{-1}(\Omega)}. \tag{41}$$

We mimic the proof of the estimate (35) of Theorem 23, and use a compactness uniqueness argument to prove that (38) holds for any initial data (u_0, u_1) with $u_0 = 0$ and $u_1 \in L^2(\Omega)$.

By contradiction and following the proof of the estimate (35) of Theorem 23, we get a sequence $u_{1,k}$ such that $((I - \psi(x, D_x)) u_{1,k})_{k \in \mathbb{N}}$ goes to 0 in $H^{-1}(\Omega)$, $\psi(x, D_x) u_{1,k}$ is of unit norm in $H^{-1}(\Omega)$, and such that the corresponding solutions u_k of (1) satisfies that $(\partial_t u_k)_{k \in \mathbb{N}}$ goes to 0 in $L^2(\omega_T)$. Consequently, up to a subsequence, we get u_1 such that $(u_{1,k})$ converges weakly to u_1 in $H^{-1}(\Omega)$, and $(I - \psi(x, D_x)) u_1 = 0$, and such that the corresponding solution of (1) with initial data $(0, u_1)$ satisfies $\partial_t u = 0$ in $(0, T) \times \omega$. By the above uniqueness result, we thus get that $u_1 = 0$ in the set $O^T = \{x \in \overline{\Omega}, \, d(x, \omega) < T\}$. Finally, since $\psi = \psi(x, D_x) \in \mathcal{B}^0$ satisfies $\text{Supp}(\psi) \cap T_b^* \Omega \subset \mathcal{R}_0(\omega_T)$ and the x-projection of $\mathcal{R}_0(\omega_T)$ is included in O^T, $u_1 = \psi(x, D_x) u_1$ implies that u_1 is supported in O^T. Therefore, u_1 vanishes everywhere. Now, using (26), $\psi(x, D_x)(u_{1,k})$ is bounded in $L^2(\Omega)$, so we also obtain by compactness that $(\psi(x, D_x) u_{1,k})_{k \in \mathbb{N}}$ strongly converges to 0 in $H^{-1}(\Omega)$, thus getting a contradiction. $\qquad \square$

5. Control theoretical consequences

Each of the new observability results we have presented have their counterpart at the control level. This can be seen systematically by the employment of the duality arguments as in [13, 14].

Note, however, that duality transfers the observability of the adjoint backward wave equation into the control of the forward wave process. Thus, attention has to be paid to rewriting the needed microlocal assumptions reversing the sense of time. This is a purely technical minor aspect since we are dealing with time-independent variable coefficients and the geometry of the relevant pairs $(\omega, \overline{\mathcal{O}})$ is independent of the sense of time. Waves with time-dependent coefficients pose new technical difficulties, as we will discuss in the last section.

5.1. *Controllable* $(\omega, \overline{\mathscr{O}})$ *pairs*

The following result is the counterpart of Theorem 1 from the control point of view.

Theorem 28. *Within the setting of Theorem 1, for every data* $(y_0^T, y_1^T) \in H_0^1(\Omega) \times L^2(\Omega)$, *there exists a control* $v \in L^2\big(0, T; L^2(\omega)\big)$ *such that the solution* y *of*

$$\begin{cases} \partial_t^2 y - \Delta_A y = v 1_\omega & \text{in } (0, T) \times \Omega, \\ y(t, \cdot) = 0 & \text{on } (0, T) \times \partial\Omega, \\ \big(y(0, \cdot), \partial_t y(0, \cdot)\big) = (0, 0) & \text{in } \Omega, \end{cases} \tag{42}$$

satisfies

$$y(T, \cdot) = y_0^T \quad \text{and} \quad \partial_t y(T, \cdot) = y_1^T \quad \text{in } \overline{\mathscr{O}}. \tag{43}$$

Furthermore, there exists $C > 0$ *such that*

$$\|v\|_{L^2(\omega_T)} \leq C \big\|(y_0^T, y_1^T)\big\|_{H_0^1(\Omega) \times L^2(\Omega)}. \tag{44}$$

Proof. Since the set $\mathcal{O}(\omega_T)$ is open and $\overline{\mathscr{O}} \subset \mathcal{O}(\omega_T)$ by assumption, there exists an open set $\mathscr{O}_1$ such that $\overline{\mathscr{O}} \subset \mathscr{O}_1$ and $\overline{\mathscr{O}_1} \subset \mathcal{O}(T)$. We then take $\chi \in \mathscr{C}_c^\infty(\mathscr{O}_1)$ which equals to 1 in $\overline{\mathscr{O}}$.

Applying the observability inequality (4) of Theorem 1 on $\mathscr{O}_1$, we get that for any initial data $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ supported in $\overline{\mathscr{O}_1}$,

$$\big\|(u_0, u_1)\big\|_{L^2(\Omega) \times H^{-1}(\Omega)} \leq C \|u\|_{L^2(\omega_T)}, \tag{45}$$

where u is the corresponding solution of (1).

By time reversal $(t \mapsto T - t)$, for any initial data $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ supported in $\overline{\mathscr{O}_1}$, the solution of

$$\begin{cases} \partial_t^2 u - \Delta_A u = 0 & \text{in } \mathcal{L}, \\ u(t, \cdot) = 0 & \text{on } \partial\mathcal{L}, \\ \big(u(T, \cdot), \partial_t u(T, \cdot)\big) = (u_0^T, u_1^T), \end{cases} \tag{46}$$

satisfies

$$\big\|(u_0^T, u_1^T)\big\|_{L^2(\Omega) \times H^{-1}(\Omega)} \leq C \|u\|_{L^2(\omega_T)}. \tag{47}$$

We then introduce the set

$$X = \Big\{ (u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega) \text{ supported in } \overline{\mathscr{O}_1} \Big\},$$

which is obviously closed for the $L^2 \times H^{-1}$ topology.

Take $(y_0^T, y_1^T) \in H_0^1(\Omega) \times L^2(\Omega)$, and introduce the functional J defined for $(u_0^T, u_1^T) \in X$ by

$$J(u_0^T, u_1^T) = \frac{1}{2} \int_0^T \int_\omega |u|^2 \, dx \, dt - \int_\Omega \chi u_0^T y_1^T \, dx + \langle \chi u_1^T, y_0^T \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \tag{48}$$

where u is the corresponding solution of (46).

Here and in what follows, $\langle \cdot, \cdot \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$ stands for the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

It is obvious from the estimate (45) that J is continuous, strictly convex and coercive on X. Therefore, there exists a minimizer $(U_0, U_1) \in X$ of J such that

$$\|U\|_{L^2(\omega_T)} \leq C \big\|(y_0, y_1)\big\|_{H_0^1(\Omega) \times L^2(\Omega)}.$$

The Euler–Lagrange equation then gives that for all $(u_0, u_1) \in X$,

$$0 = \int_0^T \int_\omega U u \, dx \, dt - \int_\Omega \chi u_0 y_1 \, dx + \langle \chi u_1, y_0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

Since the solution y of (42) corresponding to a control function $v \in L^2\left(0, T; L^2(\omega)\right)$ satisfies that for all $(u_0, u_1) \in X$,

$$0 = \int_0^T \int_\omega v u \, dx \, dt - \int_\Omega u_0 \partial_t y(T) \, dx + \left\langle u_1, y(T, \cdot) \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)},$$

by setting

$$v = U 1_\omega,$$

we observe that the corresponding solution y of (42) satisfies

$$y(T, \cdot) = \chi y_0 \quad \text{and} \quad \partial_t y(T, \cdot) = \chi y_1 \quad \text{in } \mathcal{O}_1.$$

This concludes the proof of Theorem 28. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 29. Starting from Corollary 19, we can improve the result of Theorem 28 as follows. For $\chi \in \mathscr{C}_c^\infty\left(\mathcal{O}(\omega_T)\right)$, for every data $(y_0^T, y_1^T) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a control $v \in L^2\left(0, T; L^2(\omega)\right)$ such that the solution y of (42) satisfies

$$y(T, \cdot) = y_0^T \quad \text{and} \quad \partial_t y(T, \cdot) = y_1^T \quad \text{in } \{\chi = 1\}, \tag{49}$$

and there exists $C > 0$ such that

$$\|v\|_{L^2(\omega_T)} + \left\|\left(y(T), \partial_t y(T)\right) - \chi(y_0^T, y_1^T)\right\|_{H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)} \leq C \left\|(y_0^T, y_1^T)\right\|_{H_0^1(\Omega) \times L^2(\Omega)}. \tag{50}$$

Note that, since v belongs to $L^2(\omega_T)$, we should rather expect the solution y of (42) to be in $\mathscr{C}^0\left([0, T]; H_0^1(\Omega)\right) \cap \mathscr{C}^1\left([0, T]; L^2(\Omega)\right)$. In other words, such improvement means that we can construct a control process that controls exactly the solution y at time T on $\overline{\mathcal{O}}$ and does not create H^1 singularities outside of the support of $1 - \chi$.

In order to prove such result, simply replace the functional J above by J_χ defined by

$$J_\chi(u_0^T, u_1^T) = \frac{1}{2} \int_0^T \int_\omega |u|^2 \, dx \, dt + \frac{1}{2} \left\|(1 - \chi)(u_0^T, u_1^T)\right\|_{H^{-1} \times H^{-2}}^2$$
$$- \int_\Omega \chi u_0^T y_1^T \, dx + \left\langle \chi u_1^T, y_0^T \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \tag{51}$$

for $(u_0^T, u_1^T) \in L^2(\Omega) \times H^{-1}(\Omega)$, where u is the corresponding solution of (46), and H^{-2} is a short notation for $\left(H^2 \cap H_0^1(\Omega)\right)'$.

The observability estimate (32) easily provides the coercivity and strict convexity of the functional J_χ on the space $X_{\mathrm{obs}} = \overline{L^2(\Omega) \times H^{-1}(\Omega)}^{\|\cdot\|_{\mathrm{obs}}}$, where the norm $\|\cdot\|_{\mathrm{obs}}$ is given by

$$\left\|(u_0^T, u_1^T)\right\|_{\mathrm{obs}}^2 = \int_0^T \int_\omega |u|^2 \, dx \, dt + \left\|(1 - \chi)(u_0^T, u_1^T)\right\|_{H^{-1} \times H^{-2}}^2.$$

There is therefore a unique minimizer $(U_0^T, U_1^T) \in X_{\mathrm{obs}}$ of J_χ, which satisfies

$$\left\|(U_0^T, U_1^T)\right\|_{\mathrm{obs}} \leq C \left\|(y_0^T, y_1^T)\right\|_{H_0^1(\Omega) \times L^2(\Omega)}.$$

Following the above proof, one then easily derives that

$$y(T) = \chi y_0 + (1 - \chi)(-\Delta)^{-2}(1 - \chi) U_1^T,$$
$$\partial_t y(T) = \chi y_1 - (1 - \chi)(-\Delta)^{-1}(1 - \chi) U_0^T,$$

from which we directly conclude the proof of the above statement.

Remark 30. When, in addition to the geometric conditions of Theorem 1, the time horizon T is long enough so that unique continuation holds i.e., condition (33), the control result above can be improved to guarantee the simultaneous approximate controllability and the control of the projections as in (43). More precisely, for all $\varepsilon > 0$ there exists a control v_ε such that the solution satisfies both (43) and

$$\left\|y(T, \cdot) - y_0^T\right\|_{H_0^1(\Omega)} + \left\|\partial_t y(T, \cdot) - y_1^T\right\|_{L^2(\Omega)} \leq \varepsilon.$$

To prove it, it suffices to minimise the functional J_ε defined by

$$J_\varepsilon(u_0^T, u_1^T) = \frac{1}{2} \int_0^T \int_\omega |u|^2 \, \mathrm{d}x \, \mathrm{d}t + \varepsilon \left\| \left((1-\chi) u_0^T, (1-\chi) u_1^T \right) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)}$$
$$- \int_\Omega u_0^T y_1^T \, \mathrm{d}x + \langle u_1^T, y_0^T \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad (52)$$

on $L^2(\Omega) \times H^{-1}(\Omega)$, following the arguments in [25, Section 2], to prove the coercivity of the functional J_ε in $L^2(\Omega) \times H^{-1}(\Omega)$ and then writing the Euler–Lagrange equation for the minimizer to deduce the control. Note, however, that this approach does not provide a quantitative estimate for the cost of controllability in this setting, i.e., on the norm of the control in terms of ε.

5.2. *Pseudodifferential control when unique continuation holds*

Rather than presenting all the control results that can be derived by duality from the observability estimates in Section 4, we focus below on a representative control result of microlocal nature, which serves as the counterpart to items (2a) and (2b) of Theorem 23.

In order to do so, for $T > 0$ we introduce the set

$$\widetilde{\mathscr{R}_0(\omega_T)} = \left\{ (x, \xi) \in T_b^* \Omega \backslash 0 \left| \begin{array}{l} \text{such that bicharacteristics } \gamma_\rho \\ \text{issued from } (x, \xi) \text{ at time } T \\ \text{satisfy } \gamma_\rho(\mathbb{R}) \cap T^*(\omega_T) \neq \emptyset \end{array} \right. \right\}. \quad (53)$$

Note that the set $\widetilde{\mathscr{R}_0(\omega_T)}$ differs from $\mathscr{R}_0(\omega_T)$ in that it considers bicharacteristics originating from (x, ξ) at time T, rather than at the initial time. By a simple time-reversal argument (i.e., the change of variable $t \mapsto T - t$), this is precisely the relevant set when the goal is to obtain information about $(u, \partial_t u)$ at time $t = T$ rather than at the initial time $t = 0$, for solutions of (1).

Theorem 31 (Pseudodifferential control). *Assuming the uniqueness condition* (8), *for every operator* $\psi(x, D_x) \in \mathcal{B}^0$ *with* $\mathrm{Supp}(\psi) \cap T_b^* \Omega \subset \widetilde{\mathscr{R}_0(\omega_T)}$, *there exists* $C > 0$ *such that for any initial data* $(y_0^T, y_1^T) \in H_0^1(\Omega) \times L^2(\Omega)$, *there exists* $v \in L^2(0, T; L^2(\omega))$ *such that the control* v *and the corresponding solution* y *of* (42) *satisfies the following estimates:*

$$\left\| \left(y(T, \cdot), \partial_t y(T, \cdot) \right) - \psi(x, D_x)^\star (y_0^T, y_1^T) \right\|_{H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)} + \|v\|_{L^2(\omega_T)}$$
$$\leqslant C \left\| (y_0^T, y_1^T) \right\|_{H_0^1(\Omega) \times L^2(\Omega)}. \quad (54)$$

Remark 32. Let us briefly comment the control requirement (54). Here, let us emphasize that the target state (y_0^T, y_1^T) belongs to $H_0^1(\Omega) \times L^2(\Omega)$ and the control function v belongs to $L^2(\omega_T)$, so that the solution $(y, \partial_t y)$ of (42) belongs to $\mathscr{C}^0([0, T]; H_0^1(\Omega)) \times \mathscr{C}^1([0, T]; L^2(\Omega))$. The relevant information of (54) is thus that we can choose a control function v such that $\left(y(T, \cdot), \partial_t y(T, \cdot) \right) - \psi(x, D_x)^\star (y_0^T, y_1^T)$ belongs to $H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)$, that is such that the $H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)$ singularities of $\left(y(T), \partial_t y(T) \right)$ coincide with the ones of $\psi(x, D_x)^\star (y_0^T, y_1^T)$.

Proof. Let $(y_0^T, y_1^T) \in H_0^1(\Omega) \times L^2(\Omega)$. We introduce the functional

$$J(u_0^T, u_1^T) = \frac{1}{2} \int_0^T \int_\omega |\partial_t u|^2 \, \mathrm{d}x \, \mathrm{d}t + \frac{1}{2} \left\| (I - \psi(x, D_x))(u_0^T, u_1^T) \right\|_{L^2 \times H^{-1}}^2$$
$$- \int_\Omega A \nabla u_0^T \cdot \nabla \psi(x, D_x)^\star y_0^T \, \mathrm{d}x - \int_\Omega u_1^T \psi(x, D_x)^\star y_1^T \, \mathrm{d}x, \quad (55)$$

defined for $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, where u is the corresponding solution of (46).

Here, to be precise, we define the $H^{-1}(\Omega)$-norm by the formula

$$\|f\|_{H^{-1}(\Omega)}^2 = \int_\Omega A \nabla (-\Delta_A)^{-1} f \cdot \nabla (-\Delta_A)^{-1} f \, \mathrm{d}x,$$

where $-\Delta_A$ is the operator $-\operatorname{div}(A\nabla \cdot)$ in Ω with domain $H_0^1(\Omega)$ on $H^{-1}(\Omega)$.

From (35), it is clear that the quantity

$$\big\|(u_0^T, u_1^T)\big\|_{\mathrm{obs}}^2 = \int_0^T \int_\omega |\partial_t u|^2 \,\mathrm{d}x\,\mathrm{d}t + \big\|\big(I - \psi(x, D_x)\big)(u_0^T, u_1^T)\big\|_{L^2 \times H^{-1}}^2$$

defines a norm on $H_0^1(\Omega) \times L^2(\Omega)$, and we consider the closure X of $H_0^1(\Omega) \times L^2(\Omega)$ with respect to this norm. Note that we easily have

$$\big\|(u_0^T, u_1^T)\big\|_{L^2 \times H^{-1}} \leqslant C \big\|(u_0^T, u_1^T)\big\|_{\mathrm{obs}}.$$

We then check that the linear maps

$$(u_0^T, u_1^T) \longmapsto \int_\Omega A\nabla u_0^T \cdot \nabla \psi(x, D_x)^\star y_0^T \,\mathrm{d}x \qquad \text{and} \qquad (u_0^T, u_1^T) \longmapsto \int_\Omega u_1 \psi(x, D_x)^\star y_1 \,\mathrm{d}x$$

are continuous with respect to the norm $\|\cdot\|_{\mathrm{obs}}$: indeed,

$$\left| \int_\Omega u_1^T \psi(x, D_x)^\star y_1^T \,\mathrm{d}x \right| = \left| \int_\Omega \psi(x, D_x) u_1^T y_1^T \,\mathrm{d}x \right| \leqslant \big\|\psi(x, D_x) u_1^T\big\|_{L^2} \|y_1^T\|_{L^2} \leqslant C \big\|(u_0^T, u_1^T)\big\|_{\mathrm{obs}} \|y_1^T\|_{L^2},$$

and

$$\left| \int_\Omega A\nabla u_0^T \cdot \nabla \psi^\star(x, D_x) y_0^T \,\mathrm{d}x \right| \leqslant \left| \int_\Omega A\nabla \psi(x, D_x) u_0^T \cdot \nabla y_0^T \,\mathrm{d}x \right| + \big| \big\langle u_0^T, [\psi(x, D_x)^\star, \operatorname{div}(A\nabla \cdot)] y_0^T \big\rangle \big|$$

$$\leqslant C \big\|\psi(x, D_x) u_0^T\big\|_{H_0^1} \|y_0^T\|_{H_0^1} + C \|u_0^T\|_{L^2} \|y_0^T\|_{H_0^1}$$

$$\leqslant C \big\|(u_0^T, u_1^T)\big\|_{\mathrm{obs}} \|y_0^T\|_{H_0^1}.$$

Accordingly, the functional J can be extended uniquely as a continuous coercive functional on X, and it has a unique minimizer $(U_0^T, U_1^T) \in X$, which satisfies

$$\big\|(U_0^T, U_1^T)\big\|_{\mathrm{obs}} \leqslant C \big\|(y_0^T, y_1^T)\big\|_{H_0^1 \times L^2}.$$

The Euler–Lagrange equation satisfied by $(U_0, {}^T U_1^T)$ then gives that for all $(u_0^T, u_1^T) \in H_0^1(\Omega) \times L^2(\Omega)$,

$$0 = \int_0^T \int_\omega \partial_t U \partial_t u \,\mathrm{d}x\,\mathrm{d}t + \big\langle \big(I - \psi(x, D_x)\big)(U_0^T, U_1^T), \big(I - \psi(x, D_x)\big)(u_0^T, u_1^T) \big\rangle_{L^2 \times H^{-1}}$$

$$- \int_\Omega A\nabla u_0^T \cdot \nabla \psi(x, D_x)^\star y_0^T \,\mathrm{d}x - \int_\Omega u_1^T \psi(x, D_x)^\star y_1^T \,\mathrm{d}x.$$

It is then convenient to notice that the solution y of (42) corresponding to a control function v satisfies, for all $(u_0^T, u_1^T) \in H_0^1(\Omega) \times L^2(\Omega)$,

$$0 = \int_0^T \int_\omega v \partial_t u \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega A\nabla u_0^T \cdot \nabla y(T, \cdot) \,\mathrm{d}x - \int_\Omega u_1 \partial_t y(T, \cdot) \,\mathrm{d}x.$$

Therefore, setting $v = \partial_t U\big|_{(0,T) \times \omega}$, the corresponding solution y of (42) satisfies:

$$-\operatorname{div}\big(A\nabla\big(y(T, \cdot) - \psi(x, D_x)^\star y_0^T\big)\big) = -\big(I - \psi(x, D_x)\big)^\star \big(I - \psi(x, D_x)\big) U_0^T, \qquad \text{in } \Omega, \qquad (56)$$

$$\partial_t y(T, \cdot) - \psi(x, D_x)^\star y_1 = -\big(I - \psi(x, D_x)\big)^\star (-\Delta_A)^{-1} \big(I - \psi(x, D_x)\big) U_1^T, \quad \text{in } \Omega. \qquad (57)$$

Accordingly, by elliptic regularity, $\big(y(T, \cdot) - \psi(x, D)^\star y_0^T\big) \in H^2 \cap H_0^1(\Omega)$ and $\partial_t y(0, \cdot) - \psi(x, D)^\star y_1 \in H_0^1(\Omega)$ and we get:

$$\big\|y(T, \cdot) - \psi(x, D)^\star y_0^T\big\|_{H^2 \cap H_0^1(\Omega)} \leqslant C \|U_0^T\|_{L^2} \leqslant C \big\|(U_0^T, U_1^T)\big\|_{\mathrm{obs}} \leqslant C \big\|(y_0^T, y_1^T)\big\|_{H_0^1 \times L^2},$$

$$\big\|\partial_t y(T, \cdot) - \psi(x, D)^\star y_1^T\big\|_{H_0^1(\Omega)} \leqslant C \|U_1^T\|_{H^{-1}} \leqslant C \big\|(U_0^T, U_1^T)\big\|_{\mathrm{obs}} \leqslant C \big\|(y_0^T, y_1^T)\big\|_{H_0^1 \times L^2}.$$

This concludes the proof of Theorem 31. $\hfill\square$

Remark 33. In the above proof of Theorem 31, we use the duality between the observability and controllability with respect to the pivot space $H_0^1(\Omega) \times L^2(\Omega)$ instead of the usual one developed in [13,14] with respect to the pivot space $L^2(\Omega)$ that we were using in the proof of Theorem 28. This is indeed slightly simpler to handle in the proof of Theorem 31 since it involves less singular spaces.

One may wonder why this approach was not used in the proof of Theorem 28. The reason lies in the structure of formulas (56)–(57), which involve commutators with the operator $-\Delta_A$. While these commutators do not affect the regularity of the solutions, they significantly alter their support properties — particularly in the case of formula (56). As a result, this method is not well-suited for establishing Theorem 28.

6. Extensions, open problems and perspectives

6.1. *Time-dependent coefficients*

It would be interesting to investigate the extension of our results to wave equations with time-dependent coefficients. Under a suitable reformulation of the microlocal geometric condition on the pair $(\omega, \mathcal{O})$, the high-frequency propagation results remain valid, and relaxed observability inequalities, similar to those in Lemma 7, can still be established.

However, removing the compact remainder term in this setting requires a unique continuation result. Assuming analyticity with respect to the time variable, one can obtain a refined observability inequality under a time condition analogous to (8). Nevertheless, the compactness-uniqueness argument used in Lemma 8 is no longer applicable, as the wave equation with time-dependent coefficients is not invariant under time differentiation.

As a result, obtaining sharp observability results analogous to Theorem 1 becomes significantly more challenging in the time-dependent case. This limitation is particularly critical when addressing control problems for semilinear or quasilinear wave equations, where time-dependent coefficients naturally arise when applying fixed point techniques.

6.2. *Other observability techniques*

Other than the microlocal tools employed in this article, the observability of waves has been often addressed employing multiplier methods [13] or Carleman estimates (see for instance [6]). Although they allow to refine global observability estimates when imposing conditions on the support of the initial data (see [13, Chapitre I, Section 9]) by reducing the observability time, these methods do not allow to get the sharp microlocal results in this paper.

6.3. *Schrödinger and plate equations*

There exists an extensive literature on the observability and control of Schrödinger and plate equations. These models can be roughly viewed as wave-type equations with infinite speed of propagation, which implies that whenever the wave equation is observable or controllable in finite time, the same property holds for the Schrödinger or plate equation in arbitrarily small time, using the same observation and/or control region.

Extending the microlocal and geometric results developed in the present work to such equations remains an interesting and challenging open problem.

6.4. *Control of the heat equation for some specific data*

It would be interesting to develop analogues of the results presented in this article for heat-type equations. For instance, one could investigate the cost of controllability in small time for initial data localized in an open subset $\mathcal{O}$, using controls supported on $(0, T) \times \omega$, under the same geometric setting as in Theorem 1. It is natural to conjecture that, in such a case, the controllability cost as $T \to 0$ should be related to the time threshold T_0 identified in Theorem 1, and behave like $C \exp(C T_0^2 / T)$ for some constant $C > 0$.

Indeed, it was shown using the transmutation technique (see [17]) that one can leverage the controllability properties of the wave equation to derive estimates for the cost of controlling the heat equation in small time. However, the arguments developed in [17] do not seem directly applicable to the microlocal or geometric setting considered here, and the question remains an open problem.

We also refer to the work [18] for a related open question, approached from a different perspective.

Additionally, we note that the transmutation method has also been employed to describe the reachable set for the heat equation (see [5]), based on the observability properties of the wave equation. It would be interesting to investigate whether the results of the present work could lead to new estimates on the reachable set for the heat equation, especially in multi-dimensional settings, where this question remains largely open. To our knowledge, the reachable set is fully understood only in the specific case of a ball controlled from its entire boundary, as studied in [23].

6.5. *Numerical approximation*

The numerical analysis of the observability and controllability properties of the wave equation has been also thoroughly investigated. The adaptation of the results in this paper to the discrete context is of interest and would probably require either some suitable filtering processes to avoid the spurious rays (see [26]) and/or some suitable meshes to bend the spurious discrete high-frequency rays (see [15]).

6.6. *Stabilisation*

It is well known that classical observability and controllability properties are closely linked to the exponential stabilizability of the system. Investigating the stabilization implications of the results developed in this paper thus constitutes an interesting and promising direction for future research.

Acknowledgments

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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