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Two quantitative versions of the Nonlinear Carleson Conjecture

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Abstract. In this note, we state and compare two quantitative versions of the Nonlinear Carleson Conjecture (NCC). We provide motivations for our conjectures and show that they both imply the NCC. We also obtain some applications to the zero distribution of polynomials orthogonal on the unit circle and to their pointwise asymptotics.

Keywords. Maximal function, variational norm, product of matrices, nonlinear Carleson conjecture, orthogonal polynomials, Schur functions.

2020 Mathematics Subject Classification. 42C05, 30H35.

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We will start with notation: $\mathbb{N} := \{1,2,\dots\}$, $\mathbb{Z}_+ := \{0,1,2,\dots\}$, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. $M(d,\mathbb{C})$ stands for the space of the $d \times d$ complex matrices. For $A \in M(d,\mathbb{C})$, the symbol $\|A\|_2$ denotes the Frobenius norm: $\|A\|_2 := \sqrt{\operatorname{tr}(A^*A)}$ and $\|A\|$ denotes the operator norm. For $p \in [1,\infty]$, the symbol p' is the dual exponent: $p' = p/(p-1)$. For a set $S \subset \mathbb{T}$, the symbol $|S|$ indicates its Lebesgue measure.

1. OPUC, SU(1,1), and matrix products

Let $\mathcal{M}(\mathbb{T})$ denote the set of probability measures on the unit circle $\mathbb{T}$ whose support is not a finite subset of $\mathbb{T}$. For $\sigma \in \mathcal{M}(\mathbb{T})$, define $F_\sigma(z) := \int_{\mathbb{T}} \frac{\xi+z}{\xi-z}\, d\sigma$, $\xi = e^{i\theta}$, $\theta \in [0,2\pi)$. The function F_σ is analytic in $\mathbb{D}$, $\Re F > 0$ in $\mathbb{D}$, and $F(0) = 1$. The same properties hold for the function $1/F_\sigma$ and therefore [7] there is a probability measure $\sigma_d \in \mathcal{M}(\mathbb{T})$ such that

$$F_\sigma^{-1}(z) = F_{\sigma_d}(z) = \int_{\mathbb{T}} \frac{\xi+z}{\xi-z}\, d\sigma_d.$$

We will call such a measure σ_d dual to σ. For each $\sigma \in \mathcal{M}(\mathbb{T})$, one can define the monic orthogonal polynomials $\{\Phi_n(z,\sigma)\}$, $n \in \mathbb{Z}_+$, on the unit circle (OPUC) as algebraic polynomials that satisfy the following conditions:

$$\Phi_n(z,\sigma) = z^n + \cdots, \quad \langle \Phi_n, z^j\rangle_\sigma = 0, \quad \forall\, j \in \{0,\dots,n-1\}, \quad \langle f,g\rangle_\sigma := \int_{\mathbb{T}} f\bar{g}\, d\sigma.$$

Denote $\Phi_n := \Phi_n(z,\sigma)$, $\Psi_n := \Phi_n(z,\sigma_d)$. For every polynomial Q of degree at most n, let $Q^*(z) = z^n\overline{Q(\bar{z}^{-1})}$. Notice that the map $Q \mapsto Q^*$ depends on n. It is known [16] that

$$\begin{bmatrix}\Phi_{n+1} & -\Psi_{n+1}\\ \Phi^*_{n+1} & \Psi^*_{n+1}\end{bmatrix} = \begin{bmatrix} z & -\bar{\gamma}_n\\ -z\gamma_n & 1\end{bmatrix}\begin{bmatrix}\Phi_n & -\Psi_n\\ \Phi^*_n & \Psi^*_n\end{bmatrix}, \qquad \begin{bmatrix}\Phi_0 & -\Psi_0\\ \Phi^*_0 & \Psi^*_0\end{bmatrix} = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}, \tag{1}$$

where $\{\gamma_g\}$ are the so-called Schur (Verblunsky) parameters and they satisfy $\gamma_g \in \mathbb{D}$, $\forall\ g \in \mathbb{Z}_+$. There is a bijection [16] between the set of measures $\mathcal{M}(\mathbb{T})$ and the set of Schur parameters $\mathbb{D}^\infty$. If $\{\gamma_g\}$ are Schur parameters for σ, then the Schur parameters for σ_d are $\{-\gamma_g\}$. Denote $\rho_n := (1-|\gamma_n|^2)^{\frac{1}{2}}$ and $\mathcal{R}_n := \prod_{j\le n}\rho_j$, $\mathcal{R}_\infty := \lim_{n\to\infty}\mathcal{R}_n$. Since $\mathcal{R}_n$ is decreasing, $\mathcal{R}_\infty$ always exists and it is zero iff $\{\gamma_g\} \notin \ell^2(\mathbb{Z}_+)$.

The Szegő class of measures is defined as the set $\mathrm{Sz}(\mathbb{T}) := \{\sigma \in \mathcal{M}(\mathbb{T}) : \int_{\mathbb{T}} \log w\, dm > -\infty\}$ where m is normalized Lebesgue measure on $\mathbb{T}$, i.e., $dm := d\theta/(2\pi)$, and $d\sigma = w\,dm + d\sigma_s$ with σ_s being the singular measure. The Szegő theorem in the theory of OPUC states that $\sigma \in \mathrm{Sz}(\mathbb{T})$ iff $\{\gamma_g\} \in \ell^2(\mathbb{Z}_+)$ and that can be quantified by the following identity [16]:

$$\exp\left(\int_{\mathbb{T}} \log w\, dm\right) = \prod_{n\ge 0}\rho_n^2 = \mathcal{R}_\infty^2, \tag{2}$$

which holds for any σ and $\{\gamma_g\}$. Clearly, $\sigma \in \mathrm{Sz}(\mathbb{T})$ iff $\sigma_d \in \mathrm{Sz}(\mathbb{T})$. For measures $\sigma \in \mathrm{Sz}(\mathbb{T})$, the Szegő function is defined as $D_\sigma(z) := \exp\left(\frac{1}{2}\int_{\mathbb{T}} \frac{\xi+z}{\xi-z}\log w\, dm\right)$. This is the outer function in $H^2(\mathbb{D})$ that satisfies $D_\sigma(0) > 0$ and $|D_\sigma(\xi)|^2 = w(\xi)$ for a.e. $\xi \in \mathbb{T}$.

Sometimes it is more convenient to work with polynomials orthonormal on the unit circle with respect to measure σ. These polynomials are given by the formula $\phi_n(z,\sigma) = \Phi_n(z,\sigma)/\|\Phi_n(\xi,\sigma)\|_{L^2_\sigma}$. Since (see [16, formula (1.5.13)]) $\|\Phi_n(\xi,\sigma)\|_{L^2_\sigma} = \mathcal{R}_{n-1}$, we get

$$\sup_{z,n}\left|\frac{\phi_n(z,\sigma)}{\Phi_n(z,\sigma)} - 1\right| \lesssim \|\{\gamma_g\}\|^2_{\ell^2(\mathbb{Z}_+)},$$

when $\|\{\gamma_g\}\|_{\ell^2(\mathbb{Z}_+)} \le \frac{1}{2}$. Hence, the problems of studying the size of Φ_n or ϕ_n are identical in the Szegő class when $\|\{\gamma_g\}\|_{\ell^2(\mathbb{Z}_+)}$ is small. For Szegő measures, it is known that

$$\lim_{n\to\infty}\Phi^*_n(z,\sigma) = D_\sigma(0)/D_\sigma(z), \qquad \lim_{n\to\infty}\phi^*_n(z,\sigma) = 1/D_\sigma(z), \tag{3}$$

locally uniformly in $\mathbb{D}$ (see [16, Theorem 2.4.1, p. 144]).

Given any sequence $\{\gamma_n\}$ such that $\gamma_n \in \mathbb{D}$, $n \in \mathbb{Z}_+$, define

$$\Omega_n := \rho_n^{-1}\begin{bmatrix} 1 & \bar{\gamma}_n z^{-n}\\ \gamma_n z^n & 1\end{bmatrix} \tag{4}$$

and $\Pi_n := \Omega_n \cdots \Omega_0$. It is a simple fact that $\Pi_n \in \mathrm{SU}(1,1)$ for $z \in \mathbb{T}$ and therefore

$$\Pi_n = \begin{bmatrix}\bar{a}_n & \bar{b}_n\\ b_n & a_n\end{bmatrix}, \quad |a_n|^2 = 1 + |b_n|^2, \quad z \in \mathbb{T}. \tag{5}$$

The polynomials a_n and b_n have degrees at most n, a_n has no roots in $\mathbb{D}$ and $a_n(0) > 0$ (see [11, Lemma 4.5] or [20]). If γ_n are Schur parameters for σ, then (see [11, formulas (4.12) and (4.13)])

$$\begin{bmatrix}\Phi_{n+1} & -\Psi_{n+1}\\ \Phi^*_{n+1} & \Psi^*_{n+1}\end{bmatrix} = \mathcal{R}_n\begin{bmatrix} za^*_n & -b^*_n\\ -zb_n & a_n\end{bmatrix}\cdot\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} \tag{6}$$

and, therefore,

$$a_n = (\Phi^*_{n+1} + \Psi^*_{n+1})/(2\mathcal{R}_n), \qquad -zb_n = (\Phi^*_{n+1} - \Psi^*_{n+1})/(2\mathcal{R}_n). \tag{7}$$

Denote

$$o_1(\xi) := \sup_{n\ge 0}|a_n(\xi) - 1|, \quad o_2(\xi) := \sup_{n\ge 0}|b_n(\xi)|, \quad O(\xi) := \sup_{n\ge 0}\|\Pi_n(\xi) - I\|_2, \quad \xi \in \mathbb{T}, \tag{8}$$

and we clearly have

$$\|\Pi_n - I\|_2^2 = 2\left(|a_n - 1|^2 + |b_n|^2\right) \le 2(o_1^2 + o_2^2). \tag{9}$$

2. Two versions of quantified NCC and motivations for them

The famous Carleson–Hunt theorem [9] says that

$$\|M_0 f\|_{L^p[0,1)} \le_p \|f\|_{L^p[0,1)}, \qquad M_0 f := \sup_{N\in\mathbb{N}} \left| \sum_{|n|<N} \widehat{f}_n e^{2\pi i n x} \right|, \tag{10}$$

for $p \in (1,\infty)$, where $\widehat{f}_n = \int_0^1 f e^{-2\pi i n x}\,\mathrm{d}x$ is the Fourier coefficient of a function f. Function M_0 is an associated maximal function. The bound (10) was used to settle Lusin's conjecture, i.e., it showed that *for every* $f \in L^p(0,1)$, $p \in (1,\infty]$, *we have* $\lim_{N\to\infty} \sum_{|n|<N} \widehat{f}_n e^{2\pi i n x} = f(x)$ *for a.e.* $x \in [0,1)$. The analog of Lusin's conjecture in the OPUC theorem is called Nonlinear Carleson Conjecture (NCC) and it says: *is it true that asymptotics* (3) *holds for a.e.* $z \in \mathbb{T}$ *assuming that* $\sigma \in \mathrm{Sz}(\mathbb{T})$*?* This problem has a long history and it can also be formulated for Dirac equations, Schrödinger operators and Krein systems (nonlinear Fourier transform) [3,14,15,18,20]. The general theory of the nonlinear Fourier transform is an active field with multiple applications [1,6,12,19,20]. One purpose of the current note is to put forward two quantifications of such a conjecture, motivate them and compare. For $\sigma \in \mathcal{M}(\mathbb{T})$, define the maximal function $M(\xi,\sigma) := \sup_n \left|\Phi_n^*(\xi,\sigma) - 1\right|$, $\xi \in \mathbb{T}$. From (7), we get

$$o_{1(2)}(\xi) \le \big(M(\xi,\sigma) + M(\xi,\sigma_d)\big)/(2\mathscr{R}_\infty) + \mathscr{R}_\infty^{-1} - 1, \quad \xi \in \mathbb{T}. \tag{11}$$

Conjecture (qNCC-I). *There is* $\epsilon > 0$ *such that* $\|\{\gamma_g\}\|_{\ell^2(\mathbb{Z}_+)} \le \epsilon$ *implies*

$$\int_{\mathbb{T}} M^2(\xi,\sigma)\,\mathrm{d}\sigma \lesssim \|\{\gamma_g\}\|^2_{\ell^2(\mathbb{Z}_+)}. \tag{12}$$

Conjecture (qNCC-II). *There is* $\epsilon > 0$ *such that* $\|\{\gamma_g\}\|_{\ell^2(\mathbb{Z}_+)} \le \epsilon$ *implies*

$$\int_{\mathbb{T}} \log\big(1 + O^2\big(\xi,\{\gamma_g\}\big)\big)\,\mathrm{d}m \lesssim \|\{\gamma_g\}\|^2_{\ell^2(\mathbb{Z}_+)}. \tag{13}$$

We will see later that the assumption on ℓ^2-norm of $\{\gamma_g\}$ being small is not restrictive when studying the problem of pointwise asymptotics. Our first result compares these two conjectures.

Theorem 1. *We have qNCC-I* $\Rightarrow$ *qNCC-II.*

Proof. Assume qNCC-I holds. Due to (9), we have $\log(1+O^2) \lesssim \log(1+o_1^2+o_2^2)$ and it is sufficient to show that

$$\int_{\mathbb{T}} \log(1+o_1^2+o_2^2)\,\mathrm{d}m \lesssim \|\{\gamma_g\}\|^2_{\ell^2(\mathbb{Z}_+)}. \tag{14}$$

Now, (11) implies $\log\big(1+o_j^2(\xi)\big) \lesssim \log\big(1+M^2(\xi,\sigma)\big) + \log\big(1+M^2(\xi,\sigma_d)\big) + \|\{\gamma_g\}\|^2_{\ell^2(\mathbb{Z}_+)}$, $j \in \{1,2\}$. We can write

$$\begin{aligned} \int_{w>1} \log\big(1+M^2(\xi,\sigma)\big)\,\mathrm{d}m &\le \int_{w>1} \log\big(1+M^2(\xi,\sigma)\,w\big)\,\mathrm{d}m \\ &\le \int \log\big(1+M^2(\xi,\sigma)\,w\big)\,\mathrm{d}m \\ &\le \int M^2(\xi,\sigma)\,w\,\mathrm{d}m \\ &\overset{\text{(qNCC-I)}}{\lesssim} \|\{\gamma_g\}\|^2_{\ell^2(\mathbb{Z}_+)}. \end{aligned} \tag{15}$$

Then,

$$\begin{aligned}\int_{w\le 1}\log\big(1+M^2(\xi,\sigma)\big)\,\mathrm{d}m &= -\int_{w\le 1}\log w\,\mathrm{d}m+\int_{w\le 1}\log w\,\mathrm{d}m+\int_{w\le 1}\log\big(1+M^2(\xi,\sigma)\big)\,\mathrm{d}m\\ &= -\int_{w\le 1}\log w\,\mathrm{d}m+\int_{w\le 1}\log\big(w+M^2(\xi,\sigma)\,w\big)\,\mathrm{d}m\\ &\le -\int_{w\le 1}\log w\,\mathrm{d}m+\int_{w\le 1}\log\big(1+M^2(\xi,\sigma)\,w\big)\,\mathrm{d}m\\ &\overset{(2)+(15)}{\lesssim} \big\|\{\gamma_g\}\big\|^2_{\ell^2(\mathbb{Z}_+)}.\end{aligned}$$

By qNCC-I applied to σ_d, we get $\int_{\mathbb{T}} M^2(\xi,\sigma_d)\,w_d\,\mathrm{d}m \lesssim \big\|\{\gamma_g\}\big\|^2_{\ell^2(\mathbb{Z}_+)}$ and we similarly have

$$\int \log\big(1+M^2(\xi,\sigma_d)\big)\,\mathrm{d}m \lesssim \big\|\{\gamma_g\}\big\|^2_{\ell^2(\mathbb{Z}_+)}$$

for $\big\|\{\gamma_g\}\big\|_{\ell^2(\mathbb{Z}_+)}$ small enough. Hence, (14) follows. □

Next, we discuss the motivations for our conjectures. The Menshov–Rademacher theorem [10,13] states:

Theorem 2 (Menshov–Rademacher). *Suppose* $\{\chi_n(\xi)\}$, $n\ge 1$ *is an orthonormal system in* $L^2_\sigma(\mathbb{T})$, *then*

$$\int_{\mathbb{T}}\sup_n\Big|\sum_{j=1}^n \alpha_j\chi_j(\xi)\Big|^2\,\mathrm{d}\sigma \lesssim \sum_{j\ge 1}\big|\alpha_j\log(1+j)\big|^2$$

for every sequence $\{\alpha_j\}$.

The connection between qNCC-I and the previous result is almost immediate (see [18, Section 8]) if we take the recursion $\Phi^*_{n+1}=\Phi^*_n-\gamma_n z\Phi_n$ and $\Phi^*_0=1$ into account (see (1)). Indeed,

$$\Phi^*_n(z,\sigma)=1-z\sum_{j=0}^{n-1}\gamma_j\|\Phi_j\|_{2,\sigma}\phi_j(z,\sigma)$$

so

$$M(\xi,\sigma)\le\Big|\sup_{n\ge1}\sum_{j=0}^{n-1}\gamma_j\|\Phi_j\|_{2,\sigma}\phi_j(\xi,\sigma)\Big|$$

and $\|M\|_{2,\sigma}\lesssim\big(\sum_{j\ge0}|\gamma_j|^2\log^2(2+j)\big)^{\frac12}$. The qNCC-I suggests that for OPUC the logarithm in the last sum can be dropped provided that $\big\|\{\gamma_g\}\big\|_{\ell^2(\mathbb{Z}_+)}$ is small.

We continue with the motivation for qNCC-II. Take $d\ge 2$ and consider $\gamma(t)\colon[0,1]\mapsto \mathrm{GL}(d,\mathbb{C})$, a smooth curve. Let $\gamma_r(t):=\int_0^t\gamma'\gamma^{-1}\,\mathrm{d}\tau\colon[0,1]\mapsto M(d,\mathbb{C})$ be its right trace. One can define the distance between $A,B\in\mathrm{GL}(d,\mathbb{C})$ by

$$d(A,B):=\inf_{\gamma(0)=A,\ \gamma(1)=B}\int_0^1\|\gamma'\gamma^{-1}\|\,\mathrm{d}t.$$

Given $q\in[1,\infty]$, the variational norm of a continuous curve $\Gamma\colon[0,1]\mapsto M(d,\mathbb{C})$ is defined as

$$\|\Gamma\|_{V^q}:=\begin{cases}\sup_{n\in\mathbb{N}}\sup_{0=t_0<t_1<\cdots<t_n=1}\Big(\sum_{j=0}^{n-1}\big\|\Gamma(t_{j+1})-\Gamma(t_j)\big\|^q\Big)^{1/q}, & q<\infty,\\ \mathrm{diam}\,(\Gamma), & q=\infty.\end{cases}$$

In [15, Lemma C.3], the authors proved, in particular, that

$$\|\gamma\|_{V^q}\le\|\gamma_r\|_{V^q}+C_q\min\big(\|\gamma_r\|^q_{V^q},\|\gamma_r\|^2_{V^q}\big),\qquad \|\gamma_r\|_{V^q}\le\|\gamma\|_{V^q}+C_q\min\big(\|\gamma\|^q_{V^q},\|\gamma\|^2_{V^q}\big),\tag{16}$$

for every $q\in(1,2)$ which gives

$$\|\gamma\|_{V^q}\le\|\gamma_r\|_{V^q}+C_q\|\gamma_r\|^2_{V^q},\qquad \|\gamma_r\|_{V^q}\le\|\gamma\|_{V^q}+C_q\|\gamma\|^2_{V^q},\tag{17}$$

for all $q\in(1,2)$.

In [15], the authors use (16) along with a variational version of Menshov–Paley–Zygmund theorem to improve earlier results of Christ and Kiselev (see [3,4] and [5, formula (1.3)]). Now, consider the sequence $\{\Upsilon_n\}$, $\Upsilon_n \in \mathrm{GL}(d,\mathbb{C})$, $n \in \mathbb{Z}_+$, where $\sup_n \|\Upsilon_n - I\| \le \frac{1}{2}$. One can define the piecewise smooth curve $\gamma(t)$, $t \ge 0$ as the solution to $\gamma' = V\gamma$, $\gamma(0) = I$, where

$$V(t) := \log \Upsilon_j, \quad t \in [j, j+1),\ j \in \mathbb{Z}_+.$$

We notice that $V(t) = \Upsilon_j - I + \Delta_j$, $t \in [j, j+1)$, where $\|\Delta_j\| \sim \|\Upsilon_j - I\|^2$. Then, $\Upsilon_n \cdot \ldots \cdot \Upsilon_0 = \gamma(n+1)$ and (16) can be applied to the product of matrices (the assumption made in [15, Lemma C.3] that the curve γ is smooth can be relaxed to piecewise smooth). Recall that $\mathrm{dist}(I, A) \sim \log\big(1 + \|I - A\|\big)$ for $A \in \mathrm{SU}(1,1)$. Taking $\Upsilon_n = \Omega_n\big(\xi, \{\gamma_g\}\big)$ as in (4), applying (17) to γ, and the variational Menshov–Paley–Zygmund theorem for Fourier series (see [15, Section B] for Fourier integral version) gives us, in particular (compare with [15, p. 461]),

$$\big\|\log^{\frac{1}{2}}(1 + O^2)\big\|_{L^{p'}(\mathbb{T})} \lesssim_p \big\|\{\gamma_g\}\big\|_{\ell^p(\mathbb{Z}_+)}$$

for $p \in (1,2)$. Setting $p = 2$ in this estimate gives the conjectured bound (13).

3. Some applications of qNCC

We have the following result.

Theorem 3. *Suppose* $\{\gamma_g\} \in \ell^2(\mathbb{Z}_+)$ *and qNCC-II holds, then* $\lim_{n\to\infty} \Pi_n(\xi)$ *exists for a.e.* $\xi \in \mathbb{T}$ *and* $\lim_{n\to\infty} \phi_n^*(\xi, \sigma) = D_\sigma^{-1}(\xi)$ *for a.e.* $\xi \in \mathbb{T}$.

Proof. Assume qNCC-II holds. We will show that the sequence $\big\{\Pi_n\big(\xi, \{\gamma_g\}\big)\big\}$ is Cauchy for a.e. $\xi \in \mathbb{T}$. We have $\|\Pi_{n+p} - \Pi_n\| = \big\|(\Omega_{n+p} \cdots \Omega_{n+1} - I)\Pi_n\big\| \le O \cdot \big\|(\Omega_{n+p} \cdots \Omega_{n+1} - I)\big\|$. Since $O(\xi) < \infty$ for a.e. $\xi \in \mathbb{T}$, we only need to show that

$$\limsup_{N\to\infty} O_N(\xi) = 0, \quad O_N := \sup_{n \ge N,\ p \ge 1} \big\|\Omega_{n+p} \cdots \Omega_{n+1} - I\big\|$$

for a.e. $\xi \in \mathbb{T}$. Since

$$\int \log(1 + O_N^2)\, \mathrm{d}m \overset{\text{(qNCC-II)}}{\lesssim} \big\|\{\gamma_g\}\big\|^2_{\ell^2(\ge N)},$$

we can apply Markov's inequality: for every $\epsilon > 0$ and $N \in \mathbb{N}$, we have

$$\big|\big\{\xi \in \mathbb{T} : O_N(\xi) > \epsilon\big\}\big| \lesssim \frac{\big\|\{\gamma_g\}\big\|^2_{\ell^2(\ge N)}}{\log(1 + \epsilon^2)}.$$

Since $\{O_N\} \searrow \limsup_{N\to\infty} O_N$ as $N \to \infty$, we have $\big|\big\{\xi \in \mathbb{T} : \limsup_{N\to\infty} O_N(\xi) \ge \epsilon\big\}\big| = 0$ for every $\epsilon > 0$. Therefore, $\limsup_{N\to\infty} O_N(\xi) = 0$ for a.e. $\xi \in \mathbb{T}$ and we have convergence of $\lim_{n\to\infty} \Pi_n$ by applying the Cauchy criterion. The formula (6) provides convergence of $\big\{\phi_n^*(\sigma, \xi)\big\}$ for a.e. $\xi \in \mathbb{T}$. Since (see [11, Corollary 5.11])

$$\lim_{n\to\infty} \int \left| \frac{1}{\phi_n^*} - D_\sigma \right|^2 \mathrm{d}m = 0,$$

we get $\lim_{n\to\infty} \phi_n^*(\xi, \sigma) = D_\sigma^{-1}(\xi)$ a.e. □

It is known that all zeroes of $\phi_n(z, \sigma)$ are inside $\mathbb{D}$. The following theorem shows, in particular, that for the Szegő measures, the pointwise convergence of $|\phi_n|$ on $\mathbb{T}$ is equivalent to the condition that its zeroes stay away from $\mathbb{T}$. To state this result, we need some notation first. Given a parameter $\rho \in (0,1)$ and a point $\xi \in \mathbb{T}$, define the Stolz angle $S_\rho^*(\xi)$ to be the convex hull of $\rho\mathbb{D}$ and ξ. Let $\{f_n\}$ denote the Schur iterates (see [11]) for the measure σ.

Theorem 4 (Bessonov–Denisov, [2]). *Let $\sigma \in \mathrm{Sz}(\mathbb{T})$ and $Z(\phi_n) = \{z \in \mathbb{D} : \phi_n(z,\sigma) = 0\}$. Take any $a > 0$ and denote $r_{a,n} = 1 - a/n$. Then, for almost every $\xi \in \mathbb{T}$, the following assertions are equivalent:*

(a) $\lim_{n\to\infty} |\phi_n^*(\xi)|^2 = |D_\sigma^{-1}(\xi)|^2$;
(b) $\lim_{n\to\infty} \mathrm{dist}(Z(\phi_n), \xi)\, n = +\infty$;
(c) $\lim_{n\to\infty} f_n(r_{a,n}\xi) = 0$;
(d) $\lim_{n\to\infty} \sup_{z\in S_\rho^*(\xi)} |f_n(z)| = 0$ *for every* $\rho \in (0,1)$.

Our previous results imply the following corollary.

Corollary 5. *Suppose that either $\{\gamma_g\} \in \ell^p(\mathbb{Z}_+)$ for some $p \in [1,2)$ or $\{\gamma_g \log(g+2)\} \in \ell^2(\mathbb{Z}_+)$, then* (a)–(d) *from the previous theorem hold for a.e. $\xi \in \mathbb{T}$.*

4. Connections to Carleson–Hunt maximal function and SU(1,1) version of a theorem of Calderon and Stein

In conclusion, we put forward the weakened versions of qNCC-I and qNCC-II:

$$\int \sup_{n\ge 0} |\Phi_n(\xi,\sigma)|^2 \, d\sigma \le 1 + \nu_1(\|\{\gamma_g\}\|_{\ell^2(\mathbb{Z}_+)}) \qquad \text{(wqNCC-I)}$$

and

$$\int \log \sup_{n\ge 0} \|\Pi_n(\xi,\{\gamma_g\})\| \, dm \le \nu_2(\|\{\gamma_g\}\|_{\ell^2(\mathbb{Z}_+)}), \qquad \text{(wqNCC-II)}$$

assuming that $\|\{\gamma_g\}\|_{\ell^2(\mathbb{Z}_+)} < \epsilon$ with some $\epsilon > 0$ and ν_1, ν_2 are certain functions that satisfy $\lim_{t\downarrow 0} \nu_{1(2)}(t) = 0$. It is not hard to see that qNCC-I implies wqNCC-I, qNCC-II implies wqNCC-II, and wqNCC-I implies wqNCC-II. The converse statements are not known. It is also not known if these weak versions imply the NCC. The inequality in (9) shows that $O^2 \ge 2o_2^2$ and, therefore, $\int \log(1+2o_2^2)\, dm \le \int \log(1+O^2)\, dm$. The next result shows that the weakened version of qNCC implies the known Carleson–Hunt bound (10) for $p = 2$.

Theorem 6. *If there is $\epsilon > 0$ so that*

$$\int \log(1+o_2^2)\, dm \lesssim \|\{\gamma_g\}\|^2_{\ell^2(\mathbb{Z}_+)} \qquad (18)$$

for every sequence $\{\gamma_g\}$ such that $\|\{\gamma_g\}\|_{\ell^2(\mathbb{Z}_+)} \le \epsilon$, then

$$\int \sup_n \left| \sum_{j\le n} \gamma_j e^{ij\theta} \right|^2 dm \lesssim \|\{\gamma_g\}\|^2_{\ell^2(\mathbb{Z}_+)} \qquad (19)$$

for every $\{\gamma_g\} \in \ell^2(\mathbb{Z}_+)$.

Proof. Fix any $\{\gamma_g\} \in \ell^2(\mathbb{Z}_+)$. The map $b_n(\xi, \{\lambda\gamma_j\}) : (\xi, \lambda, \{\gamma_j\}_{j\le n}) \in \mathbb{T}\times\mathbb{D}\times\mathbb{D}^{n+1} \mapsto \mathbb{C}$ satisfies [20] the bound

$$\left| b_n(\xi, \{\lambda\gamma_j\}) - \lambda \sum_{j\le n} \gamma_j \xi^j \right| \le C(\{\gamma_g\})|\lambda|^2$$

as $\lambda \to 0$. Given any $N \in \mathbb{N}$ and a measurable map $N(\xi) : \xi \in \mathbb{T} \mapsto \{0,\dots,N\}$, we get

$$\int \log\left(1 + |b_{N(\xi)}(\xi, \{\lambda\gamma_j\})|^2\right) dm \lesssim |\lambda|^2 \|\{\gamma_g\}\|^2_{\ell^2(\mathbb{Z}_+)}$$

from (18). As $\lambda \to 0$, we get

$$|\lambda|^2 \int \left| \sum_{j\le N(\xi)} \gamma_j \xi^j \right|^2 dm + O(|\lambda|^3) \lesssim |\lambda|^2 \|\{\gamma_g\}\|^2_{\ell^2(\mathbb{Z}_+)}.$$

Hence,

$$\int\Big|\sum_{j\le N(\xi)}\gamma_j\xi^j\Big|^2\,\mathrm{d}m\lesssim\|\{\gamma_g\}\|^2_{\ell^2(\mathbb{Z}_+)}.$$

Since N and $N(\xi)$ are arbitrary, we obtain (19). □

The following result is classical (see [8, Section 2.1]) and has many generalizations [17]. It predates Carleson's proof of Lusin's conjecture and is attributed to Calderon and to Stein.

Theorem 7. *The following statements are equivalent:*

(A) *For every $f\in L^2(0,1)$, the sequence $\{\sum_{j=-n}^n\widehat{f}_je^{2\pi ijx}\}$ of partial Fourier sums converges for a.e. $x\in[0,1)$.*

(B) *We have the weak $(2,2)$ type estimate for the maximal function M_0, i.e.,*

$$\Big|\Big\{x\in[0,1):\big|(M_0f)(x)\big|\ge\lambda\Big\}\Big|\lesssim\frac{\|f\|_2^2}{\lambda^2}$$

for all $\lambda>0$ and all $f\in L^2(0,1)$.

Below, we will adjust its proof to obtain an SU(1,1) nonlinear version. We list some properties of the transform $\{\gamma_g\}\mapsto\{\Pi_n(\xi,\{\gamma_g\})\}$ we need. They are immediate from the definition (5). Firstly, for every $\eta\in\mathbb{T}$, we have

$$\Pi_n(\xi,\{\eta\gamma_g\})=\begin{bmatrix}\bar a_n & \overline{\eta b_n}\\ \eta b_n & a_n\end{bmatrix},\qquad \|\Pi_n(\xi,\{\eta\gamma_g\})-I\|_2=\|\Pi_n(\xi,\{\gamma_g\})-I\|_2. \tag{20}$$

Secondly, suppose $\xi_0=e^{i\alpha_0}$. Then,

$$\text{rotation by }\alpha_0\text{:}\quad \Pi_n(\xi,\{\gamma_g\xi_0^{-g}\})=\Pi_n(\xi/\xi_0,\{\gamma_g\}). \tag{21}$$

For each $N\in\mathbb{N}$, define the $\{\gamma_g^{[N]}\}$ by

$$\gamma_g^{[N]}:=\begin{cases}0, & g<N,\\ \gamma_{g-N}, & g\ge N.\end{cases}$$

The last property we need is

$$\text{the right shift by }N\text{ coordinates:}\quad \Pi_{n+N}(\xi,\{\gamma_g^{[N]}\})=\begin{bmatrix}\bar a_n & \bar b_n\xi^{-N}\\ b_n\xi^N & a_n\end{bmatrix}.$$

Notice that

$$\left\|\begin{bmatrix}\bar a_n & \bar b_n\xi^{-N}\\ b_n\xi^N & a_n\end{bmatrix}-I\right\|_2=\left\|\begin{bmatrix}\bar a_n & \bar b_n\\ b_n & a_n\end{bmatrix}-I\right\|_2 \tag{22}$$

and, therefore, for each $\eta,\xi_0\in\mathbb{T}$, we get

$$\|\Pi_{n+N}(\xi,\{\eta\gamma_g^{[N]}\xi_0^{-g}\})-I\|_2\overset{(20)+(21)}{=}\|\Pi_n(\xi/\xi_0,\{\gamma_g\})-I\|_2. \tag{23}$$

Consider the following function (recall (8) and our second conjecture in (13))

$$G(\alpha,\beta):=\sup_{\{\gamma_g\}\text{ s.t. }\|\{\gamma_g\}\|_{\ell^2(\mathbb{Z}_+)}\le\beta}\Big|\Big\{\xi\in\mathbb{T}:\log\big(1+O^2(\xi,\{\gamma_g\})\big)\ge\alpha\Big\}\Big|.$$

Clearly, $G\le|\mathbb{T}|=2\pi$, it is increasing in β and decreasing in α.

Theorem 8. *Assume that for every $\{\gamma_g\}$ such that $\|\{\gamma_g\}\|_{\ell^2(\mathbb{Z}_+)}\le\frac12$, the sequence $\{\Pi_n(\xi,\{\gamma_g\})\}$ converges for $\xi\in W(\{\gamma_g\})\subset\mathbb{T}$, where the set $W(\{\gamma_g\})$ can depend on $\{\gamma_g\}$ and has positive Lebesgue measure. Then,*

$$G(\alpha,\beta)\le_\alpha\beta^2 \tag{24}$$

for every $\beta\le\frac12$. Conversely, if (24) holds for every $\alpha>0$ and $\beta\le\frac12$, then the sequence $\{\Pi_n(\xi,\{\gamma_g\})\}$ converges for a.e. $\xi\in\mathbb{T}$.

Proof. Recall that the matrices Π_n in (5) satisfy $\|\Pi_n\| = \|\Pi_n^{-1}\|$ since $\Pi_n \in \mathrm{SU}(1,1)$. Therefore, we have

$$\lim_{N\to\infty} \sup_{n\ge N,\ p\in\mathbb{N}} \left\|\Omega_{n+p}\big(\xi,\{\gamma_g\}\big)\cdots\Omega_{n+1}\big(\xi,\{\gamma_g\}\big) - I\right\| = 0 \tag{25}$$

for each $\xi \in W\big(\{\gamma_g\}\big)$. We continue the proof by assuming that (24) fails for some α. Then, choosing the subsequence if necessary, we can find a sequence $\{\beta_j\}\downarrow 0$, $\beta_j \le 2^{-j}$, such that

$$G(\alpha,\beta_j) \ge \beta_j^2 j^2. \tag{26}$$

Hence, for each j, there is $\{\gamma_g^{(j)}\}$ with finite support, i.e., $\gamma_g^{(j)} = 0$ for all $g \ge N_j$, such that $\|\gamma_g^{(j)}\|_{\ell^2(\mathbb{Z}_+)} \le \beta_j$ and

$$|E_j| \ge \beta_j^2 j^2/2, \tag{27}$$

where

$$E_j := \left\{\xi\in\mathbb{T} : \log\big(1+O^2\big(\xi,\{\gamma_g^{(j)}\}\big)\big) \ge \alpha/2\right\}. \tag{28}$$

Notice that, since $\gamma_g^{(j)} = 0$ for $g \ge N_j$, we get $\sup_{n\ge 0}\log\big(1+\big\|\Pi_n\big(\xi,\{\gamma_g^{(j)}\}\big)-I\big\|_2^2\big) = \sup_{n<N_j}\log\big(1+\big\|\Pi_n\big(\xi,\{\gamma_g^{(j)}\}\big)-I\big\|_2^2\big)$. We claim that there is a sequence $\{d_j\}$ of natural numbers such that

$$\sum_{j\ge 1} d_j\beta_j^2 < \infty, \qquad \sum_{j\ge 1} d_j\beta_j^2 j^2 = \infty. \tag{29}$$

Indeed, it suffices to chose d_j such that $d_j \sim \beta_j^{-2} j^{-2}$ and recall that $\beta_j \le 2^{-j}$. We consider the sequence of sets $\{E_p^*\}$ in which each set E_j is repeated d_j times, i.e.,

$$\{E_p^*\} := \{\underbrace{E_1,\dots,E_1}_{d_1},\underbrace{E_2,\dots,E_2}_{d_2},\dots\},$$

so that $\sum_{p\ge 1}|E_p^*| \overset{(27)+(29)}{=} \infty$ and, therefore (see [8, Lemma 2.1.2]), there is a sequence $\{\xi_p^*\}$, $\xi_p^* = e^{i\alpha_p^*} \in \mathbb{T}$ such that the shifted sets $\widehat{E}_p := \big\{e^{i(\alpha_p^*+\beta)} : e^{i\beta}\in E_p^*\big\}$ satisfy $\big|\limsup_p \widehat{E}_p\big| = |\mathbb{T}| = 2\pi$, where $\limsup_p \widehat{E}_p := \bigcap_{j\in\mathbb{N}}\bigcup_{s\ge j}\widehat{E}_s$.

Our next goal is to use $\{\gamma_g^{(m)}\}$, $m\in\mathbb{N}$, to construct $\{\gamma_g^*\}\in\ell^2(\mathbb{Z}_+)$ for which the assumption of the theorem is violated. Let $d_0 := 0$, $N_0 := 0$, $P_j := N_0 d_0 + \cdots + N_j d_j$, $j \ge 0$. Then, for each $\widehat{E}_p$ with $p = d_0 + \cdots + d_j + s$, $1 \le s \le d_{j+1}$, we let

$$\gamma^*_{\ell+(s-1)N_{j+1}+P_j} := \gamma_\ell^{(j+1)}\cdot\xi_p^{-\ell}, \quad 0 \le \ell \le N_{j+1}-1.$$

For such a choice, we can use (23) and (28) to obtain

$$\sup_{0\le\ell\le N_{j+1}-1}\left\|\Omega_{\ell+(s-1)N_{j+1}+P_j}\big(\xi,\{\gamma_g^*\}\big)\cdots\Omega_{(s-1)N_{j+1}+P_j}\big(\xi,\{\gamma_g^*\}\big)-I\right\|_2 \ge (e^{\alpha/2}-1)^{\frac{1}{2}}$$

for $\xi\in\widehat{E}_p$ and every p. Since $\big\|\{\gamma_g^*\}\big\|_{\ell^2(\mathbb{Z}_+)} \overset{(29)}{<} \infty$, we have a contradiction with (25) for $\xi \in W\big(\{\gamma_g^*\}\big)\cap\limsup_p\widehat{E}_p \neq \emptyset$. The first claim of the theorem is proved. The proof of the second one is standard and repeats the argument from the proof of Theorem 3. □

Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

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