Comptes Rendus

Mathématique

Miguel A. Cardona, Diego A. Mejía and Ismael E. Rivera-Madrid

Directed schemes of ideals and cardinal characteristics. I: The meager additive ideal

Research article
Mathematical logic

Directed schemes of ideals and cardinal characteristics. I: The meager additive ideal

Miguel A. Cardona [ID],[a,b], **Diego A. Mejía** [ID],[c] **and Ismael E. Rivera-Madrid** [ID],[a]

[a] Faculty of Engineering, Institución Universitaria Pascual Bravo. Calle 73 No. 73A –
226, Medellín, Colombia

[b] Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew
University of Jerusalem, Jerusalem, 91904, Israel

[c] Graduate School of System Informatics, Kobe University. 1-1 Rokkodai-cho,
Nada-ku, Kobe, Hyogo 657-8501 Japan

E-mails: miguel.cardona@pascualbravo.edu.co, damejiag@people.kobe-u.ac.jp,
ismael.rivera@pascualbravo.edu.co

Abstract. We introduce the notion of *directed scheme of ideals* to characterize peculiar ideals on the reals, which comes from a formalization of the framework of Yorioka ideals for strong measure zero sets. We prove general theorems for directed schemes and propose a directed scheme $\vec{\mathcal{M}} = \{\mathcal{M}_I : I \in \mathbb{I}\}$ for the ideal $\mathcal{M}\mathcal{A}$ of meager-additive sets of reals. This directed scheme does not only helps us to understand more the combinatorics of $\mathcal{M}\mathcal{A}$ and its cardinal characteristics, but provides us new characterizations of the additivity and cofinality numbers of the meager ideal of the reals.

In addition, we display connections between the characteristics associated with $\mathcal{M}_I$ and other classical characteristics. Furthermore, we demonstrate the consistency of $\mathrm{cov}(\mathcal{N}\mathcal{A}) < \mathfrak{c}$ and $\mathrm{cof}(\mathcal{M}\mathcal{A}) < \mathrm{non}(\mathcal{S}\mathcal{N})$. The first one answers a question raised by the authors in [14].

1. Introduction

Recent study of $\mathcal{S}\mathcal{N}$, the ideal of strong measure zero subsets of $^{\omega}2$, and its cardinal characteristics, involves its characterization via the *Yorioka ideals*. These have been used to control de cofinality of $\mathcal{S}\mathcal{N}$ in forcing models [6,10,12,13,22], as well as its additivity number [5]. These ideals have helped to overcome challenges to control the combinatorics of $\mathcal{S}\mathcal{N}$ and its cardinal characteristics, particularly in forcing models.

We aim to generalize the framework of Yorioka ideals to control de combinatorics of other very peculiar ideals. In this paper, we give special attention to the ideal of meager-additive subsets of $^{\omega}2$.

Notation 1. Before proceeding, we first fix some notation.

(1) Given a formula ϕ, $\forall^{\infty} n < \omega : \phi$ means that all but finitely many natural numbers satisfy ϕ; $\exists^{\infty} n < \omega : \phi$ means that infinitely many natural numbers satisfy ϕ.

(2) $^{\uparrow\omega}\omega$ denotes the set of all increasing functions in $^{\omega}\omega$.

(3) For $x, y \in {}^{\omega}\omega$, $x \leq^* y$ means that $\forall^{\infty} n < \omega : x(i) \leq y(i)$.

(4) The identity function on ω is denoted by id.

(5) $\mathfrak{c} := 2^{\aleph_0}$; $\mathscr{N}$ and $\mathscr{M}$ denote the ideals of Lebesgue measure zero sets and of meager sets in the Cantor space $^{\omega}2$, respectively. Let $\mathscr{E}$ be the σ-ideal generated by the closed measure zero subsets of $^{\omega}2$. It is well-known that $\mathscr{E} \subseteq \mathscr{N} \cap \mathscr{M}$. Even more, it was proved that $\mathscr{E}$ is a proper subideal of $\mathscr{N} \cap \mathscr{M}$ (see [1, Lemma 2.6.1]).

(6) A *preorder* is a pair $\langle P, \leq \rangle$ where P is a set and $\leq$ is a reflexive and transitive relation on P.

(7) A preorder $\langle S, \leq \rangle$ is *directed* if, for $s, s' \in S$, there is some $t \in S$ such that $s \leq t$ and $s' \leq t$.

(8) $\mathbb{I}$ denotes the set of partitions $I = \langle I_n : n < \omega \rangle$ of ω into consecutive non-empty intervals. We consider the directer preorder on $\mathbb{I}$ defined by $I \sqsubseteq J$ iff $\forall^{\infty} n < \omega \ \exists k < \omega : I_k \subseteq J_n$.

Given an ideal $\mathscr{I}$ of subsets of X such that $\{x\} \in \mathscr{I}$ for all $x \in X$, *the cardinal characteristics associated with $\mathscr{I}$* are defined by:

$$\mathrm{add}(\mathscr{I}) := \min\{|\mathscr{J}| \ : \ \mathscr{J} \subseteq \mathscr{I} \text{ and } \bigcup \mathscr{J} \notin \mathscr{I}\},$$
$$\mathrm{cov}(\mathscr{I}) := \min\{|\mathscr{J}| \ : \ \mathscr{J} \subseteq \mathscr{I} \text{ and } \bigcup \mathscr{J} = X\},$$
$$\mathrm{non}(\mathscr{I}) := \min\{|A| \ : \ A \subseteq X \text{ and } A \notin \mathscr{I}\},$$
$$\mathrm{cof}(\mathscr{I}) := \min\{|\mathscr{J}| \ : \ \mathscr{J} \subseteq \mathscr{I} \text{ is cofinal in } \langle \mathscr{I}, \subseteq \rangle\}.$$

These cardinals are called *additivity, covering, uniformity, and cofinality of $\mathscr{I}$*, respectively. The relationship between the cardinals defined above is illustrated in Figure 1.
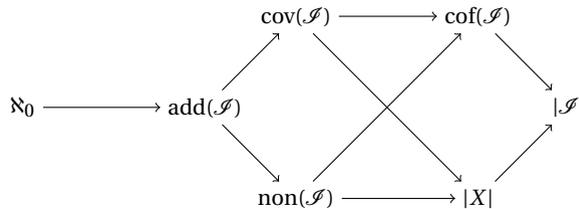
Figure 1. Diagram of the cardinal characteristics associated with $\mathscr{I}$. An arrow $\mathfrak{x} \to \mathfrak{y}$ means that (provably in ZFC) $\mathfrak{x} \leq \mathfrak{y}$.

Yorioka [22] defined ideals $\mathscr{I}_f$ on $^{\omega}2$, parametrized by $f \in {}^{\uparrow\omega}\omega$, which have Borel bases and characterize $\mathscr{SN} := \bigcap_{f \in {}^{\uparrow\omega}\omega} \mathscr{I}_f$. A further property of the Yorioka ideals is that $\mathscr{I}_g \subseteq \mathscr{I}_f$ whenever $f \leq^* g$.

These ideals led Yorioka to rediscover and significantly improve Seredyński [18] result about the cofinality of $\mathscr{SN}$, and to show that no inequality between $\mathrm{cof}(\mathscr{SN})$ and $\mathfrak{c}$ can be decided in ZFC. More research about the cofinality has been conducted by the first and second authors [6,11]. One representative result is the following, whose notation is clarified afterward.

Theorem 2 ([11, Theorem 4.6]). $\mathrm{cof}(\mathscr{SN}) \leq \mathrm{cov}\big(\big([\sup \mathrm{cof}(\vec{\mathscr{F}})]^{<\min \mathrm{add}(\vec{\mathscr{F}})}\big)^{\mathfrak{d}}\big).$

We discovered that the result above is not associated with the meaning of *strong measure zero* but of the framework given by the Yorioka ideals. We generalize this framework below, with the central notion of this paper.

Definition 3. *Let X be a set and let S be a directed preorder. We say that a sequence $\vec{\mathscr{I}} = \langle \mathscr{I}_s : s \in S \rangle$ of ideals on X is a* directed scheme of ideals on X *if $\mathscr{I}_{s'} \subseteq \mathscr{I}_s$ whether $s \leq s'$. Put $\vec{\mathscr{I}}^* = \bigcap_{s \in S} \mathscr{I}_s$, which we simply abbreviate $\mathscr{I}^*$ when clear from the context.*

These directed schemes of ideals give rise to the study of the cardinals associated with $\mathscr{I}_s$. Moreover, we associate the following cardinal characteristics to $\vec{\mathscr{I}}$:

$$\min \mathrm{add}(\vec{\mathscr{I}}) := \min_{s \in S} \mathrm{add}(\mathscr{I}_s), \qquad \sup \mathrm{add}(\vec{\mathscr{I}}) := \sup_{s \in S} \mathrm{add}(\mathscr{I}_s),$$

$$\min \mathrm{cov}(\vec{\mathscr{I}}) := \min_{s \in S} \mathrm{cov}(\mathscr{I}_s), \qquad \sup \mathrm{cov}(\vec{\mathscr{I}}) := \sup_{s \in S} \mathrm{cov}(\mathscr{I}_s),$$

$$\min \mathrm{non}(\vec{\mathscr{I}}) := \min_{s \in S} \mathrm{non}(\mathscr{I}_s), \qquad \sup \mathrm{non}(\vec{\mathscr{I}}) := \sup_{s \in S} \mathrm{non}(\mathscr{I}_s),$$

$$\min \mathrm{cof}(\vec{\mathscr{I}}) := \min_{s \in S} \mathrm{cof}(\mathscr{I}_s), \qquad \sup \mathrm{cof}(\vec{\mathscr{I}}) := \sup_{s \in S} \mathrm{cof}(\mathscr{I}_s).$$

It is clear that $\mathrm{cov}(\mathscr{I}_s) \leq \mathrm{cov}(\mathscr{I}_{s'})$ and $\mathrm{non}(\mathscr{I}_{s'}) \leq \mathrm{non}(\mathscr{I}_s)$ whenever $s \leq s'$ in S. Then, whenever S has a minimum element s_0, $\min \mathrm{cov}(\vec{\mathscr{I}}) = \mathrm{cov}(\mathscr{I}_{s_0})$ and $\sup \mathrm{non}(\vec{\mathscr{I}}) = \mathrm{non}(\mathscr{I}_{s_0})$.

For example, the Yorioka ideals form a directed scheme $\vec{\mathscr{I}} := \langle \mathscr{I}_f : f \in {}^{\uparrow \omega} \omega \rangle$ where ${}^{\uparrow \omega} \omega$ is ordered by $\leq^*$, so $\mathscr{SN} = \vec{\mathscr{I}}^*$. Since id is a $\leq^*$ minimum element of ${}^{\uparrow \omega} \omega$, $\min \mathrm{cov}(\vec{\mathscr{I}}) = \mathrm{cov}(\mathscr{I}_{\mathrm{id}})$ and $\sup \mathrm{non}(\vec{\mathscr{I}}) = \mathrm{non}(\mathscr{I}_{\mathrm{id}})$. In this context, we also have $\sup \mathrm{add}(\vec{\mathscr{I}}) = \mathrm{add}(\mathscr{I}_{\mathrm{id}})$ and $\min \mathrm{cof}(\vec{\mathscr{I}}) = \mathrm{cof}(\mathscr{I}_{\mathrm{id}})$, see [10, Theorem 3.15].

Directed schemes of ideals is a crucial notion when it comes to obtaining information on the cardinals of $\mathscr{I}^*$. In fact, the two first authors [6,11] used $\vec{\mathscr{I}}$ when studying the cofinality of $\mathscr{SN}$. Moreover, in [10], they used these ideals to investigate the rest of the cardinals associated with $\mathscr{SN}$. So one of the aims of this paper is to generalize most of the results established of the cardinals of $\mathscr{SN}$ via Yorioka ideals in the context of the directed scheme of ideals. For example, Theorem 2 is a consequence of our following main result, which we prove in Section 2.

Theorem 4. *Let $\vec{\mathscr{I}}$ be a directed scheme of ideals on S. Then:*
  (a) $\min \mathrm{add}(\vec{\mathscr{I}}) \leq \mathrm{add}(\mathscr{I}^*)$;
  (b) $\sup \mathrm{cov}(\vec{\mathscr{I}}) \leq \mathrm{cov}(\mathscr{I}^*) \leq \mathrm{cov}\big(\prod_{s \in D} \mathrm{Cv}_{\mathscr{I}_s}\big)$;
  (c) $\mathrm{non}(\mathscr{I}^*) = \min \mathrm{non}(\vec{\mathscr{I}})$;
  (d) $\mathrm{cof}(\mathscr{I}^*) \leq \mathfrak{d}\big(\prod \vec{\mathscr{I}}\big) \leq \mathrm{cov}\big(([\sup \mathrm{cof}(\vec{\mathscr{I}})]^{<\min \mathrm{add}(\vec{\mathscr{I}})})^{\mathrm{cof}(S)}\big)$.

Throughout time, numerous characterizations of $\mathscr{SN}$ were discovered. One of the very interesting characterizations of $\mathscr{SN}$ is the Galvin–Mycielski–Solovay Theorem stated below, considering ${}^{\omega} 2$ as a topological group with the standard coordinate-wise addition modulo 2.

Theorem 5 ([15]). *A set $X \subseteq {}^{\omega} 2$ is $\mathscr{SN}$ if and only if $X + M \neq {}^{\omega} 2$ for each $M \in \mathscr{M}$.*

This theorem led to the rise of a certain kind of small sets, such as the meager-additive and null-additive sets, which are defined as follows.

Definition 6. *Let $\mathscr{I} \subseteq \mathscr{P}({}^{\omega} 2)$ be an ideal.*
  (1) *$\mathscr{I}$ is* translation invariant *if $A + x \in \mathscr{I}$ for each $A \in \mathscr{I}$ and $x \in {}^{\omega} 2$.*
  (2) *A set $X \subseteq {}^{\omega} 2$ is termed $\mathscr{I}$-additive if, for every $A \in \mathscr{I}$, $A + X \in \mathscr{I}$. Denote by $\mathscr{IA}$ the collection of the $\mathscr{I}$-additive subsets of ${}^{\omega} 2$. Notice that $\mathscr{IA}$ is a (σ-)ideal and $\mathscr{IA} \subseteq \mathscr{I}$ when $\mathscr{I}$ is a translation invariant (σ-)ideal.*
  (3) *The members of $\mathscr{MA}$ are called* meager-additive, *while those in $\mathscr{NA}$ are called* null-additive.

The cardinal characteristics associated with $\mathscr{IA}$ and $\mathscr{I}$ are easily related as follows.

Lemma 7 (e.g. [14, Lemma 1.3]).   *For any translation invariant ideal $\mathscr{I}$ on $^{\omega}2$:*

   (a) $\mathrm{add}(\mathscr{I}) \leq \mathrm{add}(\mathscr{I\!A})$*;*

   (b) $\mathrm{non}(\mathscr{I\!A}) \leq \mathrm{non}(\mathscr{I})$*;*

   (c) $\mathrm{cov}(\mathscr{I}) \leq \mathrm{cov}(\mathscr{I\!A})$*.*

The ideal $\mathscr{I\!A}$ has attracted a lot of attention when $\mathscr{I}$ is either $\mathscr{M}$ or $\mathscr{N}$. For instance, they were investigated by Bartoszyński and Judah [1], Pawlikowski [17], Shelah [19], Zindulka [23], and more recently by the authors [14].

The ideals $\mathscr{M\!A}$ and $\mathscr{N\!A}$ are characterized below.

Theorem 8.   *Let $X \subseteq {}^{\omega}2$.*

   (1) ([19, Theorem 13]) *$X \in \mathscr{N\!A}$ iff for all $I = \langle I_n : n \in \omega \rangle \in \mathbb{I}$ there is some $\varphi \in \prod_{n \in \omega} \mathscr{P}(^{I_n}2)$ such that $\forall\, n \in \omega : \big| \varphi(n) \big| \leq n$ and $X \subseteq H_\varphi$, where*

$$H_\varphi := \big\{ x \in {}^{\omega}2 : \forall^{\infty}\, n \in \omega : x{\restriction}I_n \in \varphi(n) \big\}.$$

   (2) ([1, Theorem 2.2]) *$X \in \mathscr{M\!A}$ iff for all $I \in \mathbb{I}$ there are $J \in \mathbb{I}$ and $y \in {}^{\omega}2$ such that*

$$\forall\, x \in X \;\; \forall^{\infty}\, n < \omega \;\; \exists\, k < \omega : I_k \subseteq J_n \text{ and } x{\restriction}I_k = y{\restriction}I_k. \tag{$\mathbf{\mathscr{O}}$}$$

   Moreover, Shelah [19, Theorem 18] proved that J can be found coarser than I, i.e. every member of J is the union of members of I.

These characterizations led Shelah [19] to prove that every null-additive set is meager-additive, that is $\mathscr{N\!A} \subseteq \mathscr{M\!A}$. Zindulka [24], on the other hand, used combinatorial properties of meager-additive sets described by Shelah and Pawlikowski to characterize meager-additive sets in $^{\omega}2$ in a way that nicely parallels the definition of strong measure zero sets. This led him to establish the following.

Theorem 9. $\mathscr{E\!A} = \mathscr{M\!A}$.

In the present paper, motivated by ($\mathbf{\mathscr{O}}$), we offer a directed scheme of ideals on $^{\omega}2$ for the ideal of meager-additive sets, which we denote $\vec{\mathscr{M}} = \langle \mathscr{M}_I : I \in \mathbb{I} \rangle$ (where $\mathbb{I}$ is ordered by $\sqsubseteq$), i.e.

$$\mathscr{M\!A} = \bigcap \{ \mathscr{M}_I : I \in \mathbb{I} \}.$$

In terms of the ideal $\mathscr{M}_I$, we also provide a reformulation of $\mathscr{M}$. Concretely, we have $\mathscr{M}_{I^1} = \mathscr{M}$ where I^1 denotes the partition of ω into singletons, which is a $\sqsubseteq$-minimum element of $\mathbb{I}$. As a consequence, for any $I \in \mathbb{I}$, $\mathscr{M}_I \subseteq \mathscr{M}$. Details of this new definition are provided in Section 3.

As in the case of the Yorioka ideals, we greatly expect that the directed scheme $\vec{\mathscr{M}}$ is useful to understand the combinatorics of $\mathscr{M\!A}$ and its cardinal characteristics. Initially, we intended to approach the following questions stated in [14].

Questions 10.

   (1) *Is $\mathrm{add}(\mathscr{M\!A}) \leq \mathfrak{b}$?*

   (2) *Is $\mathrm{add}(\mathscr{M\!A}) = \mathrm{non}(\mathscr{M\!A})$?*

Both questions cannot have positive answers simultaneously because $\mathrm{add}(\mathscr{M\!A}) \leq \mathfrak{b}$ implies $\mathrm{add}(\mathscr{M\!A}) = \mathrm{add}(\mathscr{M})$ (considering that $\mathrm{add}(\mathscr{M}) = \min\{\mathfrak{b}, \mathrm{non}(\mathscr{M\!A})\}$, due to Pawlikowski [17]), while $\mathfrak{b} < \mathrm{non}(\mathscr{M\!A})$ is consistent with ZFC [17, Theorem 2.4].

Instead of solving the previous questions, the directed scheme $\vec{\mathscr{M}}$ gives us interesting characterizations of category.

Theorem 11 (Theorem 44).   *For all $I \in \mathbb{I}$, we have*

$$\mathrm{add}(\mathscr{M}_I) = \mathrm{add}(\mathscr{M}) = \min\{\mathfrak{b}, \min \mathrm{non}(\vec{\mathscr{M}})\},$$

$$\mathrm{cof}(\mathscr{M}_I) = \mathrm{cof}(\mathscr{M}) = \max\{\mathfrak{d}, \sup \mathrm{cov}(\vec{\mathscr{M}})\}.$$

The characterization of add($\mathcal{M}$) in this theorem is already known from Pawlikowski's add($\mathcal{M}$) = min$\{\mathfrak{b}, \text{non}(\mathcal{MA})\}$ ([17]), and by Theorem 4, min non($\vec{\mathcal{M}}$) = non($\mathcal{MA}$)).

Consequently, it is clear that min add($\vec{\mathcal{M}}$) = sup add($\vec{\mathcal{M}}$) = add($\mathcal{M}$) and min cof($\vec{\mathcal{M}}$) = sup cof($\vec{\mathcal{M}}$) = cof($\mathcal{M}$).

Thanks to the directed scheme of ideals $\vec{\mathcal{M}}$, we can estimate an upper bound of the cofinality of $\mathcal{MA}$. Combining Theorems 4 and 11 results in the following.

Corollary 12. *For any $I \in \mathbb{I}$,*

$$\text{add}(\mathcal{M}) \leq \text{add}(\mathcal{MA}) \quad and \quad \text{cof}(\mathcal{MA}) \leq \text{cov}\left(\left(\left[\text{cof}(\mathcal{M})\right]^{<\text{add}(\mathcal{M})}\right)^{\mathfrak{d}}\right)$$

Note that the first inequality in the prior theorem is already known (Lemma 7(a)).

Below, we illustrate the connections between the cardinal characteristics associated with the ideals $\mathcal{M}_I$ and some other classical characteristics (Figure 2).

Theorem 13. *For $I \in \mathbb{I}$:*

(1) *if $\sum_{k<\omega} 2^{-|I_k|} < \infty$, then non($\mathcal{M}_I$) ≤ non($\mathcal{E}$) and cov($\mathcal{E}$) ≤ cov($\mathcal{M}_I$);*
(2) *if $\sum_{k<\omega} 2^{-|I_k|} = \infty$, then cov($\mathcal{N}$) ≤ non($\mathcal{M}_I$) and cov($\mathcal{M}_I$) ≤ non($\mathcal{N}$).*

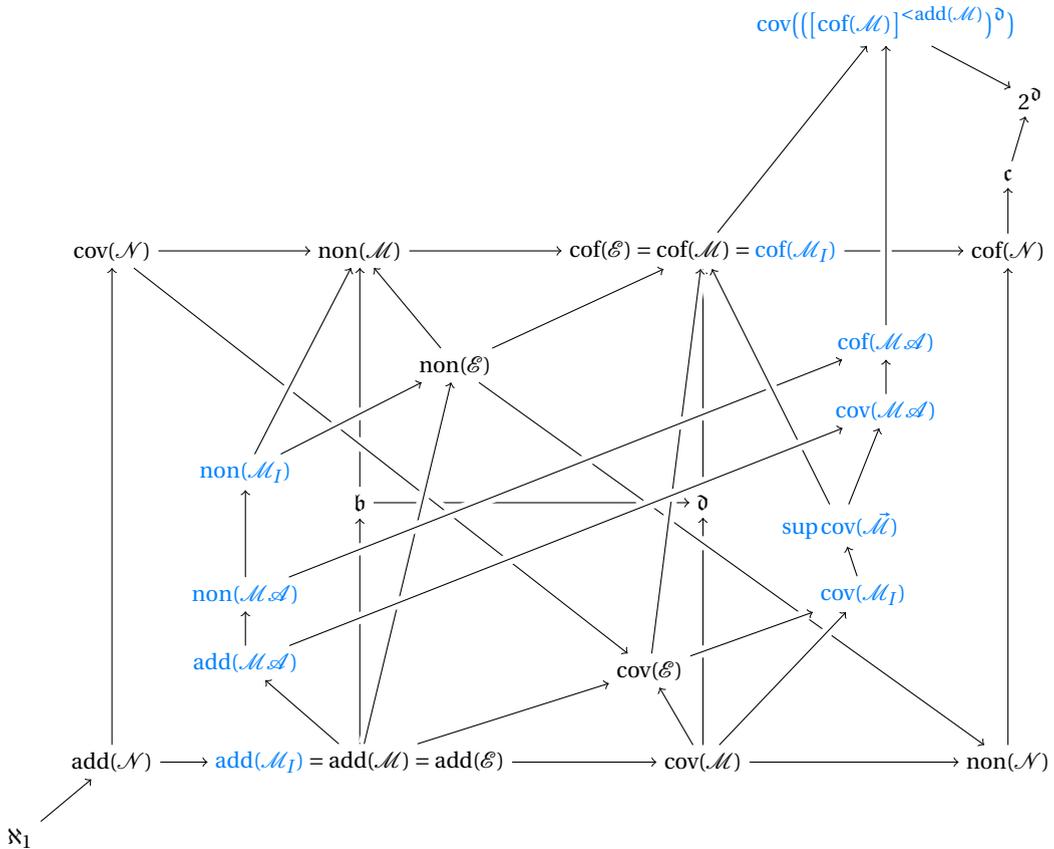**Figure 2.** Cichoń's diagram including the cardinal characteristics associated with our ideals, and add($\mathcal{E}$) = add($\mathcal{M}$) and cof($\mathcal{E}$) = cof($\mathcal{M}$) due to Bartoszyński and Shelah [3]. The inequality non($\mathcal{M}_I$) ≤ non($\mathcal{E}$) and cov($\mathcal{E}$) ≤ cov($\mathcal{M}_I$) holds whenever $\sum_{k<\omega} 2^{-|I_k|} < \infty$.

Theorems 11 and 13 will be proved in Section 3. At the end of this section, we further show the connections between $\mathscr{M}_I$ and $\mathbf{R}_b$ ($b \in {}^{\omega}\omega$), a relational system that helps us in [14] dissect and reformulate Barotszyński's and Judah's [1] characterization of non($\mathscr{M}\mathscr{A}$).

In Section 4 we present consistency results. The authors proved in [14] the consistency of cov($\mathscr{M}\mathscr{A}$) < $\mathfrak{c}$, even cov($\mathscr{M}\mathscr{A}$) < non($\mathscr{S}\mathscr{N}$). Since $\mathscr{M}\mathscr{A} \subseteq \mathscr{S}\mathscr{N}$, cov($\mathscr{S}\mathscr{N}$) ≤ cov($\mathscr{M}\mathscr{A}$), but cov($\mathscr{S}\mathscr{N}$) = $\mathfrak{c}$ holds in Sacks model (see e.g. [13]). We asked about the consistency of cov($\mathscr{N}\mathscr{A}$) < $\mathfrak{c}$ in [14], which we solve in this paper.

Theorem 14. *It is consistent with* ZFC *that* cov($\mathscr{N}\mathscr{A}$) < $\mathfrak{c}$.

To prove this, we refine Shelah's characterization of $\mathscr{N}\mathscr{A}$ from Theorem 8(1) and show a relevant Tukey connection that gives us a suitable upper bound of cov($\mathscr{N}\mathscr{A}$).

A final consistency result is the following.

Theorem 15. *It is consistent with* ZFC *that* cof($\mathscr{M}\mathscr{A}$) < $\mathfrak{c}$, *even* cof($\mathscr{M}\mathscr{A}$) < non($\mathscr{S}\mathscr{N}$).

The directed scheme $\vec{\mathscr{M}}$ helps us see that this inequality holds in our model of cov($\mathscr{M}\mathscr{A}$) < non($\mathscr{S}\mathscr{N}$) presented in [14].

2. Directed schemes of ideals

Most of the results of this section (let alone this paper) are presented in terms of *relational systems*. Our review of relational systems is based on [4,21]. We say that $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ is a *relational system* if it consists of two non-empty sets X and Y and a relation $\sqsubset$.

  (1) A set $F \subseteq X$ is $\mathbf{R}$-*bounded* if $\exists\, y \in Y\ \forall\, x \in F\colon x \sqsubset y$.
  (2) A set $E \subseteq Y$ is $\mathbf{R}$-*dominating* if $\forall\, x \in X\ \exists\, y \in E\colon x \sqsubset y$.

We associate two cardinal characteristics with this relational system:

$$\mathfrak{b}(\mathbf{R}) := \min\{|F|\colon F \subseteq X \text{ is } \mathbf{R}\text{-unbounded}\} \text{ the } \textit{bounding number of } \mathbf{R},$$

$$\text{and} \qquad \mathfrak{d}(\mathbf{R}) := \min\{|D|\colon D \subseteq Y \text{ is } \mathbf{R}\text{-dominating}\} \text{ the } \textit{dominating number of } \mathbf{R}.$$

The *dual of* $\mathbf{R}$ is the relational system $\mathbf{R}^{\perp} := \langle Y, X, \sqsubset^{\perp} \rangle$ where $y \sqsubset^{\perp} x$ means $x \not\sqsubset y$. Note that $\mathfrak{b}(\mathbf{R}^{\perp}) = \mathfrak{d}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R}^{\perp}) = \mathfrak{b}(\mathbf{R})$.

For instance, a preorder $\langle P, \leq \rangle$ (i.e. $\leq$ is reflexive and transitive in P) can be viewed as the relational system $\langle P, P, \leq \rangle$, simply denoted by P when $\leq$ is understood. In this context, $\mathfrak{d}(P)$ is usually denoted by cof(P) and cf(P), and called the *cofinality of P*.

The following are representative directed preorders, mostly present in this work.

Definition 16. *Define the relation $\leq^*$ on ${}^{\omega}\omega$ by $x \leq^* y$ means $\forall^{\infty}\, n < \omega\colon x(n) \leq y(n)$. It is clear that $\langle {}^{\omega}\omega \leq^* \rangle$ is a directed preorder and we use ${}^{\omega}\omega$ to denote its relational system. Then $\mathfrak{b} := \mathfrak{b}({}^{\omega}\omega)$ and $\mathfrak{d} := \mathfrak{d}({}^{\omega}\omega)$ are the well-known* bounding number *and* dominating number*, respectively.*

Example 17. Let $\mathscr{I}$ be an ideal on X containing $[X]^{<\aleph_0}$. As a relational system, $\mathscr{I}$ is the partial order $\langle \mathscr{I}, \subseteq \rangle$, which is directed. Another relational system associated with $\mathscr{I}$ is $\mathrm{Cv}_{\mathscr{I}} := \langle X, \mathscr{I}, \in \rangle$. The cardinal characteristics associated with $\mathscr{I}$ are obtained by these relational systems:

$$\mathfrak{b}(\mathscr{I}) = \mathrm{add}(\mathscr{I}), \qquad \mathfrak{d}(\mathscr{I}) = \mathrm{cof}(\mathscr{I}),$$

$$\mathfrak{b}(\mathrm{Cv}_{\mathscr{I}}) = \mathrm{non}(\mathscr{I}), \qquad \mathfrak{d}(\mathrm{Cv}_{\mathscr{I}}) = \mathrm{cov}(\mathscr{I}).$$

For relational systems $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ and $\mathbf{R}' = \langle X', Y', \sqsubset' \rangle$, a pair (Ψ_-, Ψ_+) is a *Tukey connection from $\mathbf{R}$ into $\mathbf{R}'$* if $\Psi_-\colon X \to X'$ and $\Psi_+\colon Y' \to Y$ are functions satisfying

$$\forall\, x \in X\ \forall\, y' \in Y'\colon \Psi_-(x) \sqsubset' y' \Rightarrow x \sqsubset \Psi_+(y').$$

We say that $\mathbf{R}$ is *Tukey below* $\mathbf{R}'$, denoted by $\mathbf{R} \leq_{\mathrm{T}} \mathbf{R}'$, if there is a Tukey connection from $\mathbf{R}$ into $\mathbf{R}'$, and we say that $\mathbf{R}$ and $\mathbf{R}'$ are *Tukey equivalent*, denoted by $\mathbf{R} \cong_{\mathrm{T}} \mathbf{R}'$, if $\mathbf{R} \leq_{\mathrm{T}} \mathbf{R}'$ and $\mathbf{R}' \leq_{\mathrm{T}} \mathbf{R}$. It is well-known that $\mathbf{R} \leq_{\mathrm{T}} \mathbf{R}'$ implies $(\mathbf{R}')^{\perp} \leq_{\mathrm{T}} \mathbf{R}^{\perp}$, $\mathfrak{b}(\mathbf{R}') \leq \mathfrak{b}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R}) \leq \mathfrak{d}(\mathbf{R}')$. Hence, $\mathbf{R} \cong_{\mathrm{T}} \mathbf{R}'$ implies $\mathfrak{b}(\mathbf{R}') = \mathfrak{b}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R}) = \mathfrak{d}(\mathbf{R}')$.

Example 18.
(1) Let $\mathscr{I}$ and $\mathscr{J}$ be ideals on X. If $\mathscr{I} \subseteq \mathscr{J}$ then $\mathrm{Cv}_{\mathscr{J}} \leq_{\mathrm{T}} \mathrm{Cv}_{\mathscr{I}}$.
(2) For any ideal $\mathscr{I}$ on X, $\mathrm{Cv}_{\mathscr{I}} \leq_{\mathrm{T}} \mathscr{I}$ and $\mathrm{Cv}_{\mathscr{I}}^{\perp} \leq_{\mathrm{T}} \mathscr{I}$. These determine some of the inequalities in Figure 1.
(3) If $\theta' \leq \theta$ are cardinals and $\theta \leq |X| \leq |X'|$, then $\mathrm{Cv}_{[X]^{<\theta}} \leq_{\mathrm{T}} \mathrm{Cv}_{[X']^{<\theta'}}$ and $[X]^{<\theta} \leq_{\mathrm{T}} [X']^{<\theta}$.
(4) For any cardinal μ, $\mathrm{Cv}_{[\mu]^{<\mu}} \leq_{\mathrm{T}} \mu \leq_{\mathrm{T}} \mathrm{Cv}_{[\mu]^{<\mu}}^{\perp} \leq_{\mathrm{T}} [\mu]^{<\mu}$. In the case when μ is regular, $[\mu]^{<\mu} \leq_{\mathrm{T}} \mathrm{Cv}_{[\mu]^{<\mu}}$, so $\mathrm{add}\big([\mu]^{<\mu}\big) = \mathrm{cof}\big([\mu]^{<\mu}\big) = \mu$.
(5) Whenever S is a directed preorder, $S \leq_{\mathrm{T}} \mathrm{Cv}_{[\mathfrak{d}(S)]^{<\mathfrak{b}(S)}}$.

Example 19 ([4]). The directed preorder $\mathbb{I} = \langle \mathbb{I}, \sqsubseteq \rangle$ is Tukey-equivalent with ${}^{\omega}\omega$. Therefore, $\mathfrak{b}(\mathbb{I}) = \mathfrak{b}$ and $\mathfrak{d}(\mathbb{I}) = \mathfrak{d}$.

To prove the results of this work, we employ the following operations of relational systems.

Definition 20. *Let* $\vec{\mathbf{R}} = \langle \mathbf{R}_i : i \in K \rangle$, $\mathbf{R}_i = \langle X_i, Y_i, \sqsubset_i \rangle$, *be a sequence of relational systems. Define the following relational systems:*

(1) $\bigvee \vec{\mathbf{R}} = \bigvee_{i \in K} \mathbf{R}_i := \big\langle \bigcup_{i \in K} \{i\} \times X_i, \prod_{i \in K} Y_i, \sqsubset^{\vee} \big\rangle$ *such that* $(i, x) \sqsubset^{\vee} y$ *iff* $x \sqsubset_i y_i$;
(2) $\prod \vec{\mathbf{R}} = \prod_{i \in K} \mathbf{R}_i := \big\langle \prod_{i \in K} X_i, \prod_{i \in K} Y_i, \sqsubset^{\otimes} \big\rangle$ *where* $x \sqsubset^{\otimes} y$ *iff* $x_i \sqsubset_i y_i$ *for all* $i \in K$. *When* $\mathbf{R}_i = \mathbf{R}$ *for all* $i \in K$, *we write* $\mathbf{R}^K := \prod \vec{\mathbf{R}}$;
(3) *when* $K = \gamma$ *is an ordinal, the* sequential composition *is*

$$\vec{\mathbf{R}}^{;} := \Big\langle \prod_{\alpha < \gamma} {}^{\prod_{i < \alpha} Y_i} X_{\alpha}, \prod_{\alpha < \gamma} Y_{\alpha}, \sqsubset^{;} \Big\rangle$$

where $\overline{f} = \langle f_{\alpha} : \alpha < \gamma \rangle \sqsubset^{;} y$ *iff* $f_{\alpha}(y{\restriction}\alpha) \sqsubset_{\alpha} y_{\alpha}$ *for all* $\alpha < \gamma$.

When $K = \{0, 1\}$, *we denote the previous relational systems by* $\mathbf{R}_0 \vee \mathbf{R}_1$, $\mathbf{R}_0 \times \mathbf{R}_1$ *and* $(\mathbf{R}_0; \mathbf{R}_1)$.

Blass [4, Theorem 4.11] states a result about the cardinal characteristics associated with the operations above for $K = \{0, 1\}$, which we generalize below.

Theorem 21. *Let* $\vec{\mathbf{R}} = \langle \mathbf{R}_i : i \in K \rangle$ *be a sequence of relational systems* $\mathbf{R}_i = \langle X_i, Y_i, \sqsubset_i \rangle$.
(a) $\bigvee \vec{\mathbf{R}}$ *is the supremum of* $\{\mathbf{R}_i : i \in K\}$ *with respect to* $\leq_{\mathrm{T}}$, *i.e.* $\mathbf{R}_i \leq_{\mathrm{T}} \bigvee \vec{\mathbf{R}}$ *for all* $i \in K$ *and, whenever* $\mathbf{R}'$ *is another relational system such that* $\mathbf{R}_i \leq_{\mathrm{T}} \mathbf{R}'$ *for all* $i \in K$, $\bigvee \vec{\mathbf{R}} \leq_{\mathrm{T}} \mathbf{R}'$.
(b) $\bigvee \vec{\mathbf{R}} \leq_{\mathrm{T}} \prod \vec{\mathbf{R}}$ *and* $\prod \vec{\mathbf{R}} \leq_{\mathrm{T}} \vec{\mathbf{R}}^{;}$, *the latter when* K *is an ordinal.*
(c) $\mathfrak{b}(\bigvee \vec{\mathbf{R}}) = \min_{i \in K} \mathfrak{b}(\mathbf{R}_i)$ *and* $\mathfrak{d}(\bigvee \vec{\mathbf{R}}) = \sup_{i \in K} \mathfrak{d}(\mathbf{R}_i)$.
(d) $\mathfrak{b}(\prod \vec{\mathbf{R}}) = \min_{i \in K} \mathfrak{b}(\mathbf{R}_i)$ *and* $\sup_{i \in K} \mathfrak{d}(\mathbf{R}_i) \leq \mathfrak{d}(\prod \vec{\mathbf{R}}) \leq \prod_{i \in K} \mathfrak{d}(\mathbf{R}_i)$ *(see [12, Fact 3.9]).*
(e) *When* K *is an ordinal,* $\mathfrak{b}(\vec{\mathbf{R}}^{;}) = \min_{i \in K} \mathfrak{b}(\mathbf{R}_i)$ *and* $\mathfrak{d}(\vec{\mathbf{R}}^{;}) \leq \prod_{i \in K} \mathfrak{d}(\mathbf{R}_i)$. *Equality holds when* K *is finite.*

Proof. Without loss of generality, we can assume that $K = \gamma$ is an ordinal.

For $i \in K$, $\mathbf{R}_i \leq_{\mathrm{T}} \bigvee \vec{\mathbf{R}}$ is obtained by the maps $x \mapsto (i, x)$ and $y \mapsto y(i)$.

If, for each $i \in K$, (Ψ_-^i, Ψ_+^i) is a Tukey connection for $\mathbf{R}_i \leq_{\mathrm{T}} \mathbf{R}'$, then $\bigvee \vec{\mathbf{R}} \leq_{\mathrm{T}} \mathbf{R}'$ is obtained by the maps $(i, x) \mapsto \Psi_-^i(x)$ and $y' \mapsto \langle \Psi_+^i(y') : i \in K \rangle$. This shows (a).

It is easy to show that $\mathbf{R}_i \leq_{\mathrm{T}} \prod \vec{\mathbf{R}}$ for all $i \in K$, so $\bigvee \vec{\mathbf{R}} \leq_{\mathrm{T}} \prod \vec{\mathbf{R}}$ by (a). On the other hand, $\prod \vec{\mathbf{R}} \leq_{\mathrm{T}} \vec{\mathbf{R}}^{;}$ via the maps $x \mapsto \langle f_{\alpha}^x : \alpha < \gamma \rangle$ such that each f_{α}^x is the constant map with value x_{α}, and $y \mapsto y$. This shows (b).

By (a) it is clear that $\mathfrak{b}(\bigvee \vec{\mathbf{R}}) \leq \mathfrak{b}(\mathbf{R}_i)$ and $\mathfrak{d}(\mathbf{R}_i) \leq \mathfrak{d}(\bigvee \vec{\mathbf{R}})$ for all $i \in K$, so $\mathfrak{b}(\bigvee \vec{\mathbf{R}}) \leq \min_{i \in K} \mathfrak{b}(\mathbf{R}_i)$ and $\sup_{i \in K} \mathfrak{d}(\mathbf{R}_i) \leq \mathfrak{d}(\bigvee \vec{\mathbf{R}})$.

For each $i \in K$, pick some $\mathbf{R}_i$-dominating $D^i := \{y^i_\xi : \xi < \mathfrak{d}(\mathbf{R}_i)\} \subseteq Y_i$. When $\mathfrak{d}(\mathbf{R}_i) \leq \xi < \sup_{i \in K} \mathfrak{d}(\mathbf{R}_i)$, set $y^i_\xi := y^i_0$. So define $y_\xi := \langle y^i_\xi : i \in K \rangle$ for $\xi < \sup_{i \in K} \mathfrak{d}(\mathbf{R}_i)$. Then the set $\{y_\xi : \xi < \sup_{i \in K} \mathfrak{d}(\mathbf{R}_i)\}$ is $\bigvee \vec{\mathbf{R}}$-dominating, so $\mathfrak{d}(\bigvee \vec{\mathbf{R}}) \leq \sup_{i \in K} \mathfrak{d}(\mathbf{R}_i)$. On the other hand, it is easy to see that $\prod_{i \in K} D_i$ is dominating for $\prod \vec{\mathbf{R}}$ and $\vec{\mathbf{R}}^{\mathbf{\cdot}}$, so $\mathfrak{d}(\prod \vec{\mathbf{R}}) \leq \mathfrak{d}(\vec{\mathbf{R}}^{\mathbf{\cdot}}) \leq \prod_{i \in K} \mathfrak{d}(\mathbf{R}_i)$.

By (b) and what we have proved so far, $\mathfrak{b}(\vec{\mathbf{R}}^{\mathbf{\cdot}}) \leq \mathfrak{b}(\prod \vec{\mathbf{R}}) \leq \mathfrak{b}(\bigvee \vec{\mathbf{R}}) \leq \min_{i \in K} \mathfrak{b}(\mathbf{R}_i)$ and $\sup_{i \in K} \mathfrak{d}(\mathbf{R}_i) = \mathfrak{d}(\bigvee \vec{\mathbf{R}}) \leq \mathfrak{d}(\prod \vec{\mathbf{R}}) \leq \mathfrak{d}(\vec{\mathbf{R}}^{\mathbf{\cdot}}) \leq \prod_{i \in K} \mathfrak{d}(\mathbf{R}_i)$. So, to conclude (c)–(d) and part of (e), it is enough to show that $\min_{i \in K} \mathfrak{b}(\mathbf{R}_i) \leq \mathfrak{b}(\vec{\mathbf{R}}^{\mathbf{\cdot}})$. Let $F \subseteq \prod_{\alpha < \gamma} \prod_{i < \alpha} {}^{Y_i} X_\alpha$ be a set of size $< \min_{\alpha < \gamma} \mathfrak{b}(\mathbf{R}_\alpha)$. Find $y_\alpha \in Y_\alpha$ by recursion on $\alpha < \gamma$ such that $f_\alpha(y \restriction \alpha) \sqsubset_\alpha y_\alpha$ for all $\overline{f} = \langle f_\alpha : \alpha < \gamma \rangle \in F$. This is fine because $|F| < \min_{\alpha < \gamma} \mathfrak{b}(\mathbf{R}_\alpha)$. Therefore, F is $\vec{\mathbf{R}}^{\mathbf{\cdot}}$-bounded by $y := \langle y_\alpha : \alpha < \gamma \rangle$.

It remains to show the equality of the dominating numbers in (e) when K is finite. Since sequential composition is associative (modulo Tukey-equivalence), it is enough to assume that $K = \{0, 1\}$. So let $D \subseteq Y_0 \times Y_1$ of size $< \mathfrak{d}(\mathbf{R}_0) \cdot \mathfrak{d}(\mathbf{R}_1)$. Set D_0 as the set of $y_0 \in Y_0$ such that $D^{y_0}_1 := \{y_1 : (y_0, y_1) \in D\}$ is $\mathbf{R}_1$-dominating. Then D_0 is not $\mathbf{R}_0$-dominating, otherwise $|D_0| \geq \mathfrak{d}(\mathbf{R}_0)$ and, since $|D^{y_0}_1| \geq \mathfrak{d}(\mathbf{R}_1)$ for $y_0 \in D_0$, we would get that $|D| \geq \mathfrak{d}(\mathbf{R}_0) \cdot \mathfrak{d}(\mathbf{R}_1)$, a contradiction.

Hence, there is some $x_0 \in X_0$ such that $x_0 \not\sqsubset_0 y_0$ for all $y_0 \in D_0$. Define $f_0 : \{\emptyset\} \to X_0$ such that $f_0(\emptyset) := x_0$ and let $f_1 : Y_0 \to X_1$ be any map such that, whenever $y_0 \notin D_0$, $f_1(y_0) \not\sqsubset_1 y_1$ for any $y_1 \in D^{y_0}_1$. We then get that $\overline{f} := (f_0, f_1) \not\sqsubset^{\mathbf{\cdot}} (y_0, y_1)$ for all $(y_0, y_1) \in D$. $\qquad\square$

Remark 22. The equality of the $\mathfrak{d}$-number in Theorem 21(e) does not hold in general when K is infinite. Consider the case when $K = \omega$ and $\mathbf{R}_i = \langle \omega, \neq \rangle$ for $i < \omega$. Then $\vec{\mathbf{R}}^{\mathbf{\cdot}} \cong_{\mathrm{T}} \prod \vec{\mathbf{R}}$ and $\mathfrak{d}(\prod \vec{\mathbf{R}}) = \mathrm{cov}(\mathcal{M})$ (this is a consequence of the well-known Bartoszyński's and Miller's characterization of $\mathrm{cov}(\mathcal{M})$, see e.g. [11, Section 5]), but $\prod_{i < \omega} \mathfrak{d}(\mathbf{R}_i) = \mathfrak{c}$.

To see this Tukey equivalence, notice that $\prod \vec{\mathbf{R}} = \langle {}^\omega \omega, \neq^\bullet \rangle$ where $x \neq^\bullet y$ means that $x(n) \neq y(n)$ for all $n < \omega$, while $\vec{\mathbf{R}}^{\mathbf{\cdot}} = \langle {}^{<\omega} \omega, {}^\omega \omega, \neq^\bullet \rangle$ where we interpret $\varphi \neq^\bullet y$ as $\varphi(y \restriction n) \neq y(n)$ for all $n < \omega$. It is clear that $\prod \vec{\mathbf{R}} \cong_{\mathrm{T}} \langle {}^{<\omega} \omega, \neq^\bullet \rangle$ where $\varphi \neq^\bullet \psi$ is interpreted as $\varphi(s) \neq \psi(s)$ for all $s \in {}^{<\omega} \omega$.

It is enough to show that $\vec{\mathbf{R}}^{\mathbf{\cdot}} \leq_{\mathrm{T}} \langle {}^{<\omega} \omega, \neq^\bullet \rangle$. Let F be the identity function on ${}^{<\omega} \omega$ and define $F' : {}^{<\omega} \omega \to {}^\omega \omega$ such that $F'(\psi) := y_\psi$ is defined recursively by $y_\psi(n) := \psi(y_\psi \restriction n)$. It is clear that (F, F') is the required Tukey connection.

Another example is $\mathbf{R}_i := \langle \omega, \leq \rangle$ for $i < \omega$, where $x \leq y$ means $x(n) \leq y(n)$ for all $n < \omega$, where we also get $\vec{\mathbf{R}}^{\mathbf{\cdot}} \cong_{\mathrm{T}} \prod \vec{\mathbf{R}}$, whose $\mathfrak{d}$-number is $\mathfrak{d}$.

The following result is a useful trick to produce Tukey connections.

Lemma 23 (cf. [21, p. 4.1.6]). *For $i \in \{0, 1\}$, let $\mathbf{R}_i = \langle X_i, Y_i, \sqsubset_i \rangle$ be a relational system, and let $\mathbf{R} = \langle S, Y, \sqsubset \rangle$ be a relational system such that S is directed and satisfying that, for $s, s' \in S$ and $y \in Y$, if $s' \sqsubset y$ and $s \leq_S s'$ then $s \sqsubset y$.*

If $\mathbf{R}_0 \leq_{\mathrm{T}} \mathbf{R}$ and $\mathbf{R}_1 \leq_{\mathrm{T}} \mathbf{R}$, then $\mathbf{R}_0 \times \mathbf{R}_1 \leq_{\mathrm{T}} \mathbf{R}$.

Proof. For $i \in \{0, 1\}$, consider the Tukey connections $\Psi^i_- : X_i \to S$ and $\Psi^i_+ : Y \to Y_i$ for $\mathbf{R}_i \leq_{\mathrm{T}} \mathbf{R}$. Define $\Psi_- : X_0 \times X_1 \to S$ such that $\Psi_-(x_0, x_1)$ is above $\Psi^0_-(x_0)$ and $\Psi^1_-(x_1)$ in S, and define $\Psi_+ : Y \to Y_0 \times Y_1$ by $\Psi_+(y) := \big(\Psi^0_+(y_0), \Psi^1_+(y_1) \big)$. $\qquad\square$

Powers of ideals can be introduced using the product of relational systems in the following ways.

Definition 24 ([11, Definition 3.14]). *Given an ideal $\mathscr{I}$ on X and a set w, define $\mathscr{I}^{(w)}$ as the ideal on ${}^w X$ generated by the sets of the form $\prod_{i \in w} A_i$ with $\langle A_i : i \in w \rangle \in {}^w \mathscr{I}$. Denote $\mathrm{add}(\mathscr{I}^w) := \mathfrak{b}(\mathscr{I}^{(w)})$, $\mathrm{cof}(\mathscr{I}^w) := \mathfrak{d}(\mathscr{I}^{(w)})$, $\mathrm{non}(\mathscr{I}^w) := \mathfrak{b}(\mathrm{Cv}_{\mathscr{I}^{(w)}})$ and $\mathrm{cov}(\mathscr{I}^w) := \mathfrak{d}(\mathrm{Cv}_{\mathscr{I}^{(w)}})$.*

Fact 25 ([11, Fact 3.15]). *Let w be a set and let $\mathscr{I}$ be an ideal on X. Then:*

    (a) $\mathscr{I}^w \cong_{\mathrm{T}} \mathscr{I}^{(w)}$;

    (b) $\mathrm{Cv}^w_{\mathscr{I}} \cong_{\mathrm{T}} \mathrm{Cv}_{\mathscr{I}^{(w)}}$;

(c) $\operatorname{add}(\mathscr{I}^w) = \operatorname{add}(\mathscr{I})$ *and* $\operatorname{non}(\mathscr{I}^w) = \operatorname{non}(\mathscr{I})$;

(d) $\operatorname{cov}(\mathscr{I}) \leq \operatorname{cov}(\mathscr{I}^w) \leq \operatorname{cov}(\mathscr{I})^{|w|}$ *and* $\operatorname{cof}(\mathscr{I}) \leq \operatorname{cof}(\mathscr{I}^w) \leq \operatorname{cof}(\mathscr{I})^{|w|}$.

We can prove Theorem 4 in terms of relational systems and Tukey connections. The following result is the starting point to deal with the covering and uniformity numbers.

Theorem 26. *Let X be a set, S a directed preorder and $D \subseteq S$ cofinal in S. Assume that $\vec{\mathscr{I}} = \langle \mathscr{I}_s : s \in S \rangle$ is a directed scheme of ideals on X. Then:*

(1) *if $s \leq s'$, then $\operatorname{Cv}_{\mathscr{I}_s} \leq_{\mathrm{T}} \operatorname{Cv}_{\mathscr{I}_{s'}} \leq_{\mathrm{T}} \operatorname{Cv}_{\mathscr{I}^*}$;*

(2) $\operatorname{Cv}_{\mathscr{I}^*} \leq_{\mathrm{T}} \prod_{s \in D} \operatorname{Cv}_{\mathscr{I}_s}$;

(3) $\sup \operatorname{cov}(\vec{\mathscr{I}}) \leq \operatorname{cov}(\mathscr{I}^*) \leq \mathfrak{d}\left(\prod_{s \in D} \operatorname{Cv}_{\mathscr{I}_s} \right)$ *and* $\operatorname{non}(\mathscr{I}^*) = \min \operatorname{non}(\vec{\mathscr{I}})$.

Proof. (1) follows directly by Example 18(1), and (3) is a consequence of (1), (2) and Theorem 21(d). So it remains to show (2). Define the functions $\Psi_- : X \to \prod_{s \in D} X$ and $\Psi_+ : \prod_{s \in D} \mathscr{I}_s \to \mathscr{I}^*$ where $\Psi_-(x)$ is the constant sequence with value x, and $\Psi_+\left(\langle A_s : s \in D \rangle \right) := \bigcap_{s \in D} A_s$, which is in $\mathscr{I}^*$ because D is cofinal in S. These maps form the desired Tukey connection. $\square$

We now look at the additivity and cofinality numbers.

Theorem 27. *Let $\vec{\mathscr{I}} = \langle \mathscr{I}_s : s \in S \rangle$ be a directed scheme of ideals on X. Then, for any cofinal $D \subseteq S$,*

$$\mathscr{I}^* \leq_{\mathrm{T}} \prod_{s \in D} \mathscr{I}_s \leq_{\mathrm{T}} \prod_{s \in D} \operatorname{Cv}_{[\operatorname{cof}(\mathscr{I}_s)]^{<\operatorname{add}(\mathscr{I}_s)}} \leq_{\mathrm{T}} \operatorname{Cv}^D_{[\operatorname{cof}(\vec{\mathscr{I}})]^{<\min \operatorname{add}(\vec{\mathscr{I}})}}.$$

As a consequence,

$$\min \operatorname{add}(\vec{\mathscr{I}}) \leq \operatorname{add}(\mathscr{I}^*)$$

and, whenever D is a witness of $\operatorname{cof}(S)$,

$$\operatorname{cof}(\mathscr{I}^*) \leq \mathfrak{d}\left(\prod_{s \in D} \mathscr{I}_s \right) \leq \operatorname{cov}\left(\left([\sup \operatorname{cof}(\vec{\mathscr{I}})]^{<\min \operatorname{add}(\vec{\mathscr{I}})} \right)^{\operatorname{cof}(S)} \right).$$

Proof. The inequality $\mathscr{I}^* \leq_{\mathrm{T}} \prod_{s \in D} \mathscr{I}_s$ is easy to show: define $\Psi_- : \mathscr{I}^* \to \prod_{s \in D} \mathscr{I}_s$ such that $\Psi_-(A)$ is the constant sequence with value A, and define $\Psi_+ : \prod_{s \in D} \mathscr{I}_s \to \mathscr{I}^*$ such that $\Psi_+\left(\langle A_s : s \in D \rangle \right) := \bigcap_{s \in D} A_s$, which is in $\mathscr{I}^*$ because D is cofinal in S. Note that (Ψ_-, Ψ_+) is the required Tukey connection.

On the other hand, using (3) and (5) of Example 18, we obtain

$$\prod_{s \in D} \mathscr{I}_s \leq_{\mathrm{T}} \prod_{s \in D} \operatorname{Cv}_{[\operatorname{cof}(\mathscr{I}_s)]^{<\operatorname{add}(\mathscr{I}_s)}} \leq_{\mathrm{T}} \operatorname{Cv}^D_{[\sup \operatorname{cof}(\vec{\mathscr{I}})]^{<\min \operatorname{add}(\vec{\mathscr{I}})}}. \qquad \square$$

Theorem 4 is a direct consequence of Theorems 26 and 27.

3. Directed scheme for the meager additive ideal

The purpose of this section is to introduce the directed scheme of ideals $\langle \mathscr{M}_I : I \in \mathbb{I} \rangle$, which reformulates $\mathscr{M}\mathscr{A} = \bigcap \{ \mathscr{M}_I : I \in \mathbb{I} \}$, and study these ideals. In particular, we prove Theorems 11 and 13.

We start with reviewing the following cofinal family of the meager ideal.

Definition 28. *Let $I \in \mathbb{I}$ and $y \in {}^\omega 2$. Define*

$$B_{y,I} := \left\{ x \in {}^\omega 2 : \forall^\infty n \in \omega : x {\restriction} I_n \neq y {\restriction} I_n \right\}.$$

A pair $(y, I) \in 2^\omega \times \mathbb{I}$ is known as a *chopped real*, and these are used to produce a cofinal family of meager sets. It is clear that $B_{y,I}$ is a meager subset of ${}^\omega 2$ (see e.g. [4]).

Theorem 29 (Talagrand [20], see e.g. [2, Proposition 13]). *For every meager set $F \subseteq {}^\omega 2$ and $I \in \mathbb{I}$ there are $y \in {}^\omega 2$ and $I' \in \mathbb{I}$ such that $F \subseteq B_{y,I'}$ and each I'_n is the union of finitely many I_k's.*

Inspired by (🌿) let us introduce a new ideal related to $\mathscr{MA}$ as follows.

Definition 30. *Fix $I \in \mathbb{I}$. For $J \in \mathbb{I}$ and $y \in {}^{\omega}2$, denote*

$$A^I_{J,y} := \left\{ x \in {}^{\omega}2 : \forall^{\infty} n < \omega \ \exists k < \omega : I_k \subseteq J_n \text{ and } x{\upharpoonright}I_k = y{\upharpoonright}I_k \right\},$$

which is an F_{σ}-meager set, and define

$$\mathscr{M}_I := \left\{ X \subseteq {}^{\omega}2 : \exists (J,y) \in \mathbb{I} \times {}^{\omega}2 : X \subseteq A^I_{J,y} \right\}.$$

Also denote $\vec{\mathscr{M}} = \langle \mathscr{M}_I : I \in \mathbb{I} \rangle$.

By Theorem 8(2), it is clear that $\mathscr{MA} = \bigcap_{I \in \mathbb{I}} \mathscr{M}_I$, but we will offer a proof in Corollary 40. To make sense of the previous definition and conclude that $\vec{\mathscr{M}}$ is a directed scheme of ideals, we aim to show that $\mathscr{M}_I$ is a σ-ideal and $\mathscr{M}_{I'} \subseteq \mathscr{M}_I$ whenever $I \sqsubseteq I'$ in $\mathbb{I}$. The former requires some work, for which we introduce the following directed preorder.

Definition 31. *Fix $I \in \mathbb{I}$. For $J, J' \in \mathbb{I}$ and $y, y' \in {}^{\omega}2$ define the relation $(J, y) \sqsubseteq^I (J', y')$ iff either $I \not\sqsubseteq J$ or, for all but finitely many $n < \omega$, there is some $\ell < \omega$ satisfying the following conditions:*

  (i) *J_ℓ contains some I_k;*
  (ii) *for all $k < \omega$, $I_k \subseteq J_\ell$ implies $I_k \subseteq J'_n$ and $y{\upharpoonright}I_k = y'{\upharpoonright}I_k$.*

Notice that $I \sqsubseteq J$ and $(J, y) \sqsubseteq^I (J', y')$ implies $I \sqsubseteq J'$.

Lemma 32. *$\langle \mathbb{I} \times {}^{\omega}2, \sqsubseteq^I \rangle$ is a directed preorder and $\mathfrak{b}(\mathbb{I} \times {}^{\omega}2, \sqsubseteq^I)$ is uncountable.*

Proof. It is easy to show that $\langle \mathbb{I} \times {}^{\omega}2, \sqsubseteq^I \rangle$ is a preorder, so it is enough to prove that $\mathfrak{b}(\mathbb{I} \times {}^{\omega}2, \sqsubseteq^I)$ is uncountable (which implies directedness). Let $(J^m, y^m) \in \mathbb{I} \times {}^{\omega}2$ for $m < \omega$ and, wlog, assume that $I \sqsubseteq J^m$ for all $m < \omega$. Construct a $J = \langle J_n : n < \omega \rangle \in \mathbb{I}$ such that each J_n contains one interval from J^m, say $J^m_{\ell_{n,m}}$, for each $m \leq n$, also demanding that $\{J^m_{\ell_{n,m}} : m \leq n\}$ is pairwise disjoint and that each member of this set contains an interval from I. So we can define a $y \in {}^{\omega}2$ such that $y{\upharpoonright}J^m_{\ell_{m,n}} = y^m{\upharpoonright}J^m_{\ell_{m,n}}$ for all $m \leq n < \omega$. Then $(J^m, y^m) \sqsubseteq^I (J, y)$ for all $m < \omega$. $\qquad\square$

Main Lemma 33. *Let I, J, J' and y, y' as in Definition 31. Then, we have that $A^I_{J,y} \subseteq A^I_{J',y'}$ iff $(J, y) \sqsubseteq^I (J', y')$.*

Proof.

"$\Leftarrow$". Assume that $x \in A^I_{J,y}$, so $\forall^{\infty} n < \omega \ \exists k < \omega : I_k \subseteq J_n$ and $x{\upharpoonright}I_k = y{\upharpoonright}I_k$. Since $(J, y) \sqsubseteq^I (J', y')$, for large enough n we can choose an $\ell < \omega$ such that:

  ($\oplus_1$) $\exists k < \omega : I_k \subseteq J_\ell$ and $x{\upharpoonright}I_k = y{\upharpoonright}I_k$;
  ($\oplus_2$) $\forall k < \omega : I_k \subseteq J_\ell \Rightarrow I_k \subseteq J'_n$ and $y{\upharpoonright}I_k = y'{\upharpoonright}I_k$.

Observe that ($\oplus_2$) implies $I_k \subseteq J'_n$ and $x{\upharpoonright}I_k = y'{\upharpoonright}I_k$ for a k as in ($\oplus_1$). Hence, $x \in A^I_{J',y'}$.

"$\Rightarrow$". We prove the contrapositive. Assume $(J, y) \not\sqsubseteq^I (J', y')$, so $I \sqsubseteq J$, i.e. there is some $\ell_0 < \omega$ such that J_ℓ contains some I_k for all $\ell \geq \ell_0$, and there are infinitely many $n < \omega$ such that:

    for all $\ell \geq \ell_0$ there is some $k < \omega$ such that $I_k \subseteq J_\ell$ and either $I_k \not\subseteq J'_n$ or $y{\upharpoonright}I_k \neq y'{\upharpoonright}I_k$.   ($\star$)

Let w be an infinite set of $n < \omega$ satisfying ($\star$) and such that $\left| \{n \in w : J_\ell \cap J'_n \neq \emptyset\} \right| \leq 1$ for any $\ell < \omega$. Denote by n_ℓ the unique member of this set, in case it exists.

Define $x \in {}^{\omega}2$ at each I_k as follows: first consider the case when $I_k \subseteq J_\ell$ for some $\ell \geq \ell_0$ (which is unique). In the case n_ℓ exists, there is some $k_\ell < \omega$ such that $I_{k_\ell} \subseteq J_\ell$ and either $I_{k_\ell} \not\subseteq J'_{n_\ell}$ or $y{\upharpoonright}I_{k_\ell} \neq y'{\upharpoonright}I_{k_\ell}$. Define $x{\upharpoonright}I_k := y{\upharpoonright}I_k$ when $k = k_\ell$, otherwise $x{\upharpoonright}I_k$ can be anything different from $y'{\upharpoonright}I_k$; in the case that n_ℓ does not exist, set $x{\upharpoonright}I_k := y{\upharpoonright}I_k$. On the other hand, in the case that I_k is not contained in any J_ℓ for $\ell \geq \ell_0$, let $x{\upharpoonright}I_k$ be anything different from $y'{\upharpoonright}I_k$.

Firstly, it is clear that $x \in A_{J,y}^I$. On the other hand, $\forall\, n \in w \;\; \forall\, k < \omega : I_k \subseteq J_n' \Rightarrow x{\upharpoonright}I_k \neq y'{\upharpoonright}I_k$, i.e., $x \notin A_{J',y'}^I$. Indeed, for $n \in w$ and $k < \omega$ such that $I_k \subseteq J_n'$, in case $I_k \subseteq J_\ell$ for some $\ell \geq \ell_0$, we have $n = n_\ell$ and, if $k = k_\ell$, then $x{\upharpoonright}I_{k_\ell} = y{\upharpoonright}I_{k_\ell} \neq y'{\upharpoonright}I_{k_\ell}$, else $x{\upharpoonright}I_k \neq y'{\upharpoonright}I_k$ by the definition of $x{\upharpoonright}I_k$; otherwise, I_k is not contained in any J_ℓ for $\ell \geq \ell_0$, in which case $x{\upharpoonright}I_k \neq y'{\upharpoonright}I_k$. $\square$

As a direct consequence of Lemma 33, not only do we have that any $\mathcal{M}_I$ is a σ-ideal but also that $\mathcal{M}_I$ is Tukey-equivalent to a clean directed preorder.

Theorem 34. *For $I \in \mathbb{I}$, $\mathcal{M}_I$ is a σ-ideal and $\mathcal{M}_I \cong_T \langle \mathbb{I} \times {}^\omega 2, \sqsubset^I \rangle$.*

We must show, of course, that ${}^\omega 2 \notin \mathcal{M}_I$, but this is immediate from $\mathcal{M}_I \subseteq \mathcal{M}$, which we prove in Lemma 39.

It is clear that the set $\big\{ (J, y) \in \mathbb{I} \times {}^\omega 2 : J \text{ is coarser that } I \big\}$ is cofinal in $\langle \mathbb{I} \times {}^\omega 2, \sqsubset^I \rangle$. As a consequence, we have the following.

Corollary 35. *We can rewrite*
$$\mathcal{M}_I = \big\{ x \in {}^\omega 2 : \exists\, (J, y) \in \mathbb{I} \times {}^\omega 2 : X \subseteq A_{J,y}^I \text{ and } J \text{ is coarser than } I \big\}.$$

We point out some simple facts below, which let us conclude that $\vec{\mathcal{M}}$ is a directed scheme of ideals.

Fact 36. *Let $I, I', J, J' \in \mathbb{I}$ and $x, y, y' \in {}^\omega 2$. Then:*
   (a) $x \in A_{J,y}^I$ *iff* $(I, x) \sqsubset^I (J, y)$;
   (b) *if* $I \sqsubseteq I'$ *then* $A_{J,y}^{I'} \subseteq A_{J,y}^I$; *in particular, $\mathcal{M}_{I'} \subseteq \mathcal{M}_I$.*

We ask the following.

Question 37. *Does $I \sqsubseteq I'$ imply $\mathcal{M}_{I'} \leq_T \mathcal{M}_I$?*

Although the answer to the previous question is unknown to us, we have a stronger relation on $\mathbb{I}$ from where $\mathcal{M}_{I'} \leq_T \mathcal{M}_I$ holds. Namely, for $I, I' \in \mathbb{I}$, $I \sqsubseteq^+ I'$ means that, for all but finitely many $n < \omega$, I_n' is the union of some I_k's. Notice that $\langle I^+, \sqsubseteq^+ \rangle$ is a preorder but not directed, even more, $\mathrm{cof}(\mathbb{I}, \sqsubseteq^+) = \mathfrak{c}$ (see [9, Proposition 3.7]).

Fact 38. *For $I, I' \in \mathbb{I}$, if $I \sqsubseteq^+ I'$ then $(J, y) \sqsubset^I (J', y')$ implies $(J, y) \sqsubset^{I'} (J', y')$, In particular, $\langle \mathbb{I} \times {}^\omega 2, \sqsubset^{I'} \rangle \leq_T \langle \mathbb{I} \times {}^\omega 2, \sqsubset^I \rangle$ and $\mathcal{M}_{I'} \leq_T \mathcal{M}_I$.*

Recall that I^1 denotes the partition of ω into singletons.

Lemma 39.
   (a) $\mathcal{M}_{I^1} = \mathcal{M}$.
   (b) *For all $I \in \mathbb{I}$, $\mathcal{M}_I \subseteq \mathcal{M}$ and $\mathcal{M}_I \leq_T \mathcal{M}$.*

Proof. We just prove (a) because (b) is consequence of (a), Facts 36 and 38. For $x \in {}^\omega 2$, $x \in A_{J,y}^{I^1}$ iff $\forall^\infty n \; \exists\, k \in J_n : x(k) = y(k)$, which is equivalent to $\forall^\infty n : x{\upharpoonright}J_n \neq y'{\upharpoonright}J_n$ where $y' := y + 1 \mod 2$. This actually shows $A_{J,y}^{I^1} = B_{y',J}$ (see Definition 28). Therefore, $\mathcal{M}_{I^1} = \mathcal{M}$. $\square$

The previous representation of $\mathcal{M}$ can be used to prove the following.

Theorem 40 ([1]). $\mathcal{M}\mathcal{A} = \bigcap_{I \in \mathbb{I}} \mathcal{M}_I$.

Proof. For $X \subseteq {}^\omega 2$,
$$X \in \mathcal{M}\mathcal{A} \text{ iff } \forall\, A \in \mathcal{M} : X + A \in \mathcal{M}$$
$$\text{iff } \forall\, (I, y) \in \mathbb{I} \times {}^\omega 2 \; \exists\, (J, y') \in \mathbb{I} \times {}^\omega 2 : X + A_{I,y}^{I^1} = \underbrace{\bigcup_{x \in X} x + A_{I,y}^{I^1}}_{\bigcup_{x \in X} A_{I,x+y}^{I^1}} \subseteq A_{J,y'}^{I^1}$$

$$\text{iff } \forall\, (I, y) \in \mathbb{I} \times {}^{\omega}2 \; \exists\, (J, y') \in \mathbb{I} \times {}^{\omega}2 \; \forall\, x \in X : A_{I, x+y}^{I^1} \subseteq A_{J, y'}^{I^1}$$

$$\text{iff } \forall\, I \in \mathbb{I} \; \exists\, (J, y) \in \mathbb{I} \times {}^{\omega}2 \; \forall\, x \in X : A_{I, x}^{I^1} \subseteq A_{J, y}^{I^1} \text{ (using a translation)}$$

$$\text{iff } \forall\, I \in \mathbb{I} \; \exists\, (J, y) \in \mathbb{I} \times {}^{\omega}2 \; \forall\, x \in X \; \forall^{\infty} n \in \omega \; \exists\, \ell < \omega : I_\ell \subseteq J_n \wedge x{\restriction}I_\ell = y{\restriction}I_\ell$$

$$\text{iff } \forall\, I \in \mathbb{I} \; \exists\, (J, y) \in \mathbb{I} \times {}^{\omega}2 : X \subseteq A_{J, y}^{I}$$

$$\text{iff } \forall\, I \in \mathbb{I} : X \in \mathcal{M}_I$$

$$\text{iff } X \in \bigcap_{I \in \mathbb{I}} \mathcal{M}_I. \qquad\qquad\qquad\qquad \square$$

We now look at the cardinal characteristics associated with $\vec{\mathcal{M}}$ and its influence on those with $\mathcal{M}\mathcal{A}$ as well. Since $\vec{\mathcal{M}}$ is a directed scheme of ideals and $\mathcal{M}\mathcal{A} = \vec{\mathcal{M}}^*$, by Theorems 26 and 27 we immediately obtain the following.

Lemma 41. *Let $I, J \in \mathbb{I}$ and $D \subseteq \mathbb{I}$ cofinal.*

(a) *If $I \sqsubseteq J$ then $\mathrm{Cv}_{\mathcal{M}} \leq_{\mathrm{T}} \mathrm{Cv}_{\mathcal{M}_I} \leq_{\mathrm{T}} \mathrm{Cv}_{\mathcal{M}_J} \leq_{\mathrm{T}} \mathrm{Cv}_{\mathcal{M}\mathcal{A}} \leq_{\mathrm{T}} \prod_{I \in D} \mathrm{Cv}_{\mathcal{M}_I}$. In particular, $\mathrm{cov}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{M}_I) \leq \mathrm{cov}(\mathcal{M}_J) \leq \sup\mathrm{cov}(\vec{\mathcal{M}}) \leq \mathrm{cov}(\mathcal{M}\mathcal{A}) \leq \mathfrak{d}\big(\prod_{I \in D} \mathrm{Cv}_{\mathcal{M}_I}\big)$ and $\mathrm{non}(\mathcal{M}\mathcal{A}) = \min\mathrm{non}(\vec{\mathcal{M}}) \leq \mathrm{non}(\mathcal{M}_I) \leq \mathrm{non}(\mathcal{M}_J) \leq \mathrm{non}(\mathcal{M})$.*

(b) *$\mathcal{M}\mathcal{A} \leq_{\mathrm{T}} \prod_{I \in D} \mathcal{M}_I$.*

It is also clear that $\min\mathrm{cov}(\vec{\mathcal{M}}) = \mathrm{cov}(\mathcal{M})$ and $\sup\mathrm{non}(\vec{\mathcal{M}}) = \mathrm{non}(\mathcal{M})$. Although Lemma 41(b) gives us some information about the additivity and cofinality numbers, we are going to prove much more. Towards this direction, we first show a connection between ${}^{\omega}\omega$ and $\mathcal{M}_I$.

Lemma 42. ${}^{\omega}\omega \leq_{\mathrm{T}} \mathcal{M}_I$.

Proof. It suffices to show that $\mathbb{I} \leq_{\mathrm{T}} \langle \mathbb{I} \times {}^{\omega}2, \sqsubset^{I} \rangle$ because $\mathbb{I} \cong_{\mathrm{T}} {}^{\omega}\omega$ (see Example 19). Define the functions $\Psi_- : \mathbb{I} \to \mathbb{I} \times {}^{\omega}2$ and $\Psi_+ : \mathbb{I} \times {}^{\omega}2 \to \mathbb{I}$ as follows. For $J \in \mathbb{I}$, pick some $J^+ \in \mathbb{I}$, coarser that I, such that $J \sqsubseteq J^+$.

So define

$$\Psi_-(J) := (J^+, \bar{0}),$$
$$\Psi_+(J', y') := J',$$

where $\bar{0}$ is the constant sequence with value 0.

The pair (Ψ_-, Ψ_+) is the desired Tukey connection: assume $(J^+, \bar{0}) \sqsubset^{I} (J', y')$. Since J^+ is coarser than I, we obtain $J^+ \sqsubseteq J'$, so $J \sqsubseteq J'$. $\qquad\qquad \square$

Another relevant result is the following.

Lemma 43. *For any $I, I' \in \mathbb{I}$, $\mathrm{Cv}_{\mathcal{M}_I} \leq_{\mathrm{T}} \mathcal{M}_{I'}$.*

Proof. Find $I'' \in \mathbb{I}$ coarser than I' such that $I \sqsubseteq I''$. Using Fact 38 and Lemma 41(a), we get $\mathrm{Cv}_{\mathcal{M}_I} \leq_{\mathrm{T}} \mathrm{Cv}_{\mathcal{M}_{I''}} \leq_{\mathrm{T}} \mathcal{M}_{I''} \leq_{\mathrm{T}} \mathcal{M}_{I'}$ $\qquad\qquad \square$

We are ready to prove Theorem 11. Since $\min\mathrm{non}(\vec{\mathcal{M}}) = \mathrm{non}(\mathcal{M}\mathcal{A})$, it is already known from [17, Lemma 2.3] that $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \min\mathrm{non}(\vec{\mathcal{M}})\}$.

Theorem 44. *For all $I \in \mathbb{I}$,*

$$\mathrm{add}(\mathcal{M}_I) = \mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \min\mathrm{non}(\vec{\mathcal{M}})\},$$
$$\mathrm{cof}(\mathcal{M}_I) = \mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \sup\mathrm{cov}(\vec{\mathcal{M}})\}.$$

In terms of Tukey connections, ${}^{\omega}\omega \times \bigvee_{J \in \mathbb{I}} \mathrm{Cv}_{\mathcal{M}_J} \leq_{\mathrm{T}} \mathcal{M}_I \leq_{\mathrm{T}} \mathcal{M} \leq_{\mathrm{T}} \big(\mathbb{I}; \bigvee_{J \in \mathbb{I}} \mathrm{Cv}_{\mathcal{M}_J}\big)$.

Proof. Thanks to Theorem 21, it is enough to prove the Tukey connections. Since we already have $\mathcal{M}_I \leq_{\mathrm{T}} \mathcal{M}$ by Lemma 39, the following inequalities remain.

- ${}^{\omega}\omega \times \bigvee_{J\in\mathbb{I}} \mathrm{Cv}_{\mathcal{M}_J} \leq_{\mathrm{T}} \mathcal{M}_I$: By Lemma 43 and Theorem 21, $\bigvee_{J\in\mathbb{I}} \mathrm{Cv}_{\mathcal{M}_J} \leq_{\mathrm{T}} \mathcal{M}_I$. On the other hand, ${}^{\omega}\omega \leq_{\mathrm{T}} \mathcal{M}_I$ by Lemma 42, so we can use Lemma 23 to conclude the desired inequality.
- $\mathcal{M} \leq_{\mathrm{T}} \big(\mathbb{I}; \bigvee_{J\in\mathbb{I}} \mathrm{Cv}_{\mathcal{M}_J}\big)$: We can use $\langle \mathbb{I} \times {}^{\omega}2, \sqsubset^{I^1}\rangle$ instead of $\mathcal{M}$. Define $\Psi_- : \mathbb{I} \times {}^{\omega}2 \to \mathbb{I} \times {}^{\mathbb{I}}(\mathbb{I} \times {}^{\omega}2)$ by $\Psi_-(I,y) := (I,\bar{y})$, where $\bar{y}: \mathbb{I} \to \mathbb{I} \times {}^{\omega}2$, $J \mapsto (J,y)$, and define $\Psi_+ : \mathbb{I} \times \prod_{J\in\mathbb{I}} \mathcal{M}_J \to \mathbb{I} \times {}^{\omega}2$ by $\Psi_+(J,\bar{B}) := (J',y')$, where (J',y') is chosen such that $B_J \subseteq A^J_{J',y'}$. This is the required Tukey connection: if $I \sqsubseteq J$ and $\bar{y}(J) \sqsubset^{\vee} \bar{B}$, i.e. $y \in B_J$, then $y \in A^J_{J',y'}$, so $(I,y) \sqsubset^{I^1} (J',y')$. $\qquad\square$

As a consequence, we obtain the following inequalities (although $\mathrm{add}(\mathcal{M}) \leq \mathrm{add}(\mathcal{M}\mathcal{A})$ is easy to prove directly).

Corollary 45. *For any dominating $D \subseteq \mathbb{I}$,*

$$\mathcal{M}\mathcal{A} \leq_{\mathrm{T}} \prod_{I\in D} \mathcal{M}_I \leq_{\mathrm{T}} \mathcal{M}^D.$$

In particular, $\mathcal{M}\mathcal{A} \leq_{\mathrm{T}} \mathcal{M}^{\mathfrak{d}} \leq_{\mathrm{T}} \mathrm{Cv}^{\mathfrak{d}}_{[\mathrm{cof}(\mathcal{M})]^{<\mathrm{add}(\mathcal{M})}}$, $\mathrm{add}(\mathcal{M}) \leq \mathrm{add}(\mathcal{M}\mathcal{A})$ and $\mathrm{cof}(\mathcal{M}\mathcal{A}) \leq \mathfrak{d}(\mathcal{M}^{\mathfrak{d}}) \leq \mathrm{cov}\big(\big([\mathrm{cof}(\mathcal{M})]^{<\mathrm{add}(\mathcal{M})}\big)^{\mathfrak{d}}\big)$.

Next, we show how $\mathcal{M}_I$ is related with the measure zero ideal. From now on, Lb denotes the Lebesgue measure on ${}^{\omega}2$.

Lemma 46. *For $I \in \mathbb{I}$, the following statements are equivalent.*

(i) $\mathcal{M}_I \subseteq \mathcal{N}$.
(ii) $\mathcal{M}_I \subseteq \mathcal{E}$.
(iii) $\sum_{k<\omega} 2^{-|I_k|} < \infty$.

Proof. Since $\mathcal{M}_I$ is generated by F_σ sets, (i) $\Leftrightarrow$ (ii) is clear. So it suffices to prove (i) $\Leftrightarrow$ (iii). Notice that

$$A^I_{J,y} = \bigcup_{m<\omega} \bigcap_{n\geq m} \bigcup_{\substack{k<\omega \\ I_k\subseteq J_n}} [y\restriction I_k].$$

"$\Leftarrow$". Assume that $\sum_{k<\omega} 2^{-|I_k|} < \infty$. Let $X \in \mathcal{M}_I$, so there are $J \in \mathbb{I}$ and $y \in {}^{\omega}2$ such that $X \subseteq A^I_{J,y}$. To see that $X \in \mathcal{N}$, it is sufficient to show that

$$\mathrm{Lb}\left(\bigcup_{m<\omega} \bigcap_{n\geq m} \bigcup_{\substack{k<\omega \\ I_k\subseteq J_n}} [y\restriction I_k] \right) = 0.$$

Observe that

$$\mathrm{Lb}\left(\bigcup_{\substack{k<\omega \\ I_k\subseteq J_n}} [y\restriction I_k] \right) \leq \sum_{\substack{k<\omega \\ I_k\subseteq J_n}} 2^{-|I_k|}.$$

Since $\sum_{k<\omega} 2^{-|I_k|} < \infty$, $\bigcap_{n\geq m} \bigcup_{k<\omega,\, I_k\subseteq J_n} [y\restriction I_k]$ has measure zero. Therefore, $A^I_{J,y} \in \mathcal{N}$.

"$\Rightarrow$". Assume that $\sum_{k<\omega} 2^{-|I_k|} = \infty$. Let $J \in \mathbb{I}$ be coarser than I such that

$$\sum_{\substack{k<\omega \\ I_k\subseteq J_n}} 2^{-|I_k|} \geq n.$$

We prove that $A^I_{J,y} \notin \mathcal{N}$, in fact $\mathrm{Lb}({}^{\omega}2 \smallsetminus A^I_{J,y}) = 0$. Notice that

$${}^{\omega}2 \smallsetminus A^I_{J,y} = \bigcap_{m<\omega} \bigcup_{n\geq m} \bigcap_{\substack{k<\omega \\ I_k\subseteq J_n}} \big({}^{\omega}2 \smallsetminus [y\restriction I_k]\big).$$

To show $\mathrm{Lb}(^{\omega}2 \smallsetminus A_{J,y}^I) = 0$, it is enough to see that

$$\lim_{m \to \infty} \mathrm{Lb}\left(\bigcup_{n \geq m} \bigcap_{\substack{k < \omega \\ I_k \subseteq J_n}} {}^{\omega}2 \smallsetminus [y \restriction I_k]\right) = 0.$$

Indeed,

$$\mathrm{Lb}\left(\bigcap_{\substack{k < \omega \\ I_k \subseteq J_n}} {}^{\omega}2 \smallsetminus [y \restriction I_k]\right) = \prod_{\substack{k < \omega \\ I_k \subseteq J_n}} (1 - 2^{-|I_n|})$$

$$\leq e^{-\sum_{k < \omega, I_k \subseteq J_n} 2^{-|I_k|}}$$

$$\leq e^{-n}$$

Since $\sum_{n < \omega} e^{-n} < \infty$, we conclude the desired limit. $\hspace{2cm}\square$

The previous proof allows to say a bit more in the case $\sum_{k < \omega} 2^{-|I_k|} = \infty$.

Lemma 47. *If* $\sum_{k < \omega} 2^{-|I_k|} = \infty$ *then* $\mathrm{Cv}_{\mathscr{N}} \leq_{\mathrm{T}} \mathrm{Cv}_{\mathscr{M}_I}^{\perp}$. *In particular,* $\mathrm{cov}(\mathscr{N}) \leq \mathrm{non}(\mathscr{M}_I)$ *and* $\mathrm{cov}(\mathscr{M}_I) \leq \mathrm{non}(\mathscr{N})$.

Proof. In the proof of Lemma 46, we produced a set $X \in \mathscr{M}_I$ such that $^{\omega}2 \smallsetminus X \in \mathscr{N}$. Since both ideals $\mathscr{M}_I$ and $\mathscr{N}$ are translation invariant, we obtain $\mathrm{Cv}_{\mathscr{N}} \leq_{\mathrm{T}} \mathrm{Cv}_{\mathscr{M}_I}^{\perp}$ as in Rothberger's proof of $\mathrm{Cv}_{\mathscr{N}} \leq_{\mathrm{T}} \mathrm{Cv}_{\mathscr{M}}^{\perp}$, namely, using the maps $x \mapsto x + X$ and $y \mapsto y + (^{\omega}2 \smallsetminus X)$. $\hspace{1cm}\square$

Using Lemmas 46 and 47, Theorem 13 is concluded.

We finish this section with connections between our ideals and the relational systems introduced in [14] to study $\mathrm{Cv}_{\mathscr{M}\mathscr{A}}$.

Definition 48 ([14, Definition 2.7]). *Fix* $b \in {}^{\omega}\omega$.

   (1) *For* $I \in \mathbb{I}$ *and* $f, h \in {}^{\omega}\omega$, *define*

$$f \sqsubset^{\bullet} (I, h) \quad \textit{iff} \quad \forall^{\infty} n \in \omega \ \exists k \in I_n \colon f(k) = h(k).$$

   (2) *Define the relational system* $\mathbf{R}_b := \left\langle \prod b, \mathbb{I} \times \prod b, \sqsubset^{\bullet} \right\rangle$, *where* $\prod b := \prod_{n < \omega} b(n)$.

In the context of $\mathbf{R}_b$, *we will always consider that* $b(n) > 0$ *for all* n, *even if we just write "$b \in {}^{\omega}\omega$".*

Notice that, for fixed $(I, h) \in \mathbb{I} \times \prod b$, $\left\{ f \in \prod b : f \sqsubset^{\bullet} (I, h) \right\}$ is meager whenever $b \geq^* 2$, so $\mathrm{Cv}_{\mathscr{M}} \leq_{\mathrm{T}} \mathbf{R}_b$, which implies $\mathfrak{b}(\mathbf{R}_b) \leq \mathrm{non}(\mathscr{M})$ and $\mathrm{cov}(\mathscr{M}) \leq \mathfrak{d}(\mathbf{R}_b)$.[1] On the other hand, if $b \not\geq^* 2$ then we can find some $(I, h) \in \mathbb{I} \times \prod b$ such that $f \sqsubset^{\bullet} (I, h)$ for all $f \in \prod b$, so $\mathfrak{d}(\mathbf{R}_b) = 1$ and $\mathfrak{b}(\mathbf{R}_b)$ is undefined.

Fact 49. *For* $b \in \omega^{\omega}$, $\mathbf{R}_b \cong_{\mathrm{T}} \left\langle \prod b, \mathbb{I} \times {}^{\omega}\omega, \sqsubset^{\bullet} \right\rangle$. *As a consequence, if* $b' \in \omega^{\omega}$ *and* $b \leq^* b'$, *then* $\mathbf{R}_b \leq_{\mathrm{T}} \mathbf{R}_{b'}$. *In particular,* $\mathfrak{b}(\mathbf{R}_{b'}) \leq \mathfrak{b}(\mathbf{R}_b)$ *and* $\mathfrak{d}(\mathbf{R}_b) \leq \mathfrak{d}(\mathbf{R}_{b'})$.

In [14] we have proved that $\mathbf{R}_b \leq_{\mathrm{T}} \mathrm{Cv}_{\mathscr{M}\mathscr{A}} \leq_{\mathrm{T}} \prod_{b \in D} \mathbf{R}_b$ for any $b \in {}^{\omega}\omega$ and any dominating $D \subseteq {}^{\omega}\omega$, and also that $\mathfrak{d}(\mathbf{R}_b) \leq \mathrm{cof}(\mathscr{M})$. These facts can be deduced from the following result that connects $\mathbf{R}_b$ and $\mathrm{Cv}_{\mathscr{M}_I}$.

Lemma 50. *For* $I \in \mathbb{I}$, *if* $\forall^{\infty} k : b(k) = 2^{|I_k|}$ *then* $\mathbf{R}_b \cong_{\mathrm{T}} \mathrm{Cv}_{\mathscr{M}_I}$.

Proof. Assume wlog that, for all $k \in \omega$, $b(k) = 2^{|I_k|}$, so $\prod b = 2^I := \prod_{k < \omega} {}^{I_k}2$. Define

$$f : 2^I \longrightarrow {}^{\omega}2$$

$$\eta \longmapsto \bigcup_{n < \omega} \eta(n)$$

---

[1] In [7], these are denoted by $\mathfrak{b}_b^{\mathrm{eq}}$ and $\mathfrak{d}_b^{\mathrm{eq}}$, respectively.

Note that f is a bijection.

Let $\mathbb{I}_I$ be the set of $J \in \mathbb{I}$ coarser than I. Define the maps

$$g\colon \mathbb{I}_I \longrightarrow \mathbb{I}$$

and $\qquad h\colon \mathbb{I} \times 2^I \longrightarrow \mathbb{I}_I \times {}^{\omega}2$

by the assignments

$$J \longmapsto \big\langle \big\{k < \omega \,:\, I_k \subseteq J_n\big\} \,:\, n < \omega \big\rangle$$

and $\qquad (J, y) \longmapsto \big(g^{-1}(J), f(y)\big),$

respectively. These maps are bijections as well.

To define $\Psi_+\colon \mathscr{M}_I \to \mathbb{I} \times 2^I$, for $X \in \mathscr{M}_I$, pick an $(J, y) \in \mathbb{I}_I \times {}^{\omega}2$ such that $X \subseteq A^I_{J,y}$ and put $\Psi_+(X) \coloneqq h^{-1}(J, y)$. Notice that $f(\eta) \in A^I_{J,y}$ iff $\forall^{\infty} n\ \exists k\colon I_k \subseteq J_n$ and $\eta(k) = y{\restriction}I_k$, which is equivalent to $\forall^{\infty} n\ \exists k \in J'_n\colon \eta(k) = y{\restriction}I_k$ where $J' = g(J)$, i.e. $\eta \sqsubset^{\bullet} \Psi_+(X)$. Then (f, Ψ_+) witnesses $\mathbf{R}_b \leq_{\mathrm{T}} \mathrm{Cv}_{\mathscr{M}_I}$.

To prove $\mathrm{Cv}_{\mathscr{M}_I} \leq_{\mathrm{T}} \mathbf{R}_b$, define

$$\Psi'_+\colon \mathbb{I} \times 2^I \longrightarrow \mathscr{M}_I$$

$$(J', \eta) \longmapsto A^I_{(g^{-1}(J'), f(\eta))}.$$

It is clear that (f^{-1}, Ψ'_+) witnesses $\mathrm{Cv}_{\mathscr{M}_I} \leq_{\mathrm{T}} \mathbf{R}_b$. Indeed, $f^{-1}(x) \sqsubset^{\bullet} (J', \eta)$ iff $\forall^{\infty} n\ \exists k \in J'_n\colon x{\restriction}I_k = \eta(k)$, which is equivalent to $\forall^{\infty} n\ \exists k\colon I_k \subseteq J_n$ and $x{\restriction}I_k = \eta(k)$, i.e. $x \in A^I_{(g^{-1}(J'), f(\eta))}$. $\qquad\square$

Because of the above result, we obtain the following.

Corollary 51.
 (1) *If $\forall^{\infty} k\colon \log_2 b(k) \leq |I_k|$, then $\mathbf{R}_b \leq_{\mathrm{T}} \mathrm{Cv}_{\mathscr{M}_I}$.*
 (2) *If $\forall^{\infty} k\colon |I_k| \leq \log_2 b(k)$, then $\mathrm{Cv}_{\mathscr{M}_I} \leq_{\mathrm{T}} \mathbf{R}_b$.*

Corollary 52. $\min_{b \in {}^{\omega}\omega} \mathfrak{b}(\mathbf{R}_b) = \min \mathrm{non}(\vec{\mathscr{M}})$ *and* $\sup_{b \in {}^{\omega}\omega} \mathfrak{d}(\mathbf{R}_b) = \sup \mathrm{cov}(\vec{\mathscr{M}})$.[2]

4. Consistency results

In this section, we prove our consistency results, i.e. Theorems 14–15. We start by reviewing some notation.

Definition 53. *Given a sequence of non-empty sets $b = \big\langle b(n)\colon n \in \omega \big\rangle$ and $h\colon \omega \to \omega$, define*

$$\prod b \coloneqq \prod_{n \in \omega} b(n),$$

and $\qquad \mathscr{S}(b, h) \coloneqq \prod_{n \in \omega} \big[b(n)\big]^{\leq h(n)}.$

For two functions $x \in \prod b$ and $\varphi \in \mathscr{S}(b, h)$ write

$$x \in^* \varphi \quad \textit{iff} \quad \forall^{\infty} n \in \omega\colon x(n) \in \varphi(n).$$

Let $\mathrm{Lc}(b, h) \coloneqq \big\langle \prod b, \mathscr{S}(b, h), \in^* \big\rangle$, *which is a relational system. Denote* $\mathfrak{b}^{\mathrm{Lc}}_{b,h} = \mathfrak{b}\big(\mathrm{Lc}(b, h)\big)$ *and* $\mathfrak{d}^{\mathrm{Lc}}_{b,h} = \mathfrak{d}\big(\mathrm{Lc}(b, h)\big)$.[3] *Also define*

$$\min \mathrm{Lc} \coloneqq \min\big\{\mathfrak{b}^{\mathrm{Lc}}_{b,\mathrm{id}} \,:\, b \in {}^{\omega}\omega\big\} \quad \textit{and} \quad \sup \mathrm{Lc} \coloneqq \sup\big\{\mathfrak{d}^{\mathrm{Lc}}_{b,\mathrm{id}} \,:\, b \in {}^{\omega}\omega\big\}.$$

Recall the following characterization. Pawlikowski [17] characterized $\mathrm{add}(\mathscr{N})$, while $\mathrm{cof}(\mathscr{N})$ is due to the first and second author [10, Lemma 3.11].

---

[2] The equality $\min_{b \in {}^{\omega}\omega} \mathfrak{b}(\mathbf{R}_b) = \mathrm{non}(\mathscr{M}\mathscr{A})$ is originally due to Bartoszyński and Judah [1].

[3] In [8,16], these are denoted by $\mathfrak{b}^{\mathrm{eq}}_b$ and $\mathfrak{d}^{\mathrm{eq}}_b$, respectively.

Theorem 54. $\mathrm{add}(\mathcal{N}) = \min\{\mathfrak{b}, \min \mathrm{Lc}\}$ *and* $\mathrm{cof}(\mathcal{N}) = \max\{\mathfrak{d}, \sup \mathrm{Lc}\}$.

Below, we strengthen the characterization of $\mathcal{N\!A}$ from Theorem 8(1).

Theorem 55. *Let* $X \subseteq {}^{\omega}2$ *and* $D \subseteq \mathbb{I}$ *a dominating family. Then* $X \in \mathcal{N\!A}$ *iff, for all* $I \in D$*, there is some* $\varphi \in \prod_{n\in\omega} \mathscr{P}({}^{I_n}2)$ *such that* $\forall\, n \in \omega : |\varphi(n)| \le n$ *and* $X \subseteq H_{\varphi}$.

Proof. Under Theorem 8(1), we only need to prove "$\Leftarrow$", that is, the statement about D implies the same statement but for $\mathbb{I}$ in the place of D.

Let $J \in \mathbb{I}$ be such that $|J_n| = (n+1)^2$ for all $n < \omega$, and fix $I \in \mathbb{I}$. Set $I'_n := \bigcup_{k \in J_n} I_k$, so $I' \in \mathbb{I}$. Since D is $\mathbb{I}$-dominating, we can find an $I^* \in D$ such that, for all but finitely many $n < \omega$, I^* contains at least two intervals from I'. Therefore, there is some $\varphi^* \in \mathscr{S}({}^{I^*}2, \mathrm{id})$ such that $X \subseteq H_{\varphi^*}$. It is enough to construct a $\varphi \in \mathscr{S}({}^{I}2, \mathrm{id})$ such that $H_{\varphi^*} \subseteq H_{\varphi}$.

We first show that, for all but finitely many $n < \omega$ and for any $k < \omega$, if $I_k \subseteq I^*_n$ then $k \ge n(n+1)$: For any $\ell < \omega$, I'_{ℓ} is composed of $(\ell+1)^2$-many intervals from I, so $I_k \subseteq I'_{\ell}$ iff $\frac{\ell(\ell+1)(2\ell+1)}{6} \le k < \frac{(\ell+1)(\ell+2)(2\ell+3)}{6}$. Now, for all but finitely many n, we have that whenever I'_{ℓ} intersects I^*_n, $\ell \ge n$ and $\ell(\ell+1) < \frac{(\ell-1)\ell(2\ell-1)}{6}$. Therefore, for any $I_k \subseteq I^*_n$, $n(n+1) \le \ell'(\ell'+1) < \frac{(\ell'-1)\ell'(2\ell'-1)}{6} \le k$, where ℓ' is the smallest ℓ such that $I'_{\ell} \cap I^*_n \ne \emptyset$.

Fix an n as above. Whenever $I_k \subseteq I^*_n$, define $\varphi(k) := \{t \!\restriction\! I_k : t \in \varphi^*(n)\}$, so $|\varphi(k)| \le |\varphi^*(n)| \le n \le n(n+1) \le k$; and whenever I_k intersects both I^*_n and I^*_{n+1}, define

$$\varphi(k) := \{s \!\restriction\! (I_k \cap I^*_n) \cup t \!\restriction\! (I_k \cap I^*_{n+1}) \, : \, s \in \varphi^*(n),\ t \in \varphi^*(n+1)\},$$

so $|\varphi(k)| \le |\varphi^*(n)| \cdot |\varphi^*(n+1)| \le n(n+1) \le k$.

So far, we have defined $\varphi(k)$ for all but finitely many $k < \omega$, so for the remaining ones set $\varphi(k) := \emptyset$. Therefore, $\varphi \in \mathscr{S}({}^{I}2, \mathrm{id})$.

Lastly, it is clear that $H_{\varphi^*} \subseteq H_{\varphi}$, as required. $\square$

Thanks to the above, we have the following convenient Tukey connection.

Theorem 56. *For any dominating family* $D \subseteq \omega^{\omega}$*, we have* $\mathrm{Cv}_{\mathcal{N\!A}} \le_{\mathrm{T}} \prod_{b \in D} \mathrm{Lc}(b, \mathrm{id})$.

Proof. Without loss of generality, we may assume that there is some $\mathbb{I}$-dominating family D_0 such that, for each $b \in D$, there is some $I \in D_0$ such that $b = {}^{I}2$, i.e. $b(n) = {}^{I_n}2$ for all $n < \omega$.

We define $\Psi_- : {}^{\omega}2 \to \prod_{I \in D_0} {}^{I}2$ and $\Psi_+ : \prod_{I \in D_0} \mathscr{S}({}^{I}2, \mathrm{id}) \to \mathcal{N\!A}$ as follows: for $x \in {}^{\omega}2$ and $I \in D_0$, set $x_I := \langle x \!\restriction\! I_n : n \in \omega \rangle$ and define $\Psi_-(x) := \langle x_I : I \in D_0 \rangle$; for $\overline{\varphi} = \langle \varphi_I : I \in D_0 \rangle \in \prod_{I \in D_0} \mathscr{S}({}^{I}2, \mathrm{id})$ define

$$\Psi_+(\overline{\varphi}) := \bigcap_{I \in D_0} \{x \in {}^{\omega}2 : \forall^{\infty}\, n \in \omega : x \!\restriction\! I_n \in \varphi_I(n)\}.$$

Notice that $\Psi_+(\overline{\varphi}) \in \mathcal{N\!A}$ by Theorem 55. It is clear that, for any $x \in {}^{\omega}2$ and $\overline{\varphi} = \langle \varphi_I : I \in D_0 \rangle \in \prod_{I \in D_0} \mathscr{S}({}^{I}2, \mathrm{id})$, $\Psi_-(x) \sqsubseteq^{\circledast} \overline{\varphi}$ implies $x \in \Psi_+(\overline{\varphi})$. $\square$

We are ready to prove our consistency results. First, we review some terminology.

Definition 57. *Define the following forcing notions.*

(1) *Fix* b *and* h *as in Definition 53. Define* $\mathbb{LOC}_{b,h}$ *as the poset whose conditions are pairs* $p = (\varphi_p, n_p)$ *where* $\varphi_p \in \mathscr{S}(b, h)$ *and* $n_p < \omega$ *such that, for some* $m_p < \omega$, $|\varphi_p(i)| \le m_p$ *for all* $i < \omega$. *The order is defined by* $q \le p$ *iff* $n_p \le n_q$, $\varphi_q(i) = \varphi_p(i)$ *for all* $i < n_p$ *and* $\forall\, i < \omega : \varphi_p(i) \subseteq \varphi_q(i)$.

(2) *Hechler forcing is defined by* $\mathbb{D} = \omega^{<\omega} \times {}^{\omega}\omega$*, ordered by* $(t, g) \le (s, f)$ *if* $s \subseteq t$, $f \le g$ *and* $f(i) \le t(i)$ *for all* $i \in |t| \smallsetminus |s|$. *This forcing is used to increase* $\mathfrak{b}$. *Recall that* $\mathbb{D}$ *is* σ*-centered.*

(3) *For an infinite cardinal* θ*,* $\mathrm{Fn}_{<\theta}(A, B)$ *denotes the poset of partial functions from* A *into* B *of size* $< \theta$*, ordered by* $\supseteq$.

Theorem 58. *Let $\theta \leq \lambda$ and ν be uncountable cardinals such that θ is regular, $\lambda^{\aleph_0} = \lambda$ and $\mathfrak{d}_\theta = \nu$.[4] Then there is a ccc poset forcing*

$$\operatorname{add}(\mathcal{N}) = \operatorname{cof}(\mathcal{N}) = \theta \leq \operatorname{cov}(\mathcal{NA}) \leq \nu$$

and $\qquad \mathfrak{c} = \lambda.$

In particular, it is consistent with ZFC *that* $\operatorname{cov}(\mathcal{NA}) < \lambda$ *(starting with $\nu < \lambda$).*

Here, $\mathfrak{d}_\theta$ is the canonical dominating number of $^\theta\theta$, which coincides with $\mathfrak{d}\big(\langle\theta,\theta,\leq\rangle^\theta\big)$. It is easy to force $\mathfrak{d}_\theta = \nu$, e.g. with $\operatorname{Fn}_{<\theta}(\nu,2)$ (by assuming $\theta^{<\theta} = \theta$ and $\nu^\theta = \nu$ in the ground model).

Proof. Perform a FS iteration $\mathbb{P} = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi : \xi < \lambda\theta \rangle$ ($\lambda\theta$ as product of ordinal numbers) where $\mathbb{Q}_\xi$ is a $\mathbb{P}_\xi$-name of $\mathbb{D} * \mathbb{LOC}_{\dot{d}_\xi,\mathrm{id}}$ where $\dot{d}_\xi$ is the name of the dominating real over $V_\xi := V^{\mathbb{P}_\xi}$ added by $\mathbb{D}$. Using the cofinaly-many Cohen and dominating reals $\langle \dot{d}_{\lambda\rho} : \rho < \theta \rangle$ added along the iteration and $\operatorname{cf}(\lambda\theta) = \theta$, we obtain $\mathfrak{b} = \operatorname{non}(\mathcal{M}) = \operatorname{cov}(\mathcal{M}) = \mathfrak{d} = \theta$. which implies $\operatorname{add}(\mathcal{M}) = \operatorname{cof}(\mathcal{M}) = \theta$. Even more, we obtain $\omega^\omega \cong_\mathrm{T} \operatorname{Cv}_\mathcal{M} \cong_\mathrm{T} \theta$.

In the final generic extension $V_{\lambda\theta}$, it is clear that $D := \{d_{\lambda\rho} : \rho < \theta\}$ is $\leq^*$-increasing and dominating in $^\omega\omega$. Denote $d'_\rho := d_{\lambda\rho}$ for $\rho < \theta$. We show that $\operatorname{Lc}(d'_\rho, \mathrm{id}) \cong_\mathrm{T} \theta$. On the one hand, $\theta \cong_\mathrm{T} \operatorname{Cv}_\mathcal{M} \leq_\mathrm{T} \operatorname{Lc}(d'_\rho, \mathrm{id})$. For the converse, define $F: \prod d'_\rho \to \theta$ such that, for $x \in \prod d'_\rho$, $F(x)$ is some ordinal $\eta > \rho$ such that $x \in V_{\lambda\eta}$; and define $F': \theta \to \mathscr{S}(d'_\rho, \mathrm{id})$ such that $F'(\varrho)$ is the $\mathbb{LOC}_{d'_\varrho,\mathrm{id}}$-generic real added by $\mathbb{Q}_{\lambda\varrho}$ when $\varrho \geq \rho$, otherwise $F'(\varrho) := F'(\rho)$. It is clear that (F, F') is the desired Tukey connection.

Since $\operatorname{Lc}(d'_\rho, \mathrm{id}) \cong_\mathrm{T} \theta$, we get $\min \operatorname{Lc} = \sup \operatorname{Lc} = \theta$, so by Theorem 54 $\operatorname{add}(\mathcal{N}) = \operatorname{cof}(\mathcal{N}) = \theta$. On the other hand, given that $\mathbb{P}$ is ccc, the equality $\mathfrak{d}_\theta = \nu$ is preserved (see e.g. [12, Lemma 6.6]). Now, by Theorem 56,

$$\operatorname{Cv}_{\mathcal{NA}} \leq_\mathrm{T} \prod_{b \in D} \operatorname{Lc}(b, \mathrm{id}) \cong_\mathrm{T} \langle\theta, \leq\rangle^\theta,$$

so $\operatorname{cov}(\mathcal{NA}) \leq \mathfrak{d}_\theta = \nu$. $\qquad\qquad \square$

Finally, we show that $\operatorname{cof}(\mathcal{MA}) < \operatorname{non}(\mathcal{SN})$ is valid in our model from [14, Theorem 4.2].

Theorem 59. *Let $\theta \leq \lambda$ and ν be uncountable cardinals such that θ is regular, $\mathfrak{d}_\theta = \nu$ and $\lambda^{\aleph_0} = \lambda$. Then there is a ccc poset forcing*

$$\operatorname{cov}(\mathcal{N}) = \aleph_1 \leq \operatorname{add}(\mathcal{M}) = \operatorname{cof}(\mathcal{M}) = \theta \leq \operatorname{non}(\mathcal{N}) = \mathfrak{c} = \lambda$$

and $\qquad \operatorname{cof}(\mathcal{MA}) \leq \nu.$

In particular, it is consistent with ZFC *that* $\operatorname{cof}(\mathcal{MA}) < \operatorname{non}(\mathcal{N})$.

Proof. Let $\mathbb{P}$ be the FS iteration of ccc posets constructed in the proof of [14, Theorem 4.2]. It remains to show that $\mathbb{P}$ forces $\operatorname{cof}(\mathcal{MA}) \leq \nu$. Since $\mathbb{P}$ has the ccc, the identity $\mathfrak{d}_\theta = \nu$ is preserved in the generic extension. Now, in the generic extension, by Corollary 45, $\operatorname{cof}(\mathcal{MA}) \leq \operatorname{cov}\big(([\theta]^{<\theta})^\theta\big) = \mathfrak{d}_\theta = \nu$. $\qquad\qquad \square$

We still do not know how to control the values of $\operatorname{cov}(\mathcal{NA})$, $\operatorname{cov}(\mathcal{MA})$ and $\operatorname{cof}(\mathcal{MA})$ in the previous models. In the case of the cofinality, we may need to develop some framework for lower bounds as we did in [12] for the Yorioka ideals. In particular, we ask the following.

Question 60. *Is it consistent with* ZFC *that* $\operatorname{cof}(\mathcal{MA}) > \mathfrak{c}$?

Concerning $\mathcal{NA}$ we may ask the following.

Question 61. *Does* ZFC *proves an inequality between* $\operatorname{cof}(\mathcal{NA})$ *and* $\mathfrak{c}$?

We have some ideas to construct a directed scheme for $\mathcal{NA}$ to prove the consistency of $\operatorname{cof}(\mathcal{NA}) < \mathfrak{c}$, but we will develop them in the second part of this work.

---

[4]Notice that no inequality is assumed between ν and λ.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

References

[1]   T. Bartoszyński and H. Judah, "Borel images of sets of reals", *Real Anal. Exch.* **20** (1994/95), no. 2, pp. 536–558.

[2]   T. Bartoszyński, W. Just and M. Scheepers, "Covering games and the Banach–Mazur game: K-tactics", *Can. J. Math.* **45** (1993), no. 5, pp. 897–929.

[3]   T. Bartoszyński and S. Shelah, "Closed measure zero sets", *Ann. Pure Appl. Logic* **58** (1992), no. 2, pp. 93–110.

[4]   A. Blass, "Combinatorial cardinal characteristics of the continuum", in *Handbook of set theory. Vols. 1, 2, 3* (M. Foreman and A. Kanamori, eds.), Springer, 2010, pp. 395–489.

[5]   J. Brendle, M. A. Cardona and D. A. Mejía, "Separating cardinal characteristics of the strong measure zero ideal", *J. Math. Log.* (2025). Online ready.

[6]   M. A. Cardona, "On cardinal characteristics associated with the strong measure zero ideal", *Fundam. Math.* **257** (2022), no. 3, pp. 289–304.

[7]   M. A. Cardona, "Cardinal invariants associated with the combinatorics of the uniformity number of the ideal of meager-additive sets", *RIMS Kokyuroku* **2315** (2025), pp. 1–24.

[8]   M. A. Cardona, L. D. Klausner and D. A. Mejía, "Continuum many different things: localisation, anti-localisation and Yorioka ideals", *Ann. Pure Appl. Logic* **175** (2024), no. 7, article no. 103453 (58 pages).

[9]   M. A. Cardona, A. Marton and J. Šupina, "Cardinal characteristics associated with small subsets of reals", 2024. Online at https://arxiv.org/abs/2405.11312. To appear in *Fundam. Math.*

[10] M. A. Cardona and D. A. Mejía, "On cardinal characteristics of Yorioka ideals", *Math. Log. Q.* **65** (2019), no. 2, pp. 170–199.

[11] M. A. Cardona and D. A. Mejía, "Localization and anti-localization cardinals", *RIMS Kokyuroku* **2261** (2023), pp. 47–77.

[12] M. A. Cardona and D. A. Mejía, "More about the cofinality and the covering of the ideal of strong measure zero sets", *Ann. Pure Appl. Logic* **176** (2025), no. 4, article no. 103537 (31 pages).

[13] M. A. Cardona, D. A. Mejía and I. E. Rivera-Madrid, "The covering number of the strong measure zero ideal can be above almost everything else", *Arch. Math. Logic* **61** (2022), no. 5–6, pp. 599–610.

[14] M. A. Cardona, D. A. Mejía and I. E. Rivera-Madrid, "Uniformity numbers of the null-additive and meager-additive ideals", *J. Symb. Log.* (2025). Online first.

[15] F. Galvin, J. Mycielski and R. Solovay, "Strong measure zero sets", *Notices Am. Math. Soc.* **26** (1979), no. 3, pp. A–280. Abstract 79T-E25.

[16] L. D. Klausner and D. A. Mejía, "Many different uniformity numbers of Yorioka ideals", *Arch. Math. Logic* **61** (2022), no. 5–6, pp. 653–683.

[17] J. Pawlikowski, "Powers of transitive bases of measure and category", *Proc. Am. Math. Soc.* **93** (1985), no. 4, pp. 719–729.

[18] W. Seredyński, "Some operations related with translation", *Colloq. Math.* **57** (1989), no. 2, pp. 203–219.

[19] S. Shelah, "Every null-additive set is meager-additive", *Isr. J. Math.* **89** (1995), no. 1–3, pp. 357–376.

[20] M. Talagrand, "Compacts de fonctions mesurables et filtres non mesurables", *Stud. Math.* **67** (1980), no. 1, pp. 13–43.

[21] P. Vojtáš, "Generalized Galois–Tukey-connections between explicit relations on classical objects of real analysis", in *Set theory of the reals (Ramat Gan, 1991)* (H. Judah, ed.), Israel Mathematical Conference Proceedings, vol. 6, Bar-Ilan University, 1993, pp. 619–643.

[22] T. Yorioka, "The cofinality of the strong measure zero ideal", *J. Symb. Log.* **67** (2002), no. 4, pp. 1373–1384.

[23] O. Zindulka, "Strong measure zero and meager-additive sets through the prism of fractal measures", *Commentat. Math. Univ. Carol.* **60** (2019), no. 1, pp. 131–155.

[24] O. Zindulka, "Meager-additive sets in topological groups", *J. Symb. Log.* **87** (2022), no. 3, pp. 1046–1064.