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The derivative of the fractional discrete Laplacian is an exotic Riesz potential

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Abstract. Let Δ_N be the multidimensional discrete Laplacian on $\mathbb{Z}^N$ $(N \geq 1)$. In this note, we prove that, when $N = 1$, the right-hand derivative of $(-\Delta_1)^s$ at 0 is an exotic discrete Riesz potential (namely, the endpoint case: the order is 0) in Stein–Wainger sense (*J. Anal. Math.*, 2000), and when $N \geq 2$, the corresponding derivative is also an exotic discrete Riesz potential with an additional corrector. A similar conclusion for the left-hand derivative case is also considered. All results obtained in this note extend the logarithmic Laplacian of Chen–Weth (*Commun. Partial Differ. Equations*, 2019) to the discrete setting.

Keywords. Discrete harmonic analysis, fractional Laplacian, Riesz potential.

2020 Mathematics Subject Classification. 39A12, 35R11, 39A12.

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1. Introduction

In recent years, a great deal of mathematical effort in boundary value problems involving (non-)linear integro-differential operators has been devoted to the study of the fractional power of the Laplace operator. For any $0 < s < 1$, the fractional Laplacian $(-\Delta)^s$ on $\mathbb{R}^d$ is defined via Fourier transform as

$$\widehat{(-\Delta)^s f}(\xi) = |\xi|^{2s}\widehat{f}(\xi), \quad f \in C_0^\infty(\mathbb{R}^d),$$

and it can be expressed by the pointwise formula

$$(-\Delta)^s f(x) = c_{d,s}\text{P.V.}\int_{\mathbb{R}^d} \frac{f(x)-f(y)}{|x-y|^{d+2s}}\,\mathrm{d}y.$$

It is well known that the fractional Laplacian $(-\Delta)^s$ admits the following limiting property

$$(-\Delta)^s f(x) \longrightarrow f(x) \tag{1}$$

[*] Corresponding author

when s converges to zero. A challenging problem is

whether the expansion (1) *could be extended to the first order or high order.*

A partial affirmative answer for this problem is given in [9] that, for any C^2-function f on $\mathbb{R}^d$ with compact support,

$$(-\Delta)^s f(x) = f(x) + sL_\Delta f(x) + o(s), \quad s \to 0^+,$$

where the operator L_Δ is given as a logarithmic Laplacian formally, and is regarded as the right derivative of $(-\Delta)^s$ at $s = 0$

$$L_\Delta = \frac{\mathrm{d}}{\mathrm{d}s}(-\Delta)^s\Big|_{s=0^+}.$$

Some qualitative properties for L_Δ are obtained as follows.

Theorem 1 ([9, Theorem 1.1]). *Let $f \in C^\alpha(\mathbb{R}^d)$ for some $\alpha > 0$ with compact support. Then we have*

$$L_\Delta f(x) = \frac{\mathrm{d}}{\mathrm{d}s}(-\Delta)^s\Big|_{s=0^+} f(x) = c_d \int_{\mathbb{R}^d} \frac{f(x)\mathbb{1}_{B_1(x)}(y) - f(y)}{|x-y|^d}\,\mathrm{d}y + \rho_d f(x).$$

Here $c_d = \pi^{-d/2}\Gamma(d/2)$ and $\rho_d = \log 4 + \psi(d/2) - \gamma$, where $\psi = \Gamma'/\Gamma$ is the Digamma function and γ is the Euler–Mascheroni constant. Moreover,

(1) *for $1 < p \le \infty$, we have $L_\Delta f \in L^p(\mathbb{R}^d)$ and $\big[(-\Delta)^s f - f\big]/s \to L_\Delta f$ in $L^p(\mathbb{R}^d)$ as $s \to 0^+$;*
(2) *$\widehat{L_\Delta f}(\xi) = \big(2\log|\xi|\big)\widehat{f}(\xi)$ for a.e. $\xi \in \mathbb{R}^d$.*

The study of the fractional concept has been widely applied to different settings such as the obstacle problem [2,5,19], conformal geometry [7], trace/extension problem [6,23,29], Fourier transform [11], elliptic/parabolic equation [10,12,17], Nirenberg problem [21], Yamabe problem [1,22], differential operation [13,24,26].

This note investigates a discrete version of Theorem 1 on $\mathbb{Z}^N$ ($N \ge 1$). When $N = 1$, we prove that the right derivative of the fractional discrete Laplacian at zero is the discrete Riesz potential of order zero (see Section 2 below), and when $N \ge 2$, it is an exotic discrete Riesz potential with a corrector (see Section 3 below). Here we remark that, when $N = 1$, the fractional discrete Laplacian is determined by the Gamma function, and it has an explicit representation. However, when $N \ge 2$, the kernel of the fractional discrete Laplacian is not explicit, and hence some methods and techniques in [9] are no longer applicable due to the complex structure of the underlying space $\mathbb{Z}^N$. Finally, all results of this note as regards $(-\Delta_N)^s$ $(0 < s < 1)$ can be extended to the reversed case $(-\Delta_N)^s$ $(-N/2 < s < 0)$, namely, in the discrete case $\mathbb{Z}^N$, the left derivative of the Riesz potential at zero is still the discrete Riesz potential of order zero; see Section 4 below for more details.

All constants with subscripts, such as ρ_N (which depends on N only), do not change in different occurrences.

2. The derivative of the fractional Laplacian on $\mathbb{Z}$

2.1. *Preliminaries*

In this section, we consider a mesh of fixed size $h > 0$ on $\mathbb{R}$ given by $\mathbb{Z}_h = \{hn : n \in \mathbb{Z}\}$. For a function $f\colon \mathbb{Z}_h \to \mathbb{R}$, we use the notation $f_h(n) = f(hn)$ to denote the value of f at the mesh point $hn \in \mathbb{Z}_h$. The discrete Laplacian Δ_h on $\mathbb{Z}_h$ is then given by

$$-\Delta_h f_h(n) = -\frac{1}{h^2}\big(f_h(n+1) + f_h(n-1) - 2f_h(n)\big),$$

and its fractional power $(-\Delta_h)^s$ $(0 < s < 1)$ can be defined by the heat semigroup method [16,29]

$$(-\Delta_h)^s f_h = \frac{1}{\Gamma(-s)}\int_0^\infty (e^{t\Delta_h} f_h - f_h)\frac{\mathrm{d}t}{t^{s+1}},$$

where $e^{t\Delta_h}$ is the heat semigroup associated with $-\Delta_h$, and $u_h(n,t) = e^{t\Delta_h} f_h(n)$ is the solution to the semidiscrete heat equation

$$\begin{cases} \partial_t u_h = \Delta_h u_h, & \text{in } \mathbb{Z}_h \times (0,\infty), \\ u_h(\cdot,0) = f_h, & \text{on } \mathbb{Z}_h. \end{cases}$$

For any $0 \le s \le 1$, the weighted Lebesgue space, denoted by $\ell_s(\mathbb{Z})$, is defined as the class of all functions $f\colon \mathbb{Z}_h \to \mathbb{R}$ such that their norms

$$\|f_h\|_{\ell_s(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} \frac{|f_h(n)|}{\big(1+|n|\big)^{1+2s}} < \infty.$$

A result about the fractional Laplacian in [16] tells us that, if $f_h \in \ell_s(\mathbb{Z})$ with $0 < s < 1$, then

$$(-\Delta_h)^s f_h(n) = \sum_{m\in\mathbb{Z},\, m\neq n} \mathcal{K}_s^h(n-m)\big(f_h(n) - f_h(m)\big),$$

where the discrete kernel $\mathcal{K}_s^h(m)$ is given by

$$\mathcal{K}_s^h(m) = \pi^{-1/2} s \left(\frac{2}{h}\right)^{2s} \frac{\Gamma(1/2+s)}{\Gamma(1-s)} \frac{\Gamma\big(|m|-s\big)}{\Gamma\big(|m|+1+s\big)} \mathbb{1}_{\mathbb{Z}\setminus\{0\}}(m) = C_h(s) \frac{\Gamma\big(|m|-s\big)}{\Gamma\big(|m|+1+s\big)} \mathbb{1}_{\mathbb{Z}\setminus\{0\}}(m).$$

Denote by $c_h(s)$ the quotient $C_h(s)/s$ for simplicity. Obviously, one has $c_h(0) = 1$ and

$$c_h'(0) = \log 4 - \log h^2 + \psi(1/2) + \psi(1) = -2\gamma - \log h^2. \tag{2}$$

If $h = 1$, then we throw away the superscript of $\mathcal{K}_s^h(m)$ and subscript of $\mathbb{Z}_h$.

Moreover, the corresponding logarithmic Laplacian $\log(-\Delta_h)$ is defined by the right-hand derivative of $(-\Delta_h)^s$ at 0, namely,

$$\log(-\Delta_h) = \frac{\mathrm{d}}{\mathrm{d}s}(-\Delta_h)^s\bigg|_{s=0^+}.$$

2.2. *Estimates for Gamma functions*

This section collects some properties for Gamma functions. Here we provide the proofs for the sake of completeness though their arguments are standard.

Proposition 2. *Let $s > 0$ be sufficiently small. For any integer $k \ge 1$, the following formulas hold.*

(1) $\dfrac{\Gamma(k-s)}{\Gamma(k+1+s)} \le \dfrac{1}{(k-s)^{1+2s}}$;

(2) $\displaystyle\sum_{|m|\ge k} \frac{\Gamma\big(|m|-s\big)}{\Gamma\big(|m|+1+s\big)} = \frac{\Gamma(k-s)}{s\Gamma(k+s)}$;

(3) $\dfrac{\Gamma(k-s)}{\Gamma(k+s)} = 1 - 2s\psi(k) + o(s)$.

Proof. **(1).** It follows from the identity for the quotient of the Gamma function (see [16, (5.1)]) that

$$\frac{\Gamma(k-s)}{\Gamma(k+1+s)} = \frac{1}{\Gamma(1+2s)} \int_0^\infty e^{-(k-s)t}(1-e^{-t})^{2s}\,\mathrm{d}t \le \frac{1}{\Gamma(1+2s)} \int_0^\infty e^{-(k-s)t} t^{2s}\,\mathrm{d}t = \frac{1}{(k-s)^{1+2s}},$$

where we used the basic inequality $1 - e^{-t} \le t$ for any $t \ge 0$.

(2). One may invoke the relationship between the Gamma function $\Gamma(\cdot)$ and the Beta function $B(\cdot,\cdot)$ to obtain

$$\begin{aligned} \sum_{|m|\ge k} \frac{\Gamma\big(|m|-s\big)}{\Gamma\big(|m|+1+s\big)} &= \frac{2}{\Gamma(1+2s)} \sum_{m=k}^\infty B(m-s, 1+2s) \\ &= \frac{1}{s\Gamma(2s)} \int_0^1 \sum_{m=k}^\infty t^{m-s-1}(1-t)^{2s}\,\mathrm{d}t = \frac{\Gamma(k-s)}{s\Gamma(k+s)}. \end{aligned}$$

(3). The behavior of the quotient is a consequence of L'Hospital's rule. This completes the proof. □

2.3. *The main result and its proof*

In this section, we will prove the first main result of this note.

Theorem 3. *Let f be a real-valued function on $\mathbb{Z}_h$ ($h>0$) with bounded support. Then we have*

$$\log(-\Delta_h)f_h(n)=\frac{\mathrm{d}}{\mathrm{d}s}(-\Delta_h)^s\Big|_{s=0^+}f_h(n)=-\sum_{m\in\mathbb{Z},\,m\neq n}\frac{f_h(m)}{|n-m|}-(\log h^2)f_h(n). \tag{3}$$

Moreover, we have $\log(-\Delta_h)f_h\in\ell^\infty(\mathbb{Z})$ *and*

$$\lim_{s\to0^+}\frac{(-\Delta_h)^sf_h-f_h}{s}=\log(-\Delta_h)f_h$$

in L^∞-sense.[1]

Remark 4. In Theorem 3, we adopt the notation "log" to denote the derivative of the fractional discrete Laplacian at zero, namely,

$$\log(-\Delta_h)=\frac{\mathrm{d}}{\mathrm{d}s}(-\Delta_h)^s\Big|_{s=0^+}.$$

There are two reasons for this notation. The first one is that the logarithmic function appears naturally in the differential operation (at least formally)

$$\frac{\mathrm{d}}{\mathrm{d}s}(-\Delta_h)^s\Big|_{s=0^+}=(-\Delta_h)^s\log(-\Delta_h)\big|_{s=0^+}=\log(-\Delta_h).$$

The other is that, in the Euclidean case, the (weakly) singular integral operator L_Δ has a Fourier symbol of the form $\xi\mapsto2\log|\xi|$; see Theorem 1. Moreover, in the discrete case, the operator $\log(-\Delta_h)$ is nonsingular (since $K_s^h(0)=0$) and admits a Fourier symbol of the form $\xi\mapsto 2\log\big|(2\sin\pi\xi)/h\big|$ at least formally. Indeed, for any $f_h\in\ell^1(\mathbb{Z})$, its Fourier transform is given by

$$\widehat{f_h}(\xi)=\sum_{n\in\mathbb{Z}}f_h(n)e^{2\pi in\xi},\quad -1/2\le\xi\le1/2;$$

see [14, Section 3] for more details. A routine computation gives rise to

$$\begin{aligned}\widehat{(-\Delta_hf_h)}(\xi)&=\sum_{n\in\mathbb{Z}}\frac{-1}{h^2}\big(f_h(n+1)+f_h(n-1)-2f_h(n)\big)e^{2\pi in\xi}\\&=\sum_{n\in\mathbb{Z}}f_h(n)e^{2\pi in\xi}\frac{-1}{h^2}(e^{2\pi i\xi}+e^{-2\pi i\xi}-2)\\&=\left(\frac{2}{h}\sin\pi\xi\right)^2\widehat{f_h}(\xi).\end{aligned}$$

From this it is only one step to replace the exponent 2 by a general exponent $2s$, and thus to define (at least formally) the fractional power of the discrete Laplacian by

$$\widehat{(-\Delta_h)^sf_h}(\xi)=\left(\frac{2}{h}\sin\pi\xi\right)^{2s}\widehat{f_h}(\xi).$$

The sine function of the Fourier symbol above does not appear in the Euclidean case. We believe that this phenomenon is caused by the Fourier transform of the heat kernel

$$\widehat{G_{t,1}}(\xi)=e^{-4t\sin^2\pi\xi}.$$

However, we still employ the notation $\log(-\Delta_h)$ for simplicity, rather than $\log\sin(-\Delta_h)$.

[1] In fact, the L^∞-norm can be replaced by the L^p-norm with $1<p\le\infty$, and the proof is left to the readers.

Remark 5. It is convenient to introduce the discrete kernel function

$$\begin{aligned} K\colon \mathbb{Z} &\longrightarrow \mathbb{R} \\ m &\longmapsto |m|^{-1}\mathbb{1}_{\mathbb{Z}\setminus\{0\}}(m). \end{aligned}$$

Then the summation representation (3) can be rewritten as

$$\begin{aligned} \log(-\Delta_h) f_h(n) &= -\sum_{m\in\mathbb{Z},\, m\neq n} \frac{f_h(m)}{|n-m|} - (\log h^2) f_h(n) \\ &= -K * f_h(n) - (\log h^2) f_h(n) \\ &= -\mathscr{I}_0 f_h(n) - (\log h^2) f_h(n), \end{aligned}$$

where $\mathscr{I}_0$ denotes an exotic discrete Riesz potential (namely, the discrete Riesz potential of order zero) with the kernel $K(n) = |n|^{-1}$; see [28] for example. Moreover, we remark that $h = 1$ is a special and important case in (3). In this case, the derivative of the fractional discrete Laplacian reduces to the exotic discrete Riesz potential authentically.

Proof of Theorem 3. Step 1: Decompose $(-\Delta_h)^s f_h$. Pick an integer $\ell \geq 1$ sufficiently large such that $\operatorname{supp} f_h \subset (-\ell, \ell)$. For any $s > 0$ sufficiently small, one writes

$$\begin{aligned} (-\Delta_h)^s f_h(n) &= \sum_{0<|m-n|<4\ell} \mathcal{K}_s^h(n-m)\big(f_h(n) - f_h(m)\big) + \sum_{|m-n|\geq 4\ell} \mathcal{K}_s^h(n-m)\big(f_h(n) - f_h(m)\big) \\ &= \left(\sum_{0<|m-n|<4\ell} \mathcal{K}_s^h(n-m)\big(f_h(n) - f_h(m)\big) - \sum_{|m-n|\geq 4\ell} \mathcal{K}_s^h(n-m) f_h(m)\right) \\ &\qquad + \sum_{|m-n|\geq 4\ell} \mathcal{K}_s^h(n-m) f_h(n) \\ &= A_\ell^h(s,n) + D_\ell^h(s) f_h(n). \end{aligned}$$

Step 2: Asymptotic behavior of $A_\ell^h(s,n)/s$ near the origin. When $|n| < 2\ell$, it follows from the support of f_h that

$$A_\ell^h(s,n) = \sum_{0<|m-n|<4\ell} \mathcal{K}_s^h(n-m)\big(f_h(n) - f_h(m)\big).$$

Note that

$$\frac{\mathcal{K}_s^h(n-m)}{s} = c_h(s)\frac{\Gamma\big(|n-m|-s\big)}{\Gamma\big(|n-m|+1+s\big)} \longrightarrow \frac{1}{|n-m|}, \quad s \to 0^+,$$

and hence

$$\frac{A_\ell^h(s,n)}{s} \longrightarrow \sum_{0<|m-n|<4\ell} \frac{f_h(n) - f_h(m)}{|n-m|} = a_\ell^h(n), \quad s \to 0^+.$$

Step 3: Estimate $A_\ell^h(s,n)/s$ away from the origin. When $|n| \geq 2\ell$, we note that $f_h(n) = 0$ (which is guaranteed by the support of f_h) implies

$$A_\ell^h(s,n) = -\sum_{m\in\mathbb{Z},\, m\neq n} \mathcal{K}_s^h(n-m) f_h(m)$$

and

$$|n-m| \geq |n| - |m| > |n|/2 \geq \ell$$

whenever $|m| < \ell$. From this, Proposition 2(1), it holds

$$\begin{aligned} \left|\frac{A_\ell^h(s,n)}{s}\right| &\leq \frac{C_h(s)}{s} \sum_{m\in\mathbb{Z},\, m\neq n} \frac{\Gamma\big(|n-m|-s\big)}{\Gamma\big(|n-m|+1+s\big)} |f_h(m)| \\ &\leq c_h(s) \sum_{m\in\mathbb{Z},\, m\neq n} \frac{1}{\big(|n-m|-s\big)^{1+2s}} |f_h(m)| \\ &\leq c_h(s) \|f_h\|_{\ell^1(\mathbb{Z})} (\ell - s)^{-1}. \end{aligned}$$

Step 4: Estimate $a_\ell^h(n)$ away from the origin. When $|n| \geq 2\ell$, proceeding as the argument of Step 3, we have $f_h(n) = 0$ (which is guaranteed by the support of f_h) and hence

$$\left|a_\ell^h(n)\right| \leq \sum_{0<|m-n|<4\ell} \frac{\left|f_h(m)\right|}{|n-m|} \leq \|f_h\|_{\ell^1(\mathbb{Z})}\ell^{-1}.$$

Step 5: Asymptotic behavior of $\left(D_\ell^h(s)-1\right)/s$ near the origin. One may invoke Proposition 2(2) to deduce

$$D_\ell^h(s) = C_h(s)\sum_{|m|\geq 4\ell} \frac{\Gamma\big(|m|-s\big)}{\Gamma\big(|m|+1+s\big)} = \frac{C_h(s)}{s}\frac{\Gamma(4\ell-s)}{\Gamma(4\ell+s)} = c_h(s)\frac{\Gamma(4\ell-s)}{\Gamma(4\ell+s)}$$

and hence by (2) and Proposition 2(3)

$$\lim_{s\to 0^+} \frac{D_\ell^h(s)-1}{s} = -2(\psi(4\ell)+\gamma) - \log h^2 = -\sum_{0<|m-n|<4\ell} \frac{1}{|n-m|} - \log h^2 = d_\ell^h,$$

where we used the property of ψ as follows:

$$\psi(4\ell) = \sum_{m=1}^{4\ell-1}\frac{1}{m} - \gamma = \frac{1}{2}\sum_{0<|m-n|<4\ell}\frac{1}{|n-m|} - \gamma.$$

Step 6: Add $d_\ell^h f_h(n)$ to $a_\ell^h(n)$. Taking $a_\ell^h(n)$ and $d_\ell^h f_h(n)$ into consideration, we arrive at

$$\begin{aligned}
a_\ell^h(n) + d_\ell^h f_h(n) &= \sum_{0<|m-n|<4\ell} \frac{f_h(n)-f_h(m)}{|n-m|} - \sum_{0<|m-n|<4\ell}\frac{f_h(n)}{|n-m|} - (\log h^2) f_h(n) \\
&= -\sum_{m\in\mathbb{Z},\, m\neq n} \frac{f_h(m)}{|n-m|} - (\log h^2) f_h(n) + \sum_{|m-n|\geq 4\ell} \frac{f_h(m)}{|n-m|} \\
&= \log(-\Delta_h) f_h(n) + g(n).
\end{aligned}$$

Step 7: Estimate g. Obviously, one has

$$\left|g(n)\right| \leq \|f_h\|_{\ell^1(\mathbb{Z})}\ell^{-1}.$$

Step 8: Convergence in L^∞-sense. By Steps 1 and 6, we decompose the desired result as

$$\begin{aligned}
\left\|\frac{(-\Delta_h)^s f_h - f_h}{s} - \log(-\Delta_h) f_h\right\|_{\ell^\infty(\mathbb{Z})} &= \left\|\frac{A_\ell^h(s,\cdot)}{s} - a_\ell^h + \frac{D_\ell^h(s) f_h - f_h}{s} - d_\ell^h f_h + g\right\|_{\ell^\infty(\mathbb{Z})} \\
&\leq \left\|\frac{A_\ell^h(s,\cdot)}{s} - a_\ell^h\right\|_{\ell^\infty(\mathbb{Z}\cap(-2\ell,2\ell))} + \left\|\frac{A_\ell^h(s,\cdot)}{s} - a_\ell^h\right\|_{\ell^\infty(\mathbb{Z}\setminus(-2\ell,2\ell))} \\
&\quad + \left|\frac{D_\ell^h(s)-1}{s} - d_\ell^h\right| \|f_h\|_{\ell^\infty(\mathbb{Z})} + \|g\|_{\ell^\infty(\mathbb{Z})}.
\end{aligned}$$

Therefore, invoke all estimates in Steps 2–5 and 7 to deduce

$$\lim_{s\to 0^+}\left\|\frac{(-\Delta_h)^s f_h - f_h}{s} - \log(-\Delta_h) f_h\right\|_{\ell^\infty(\mathbb{Z})} \leq 3\|f_h\|_{\ell^1(\mathbb{Z})}\ell^{-1}.$$

As the integer $\ell \geq |\operatorname{supp} f|/h$ is arbitrary, we derive the desired conclusion[2] by letting $\ell \to +\infty$.

Step 9: $\log(-\Delta_h) f_h$ is in $\ell^\infty(\mathbb{Z})$. Note that

$$\sum_{m\in\mathbb{Z},\, m\neq n} \frac{\left|f_h(m)\right|}{|n-m|} \leq \sum_{m\in\mathbb{Z},\, m\neq n} \left|f_h(m)\right| = \|f_h\|_{\ell^1(\mathbb{Z})}$$

and hence

$$\left\|\log(-\Delta_h) f_h\right\|_{\ell^\infty(\mathbb{Z})} \leq \|f_h\|_{\ell^1(\mathbb{Z})} + |\log h^2|\, \|f_h\|_{\ell^\infty(\mathbb{Z})} < \infty.$$

---

[2]This L^∞-convergence implies the pointwise convergence.

Step 10: Completion of the proof. Finally, by Steps 8 and 9, we know that the pointwise formula (3) holds, $\log(-\Delta_h)f_h$ is in $\ell^\infty(\mathbb{Z})$, and $\big[(-\Delta_h)^s f_h - f_h\big]/s$ approaches $\log(-\Delta_h)f_h$ in L^∞-sense and also in pointwise sense as s goes to 0^+. This completes the proof. □

3. The derivative of the fractional Laplacian on $\mathbb{Z}^N$

3.1. *Preliminaries*

Our target in this section is to introduce the multidimensional discrete Laplacian

$$\Delta_N f(\mathbf{n}) = \sum_{k=1}^{N} \Delta_{N,k} f(\mathbf{n}) = \sum_{k=1}^{N} \big[f(\mathbf{n}+\mathbf{e}_k) + f(\mathbf{n}-\mathbf{e}_k) - 2f(\mathbf{n})\big],$$

where $\{\mathbf{e}_k\}_{k=1}^N$ is an orthonormal basis in $\mathbb{Z}^N$ ($N \geq 2$). The heat semigroup $W_t = e^{t\Delta_N}$ is the solution mapping of the N-dimensional semidiscrete heat equation

$$\begin{cases} \partial_t u = \Delta_N u, & \text{in } \mathbb{Z}^N \times (0,\infty), \\ u(\cdot,0) = f, & \text{on } \mathbb{Z}^N. \end{cases}$$

The solution to the above equation is given by (see Section 2 for the one-dimensional case)

$$u(\mathbf{n},t) = W_t f(\mathbf{n}) = \sum_{\mathbf{m}\in\mathbb{Z}^N} G_{t,N}(\mathbf{n}-\mathbf{m}) f(\mathbf{m}),$$

where the heat kernel $G_{t,N}(\mathbf{m})$ is constructed by

$$G_{t,N}(\mathbf{m}) = \prod_{k=1}^{N} \big(e^{-2t} I_{m_k}(2t)\big), \quad \mathbf{m} = (m_1,\dots,m_N),$$

through the modified Bessel function of first kind I_a. By [14, (14) and (30)], we know that the heat kernel $G_{t,N}(\mathbf{m})$ admits the polynomial growth as

$$0 \leq G_{t,N}(\mathbf{m}) \leq C\left(\frac{\sqrt{t}}{\sqrt{t}+|\mathbf{m}|}\right)^2 \frac{1}{\big(\sqrt{t}+|\mathbf{m}|\big)^N}. \tag{4}$$

If $f \in \ell_s(\mathbb{Z}^N)$ with $0 < s < 1$, then

$$(-\Delta_N)^s f(\mathbf{n}) = \sum_{\mathbf{m}\in\mathbb{Z}^N, \mathbf{m}\neq\mathbf{n}} \mathcal{K}_s(\mathbf{n}-\mathbf{m})\big(f(\mathbf{n}) - f(\mathbf{m})\big),$$

where the discrete kernel $\mathcal{K}_s(\mathbf{m})$ is given by

$$\mathcal{K}_s(\mathbf{m}) = \frac{1}{\big|\Gamma(-s)\big|} \int_0^\infty G_{t,N}(\mathbf{m}) \frac{\mathrm{d}t}{t^{s+1}} \mathbb{1}_{\mathbb{Z}^N\setminus\{\mathbf{0}\}}(\mathbf{m}); \tag{5}$$

see [14, Section 5] for more details. Denote by $K(\mathbf{m})$ the value of the quotient $\mathcal{K}_s(\mathbf{m})/s$ at zero

$$K(\mathbf{m}) = \frac{\mathcal{K}_s(\mathbf{m})}{s}\bigg|_{s=0} = \frac{1}{\Gamma(1-s)} \int_0^\infty G_{t,N}(\mathbf{m}) \frac{\mathrm{d}t}{t^{s+1}} \mathbb{1}_{\mathbb{Z}^n\setminus\{\mathbf{0}\}}(\mathbf{m})\bigg|_{s=0} = \int_0^\infty G_{t,N}(\mathbf{m}) \frac{\mathrm{d}t}{t} \mathbb{1}_{\mathbb{Z}^n\setminus\{\mathbf{0}\}}(\mathbf{m}) \tag{6}$$

for simplicity. We refer the readers to [15,27] and references therein for more discrete works such as the Hardy–Littlewood function [3,25], Hardy space [4], Calderón reproducing formula [18], Radon transform [20].

3.2. *Estimates for discrete kernels*

This section establishes some upper bounds for $\mathscr{K}_s(\mathbf{m})$ and $K(\mathbf{m})$.

Proposition 6.

(1) *There exists a constant* $C_N > 0$ *such that*

$$0 \le \mathscr{K}_s(\mathbf{m}) \le \frac{C_N}{|\Gamma(-s)|}\left(\frac{1}{1-s} + \frac{2}{N+2s}\right)\frac{\mathbb{1}_{\mathbb{Z}^n\setminus\{\mathbf{0}\}}(\mathbf{m})}{|\mathbf{m}|^{N+2s}}.$$

(2) $\mathscr{K}_s(\mathbf{m}) = K(\mathbf{m})s + o(s)$, $s \to 0^+$.

(3) *There exists a constant* $C_N > 0$ *same as in* (1) *such that*

$$0 \le K(\mathbf{m}) \le C_N\left(1 + \frac{2}{N}\right)\frac{\mathbb{1}_{\mathbb{Z}^n\setminus\{\mathbf{0}\}}(\mathbf{m})}{|\mathbf{m}|^{N}}.$$

Proof. **(1).** See [14, Theorem 30] for the proof.

(2). One can write

$$\begin{aligned}\left|\frac{\mathscr{K}_s(\mathbf{m})}{s} - K(\mathbf{m})\right| &\le \left|\frac{1}{\Gamma(1-s)}\int_0^\infty G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t^{s+1}} - \int_0^\infty G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t}\right| \\ &\le \frac{1}{\Gamma(1-s)}\int_0^\infty G_{t,N}(\mathbf{m})|t^{-s}-1|\frac{\mathrm{d}t}{t} + \left|\frac{1}{\Gamma(1-s)} - 1\right|\int_0^\infty G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t}.\end{aligned}$$

On the one hand, we obtain by (4) that

$$\begin{aligned}\int_0^\infty G_{t,N}(\mathbf{m})|t^{-s}-1|\frac{\mathrm{d}t}{t} &\le C\int_0^1 \frac{t}{|\mathbf{m}|^{N+2}}\left(\frac{1}{t^s} - 1\right)\frac{\mathrm{d}t}{t} + C\int_1^\infty \frac{1}{t^{N/2}}\left(1 - \frac{1}{t^s}\right)\frac{\mathrm{d}t}{t} \\ &\le C\frac{s}{1-s}\frac{1}{|\mathbf{m}|^{N+2}} + C\frac{s}{1+s} \longrightarrow 0, \quad s \to 0^+.\end{aligned}$$

On the other hand, it follows that

$$\begin{aligned}|\Gamma(1) - \Gamma(1-s)| &\le \int_0^\infty e^{-t}|t - t^{1-s}|\frac{\mathrm{d}t}{t} \\ &\le \int_0^1\left(\frac{1}{t^s} - 1\right)\mathrm{d}t + 2\int_1^\infty\left(\frac{1}{t^2} - \frac{1}{t^{2+s}}\right)\mathrm{d}t \\ &= \frac{s}{1-s} + 2\frac{s}{1+s} \longrightarrow 0, \quad s \to 0^+,\end{aligned}$$

and from (4) that

$$\int_0^\infty G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t} \le C\int_0^\infty\left(\frac{\sqrt{t}}{\sqrt{t}+|\mathbf{m}|}\right)^2\frac{1}{\left(\sqrt{t}+|\mathbf{m}|\right)^N}\frac{\mathrm{d}t}{t} = \frac{C}{|\mathbf{m}|^N}. \tag{7}$$

Based on the above argument, we derive that

$$\mathscr{K}_s(\mathbf{m}) = K(\mathbf{m})s + o(s), \quad s \to 0^+.$$

(3). The required result follows from (7). This completes the proof. □

3.3. *From one-dimension to higher-dimension*

In this section, we go further to consider the N-dimensional version of Theorem 3 ($N \ge 2$).

In order to generalize Theorem 3 to the N-dimensional case, the main difficulty arises from the lack of the explicit representation for the discrete kernel $\mathscr{K}_s(\mathbf{m})$. In Theorem 3, all constants derived from $\mathcal{K}_s^h(m)$ such as $C_h(s)$, $c_h'(0)$, $D_\ell^h(s)$ and d_ℓ are clear and unambiguous. However, in the N-dimensional setting, the discrete kernel $\mathcal{K}_s(\mathbf{m})$ is determined by

$$\begin{aligned}\mathcal{K}_s(\mathbf{m}) &= \frac{1}{|\Gamma(-s)|}\int_0^\infty G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t^{s+1}}\mathbb{1}_{\mathbb{Z}^n\setminus\{\mathbf{0}\}}(\mathbf{m})\\ &= \frac{1}{|\Gamma(-s)|}\int_0^\infty \prod_{k=1}^N\big(e^{-2t}I_{m_k}(2t)\big)\frac{\mathrm{d}t}{t^{s+1}}\mathbb{1}_{\mathbb{Z}^n\setminus\{\mathbf{0}\}}(\mathbf{m}),\quad \mathbf{m}=(m_1,\dots,m_N).\end{aligned}$$

We do not know how to calculate accurately the product of $I_{m_k}(2t)$ with $k=1,\dots,N$, though it can be estimated as

$$\frac{\Gamma\big(\frac{m_1+\cdots+m_N}{N}+1\big)^N}{\Gamma(m_1+1)\cdots\Gamma(m_N+1)}\big(I_{\frac{m_1+\cdots+m_N}{N}}(2t)\big)^N \le I_{m_1}(2t)\cdots I_{m_N}(2t)\le \big(I_{\frac{m_1+\cdots+m_N}{N}}(2t)\big)^N$$

for every $m_1,\dots,m_N>-1$; see [14, Proposition 1]. With this estimate in hand, we can calculate the upper/lower bound for $G_{t,N}(\mathbf{m})$ and hence for $\mathcal{K}_s(\mathbf{m})$, but not the explicit representation.[3] Therefore we have to surmise some crucial constants similar as $c_h'(0)$ and d_ℓ in Theorem 3. In the following paragraph, we use the same notation as in the proof of Theorem 3.

Step 1 in the proof of Theorem 3 tells us that the term $D_\ell(s)$ in the N-dimensional setting should be defined by

$$D_\ell(s)=\sum_{|\mathbf{m}|\ge 4\ell}\mathcal{K}_s(\mathbf{m}).$$

Since the discrete kernel $\mathcal{K}_s(\mathbf{m})$ does not admit an explicit representation, we cannot even verify that $D_\ell(0)$ is equal to one. However, inspired by Step 5 in the proof of Theorem 3, we expect that

$$\frac{D_\ell(s)-1}{s}=-\sum_{0<|\mathbf{m}|<4\ell}K(\mathbf{m})+o(1),\quad s\to 0^+.$$

Unfortunately, this behavior of $D_\ell(s)$ at zero is valid for $\mathbb{Z}$, not for $\mathbb{Z}^N$ ($N\ge 2$). Here, we should introduce a corrector to find the correct form of the above identity. More precisely, hypothesize for the moment that

$$\frac{D_\ell(s)-1}{s}=-\sum_{0<|\mathbf{m}|<4\ell}K(\mathbf{m})+\rho_N+o(1)=d_\ell(s)+o(1),\quad s\to 0^+,$$

where the corrector ρ_N depends on N only to be determined later. The main difficulty of showing the identity above arises from the lack of the explicit representation for the discrete kernel. Note that

$$\sum_{\mathbf{m}\in\mathbb{Z}^N}G_{t,N}(\mathbf{m})=1$$

is valid by the conservation property of the heat kernel (see [14, (21)]). A careful analysis gives rise to

$$\begin{aligned}\frac{D_\ell(s)-1}{s} &= \frac{1}{\Gamma(1-s)}\int_0^1\sum_{|\mathbf{m}|\ge 4\ell}G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t^{s+1}}+\frac{1}{\Gamma(1-s)}\int_1^\infty\sum_{|\mathbf{m}|\ge 4\ell}G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t^{s+1}}-\frac{1}{s}\\ &= \frac{1}{\Gamma(1-s)}\int_0^1\sum_{|\mathbf{m}|\ge 4\ell}G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t^{s+1}}-\frac{1}{\Gamma(1-s)}\int_1^\infty\sum_{|\mathbf{m}|<4\ell}G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t^{s+1}}+\frac{1}{s\Gamma(1-s)}-\frac{1}{s}\\ &= \frac{1}{\Gamma(1-s)}\int_0^1\sum_{|\mathbf{m}|\ge 4\ell}G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t^{s+1}}-\frac{1}{\Gamma(1-s)}\int_1^\infty\sum_{|\mathbf{m}|<4\ell}G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t^{s+1}}+J_1\\ &= \frac{1}{\Gamma(1-s)}\int_0^1\sum_{|\mathbf{m}|\ge 4\ell}G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t^{s+1}}-\int_0^1\sum_{|\mathbf{m}|\ge 4\ell}G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t}+\int_0^1\sum_{|\mathbf{m}|\ge 4\ell}G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t}\\ &\quad -\frac{1}{\Gamma(1-s)}\int_1^\infty\sum_{|\mathbf{m}|<4\ell}G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t^{s+1}}+\int_1^\infty\sum_{|\mathbf{m}|<4\ell}G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t}-\int_1^\infty\sum_{|\mathbf{m}|<4\ell}G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t}+J_1\end{aligned}$$

[3] We thank Prof. Ó. Ciaurri for reminding us this fact.

$$
\begin{aligned}
&= J_2 + \int_0^1 \sum_{|\mathbf{m}|\geq 4\ell} G_{t,N}(\mathbf{m}) \frac{\mathrm{d}t}{t} + J_3 - \int_1^\infty \sum_{|\mathbf{m}|<4\ell} G_{t,N}(\mathbf{m}) \frac{\mathrm{d}t}{t} + J_1 \\
&= J_2 + \int_0^1 \sum_{\mathbf{m}\in\mathbb{Z}^N,\, \mathbf{m}\neq \mathbf{0}} G_{t,N}(\mathbf{m}) \frac{\mathrm{d}t}{t} - \int_0^1 \sum_{0<|\mathbf{m}|<4\ell} G_{t,N}(\mathbf{m}) \frac{\mathrm{d}t}{t} \\
&\qquad + J_3 - \int_1^\infty G_{t,N}(\mathbf{0}) \frac{\mathrm{d}t}{t} - \int_1^\infty \sum_{0<|\mathbf{m}|<4\ell} G_{t,N}(\mathbf{m}) \frac{\mathrm{d}t}{t} + J_1 \\
&= -\sum_{0<|\mathbf{m}|<4\ell} K(\mathbf{m}) + \sum_{\mathbf{m}\in\mathbb{Z}^N,\, \mathbf{m}\neq \mathbf{0}} \int_0^1 G_{t,N}(\mathbf{m}) \frac{\mathrm{d}t}{t} - \int_1^\infty G_{t,N}(\mathbf{0}) \frac{\mathrm{d}t}{t} + J_1 + J_2 + J_3.
\end{aligned}
$$

On the one hand, we claim that the last two terms J_2 and J_3 converge to zero as $s \to 0^+$. Indeed, by (4), we have

$$
\sum_{|\mathbf{m}|\geq 4\ell} G_{t,N}(\mathbf{m}) \leq C \sum_{|\mathbf{m}|\geq 4\ell} \left(\frac{\sqrt{t}}{\sqrt{t}+|\mathbf{m}|}\right)^2 \frac{1}{\left(\sqrt{t}+|\mathbf{m}|\right)^N} \leq C \sum_{|\mathbf{m}|\geq 4\ell} \frac{t}{|\mathbf{m}|^{N+2}} \leq C \frac{t}{\ell^2} \leq Ct
$$

and hence

$$
\begin{aligned}
|J_2| &\leq \frac{1}{\Gamma(1-s)} \int_0^1 \sum_{|\mathbf{m}|\geq 4\ell} G_{t,N}(\mathbf{m}) \left(\frac{1}{t^s} - 1\right) \frac{\mathrm{d}t}{t} + \left|\frac{1}{\Gamma(1-s)} - 1\right| \int_0^1 \sum_{|\mathbf{m}|\geq 4\ell} G_{t,N}(\mathbf{m}) \frac{\mathrm{d}t}{t} \\
&\leq \frac{C}{\Gamma(1-s)} \int_0^1 \left(\frac{1}{t^s} - 1\right) \mathrm{d}t + C \left|\frac{1}{\Gamma(1-s)} - 1\right| \int_0^1 \mathrm{d}t \\
&= \frac{C}{\Gamma(1-s)} \frac{s}{1-s} + C \left|\frac{1}{\Gamma(1-s)} - 1\right| \longrightarrow 0, \quad s \to 0^+.
\end{aligned}
$$

The proof of the term J_3 is similar, and is left to the readers. On the other hand, the first term J_1 can be estimated as

$$
\lim_{s\to 0^+} J_1 = \lim_{s\to 0^+} \frac{1-\Gamma(1-s)}{s\Gamma(1-s)} = \lim_{s\to 0^+} \frac{1-\Gamma(1-s)}{s} = \lim_{s\to 0^+} \Gamma'(1-s) = \Gamma'(1) = -\gamma
$$

by L'Hospital's rule. Therefore the corrector ρ_N can be chosen as

$$
\rho_N = \sum_{\mathbf{m}\in\mathbb{Z}^N,\, \mathbf{m}\neq \mathbf{0}} \int_0^1 G_{t,N}(\mathbf{m}) \frac{\mathrm{d}t}{t} - \int_1^\infty G_{t,N}(\mathbf{0}) \frac{\mathrm{d}t}{t} - \gamma. \tag{8}
$$

The remainder of the argument is easy and analogous to that in Theorem 3, and is left to the readers.

Based on the above argument, we derive the second result of this note.

Theorem 7. *Let f be a real-valued function on $\mathbb{Z}^N$ with bounded support. Then we have*

$$
\log(-\Delta_N) f(\mathbf{n}) = \frac{\mathrm{d}}{\mathrm{d}s} (-\Delta_N)^s \Big|_{s=0^+} f(\mathbf{n}) = -\sum_{\mathbf{m}\in\mathbb{Z}^N,\, \mathbf{m}\neq \mathbf{n}} K(\mathbf{n}-\mathbf{m}) f(\mathbf{m}) + \rho_N f(\mathbf{n}),
$$

where the discrete kernel $K(\mathbf{m})$ and the corrector ρ_N are as in (6) *and* (8)*, respectively. Moreover, we have* $\log(-\Delta_N) f \in \ell^\infty(\mathbb{Z}^N)$ *and*

$$
\lim_{s\to 0^+} \frac{(-\Delta_N)^s f - f}{s} = \log(-\Delta_N) f
$$

in L^∞-sense.

Remark 8. In the Euclidean space $\mathbb{R}^d$, Chen–Weth [9] proved that

$$
\frac{\mathrm{d}}{\mathrm{d}s} (-\Delta)^s \Big|_{s=0^+} f(x) = c_d \int_{\mathbb{R}^d} \frac{f(x)\mathbb{1}_{B_1(x)}(y) - f(y)}{|x-y|^d} \mathrm{d}y + \rho_d f(x); \tag{9}
$$

see Theorem 1. The observant reader might notice this result in the continuous setting is more or less different from our Theorem 7. The term $f(x)\mathbb{1}_{B_1(x)}(y)$ appears in (9), but there is no corresponding discrete term in our result, and hence the derivative of the fractional discrete

Laplacian can be regard as an exotic discrete Riesz potential. In fact, on the one hand, the open ball $B_1(\mathbf{n}) = \{\mathbf{m} \in \mathbb{Z}^N : |\mathbf{m}-\mathbf{n}| < 1\}$ in the discrete setting tells us $\mathbf{m} = \mathbf{n}$. On the other hand, the subscript in the summation indicates $\mathbf{m} \neq \mathbf{n}$. Therefore we reformulate $\log(-\Delta_N)f(\mathbf{n})$ in some sense as

$$\log(-\Delta_N)f(\mathbf{n}) = \sum_{\mathbf{m}\in\mathbb{Z}^N,\,\mathbf{m}\neq\mathbf{n}} K(\mathbf{n}-\mathbf{m})\big(f(\mathbf{n})\mathbb{1}_{B_1(\mathbf{n})}(\mathbf{m}) - f(\mathbf{m})\big) + \rho_N f(\mathbf{n}),$$

which is the discrete version of (9).

4. A final remark: the discrete Riesz potential

In Sections 2 and 3 we considered the asymptotic behavior for the *positive* power of the discrete Laplacian

$$(-\Delta_N)^s f(\mathbf{n}) = f(\mathbf{n}) + \big[-K * f(\mathbf{n}) + \rho_N f(\mathbf{n})\big]s + o(s), \quad s \to 0^+. \tag{10}$$

In fact, we can take the power s to be negative. The *negative* power of the discrete Laplacian is well known as the discrete Riesz potential. It can be defined by the heat semigroup

$$(-\Delta_N)^s = \frac{1}{\Gamma(-s)}\int_0^\infty e^{t\Delta_N}\frac{\mathrm{d}t}{t^{s+1}}, \quad -N/2 < s < 0.$$

When $N = 1$, the discrete kernel of $(-\Delta_h)^s$ with $h > 0$ can be expressed as

$$\mathscr{K}_s(m) = \pi^{-1/2}(-s)\left(\frac{2}{h}\right)^{2s}\frac{\Gamma(1/2+s)}{\Gamma(1-s)}\frac{\Gamma\big(|m|-s\big)}{\Gamma\big(|m|+1+s\big)}\mathbb{1}_{\mathbb{Z}\setminus\{0\}}(m)$$

through the Gamma function; see [16, Theorem 1.3] for more details. When $N \geq 2$, the discrete kernel of $(-\Delta_N)^s$ is not explicit, but given by

$$\mathscr{K}_s(\mathbf{m}) = \frac{1}{\Gamma(-s)}\int_0^\infty G_{t,N}(\mathbf{m})\frac{\mathrm{d}t}{t^{s+1}}\mathbb{1}_{\mathbb{Z}^N\setminus\{\mathbf{0}\}}(\mathbf{m})$$

similar to (5); see [14, Theorem 14] for more details.

A basic question arises from the discrete Riesz potential above.

Question 9. Can we derive the asymptotic behavior for the *negative* power of the discrete Laplacian similar to (10)?

Fortunately, the answer for this question is positive and is the same as the positive power case; see [8] for the continuous case. Here, we present the corresponding result for $(-\Delta_N)^s$ with $-N/2 < s < 0$ only, whose proof is analogous to that in Theorems 3 and 7, and left to the readers.

Theorem 10. *Let f be a real-valued function on $\mathbb{Z}^N$ ($N \geq 1$) with bounded support. Then we have*

$$\log(-\Delta_N)f(\mathbf{n}) = \frac{\mathrm{d}}{\mathrm{d}s}(-\Delta_N)^s\bigg|_{s=0^-} f(\mathbf{n}) = -\sum_{\mathbf{m}\in\mathbb{Z}^N,\,\mathbf{m}\neq\mathbf{n}} K(\mathbf{n}-\mathbf{m})f(\mathbf{m}) + \rho_N f(\mathbf{n}),$$

where the discrete kernel $K(\mathbf{m})$ and the corrector ρ_N are as in (6) *and* (8)*, respectively. Moreover, we have* $\log(-\Delta_N)f \in \ell^\infty(\mathbb{Z}^N)$ *and*

$$\lim_{s\to 0^-}\frac{(-\Delta_N)^s f - f}{s} = \log(-\Delta_N)f$$

in L^∞-sense.

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