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A historical perspective on parallel transport: isometric immersions and Foucault precession

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Abstract. This paper contributes to the historical understanding of the developments surrounding the Levi-Civita parallel transport problem, exploring its connections with the local problem of isometric immersions and alternative proposals. Additionally, it highlights one of its remarkable applications: the geometric interpretation of Foucault's pendulum precession. It also recalls how other geometric explanations of this phenomenon emerged in the context of Berry and Hannay phases.

Keywords. Parallel transport, isometric immersions, Foucault pendulum precession.

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1. Introduction

Parallel transport and developments in the theory of isometric immersions — between the 19th and 20th centuries — are intriguingly intertwined, as we will see later. Notably, the history of isometric immersions is not widely known, and many aspects of it remain unclear. In their recent book dedicated entirely to this subject, Han Qing and Hong Jia-Xing write:

> In 1873, Schlaefli made the following conjecture: every n-dimensional smooth Riemannian manifold admits a smooth local isometric embedding in $\mathbb{R}^{s_n}$, with $s_n = n(n+1)/2$. It was more than 50 years later that an affirmative answer was given for the analytic case successively by Janet and Cartan; they proved in 1926–1927 that any analytic n-dimensional Riemannian manifold has a local analytic isometric embedding in $\mathbb{R}^{s_n}$. Schlaefli's question for the smooth case when $n = 2$ was given renewed attention by Yau in the 1980's and 1990's. For the global isometric embedding, Nash in 1954 and Kuiper in 1955 proved the existence of a global C^1 isometric embedding of n-dimensional Riemannian manifolds in $\mathbb{R}^{2n+1}$. [23, p. XI]

The final work of Nash and Kuiper marked a turning point on the matter. Its local version answers a question asked by young Henri Lebesgue, whose partial solution led to the Lebesgue integral. This is carefully told in the Bourbaki seminar talk by Gustave Choquet [14]. The right statement about the embedding dimension is Choquet's Corollaire 1, which implies in particular the amazing fact that a flat n-torus can be C^1-embedded isometrically in $\mathbb{R}^{n+1}$.

However, going back to the origins, following Ludwig Schlaefli's (1814–1895) 1872 publication [43] and before Maurice Janet's (1888–1983) proof for $n = 2$, mathematicians had already provided at least a partial answer to this problem. Notably, as explored in Section 2.1, Gregorio Ricci Curbastro (1853–1925) made significant advances in a 1884 paper on quadratic forms [39].

Tullio Levi-Civita (1873–1941) later applied this theorem in his study of parallel transport. Significantly, his approach to introducing this concept evolved between his initial 1917 definition [29] and his treatise on absolute differential calculus, first published in Italian in 1925 [30] and later translated into English in 1926 [31] (see Section 2.2). In our view, this later approach was at least partially motivated by his attempt to provide a stronger justification for isometric immersion, for which he offered a more intuitive yet rigorous definition in the 2-dimensional case. Although the first general proof for $n > 2$ of Schlaefli's conjecture was given by Élie Cartan (1869–1951) in 1927 [13], and then — following [21] — by Celestin Burstin (1888–1938) in 1931 [8], Levi-Civita appears to have regarded this theorem as self-evident, likely based on the well-known Cauchy–Kowalevski theorem. An attitude, well supported *a *posteriori*, that Levi-Civita also demonstrated around a PDE problem in celestial mechanics (see Section 2.3).

In Section 4, we focus on the connections between the Foucault pendulum precession experiment and parallel transport, highlighting some interesting historical aspects, from Jean Bernard Léon Foucault (1819–1868) to Dragos Oprea. In 1851, Foucault conducted his famous experiment in the Panthéon of Paris, demonstrating how the precession of the plane of oscillation of a spherical pendulum highlighted Earth's diurnal rotation. In the discussions between Foucault and the mathematicians of the time, as reported in the proceedings of the Académie des Sciences, the decisive role of the Coriolis force, introduced in 1835, did not immediately emerge. Louis Poinsot (1777–1859) [38] warned the scientific community that this phenomenon was, in the final analysis, of an exquisitely geometric nature. The role of parallel transport in this geometric understanding, although within the context of an adiabatic approximation, became apparent shortly after the publication of [29] in 1918 by Johann Karl August Radon (1887–1956). As pointed out in [2, Section 6.4.3], unfortunately, there is no trace of his work in the literature. However, it is well described by Felix Christian Klein (1849–1925) in [27].[1]

Much later, a notable 1995 paper by Oprea [37] definitively linked Foucault's phenomenon to parallel transport. More specifically, in that work, parallel transport is presented using the definition introduced in (5). Over time, other approaches to parallel transport and alternative explanations of Foucault's phenomenon emerged. After Francesco Severi (1879–1961) in 1917, more recently, Michael Victor Berry and John Hannay have explored the topic within the framework of adiabatic approximation (see Section 3).

2. Parallel transport and isometric embedding

2.1. *From Riemann to Ricci*

The "Schlaefli conjecture," as referred to in [23], had been used for many years before it was rigorously proven. It is a generalization of a problem originally stated by Bernhard Riemann (1826–1866) in his *Commentatio Mathematica* [42], which concerns finding a coordinate transformation that locally maps a Riemannian manifold V_n onto a Euclidean space S_n of the same dimen-

[1] See Dritter Hauptteil, II: *J. Radons mechanische Herleitung des Parallelismus von T. Levi-Civita.*

sion. As is well known, this problem was later revisited by Elwin Bruno Christoffel (1829–1900) and Rudolf Lipschitz (1832–1903) in two articles published in the same year in the *Journal für die reine und angewandte Mathematik* [15,32]. In general, such a transformation is not possible unless V_n is already Euclidean, i.e. its curvature tensor vanishes. Riemann, in his work, only discussed the necessary conditions for this transformation. Later, in his 1884 memoir, Ricci Curbastro investigated the sufficient conditions under which the problem admits a solution [39].

This problem naturally extends with the introduction of the concept of isometric immersion. The question then becomes: may any Riemannian manifold V_n be isometrically immersed into a Euclidean space S_N for a sufficiently large N? More specifically, for which values of N is this possible?

In his 1873 memoir, Schlaefli [43] intended to complete a result obtained by Eugenio Beltrami (1835–1900) in his fundamental paper on non-Euclidean geometry [3]. More in detail, Beltrami established that in an n-dimensional space of constant curvature, one can choose the n independent variables so that every geodesic is represented by $n-1$ linear equations. Schlaefli set out to solve the inverse problem, namely to determine the type of Riemannian manifold V_n in which every geodesic can be expressed by $n-1$ linear equations, where n is the number of independent variables. This leads to the following fundamental question: under what conditions can a positive quadratic differential form in n variables be transformed into the Euclidean form $\sum_{r=1}^{N} \mathrm{d}y_r^2$, where $y_1, y_2, \ldots, y_N$ are suitable functions of n variables? According to Schlaefli, this problem is always solvable for sufficiently large N. In particular, he states that it is possible for $N \geq n(n+1)/2$, without providing a formal proof but rather relying on a simple and naive equation-to-unknowns count.

In his 1884 article [39], Ricci Curbastro built upon Schlaefli's work to establish a classification of the invariants of quadratic differential forms. He further developed this classification in several subsequent memoirs, marking an initial step toward the formulation of his absolute differential calculus (see [12,48]). If $N = n+h$ is the smallest possible value, then Ricci defined h as the "class" of V_n. He proved the necessary and sufficient conditions for 1-class forms, i.e. $h = 1$ (see [16, p. 262–266]), demonstrating that any manifold V_2 can be isometrically immersed in $\mathbb{R}^3$ [39, p. 162–163]. Ricci appears to be convinced that embedding a three-dimensional manifold V_3 in $\mathbb{R}^4$ is not possible; however, he did not explicitly formulate this claim.

In 1926, Janet clearly stated the general theorem of isometric immersion [25]: every V_n can be immersed in a space $\mathbb{R}^N$, where $N = n(n+1)/2$. He pointed out that, even when the number of unknown functions is not smaller than the number of equations, one cannot assume a priori that the system is compatible — a gap in Schlaefli's famous 1873 paper, where such a proof was lacking. Therefore, Janet traced this theorem for any n back to the algebraic condition of the implicit function theorem, employing an inductive proof. However, except for the base case $n = 2$, his proof did not seem to fully satisfy mathematicians, who generally attribute the first rigorous proof of the general theorem to Élie Cartan in 1927 [13]. For the first time, inaugurating an approach that would later become classical, Cartan considered analytic functions related to the immersion of V_n in $\mathbb{R}^N$ and successfully reformulated Janet's algebraic conditions in terms of Pfaffian systems.

As Misha Gromov and Vladimir Rokhlin [21] noted in 1970, it was Celestin Burstin (1888–1938) [8] who, in 1931, definitively consolidated the proofs of Janet and Cartan. Let us remember once again that all the theorems on isometric immersions discussed here are of a local nature. It was not until 1954 that John Nash (1928–2015) [35] proved his extraordinary C^1 isometric embedding theorem, where the obstruction of curvature does not exist, and then, two years later, he established a technically much harder result [36] solving the global smooth (C^k, $3 \leq k \leq \infty$) isometric embedding problem. This is where the so-called Nash–Moser iteration method made its first appearance.

Subsequent developments, including those by Matthias Günther [22] and others, propose alternative techniques to the Nash–Moser ones. In more detail, Günther found a clever trick enabling him to replace the Nash–Moser method by the ordinary implicit mapping theorem. Nevertheless, the Nash–Moser method remains fundamental in mathematics, contributing to show how fruitful the isometric embedding problem has been.

Other historical notices on isometric immersions can be found in [21,23].

2.2. *Levi-Civita*

The origins of parallel transport can be traced back to pioneering contributions by John Eiesland (1867–1950), Ernesto Laura (1879–1949), Ernest Vessiot (1865–1952), and Gaston Darboux (1842–1917). All these authors have been quoted by Levi-Civita in his fundamental paper [29] on the subject. Vessiot had introduced a definition of parallel transport depending on the metric but lacking a specific geometrical and analytical setting. Furthermore, Gerhard Hessenberg (1874–1925) characterized parallel transport along a geodesic in a 1916 paper but did not extend his analysis to arbitrary curves, as Levi-Civita later did (see [48], especially Chapter 3).

It is worth to note that Levi-Civita delved in the study of parallel transport after nearly fifteen years away from differential geometry. That was largely due to the limited acceptance of tensor calculus within the Italian mathematical community, despite his groundbreaking and worldwide acknowledged results emerged from the collaboration with his mentor Ricci, culminating in their seminal 1901 paper [41]. His renowned correspondence with Einstein in the spring of 1915 — where he identified and corrected an error in the initial formulation of general relativity — reignited his interest in differential geometry.

The introduction of Levi-Civita's 1917 paper [29] clearly reflects his attitude toward physics and his enthusiasm for Einstein's theory. The article aims to offer a geometric interpretation of the Riemann curvature tensor, which is central to the equations of the gravitational field, by examining how a vector or tensor changes when parallel transported around an infinitesimal closed circuit.

Levi-Civita assumed the existence of a local isometric embedding for any n-dimensional Riemannian manifold (V_n, a_{ij}) into a Euclidean space $(S_N, \delta_{\mu\nu})$ of suitable dimension N.[2] Using local charts, he expressed it as:

$$V_n \ni (x_1, \dots, x_n) \longmapsto y_\nu = y_\nu(x_1, \dots, x_n) \in S_N, \quad \nu = 1, \dots, N. \tag{1}$$

By means of the pull-back — what we would call it today — generated by (1), he assumed that the Riemannian metric a_{ij} on V_n was inherited from the Euclidean metric $\delta_{\mu\nu}$:[3]

$$a_{ij}(x) = \delta_{\mu\nu} \frac{\partial y_\mu}{\partial x_i}(x) \frac{\partial y_\nu}{\partial x_j}(x), \quad i, j = 1, \dots, n, \quad \mu, \nu = 1, \dots, N. \tag{2}$$

When introducing two infinitesimally close points, P and $\overline{P}$, on V_n, Levi-Civita defined parallelism between two given tangent directions, α at P and $\overline{\alpha}$ at $\overline{P}$, by imposing the equality of angles between these directions and an arbitrary direction f on the tangent plane to V_n at P. These angles were measured in the Euclidean ambient space S_N. He expressed this condition using the following equation (see [29, (1)]):

$$\text{angle}\,\widehat{(f)(\alpha)} = \text{angle}\,\widehat{(f)(\overline{\alpha})}, \tag{3}$$

which has a clear interpretation only in S_N, since on the r.h.s. of (3), f is considered as being translated from P to $\overline{P}$ exclusively within S_N.

[2] That is, in modern language, $\mathbb{R}^N$ equipped with its standard Euclidean structure.

[3] If an index is repeated in a product of vectors or tensors, summation is implied over the repeated index.

In other words, the Euclidean transport in S_N induces an infinitesimal parallel transport on V_n. Parallel transport is thus defined as the operation that maps a tangent vector at P to a tangent vector at $\overline{P}$, preserving both direction (in the above sense) and length. For each tangent vector f to V_n, represented in S_N as

$$\delta y_\nu = \frac{\partial y_\nu}{\partial x_k}\delta x_k$$

for arbitrary δx_k, and for a generic curve $x_k(\lambda)$ from P to $\overline{P}$ on V_n, we denote by $\alpha_\nu(\lambda)$ the parallel transported vector along $x_k(\lambda)$. Writing $\alpha'_\nu = \frac{\mathrm{d}\alpha_\nu}{\mathrm{d}\lambda}$, condition (3) translates into:

$$0 = \delta\alpha = \alpha'_\nu \delta y_\nu = \alpha'_\nu \frac{\partial y_\nu}{\partial x_k}\delta x_k, \quad \text{that is:} \quad \alpha'_\nu \frac{\partial y_\nu}{\partial x_k} = 0. \tag{4}$$

These considerations do not appear to be intrinsic, as they rely on the choice of embedding in a Euclidean space. Condition (4) is precisely equivalent to the following one, found in Oprea [37] in the case where $V_2 = S^2 \subset \mathbb{R}^3$:

$$\mathrm{proj}_{TV_n}\, \alpha' = 0. \tag{5}$$

Moreover, (4) is also equivalent to (12) below, expressed in different notation. After some computation, as Levi-Civita demonstrated, (4) can be rewritten in an equivalent intrinsic form, involving the vanishing of the *covariant derivative* ∇ (based on a_{ij}) of α along an arbitrary curve $x_k(\lambda)$.[4] Setting $\delta x_k = x'_k \delta\lambda$ and defining $\alpha_i = \alpha_\nu \frac{\partial y_\nu}{\partial x_i}$, the parallel-transported $\alpha_i(\lambda)$ must satisfy the following equation:[5]

$$0 = \nabla\alpha_i = \alpha'_i + \begin{Bmatrix} j & l \\ & i \end{Bmatrix} x'_j \alpha_l = 0, \quad i = 1, 2, \ldots, n. \tag{6}$$

This result is significant: equation (6) is fully intrinsic in V_n. Thus, the concept of parallel transport is *a posteriori* independent of the embedding, its properties hold regardless of the chosen embedding. Eventually, this shows the irrelevance of the particular choice of S_N for the local isometric embedding (1). In general, the concept of parallel transport depends on the specific path taken from P to $\overline{P}$, unlike in Euclidean spaces where it is path-independent. In fact, it is well known that the real obstruction to the independence is the *curvature* of the connection, involved in the above equation (6).

Levi-Civita then derived the following expressions:

$$x''_i + \begin{Bmatrix} j & l \\ & i \end{Bmatrix} x'_j x'_l = 0, \quad i = 1, 2, \ldots, n, \tag{7}$$

which govern the geodesics, the self-parallel curves: $\alpha = x'$.

2.3. *Some remarks*

Isometric immersions play a central role in Levi-Civita's introduction to parallel transport [29] and in his later treatises [31]. Since Ricci was both his mentor and a key collaborator, it is reasonable to assume that Levi-Civita had studied Ricci's 1884 memoir [39]. In that work, as discussed above, Ricci referenced Schlaefli's 1873 paper [43] on the existence of isometric

[4] The covariant derivative was introduced by Ricci in 1888; for more details, see [18]. This concept was revisited and further developed in the joint work [41] that represents the final version of the absolute differential calculus (the tensorial calculus) elaborated together with T. Levi-Civita.

[5] Today Christoffel's symbols of the second kind

$$\begin{Bmatrix} j & l \\ & i \end{Bmatrix} = \frac{1}{2} a^{ik} \left(\frac{\partial a_{jk}}{\partial x_l} + \frac{\partial a_{kl}}{\partial x_j} - \frac{\partial a_{jl}}{\partial x_k} \right)$$

are denoted by $\{{}_{j}{}^{i}{}_{l}\} = a^{ik}[jl,k]$, which differ from Levi-Civita's notation. Similarly, vector x'_j is today replaced by its contravariant form x'^j.

immersions of Riemannian manifolds V_n into some Euclidean space S_N, with at least $N = n(n+1)/2 =: s_n$, an integer that would later become known as the *Janet dimension*.

Schlaefli's so-called "conjecture" [23], originally based on the straightforward requirement of balancing equations and unknowns [21], had yet to be formally established as a theorem when Levi-Civita published his paper on parallel transport [29], in 1917. Actually, as on other occasions, Levi-Civita had a strong technical confidence in the power and applicability of the Cauchy–Kowalevski theorem, which will actually be the analytical engine for the subsequent proof of the isometric immersion matter [8,13,25]. In fact, he used Cauchy–Kowalevski theorem (without naming it)[6] in his important work on the three-body problem [28]. In that notable 1906 paper, Levi-Civita invoked a "general existence theorem" offering holomorphic solutions:

> ...l'équation précédente
>
> $$\frac{1}{8}\left\{\left(\frac{\partial W}{\partial \xi_1}+2\rho^2\eta_1\right)^2+\left(\frac{\partial W}{\partial \eta_1}-2\rho^2\xi_1\right)^2\right\}=\nu - C\rho^2+\frac{1}{2}\rho^6+\mu\rho^2 V_1 \qquad (10')$$
>
> où V_1 est ce que devient V on y remplaçant ξ, η par...
>
> Le théorème général d'existence relatif aux équations aux dérivées partielles du premier ordre, nous permet ainsi d'affirmer qu'il existe deux intégrales de $(10')$, holomorphes aux voicinage de $\xi_1 = \eta_1 = 0$ et se réduisant a zéro pour $\xi_1 = 0$. Leurs développements en série de puissances de ξ_1, η_1 peuvent être calculés de proche en proche, en partant de l'une ou de l'autre des expressions de $\frac{\partial W}{\partial \xi_1}$ fournies per $(10')$.

The technique of making explicit a derivative of the unknown as a function of the remaining variables and derivatives, while assigning initial data on a suitable hypersurface, falls within the framework of the Cauchy–Kowalevski theorem.[7] Given that this fundamental theorem was well known within the mathematical community at the time, it is reasonable to assume that he employed this approach.

In his 1917 construction, Levi-Civita had an immediate need for that result, as he had to inherit the Riemannian metric on V_n from the Euclidean one of some S_N (or $\mathbb{R}^N$). His proposal specifically concerned parallel transport along curves on Riemannian manifolds of arbitrary dimension n.

Over the following years, between 1925 and 1927, Persico edited Levi-Civita's lectures, leading to the publication of the volume in Italian [30], followed by its English version in 1926 [31]. It was around this time that the first true proto-proof of the local isometric immersion theorem emerged, provided by Janet, though limited to 2-dimensional manifolds V_2 in $\mathbb{R}^3$. In both the Italian and English volumes, parallel transport is now introduced based on a strictly 2-dimensional concept: *developability*.

In Chapter V of these works [31], Levi-Civita presents a more geometric and elegant definition of parallel transport. He moves away from the previously discussed infinitesimal angular equality and instead demonstrates the natural emergence of parallel transport on *developable* surfaces, inherited from the Euclidean case.[8]

He then generalizes the concept by defining parallel transport of vectors along a curve ℓ on a 2-dimensional surface Σ in terms of parallel transport along ℓ, now regarded as lying on the *envelope* surface $\overline{\Sigma}$ of the family of planes tangent to Σ along ℓ. The surface $\overline{\Sigma}$ is thus *developable*, and its tangent planes coincide with those of Σ. Since parallel transport on developable surfaces is naturally well-defined, this provides a rigorous foundation for the construction.

---

[6] This has been highlighted in [10].

[7] For details and an explanation, see [10].

[8] It is worth noting that, unlike the general case developed in 1917 for arbitrary dimensions, this argument is restricted to 2-dimensional surfaces in $\mathbb{R}^3$. Incidentally, for $n = 2$, the Janet dimension is indeed $N = 3$.

This refined notion of parallel transport leads to formulas (21) and (22) in [31, Chapter V].[9] Recognizing the intrinsic tensorial nature of this result naturally facilitates its extension to arbitrary-dimensional manifolds for $n > 2$.

3. Severi

In that same year 1917, and in the same journal *Rendiconti del Circolo Matematico di Palermo*, Severi — a close friend and colleague of Levi-Civita in Padua — published his own version of parallel transport [46]. His goal, among other seemingly fundamental aims related to curvature, was to introduce parallel transport in a fully intrinsic manner, avoiding reliance on those isometric immersions.

Severi's approach can be described as follows, using his notation. Consider a Riemannian manifold V_n with two "very close" points, A and A_1. In the tangent space to V_n at A, take a vector ξ. Severi defines the parallel transport of ξ from A to A_1, denoted ξ_1, as follows.

By considering the first-order approximation, we can think of A_1 as a point on V_n that is reached by a smooth curve originating from A. In local coordinates, $A_1 = A + vt$, where t is infinitesimal. At A, we have two well precise vectors in the tangent space to V_n: v (the tangent vector to the curve) and ξ (the vector we wish to parallel transport). Consider the 2-plane marked by these two vectors, ξ and v, and think of all the vectors between ξ and v as initial velocities for geodesics stemming from A. This set of vectors generates a 2-dimensional surface σ.

To obtain the parallel-transported vector ξ_1, Severi's method involves starting from ξ, then following the vectors attached to the curve between A and A_1. These vectors must be tangent to the surface σ, and maintain the same angle with the vectors tangent to the curve, at the first order determined exactly by v, the velocity of the curve. Severi demonstrates that it is precisely the Levi-Civita parallel transport, but now introduced in a fully intrinsic way.

Levi-Civita, in a friendly yet assertive manner, discusses the *teorema di Severi* in [31] and points out that this is, in fact, a trivial consequence of his own results. The reason for this is that in geodesic coordinates at point A, where the surface σ is a portion of a plane, the Christoffel symbols vanish at A and the metric at A is Euclidean. Therefore, the condition for the zero covariant derivative of the parallel transport reduces to the well-known Euclidean condition for parallel transport in flat space.

It is interesting to note that, aside from Severi's definition, other formulations of parallel transport do not necessarily rely on the concept of isometric immersion. For instance, Hermann Weyl (1885–1955) [49] proposed a different approach leading to his definition of "affine connection."[10]

Severi cared deeply about his definition of parallel transport. Many years later, in 1955, he revisited it in a rather long footnote on p. 315 of his own article on special relativity [47]:

> Of this deservedly famous parallelism, the Author of this article had the good fortune of being able to assign an intrinsic, very simple geometric meaning, cited several times in the works of Levi-Civita, which start from the analytical definition, as "theorem of Severi." It consists of this...

By the way, it is interesting to note that the use of geodesic 2-surfaces played a crucial role in Riemann's construction of his curvature tensor through sectional curvatures.

---

[9] Notably, in the English version, a minor typo in formula (21) of the Italian edition appears to have been corrected.

[10] Weyl's approach was referenced by Levi-Civita in a footnote on p. 135 and p. 117 of the Italian [30] and English [31] editions. For more details on Weyl's work, see Scholz's papers [44,45]; see also [48], especially Chapter 5.

4. Alternative roads

In his treatises, unlike in his 1917 paper, Levi-Civita introduced parallel transport through considerations strictly related to a 2-dimensional setting, specifically within the theory of developable surfaces. Inspired by [9, Section 7.4], we present an alternative approach to parallel transport for 2-dimensional surfaces immersed in $\mathbb{R}^3$, adopting a different cultural perspective while ultimately arriving at Levi-Civita's theory. A key novelty of this formulation is that it entirely avoids the use of infinitesimal concepts and reasoning. Similar final formulas were introduced by Berry [6], which we will consider in the next section (Section 4.2).

4.1. *A new heuristic approach to parallel transport*

Let a 2-surface $\Sigma \hookrightarrow \mathbb{R}^3$, locally described by

$$\mathbb{R}^2 \supseteq U \ni (u^1, u^2) \longmapsto P(u^1, u^2) \in \mathbb{R}^3, \quad \mathrm{rk}(\mathrm{d}P) = \max.$$

The triad of 3-vectors

$$\left(\frac{\partial P}{\partial u^1}, \frac{\partial P}{\partial u^2}, n := \frac{\partial P}{\partial u^1} \times \frac{\partial P}{\partial u^2} \Big/ \left\| \frac{\partial P}{\partial u^1} \times \frac{\partial P}{\partial u^2} \right\| \right)$$

is linearly independent in $\mathbb{R}^3$ and the pair $\left(\frac{\partial P}{\partial u^1}, \frac{\partial P}{\partial u^2}\right)$ is a base for any point of $T\Sigma$.

Take:

(i) a curve $\ell\colon [0,1] \ni \lambda \mapsto \ell(\lambda) = \left(u^\alpha(\lambda)\right)\big|_{\alpha=1,2} \in \Sigma$;

(ii) a vector $V_0 \in T_{\ell(0)}\Sigma$.

Our central problem: what is the "reasonable" curve

$$\widetilde{\ell}\colon [0,1] \ni \lambda \longrightarrow \widetilde{\ell}(\lambda) = \left(\ell(\lambda), V(\lambda)\right) \in T\Sigma$$

which *parallel transports* V_0 along ℓ, in some suitable sense?

A first (seemingly naive) tentative could be the following (recall that we want: $V \cdot n \equiv 0$, that is, $V \in T\Sigma$):

$$V(\lambda) = V_0 - (V_0 \cdot n)n, \tag{8}$$

in other words, we transport V_0 along the curve ℓ by the standard Euclidean affine structure of $\mathbb{R}^3$, erasing, point by point, the normal component of V_0, so that $V(\lambda)$ is really tangent to Σ. By differentiating with respect to λ,

$$\dot{V} = -(V_0 \cdot \dot{n})n - (V_0 \cdot n)\dot{n},$$

and since $\dot{n} \cdot n \equiv 0$, by (8): $V_0 \cdot \dot{n} = V \cdot \dot{n}$, hence we can rewrite

$$\dot{V} = -(V \cdot \dot{n})n - (V_0 \cdot n)\dot{n}. \tag{9}$$

This setting has evident drawbacks: (i) we are heavily using the host environment $\mathbb{R}^3$ in which Σ is immersed; moreover, (ii) the term $-(V_0 \cdot n)\dot{n}$ in (9) is *non local*. Our proposal is to drop out the last term in the r.h.s. of (9) and to postulate:

$$V(\lambda): \quad \dot{V} = -(V \cdot \dot{n})n, \tag{10}$$

that implies $V \cdot n \equiv 0$ yet.

We restart our construction by involving the induced Riemannian metric on Σ inherited from the Euclidean metric of $\mathbb{R}^3$:

$$g_{\alpha\beta} = \frac{\partial P}{\partial u^\alpha} \cdot \frac{\partial P}{\partial u^\beta}, \qquad V = v^\alpha \frac{\partial P}{\partial u^\alpha}, \qquad V \cdot \frac{\partial P}{\partial u^\beta} = v^\alpha \frac{\partial P}{\partial u^\alpha} \cdot \frac{\partial P}{\partial u^\beta} = v^\alpha g_{\alpha\beta} = v_\beta. \tag{11}$$

Equivalently to (10) we will ask

$$V(\lambda): \quad 0 = \dot{V} \cdot \frac{\partial P}{\partial u^\beta}, \quad \beta = 1,2. \tag{12}$$

This condition (12) is precisely the detailing for $\Sigma \subset \mathbb{R}^3$ given by (4) above, which is the general formula (8) in Levi-Civita [29] or formulas (21) and (22) in [31, Chapter V]. The above equivalence between (10) and (12) is easily gained. From here on, following Levi-Civita, this proposal (12) will be elaborated *for generic Riemannian manifold of any dimension* and eventually avoiding any reference at the host Euclidean structure:

$$0 = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(V \cdot \frac{\partial P}{\partial u^\beta}\right) - V \cdot \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial P}{\partial u^\beta}, \qquad 0 = \frac{\mathrm{d}}{\mathrm{d}\lambda} v_\beta - v^\alpha \frac{\partial P}{\partial u^\alpha} \cdot \frac{\partial^2 P}{\partial u^\gamma \partial u^\beta} \frac{\mathrm{d}}{\mathrm{d}\lambda} u^\gamma. \tag{13}$$

From $(11)_1$,

$$g_{\alpha\beta,\gamma} = \frac{\partial^2 P}{\partial u^\gamma \partial u^\alpha} \cdot \frac{\partial P}{\partial u^\beta} + \frac{\partial P}{\partial u^\alpha} \cdot \frac{\partial^2 P}{\partial u^\gamma \partial u^\beta},$$
$$g_{\beta\gamma,\alpha} = \frac{\partial^2 P}{\partial u^\alpha \partial u^\beta} \cdot \frac{\partial P}{\partial u^\gamma} + \frac{\partial P}{\partial u^\beta} \cdot \frac{\partial^2 P}{\partial u^\alpha \partial u^\gamma},$$
$$g_{\gamma\alpha,\beta} = \frac{\partial^2 P}{\partial u^\beta \partial u^\gamma} \cdot \frac{\partial P}{\partial u^\alpha} + \frac{\partial P}{\partial u^\gamma} \cdot \frac{\partial^2 P}{\partial u^\beta \partial u^\alpha},$$

hence,

$$\frac{\partial P}{\partial u^\alpha} \cdot \frac{\partial^2 P}{\partial u^\gamma \partial u^\beta} = \frac{1}{2}\left(g_{\alpha\beta,\gamma} + g_{\gamma\alpha,\beta} - g_{\beta\gamma,\alpha}\right) =: [\gamma\beta, \alpha]$$

recalling the *symbols of Christoffel of first and second kind,* $[\gamma\beta,\alpha]$ and $\left\{{}_{\gamma}{}^{\alpha}{}_{\beta}\right\} = g^{\alpha\rho}[\gamma\beta,\rho]$, relation $(13)_2$ becomes

$$0 = \frac{\mathrm{d}v_\beta}{\mathrm{d}\lambda} - v^\alpha [\gamma\beta, \alpha] \frac{\mathrm{d}\ell^\gamma}{\mathrm{d}\lambda} \qquad \text{or} \qquad 0 = \frac{\mathrm{d}v_\beta}{\mathrm{d}\lambda} - v_\alpha \begin{Bmatrix} \alpha \\ \gamma \quad \beta \end{Bmatrix} \frac{\mathrm{d}\ell^\gamma}{\mathrm{d}\lambda}, \tag{14}$$

which is precisely the condition of *vanishing covariant derivative* ∇ of v_β along ℓ, expressing intrinsically the Levi-Civita parallel transport.

4.1.1. *Foucault*

The Foucault pendulum is a device designed to experimentally demonstrate the Earth's rotation (see, e.g., [1, p. 132]). It consists of a spherical pendulum suspended over the tangent plane at a point P on Earth with latitude α. The mechanical description is formulated in the non-inertial reference frame attached to the Earth, accounting for gravity and the Coriolis force while neglecting centrifugal effects, considered absorbed into the gravitational force. The linearized analysis of small oscillations around the stable equilibrium reveals a precession of the oscillation plane, which, over 24 hours, rotates by an angle given by (17). As we discuss below, this precession precisely corresponds to the holonomy[11] angle associated with parallel transport along the terrestrial parallel γ at latitude α, applied to a generic initial vector $V(0)$. This transported vector V is interpreted as a tangent vector along the terrestrial parallel, representing the motion of the oscillation plane.

Let us try to retrace the explanation of the Foucault precession moving from the proposal (10) of the parallel transport.

Consider a unit radius sphere S^2 and on it a "parallel" γ at the latitude α. We denote the longitude by φ, hence

$$[0,T] \ni t \longmapsto \gamma(t) = \left(\cos\alpha\cos\varphi(t), \cos\alpha\sin\varphi(t), \sin\alpha\right) = P, \quad \gamma(0) = P = \gamma(T), \quad n(t) = \gamma(t).$$

[11] Holonomy is the map that assigns to each piecewise-smooth loop γ based at a point P in a manifold M the parallel transport operator obtained by transporting vectors (or frames) along γ with respect to a given connection. In the present context, the connection is precisely that of the Levi-Civita related to the Riemann metric on the sphere S^2 inherited from the Euclidean one.

Let us consider the mobile triad (e_1, e_2, n) moving with γ, where e_1 is tangent to γ, as $\dot n$, and (e_1, e_2) is a basis for $T_P S^2$:

$$e_1 = (-\sin\varphi, \cos\varphi, 0), \qquad \dot e_1 = (-\cos\varphi, -\sin\varphi, 0)\,\dot\varphi,$$
$$e_2 = (-\sin\alpha\cos\varphi, -\sin\alpha\sin\varphi, \cos\alpha), \qquad \dot e_2 = -\sin\alpha\,\dot\varphi\, e_1,$$
$$n = \gamma, \qquad \dot n = \cos\alpha\,\dot\varphi\, e_1.$$

We put in evidence that $\dot e_1$ is in the plane (e_2, n),

$$\begin{aligned}\dot e_1 &= (-\cos\varphi, -\sin\varphi, 0)\dot\varphi\\ &= \left[-(\sin^2\alpha+\cos^2\alpha)\cos\varphi, -(\sin^2\alpha+\cos^2\alpha)\sin\varphi, \sin\alpha\cos\alpha-\sin\alpha\cos\alpha\right]\dot\varphi\\ &= \sin\alpha(-\sin\alpha\cos\varphi, -\sin\alpha\sin\varphi, \cos\alpha)\dot\varphi - \cos\alpha(\cos\alpha\cos\varphi, \cos\alpha\sin\varphi, \sin\alpha)\dot\varphi\\ &= \sin\alpha\,\dot\varphi\, e_2 - \cos\alpha\,\dot\varphi\, n.\end{aligned}$$

Our aim is to pointing out that the Levi-Civita parallel transport of a vector $V_0 \in T_{\gamma(0)}S^2$ along the closed curve γ displays a holonomy, the discrepancy angle with respect V_0 by turning back V to $\gamma(0)$, exactly equal to the precession angle of the plane of oscillation of a Foucault pendulum moving over and with γ:

$$\begin{aligned}V &= |V|(\cos\theta\, e_1 + \sin\theta e_2)\\ \dot V &= |V|\left[-\sin\theta\,\dot\theta e_1 + \cos\theta\dot e_1 + \cos\theta\,\dot\theta e_2 + \sin\theta\dot e_2\right]\\ &= |V|\left[-\sin\theta\,\dot\theta e_1 + \cos\theta(-\cos\varphi, -\sin\varphi, 0)\dot\varphi + \cos\theta\,\dot\theta e_2 - \sin\theta\sin\alpha\,\dot\varphi\, e_1\right]\\ &= |V|\left[-\sin\theta\,\dot\theta e_1 + \cos\theta(\sin\alpha\,\dot\varphi\, e_2 - \cos\alpha\,\dot\varphi\, n) + \cos\theta\,\dot\theta e_2 - \sin\theta\sin\alpha\,\dot\varphi\, e_1\right]\\ &= |V|\left[(-(\dot\theta + \sin\alpha\dot\varphi)\sin\theta e_1 + (\dot\theta+\sin\alpha\dot\varphi)\cos\theta e_2 - \cos\theta\cos\alpha\dot\varphi n\right].\end{aligned} \tag{15}$$

From (10),

$$\dot V = -(V\cdot\dot n)n = -|V|\cos\theta\cos\alpha\dot\varphi n, \tag{16}$$

and by comparing $(15)_2$ with (16), it follows that $\dot\theta = -\sin\alpha\dot\varphi$, hence for $\theta(0) = 0 = \varphi(0)$ and $\varphi(T) = 2\pi$, we obtain the expected Foucault angle:

$$\theta(T) = -2\pi\sin\alpha. \tag{17}$$

The angle (17) denotes exactly, with the correct sign, how rotates in a day the plane of the oscillations of a spherical pendulum placed on Earth at the latitude α. This computation was first proposed by Oprea [37].

As recently put in evidence by Giancarlo Benettin [4], the purely geometric nature of this explanation was first intuited by Poinsot [38], in a memorable discussion on the phenomenon with Foucault and Binet at the Académie des Sciences, Paris, 17 février 1851.[12]

4.2. *Berry and Hannay*

It is a matter of fact that the parallel transport and the theory of connections are intimately related. More, each is a manifestation of the other, exactly as Levi-Civita parallel transport is linked to the Riemannian metric connection. The holonomy of a connection on a bundle is often denoted in the physical community as a *phase*. We just recall the important emergence of the Berry phase [5] in quantum mechanics and mainly the analogous Hannay phase [24] in classical mechanics, which needs a *Hamiltonian* environment, *integrable* (it is essential to use the *action-angle* variables), and generally[13] in posing the problem as an *adiabatic* phenomenon, all typical requirements of Berry and Hannay phases. Often in this setting a further *dynamic phase* is arising

[12] Poinsot: *"Ce mouvement, dis-je, est un phénomène purement géométrique, et dont l'explication doit être donnée par la simple géométrie, comme l'a fait M. Foucault, et non par des principes de dynamique, qui il n'y entrent pour rien".*

[13] Although sometimes not necessary, see [7].

beside the *Hannay phase*. Both Hannay angle and standard Levi-Civita parallel transport bring back the interpretation of the precession of the oscillation plane of the Foucault pendulum to a phenomenon of holonomy. There is literature on this, which however could be divided in a finer way into two parts.

(1) In a first line of thought, we find the description of the Foucault pendulum by means of the Hannay phase [26], in a Hamiltonian integrable setting, where the adiabatic requirement can be dropped, since the final result (see (17)) is independent of $\varphi(t)$. This elaboration has been proposed in great detail e.g. in [33,34], where also we can find a historical sketch of the matter.[14] Hannay description of Foucault pendulum is running outside the standard historical parallel transport, this explains why Levi-Civita setting is never mentioned in the above literature.

(2) In a second direction, precisely by going back to the nice 1989 survey by Berry [6], we encounter a new simple definition of parallel transport on a 2-sphere, whose difference from the setting of Section 4.1 is explained right below. The author, just after this definition, involves it towards quantum aspects, nevertheless we see that this new framework is nothing other than a new operative variant of the Levi-Civita parallel transport; it has been taken up by Oprea 1995 [37], who calculated the Foucault discrepancy angle by parallel transport, substantially along computations as above presented.

4.3. *Berry's 1989 version*

The outcome (10) of the above alternative construction in Section 4.1, can be also related to a former paper by Berry [6] as follows. In order to obtain the law for the ordinary parallel transport of a vector over the sphere S^2, he postulated that:[15]

The unit vector V be transported by changing the unit radius vector n and making two demands:

(i) *that identically*

$$V \cdot n = 0 \tag{18}$$

and that

(ii) *the orthogonal triad containing V and n must not twist about n, $\Omega \cdot n = 0$, where Ω is the angular velocity of the triad.*

These conditions define parallel transport of V and lead to the law

$$\dot{V} = \Omega \times V \qquad \textit{where} \quad \Omega = n \times \dot{n}. \tag{19}$$

Detailing (19), we see

$$\dot{V} = (n \cdot V)\,\dot{n} - (V \cdot \dot{n})n, \tag{20}$$

and then $\dot{V} \cdot n = -V \cdot \dot{n}$, or $\frac{\mathrm{d}}{\mathrm{d}t}(V \cdot n) = 0$, so that (18) is satisfied as soon as it holds at the beginning; we get that the essential dynamic law of the parallel transport is really (10): $\dot{V} = -(V \cdot \dot{n})n$.

We observe that the formula $(19)_1$ is present in the tracts [31],[16] without detail on the angular velocity Ω; it is interesting to notice that in the original paper [29, Section 11], Levi-Civita discusses at length its definition by means of *Ricci's rotation symbols*: these mathematical objects, introduced by Ricci in [40], are not widely known, although some important tracts treat with them (see e.g. [17,20]), they resume in a covariant way and at any dimension, deep aspects typical of the standard angular velocity of the 3-dim Euclidean setting. It is worthy of being remembered that Ricci's rotation symbols were utilized in an essential way by Levi-Civita to rebuild *ex novo* the 1929 unitary theory of gravitation and electromagnetism by Einstein [19], so overcoming the

[14] See [34, p. 57, "More History"].

[15] By reporting in italic verbatim.

[16] See p. 122 of the 1925 Italian version [30] or p. 105 of the 1926 English version [31].

initial restriction on its validity just for linearized theory, offering eventually a powerful non linear theoretical framework, see [11].

As highlighted in [4], formulation (19), detailed in (20), is interesting also because it allows us to take into account the motion of any vector V of the whole rigid 3-space associated to the tangent plane parallel sliding along ℓ.

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