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Comptes Rendus
Mathématique

Yuan Liu

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Volume 364 (2026), p. 39-44

Online since: 2 March 2026

https://doi.org/10.5802/crmath.811

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www.centre-mersenne.org — e-ISSN : 1778-3569

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https://doi.org/10.5802/crmath.811

Research article
Geometry and Topology

The applications of Bieri–Neumann–Strebel invariant on Kähler groups

Yuan Liu [a]

[a] The University of Hong Kong, Pokfulam Road, Hong Kong
URL: https://sites.google.com/view/yuan-lius-website
E-mail: lexliu@hku.hk

Abstract. We give several applications of the Bieri–Neumann–Strebel invariant on Kähler groups. Specifically, we provide a simpler proof of the Napier–Ramachandran theorem on the HNN decomposition of Kähler groups and show that amenable Kähler groups have an empty complement of the BNS invariant.

2020 Mathematics Subject Classification. 20F65, 32Q15.

Funding. The author is partially supported by the China Postdoctoral Science Foundation (reference 2023M744396) and China Scholarship Council (No. 202406340174).

Manuscript received 14 June 2025, revised 21 November 2025, accepted 4 December 2025, online since 2 March 2026.

1. Introduction

We say a finitely presented group G is a Kähler group if it can be realized as the fundamental group of a compact Kähler manifold. Not every finitely presented group is a Kähler group, and it is a question of Serre to characterize Kähler groups. The readers may refer to the monographs [1,8] for this topic. The overall picture for the Kähler groups is still very unclear.

For G a finitely generated group, Bieri, Neumann, and Strebel introduced a celebrated invariant $\Sigma(G)$ in [2] (hereafter, we call it the BNS invariant). Unfortunately, the computation of the BNS invariant is extremely difficult and is only known for restricted types of groups up to now. Surprisingly, the combination of these two mysterious objects, i.e. (the complement of) the BNS invariant of a Kähler group, is described completely by Delzant in [5], using Simpson's Lefschetz theorem[9].

It is thus quite natural to use this BNS invariant to put restrictions on Kähler groups. For the HNN decomposition, some results were known for a long time thanks to the work of Napier and Ramachandran [7, Theorem 0.2 and Theorem 0.3]. Now, with the help of the BNS invariant, we give a much easier explanation (Theorems 10 and 11). Moreover, we show that amenable Kähler groups have the empty complement of the BNS invariant (Proposition 6).

This note is organized as follows. In Section 2, we provide essential background on the BNS invariant. In Section 3, we prove the results on HNN decompositions about Kähler groups and on amenable Kähler groups.

2. The Bieri–Neumann–Strebel invariant

2.1. *General theory*

In 1987, a famous group theoretic invariant was introduced by Bieri, Neumann, and Strebel [2], which is now called the Bieri–Neumann–Strebel (BNS for short) invariant. For a finitely generated group G, this invariant gives the full information when the kernel of an abelian quotient of G is finitely generated (for its precise meaning, see Proposition 3). The readers may refer to [2] or the unpublished monograph [3] for more details. *From now on, we always assume that the first Betti number $b_1(G) > 0$.*

Let G be a finitely generated group and $\chi\colon G \to \mathbb{R}$ be a non-trivial group homomorphism. Denote $G_\chi = \{g \in G : \chi(g) \geqslant 0\}$. For any given set of generators S, denote the associated Cayley graph of G as $C := \mathrm{Cay}(G;S)$. We define its subgraph C_χ as follows: (i) a vertex in C belongs to C_χ if its χ-value is nonnegative; (ii) an edge in C belongs to C_χ if *both* vertices of this edge have nonnegative χ-values. We can consider if C_χ is connected and record χ when this happens. Notice that this property does not depend on the generating set S (see [3, Theorem 2.1] or [8, Proposition 11.1]) and is invariant if we multiply χ by a positive real constant. Denote $[\chi]$ to be this equivalent relation of multiplication by a positive real constant, and write $\mathrm{S}(G) = \big(\mathrm{Hom}(G;\mathbb{R}) - \{0\}\big)/\mathbb{R}^+$ for the collection of all these equivalent classes.

Definition 1. *With the same notation as above, for G being a finitely generated group, we define its BNS invariant as*

$$\Sigma(G) = \big\{[\chi] \in \mathrm{S}(G) : C_\chi \text{ is connected}\big\}.$$

We need the following properties of this invariant for later use.

Proposition 2 ([2, Theorem C]). *Let G be a finitely presented group with no nonabelian free subgroups, then*

$$\Sigma(G) \cup -\Sigma(G) = \mathrm{S}(G),$$

where $-\Sigma(G)$ is the image of $\Sigma(G)$ under the antipodal map.

Proposition 3 ([2, Theorem B1]). *If G is a finitely generated group and A is its non-trivial abelian quotient with kernel N, then N is finitely generated if and only if*

$$\mathrm{S}(G,N) \subseteq \Sigma(G),$$

where $\mathrm{S}(G,N) = \big\{[\chi] \in \mathrm{S}(G) : \chi(N) = 0\big\}$. *In particular, G' is finitely generated if and only if* $\Sigma(G) = \mathrm{S}(G)$, *where $G' = [G,G]$ is the commutator subgroup of G.*

2.2. *The BNS invariant for Kähler groups*

When this group G is the fundamental group of a finite CW-complex X, we can give a topological explanation of this invariant [2, Section 5]. We denote $\widehat{X}$ as the maximal free abelian covering of X with Galois group H. Notice that $\chi \in \mathrm{Hom}(G;\mathbb{R})$ naturally factors through the maximal free abelian quotient H of G, for which we still denote as $\chi\colon H \to \mathbb{R}$. Let $f\colon \widehat{X} \to \mathbb{R}$ be the primitive function in the sense that $f(h.x) = f(x) + \chi(h)$ for any $h \in H$ and $x \in \widehat{X}$. Now consider $\{x \in X \mid f(x) \geqslant 0\}$, it has a unique component on which f is unbounded, and we denote it as $\widehat{X}_f$ (see [2, Lemma 5.2]). Then we have $[\chi] \in \Sigma(G)$ if and only if the morphism on fundamental groups induced by the inclusion $\widehat{X}_f \hookrightarrow \widehat{X}$ is onto (see [2, Theorem 5.1]). Using this topological explanation along with Simpson's Lefschetz theorem, Delzant in [5] gives a complete description of (the complement of) the BNS invariant for Kähler groups.

Theorem 4 ([5, Theorem 1.1]). *Let X be a compact Kähler manifold with the fundamental group G. For a non-trivial $\chi \in H^1(G;\mathbb{R})$, we have χ is exceptional, i.e. $[\chi] \notin \Sigma(G)$, if and only if there exists a holomorphic fibration (i.e. a holomorphic map with connected fibers) onto a hyperbolic Riemann orbifold S of genus greater than or equal to 1, say $f\colon X \to S$, such that $\chi \in f^*\big(H^1\big(\pi_1^{\text{orb}}(S);\mathbb{R}\big)\big)$.*

The following conclusion is immediate, which is also stated in [8, Section 11.3].

Corollary 5 (Symmetric property of Kähler groups). *Any Kähler group G has symmetric BNS invariant, i.e.*

$$\Sigma(G) = -\Sigma(G).$$

Proof. For any $[\chi] \notin \Sigma(G)$, by Theorem 4, we have a holomorphic fibration $f\colon X \to S$ with $\chi = f^*(v)$ for some $v \in H^1\big(\pi_1^{\text{orb}}(S);\mathbb{R}\big)$. Then $-\chi = f^*(-v)$ and $-v \in H^1\big(\pi_1^{\text{orb}}(S);\mathbb{R}\big)$. Thus $[-\chi] \notin \Sigma(G)$, and the conclusion is proved. □

3. The applications

In this section, we give some applications to the study of Kähler groups.

The first result is the combination of [6, Proposition 3.1 and Lemma 3.2]. Their original proof uses the rank gradient, and we here prove the same result using solely the property of the BNS invariant.

Proposition 6. *Let G be a Kähler group which does not contain any free nonabelian subgroup, then $\Sigma(G) = S(G)$ and the commutator subgroup G' is finitely generated. In particular, if G is an amenable Kähler group, we have the above conclusion holds.*

Proof. By Proposition 2, we have $\Sigma(G) \cup -\Sigma(G) = S(G)$. Also, $\Sigma(G) = -\Sigma(G)$ by Corollary 5, then we have the desired equality. The property of G' follows directly from Proposition 3. □

Remark. The above conclusion for amenable Kähler groups can also be seen as follows. We may assume $b_1(G) > 0$; otherwise, it is trivially true. If $\Sigma^c(G) := S(G) - \Sigma(G)$ is nonempty, by Theorem 4, we have a holomorphic fibration onto a hyperbolic Riemann orbifold, say $f\colon X \to S$, which induces a surjection f_* on fundamental groups. Since $\pi_1(S)$ is not amenable while $\pi_1(X)$ is amenable, this contradicts the fact that the homomorphic image of an amenable group is still amenable. This argument was mentioned to me by Pierre Py.

Next, we will use the HNN-valuation introduced in [4] to reprove some classical results on Kähler groups in [7]. Notice a conflict in the ascending (resp. descending) HNN decomposition definition in the above two papers, and we will use the same notation as in [7] here. These results are also achieved by Fridel and Vidussi in [6] using the rank gradient.

Definition 7 (HNN decomposition). *Take any base group B and a stable letter t. Let $\varphi\colon B_1 \to B_2$ be an isomorphism between two subgroups B_1, B_2 of B. We say a group G* admits an HNN decomposition with base group B *if it can be written as*

$$G = \big\langle B, t \,\big|\, h^t := tht^{-1} = \varphi(h),\ \forall\, h \in B_1 \big\rangle.$$

If $B_1 = B$ (and $B_2 \neq B$), we say that G is (properly) descending; if $B_2 = B$ (and $B_1 \neq B$), we say that G is (properly) ascending.

For a group G admitting an HNN decomposition with base group B and stable letter t, we can define a surjective group homomorphism $\chi\colon G \twoheadrightarrow \mathbb{Z}$ by setting

$$\chi(g) = \begin{cases} 1 & \text{if } g = t, \\ 0 & \text{if } g \in B. \end{cases} \tag{1}$$

We will call this χ the *associated homomorphism.* Set $N = \ker(\chi)$, and obviously we have $N = \bigcup_{k\in\mathbb{Z}} t^k B t^{-k}$.

Now we need the concept of an HNN valuation introduced by Brown, which can be seen as a generalization of the HNN decompositions (see [4, Proposition 2.1(v) and the remark afterwards]).

Definition 8 (HNN valuation). *Let G be a group, and $\chi\colon G \to (\mathbb{R},+)$ be a given group homomorphism. A function $v\colon G \to \mathbb{R}_\infty := \mathbb{R}\cup\{\infty\}$ is called* an HNN valution with respect to χ *if it satisfies:*

(a) $v(g^{-1}) = v(g) + \chi(g)$;

(b) $v(gh) \geq \min\{v(g), v(h) - \chi(g)\}$.

We will say that v is non-trivial *if in addition:*

(c) $v\big|_{G_{\chi\leq 0}}$ *does not assume a minimal value.*

If $\chi \neq 0$, condition (c) is equivalent to $v\big|_N$ is not bounded below, where $N = \ker(\chi)$ (see [4, Proposition 2.3]).

Note here that $\chi\colon G \to \mathbb{R}$ can be any given group homomorphism. When G is a group admitting an HNN decomposition, we have a natural choice of $\chi\colon G \to \mathbb{Z} \hookrightarrow \mathbb{R}$, say the associated homomorphism defined by (1).

The main criterion we will apply is the following.

Proposition 9 ([4, Proposition 3.1]). *Let G be a finitely generated group and $\chi\colon G \twoheadrightarrow \mathbb{Z}$ a surjective group homomorphism. The following three conditions are equivalent:*

(1) *G admits a descending HNN decomposition with finitely generated base group B and with χ as the associated homomorphism;*

(2) *G does not admit a properly ascending HNN decomposition with χ as the associated homomorphism;*

(3) *there is no non-trivial HNN valuation v on G with respect to this χ.*

Moreover, if these conditions hold, then every HNN decomposition of G with χ as associated homomorphism is descending.

With this notation at hand, Brown in [4] defined his invariant to be the set of classes $[\chi] \in S(G) := \big(\mathrm{Hom}(G;\mathbb{R}) - \{0\}\big)/\mathbb{R}^+$ such that one representation (equivalently, all representations) χ satisfies condition (3) above with $\mathbb{Z}$ replaced by $\mathbb{R}$, i.e., there is no non-trivial HNN valuation v on G with respect to this χ. Later, in [4, Theorem 5.2], he showed that his invariant coincides with the BNS invariant in [2]. In conclusion, we have

$$\Sigma(G) = \left\{ [\chi] \in S(G) \;\middle|\; \begin{array}{l} \text{there is no non-trivial HNN valuation } v \\ \text{on } G \text{ with respect to } \chi \end{array} \right\}. \tag{2}$$

We can now reprove the theorems in [7] on HNN decompositions.

Theorem 10 ([7, Theorem 0.2(b)]). *Let G be the fundamental group of a compact Kähler manifold X. If G admits a properly ascending HNN decomposition, then X admits a holomorphic fibration onto a hyperbolic Riemann orbifold.*

Proof. Let $\chi\colon G \to \mathbb{Z}$ be the associated homomorphism for the properly ascending HNN decomposition. Since condition (2) of Proposition 9 is not satisfied, neither is condition (3) there. Then there exists a non-trivial HNN valuation v on G with respect to this χ, which means $[\chi] \notin \Sigma(G)$ by (2). Then X fibers over a hyperbolic Riemann orbifold according to Theorem 4. □

Theorem 11 ([7, Theorem 0.3(b)]). *Any group G admitting a properly ascending (or descending) HNN decomposition with a finitely generated base group is not a Kähler group.*

Proof. We first consider the case G admits a properly ascending HNN decomposition with a finitely generated base group B and stable letter t, and write

$$G = \langle B, t \mid tht^{-1} = \varphi(h) \in B,\ \forall\, h \in B_1 \rangle$$

with $B_1 \subsetneq B$ and $\varphi \colon B_1 \to B$ a group isomorphism. By assumption on G, condition (2) of Proposition 9 is not satisfied, and neither is (3) for the associated homomorphism χ with $\chi(t) = 1$. Thus, we have $[\chi] \notin \Sigma(G)$ by (2).

On the other hand, taking the stable letter $s = t^{-1}$ instead, we can write

$$G = \langle B, s \mid shs^{-1} = \varphi^{-1}(h) \in B_1,\ \forall\, h \in B \rangle,$$

which is a (properly) descending HNN decomposition of G. Note the associated homomorphism now is $-\chi$, since we need its value at $s = t^{-1}$ to be 1 as in (1). Now we need the assumption that B is finitely generated, which ensures condition (1) in Proposition 9 is satisfied with $-\chi$ as the associated homomorphism. Then (3) is also satisfied for $-\chi$, and we have $[-\chi] \in \Sigma(G)$ by (2). This proves that G cannot be a Kähler group, since it contradicts the symmetric property of Kähler groups (Corollary 5).

If G admits a properly descending HNN decomposition with a finitely generated base group B and χ is the associated homomorphism, we can similarly prove $[\chi] \in \Sigma(G)$ and $[-\chi] \notin \Sigma(G)$, and thus G is not Kähler. □

Acknowledgments

This work is inspired by the new book *Lectures on Kähler groups* by Professor Pierre Py [8], and the author would like to thank him for this aspect. The author also thanks Professor Yongqiang Liu for many useful conversations. The author thanks the referee for helpful comments that made Section 3 especially more self-contained and clearer.

Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

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