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Research article
Partial differential equations

On the convergence of solutions for the Ginzburg–Landau equation and system

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Abstract. Let (u_ε) be a family of solutions of the Ginzburg–Landau equation with boundary condition $u_\varepsilon = g$ on $\partial\Omega$ and of degree 0. Let u_0 denote the harmonic map satisfying $u_0 = g$ on $\partial\Omega$. We show that, if there exists a constant $C_1 > 0$ such that for ε sufficiently small we have $\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x \le C_1 \le \frac{1}{2}\int_\Omega |\nabla u_0|^2\,\mathrm{d}x$, then $C_1 = \frac{1}{2}\int_\Omega |\nabla u_0|^2\,\mathrm{d}x$ and $u_\varepsilon \to u_0$ in $H^1(\Omega)$. We also prove that if there is a constant C_2 such that for ε small enough we have $\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x \ge C_2 > \frac{1}{2}\int_\Omega |\nabla u_0|^2\,\mathrm{d}x$, then $|u_\varepsilon|$ does not converge uniformly to 1 on $\overline{\Omega}$. We obtain analogous results for both symmetric and non-symmetric two-component Ginzburg–Landau systems.

Keywords. Two component Ginzburg–Landau equations, non-symmetric potential, asymptotic behavior of solutions.

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1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Let

$$g : \partial\Omega \longrightarrow S^1 = \left\{z \in \mathbb{C} : |z| = 1\right\}$$

be a smooth map that has a nonnegative integer-valued degree $\deg(g, \partial\Omega) = d$. Let us define

$$H^1_g(\Omega) = \left\{u \in H^1(\Omega;\mathbb{C}) : u = g \text{ on } \partial\Omega\right\}.$$

For $\varepsilon > 0$, we consider the Ginzburg–Landau energy functional

$$G_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{1}{4\varepsilon^2}\int_\Omega \left(1 - |u|^2\right)^2\mathrm{d}x. \tag{1}$$

The Euler–Lagrange equations for G_ε are the Ginzburg–Landau equations

$$\begin{cases} -\Delta u = \frac{1}{\varepsilon^2} u\left(1 - |u|^2\right) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{2}$$

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In [1,2], Bethuel, Brezis and Hélein studied the convergence of minimizers. In particular, if $\deg(g,\partial\Omega)=0$, they proved the following.

Theorem 1 ([1]). *Let u_ε be a minimizer of G_ε over $H^1_g(\Omega)$ where Ω is a star-shaped domain. If $d=0$, then $u_\varepsilon\to u_0$ in $C^k_{\mathrm{loc}}(\Omega)$ for any nonnegative integer k as $\varepsilon\to 0$ such that u_0 is a unique solution of*

$$u_0=\operatorname*{argmin}_{u\in H^1_g(\Omega;S^1)} J_g(u)\quad \text{where } J_g(u)=\frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x. \tag{3}$$

The function u_0 satisfies

$$\begin{cases}-\Delta u=u|\nabla u|^2 & \text{on } \Omega,\\ u=g & \text{on } \partial\Omega,\\ |u|=1 & \text{on } \Omega.\end{cases} \tag{4}$$

See also [2] for the nonzero-degree case, [6] for a potential having a zero of infinite order, and [3] for the quantization effect on the whole plane. According to [2, Remark A.1], the conclusion of Theorem 1 can still hold even when u_ε is not a minimizer. Indeed, we have the following.

Theorem 2 ([2, p. 144]). *Assume* $\deg(g,\partial\Omega)=0$ *and let u_ε be a solution of* (2). *If*

$$u_\varepsilon\longrightarrow u_0\quad \text{in } H^1(\Omega), \tag{5}$$

then the conclusion of Theorem 1 is valid.

Theorem 2 tells us that the strong convergence (5) is a key ingredient in the proof of Theorem 1.

Let (u_ε) be a sequence of solutions to (2). In this work, we establish that

$$\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x$$

admits the critical lower bound

$$\frac{1}{2}\int_\Omega |\nabla u_0|^2\,\mathrm{d}x,$$

beyond which the sequence (u_ε) cannot be lifted to a smooth function, see the proof of Theorem 3.

We provide another sufficient condition for Theorem 1 by identifying an equivalent formulation of (5). We also introduce a two-component generalization of (1) and (2), from which we derive analogous results.

Two facts used in the proof of Theorem 1 will also play a central role in this paper.

First, if u_ε is a solution of (2), then

$$|u_\varepsilon|\le 1\quad \text{on } \overline{\Omega}. \tag{6}$$

We can prove the inequality (6) by applying the maximum principle to the following identity:

$$-\Delta\big(1-|u_\varepsilon|^2\big)=-\frac{2}{\varepsilon^2}|u_\varepsilon|^2\big(1-|u_\varepsilon|^2\big)+2|\nabla u_\varepsilon|^2\quad \text{on } \Omega. \tag{7}$$

See [1, Proposition 2].

Second, if the domain Ω is star-shaped, then for any solution u_ε of (2), the potential

$$\frac{1}{\varepsilon^2}\int_\Omega \big(1-|u_\varepsilon|^2\big)^2\,\mathrm{d}x$$

is bounded. See [2, Theorem III.2] and [10]. Moreover, it is proved in [8] (see also [9]) that the potential is also bounded provided that

$$G_\varepsilon(u_\varepsilon)\le k\ln\frac{1}{\varepsilon} \tag{8}$$

for some constant $k>0$.

In what follows, we suppose that (8) is valid or Ω is star-shaped. We have then

$$\frac{1}{\varepsilon^2}\int_\Omega \left(1-|u_\varepsilon|^2\right)^2 \mathrm{d}x \le \gamma_0. \tag{9}$$

Here, γ_0 depends only on Ω and g. The first result of this paper is the following theorem.

Theorem 3. *Suppose that*

$$\deg(g,\partial\Omega)=0. \tag{10}$$

Let u_ε be a solution of (2)*.*

(i) *If there exists a constant C_1 such that, for ε small enough, we have*

$$\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 \mathrm{d}x \le C_1 \le \frac{1}{2}\int_\Omega |\nabla u_0|^2 \mathrm{d}x, \tag{11}$$

then

$$C_1 = \frac{1}{2}\int_\Omega |\nabla u_0|^2 \mathrm{d}x \tag{12}$$

and

$$u_\varepsilon \longrightarrow u_0 \quad \text{in } H^1(\Omega).$$

Thus, Theorem 1 holds true by Theorem 2.

(ii) *If there exists a constant C_2 such that, for ε small enough, we have*

$$\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 \mathrm{d}x \ge C_2 > \frac{1}{2}\int_\Omega |\nabla u_0|^2 \mathrm{d}x, \tag{13}$$

then

$$|u_\varepsilon| \text{ does not converge uniformly to } 1 \text{ on } \overline{\Omega}. \tag{14}$$

By using Theorem 3, we prove the next theorem where we find a condition that is equivalent to (5).

Theorem 4. *Let us assume* (10) *and let u_ε be a solution for* (2)*. Then,*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega \left(1-|u_\varepsilon|^2\right)^2 \mathrm{d}x = 0 \tag{15}$$

if and only if

$$u_\varepsilon \longrightarrow u_0 \quad \text{in } H^1(\Omega). \tag{16}$$

As a two-component generalization of (1), let us consider

$$F_\varepsilon(u,v) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)\mathrm{d}x + \frac{1}{4\varepsilon^2}\int_\Omega V\left(|u|^2,|v|^2\right)\mathrm{d}x \tag{17}$$

for $(u_\varepsilon, v) \in H^1_{g_1}(\Omega)\times H^1_{g_2}(\Omega)$. Here, $g_1, g_2 : \partial\Omega \to S^1$ are smooth maps such that

$$d_i = \deg(g_i,\partial\Omega) \tag{18}$$

is a nonnegative integer for each $i = 1,2$. We assume that Ω is star-shaped. The potential function V is given two cases:

$$\text{symmetric case:}\quad V_s\left(|u|^2,|v|^2\right) = \left(2-|u|^2-|v|^2\right)^2,$$

$$\text{non-symmetric case:}\quad V_n\left(|u|^2,|v|^2\right) = \left(2-|u|^2-|v|^2\right)^2 + \left(1-|u|^2\right)^2.$$

In each case, F_ε has a minimizer $(u_\varepsilon, v_\varepsilon)$ over $H^1_{g_1}(\Omega)\times H^1_{g_2}(\Omega)$. The potential appears in the semi-local gauge field theories [7,11].

The Euler–Lagrange equations are given as follows: for $V = V_s$

$$\begin{cases} -\Delta u = \dfrac{1}{\varepsilon^2}u\left(2-|u|^2-|v|^2\right) & \text{in } \Omega,\\ -\Delta v = \dfrac{1}{\varepsilon^2}v\left(2-|u|^2-|v|^2\right) & \text{in } \Omega,\\ u = g_1, \quad v = g_2 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

and for $V = V_n$

$$\begin{cases} -\Delta u = \dfrac{1}{\varepsilon^2} u\left(2-|u|^2-|v|^2\right) + \dfrac{1}{\varepsilon^2} u\left(1-|u|^2\right) & \text{in } \Omega, \\ -\Delta v = \dfrac{1}{\varepsilon^2} v\left(2-|u|^2-|v|^2\right) & \text{in } \Omega, \\ u = g_1, \quad v = g_2 & \text{on } \partial\Omega. \end{cases} \tag{20}$$

Now, we want to extend Theorem 3 for solutions of (19) and (20). Since (6) and (9) play important roles in the proof Theorem 3, a natural question arises: can we have inequalities for solutions of (19) and (20) analogous to (6) and (9)? The answer is not easy. In fact, although the systems (19) and (20) appear to be simple extensions of (2), the nature of their solutions is quite different, as we shall see.

First, one may expect that if $(u_\varepsilon, v_\varepsilon)$ is a solution of (20), then

$$|u_\varepsilon| \le 1 \quad \text{and} \quad |v_\varepsilon| \le 1 \quad \text{on } \overline{\Omega}. \tag{21}$$

We recall that (6) was obtained using the maximum principle applied to the equation (7). However, since (19) and (20) are systems of equations, it is not possible to derive such an estimate by simply applying the maximum principle. Instead, weaker versions of (21) were established in [4,5].

Lemma 5 ([4, Lemma 2.2], [5, Lemma 2.1]).

(i) *If* $(u_\varepsilon, v_\varepsilon)$ *is a solution pair of* (19)*, then we have*

$$|u_\varepsilon|^2 + |v_\varepsilon|^2 \le 2 \quad \text{on } \overline{\Omega}. \tag{22}$$

(ii) *If* $(u_\varepsilon, v_\varepsilon)$ *is a solution pair of* (20)*, then we have*

$$|u_\varepsilon|^2 \le \frac{3}{2} \quad \text{and} \quad |v_\varepsilon|^2 \le 2 \quad \text{on } \overline{\Omega}. \tag{23}$$

Moreover, either $|u_\varepsilon| \le 1$ *or* $|v_\varepsilon| \le 1$ *on* $\overline{\Omega}$.

The first statement (i) gives no information on the individual upper bounds of $|u_\varepsilon|$ and $|v_\varepsilon|$ although theirs sums are bounded by 2. The second statement provides no information on the bounds of $|u_\varepsilon|^2 + |v_\varepsilon|^2$ and the upper bounds of $|u_\varepsilon|$ and $|v_\varepsilon|$ are rather rough compared to (21). Since the pointwise estimate $|u_\varepsilon| \le 1$ for solutions of (2) are crucial in various analysis of solutions, it is very interesting to prove (21) or to make analysis of solutions of (19) and (20) without appealing the property of (21).

Second difference among solutions of (2), (19) and (20) is the Pohozaev identity. Analogous to (9), we can prove that if Ω is star-shaped, then

$$(u_\varepsilon, v_\varepsilon)\text{: solution of (19)} \quad \Longrightarrow \quad \frac{1}{\varepsilon^2}\int_\Omega \left(2-|u_\varepsilon|^2-|v_\varepsilon|^2\right)^2 \mathrm{d}x \le \gamma_1, \tag{24}$$

$$(u_\varepsilon, v_\varepsilon)\text{: solution of (20)} \quad \Longrightarrow \quad \frac{1}{\varepsilon^2}\int_\Omega \left(2-|u_\varepsilon|^2-|v_\varepsilon|^2\right)^2 \mathrm{d}x + \frac{1}{\varepsilon^2}\int_\Omega \left(1-|u_\varepsilon|^2\right)^2 \mathrm{d}x \le \gamma_2, \tag{25}$$

for some constants γ_1 and γ_2. Since we do not know the signs of $1-|u_\varepsilon|^2$ and $1-|v_\varepsilon|^2$, (24) does not imply

$$\frac{1}{\varepsilon^2}\int_\Omega \left(1-|u_\varepsilon|^2\right)^2 \mathrm{d}x < \infty \qquad \text{and} \qquad \frac{1}{\varepsilon^2}\int_\Omega \left(1-|v_\varepsilon|^2\right)^2 \mathrm{d}x < \infty. \tag{26}$$

Indeed, these quantities can diverge for some solutions of (19) although they satisfy (24). See Theorem 6 below. On the other hand, solutions of (20) always satisfy not only (25) but also (26). See the proof of Theorem 8(ii) below.

To state the main results on the solutions of (19) and (20), we assume that $d_1 = d_2 = 0$ in (18) and Ω is star-shaped. We set

$$\mathscr{Y}(g_1, g_2) := H^1_{g_1}(\Omega; S^1) \times H^1_{g_2}(\Omega; S^2),$$
$$\mathscr{X}(g_1, g_2) := \Big\{(u, v) \in H^1_{g_1}(\Omega; \mathbb{C}) \times H^1_{g_2}(\Omega; \mathbb{C}) : |u|^2 + |v|^2 = 2 \text{ a.e. on } \Omega\Big\},$$

and

$$I_{(g_1,g_2)}(u, v) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right) dx = J_{g_1}(u) + J_{g_2}(v).$$

Let us consider the following minimization problems:

$$\alpha(g_1, g_2) := \inf\big\{I_{(g_1,g_2)}(u, v) : (u, v) \in \mathscr{Y}(g_1, g_2)\big\}, \tag{27}$$
$$\beta(g_1, g_2) := \inf\big\{I_{(g_1,g_2)}(u, v) : (u, v) \in \mathscr{X}(g_1, g_2)\big\}. \tag{28}$$

The problem (27) has a unique solution (u_0, v_0) on $\mathscr{Y}(g_1, g_2)$ that satisfies

$$\begin{cases} -\Delta u_0 = u_0 |\nabla u_0|^2 & \text{on } \Omega, \\ u_0 = g_1 & \text{on } \partial\Omega, \\ |u_0| = 1 & \text{on } \Omega, \end{cases} \qquad \begin{cases} -\Delta v_0 = v_0 |\nabla v_0|^2 & \text{on } \Omega, \\ v_0 = g_2 & \text{on } \partial\Omega, \\ |v_0| = 1 & \text{on } \Omega. \end{cases}$$

If (u_*, v_*) is a solution of (28), then (u_*, v_*) satisfies

$$\begin{cases} -\Delta u_* = \frac{1}{2} u_* (|\nabla u_*|^2 + |\nabla v_*|^2) & \text{on } \Omega, \qquad u_* = g_1 \text{ on } \partial\Omega, \\ -\Delta v_* = \frac{1}{2} v_* (|\nabla u_*|^2 + |\nabla v_*|^2) & \text{on } \Omega, \qquad v_* = g_2 \text{ on } \partial\Omega, \\ 2 = |u_*|^2 + |v_*|^2 \text{ a.e.} & \text{on } \Omega. \end{cases}$$

Since $\mathscr{Y}(g_1, g_2) \subset \mathscr{X}(g_1, g_2)$, it is obvious that

$$\alpha(g_1, g_2) \geq \beta(g_1, g_2). \tag{29}$$

The next theorem tells us that (29) has a close relation with some properties of solutions of (19).

Theorem 6 ([4, Theorem 1.3(iii)]). *Suppose that*

$$\deg(g_1, \partial\Omega) = \deg(g_2, \partial\Omega) = 0. \tag{30}$$

Let $(u_\varepsilon, v_\varepsilon)$ be a minimizer of (17) *with $V = V_s$. If $\alpha(g_1, g_2) > \beta(g_1, g_2)$, then*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_\Omega \left(1 - |u_\varepsilon|^2\right)^2 dx = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_\Omega \left(1 - |v_\varepsilon|^2\right)^2 dx = \infty.$$

Now, we extend Theorem 3 for solutions of (20) as follows.

Theorem 7. *Let Ω be star-shaped. Suppose that $d_1 = d_2 = 0$ such that* (30) *holds. Let $(u_\varepsilon, v_\varepsilon)$ be a solution for* (20) *and (u_0, v_0) be a unique minimizer of $I_{(g_1,g_2)}$ on $\mathscr{Y}(g_1, g_2)$.*

(i) *If there is a constant C_3 such that we have for ε small enough*

$$\frac{1}{2}\int_\Omega \left(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2\right) dx \leq C_3 \leq \frac{1}{2}\int_\Omega \left(|\nabla u_0|^2 + |\nabla v_0|^2\right) dx, \tag{31}$$

then

$$C_3 = \int_\Omega \left(|\nabla u_0|^2 + |\nabla v_0|^2\right) dx \tag{32}$$

and

$$(u_\varepsilon, v_\varepsilon) \longrightarrow (u_0, v_0) \quad \textit{in } H^1(\Omega) \times H^1(\Omega).$$

(ii) *If there is a constant C_4 such that for ε small enough we have*

$$\frac{1}{2}\int_\Omega \left(|\nabla u_\varepsilon|^2+|\nabla v_\varepsilon|^2\right)\mathrm{d}x \geq C_4 > \frac{1}{2}\int_\Omega \left(|\nabla u_0|^2+|\nabla v_0|^2\right)\mathrm{d}x, \tag{33}$$

then

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega \left[\left(2-|u_\varepsilon|^2-|v_\varepsilon|^2\right)^2+\left(1-|u_\varepsilon|^2\right)^2\right]\mathrm{d}x > 0. \tag{34}$$

Next, we deal with solutions of (19). In view of Theorem 6, we obtain the following theorem.

Theorem 8. *Let Ω be star-shaped. Suppose that $d_1 = d_2 = 0$ such that* (30) *holds. Let $(u_\varepsilon, v_\varepsilon)$ be a solution for* (19) *and (u_*, v_*) be a minimizer of $I_{(g_1,g_2)}$ on $\mathscr{X}(g_1, g_2)$.*

(i) *If there is a constant C_5 such that we have for ε small enough*

$$\frac{1}{2}\int_\Omega \left(|\nabla u_\varepsilon|^2+|\nabla v_\varepsilon|^2\right)\mathrm{d}x \leq C_5 \leq \frac{1}{2}\int_\Omega \left(|\nabla u_*|^2+|\nabla v_*|^2\right)\mathrm{d}x, \tag{35}$$

then

$$C_5 = \int_\Omega \left(|\nabla u_*|^2+|\nabla v_*|^2\right)\mathrm{d}x \tag{36}$$

and there exists $(\widetilde{u}, \widetilde{v}) \in \mathscr{X}(g_1, g_2)$ such that

$$(u_\varepsilon, v_\varepsilon) \longrightarrow (\widetilde{u}, \widetilde{v}) \quad \text{in } H^1(\Omega)\times H^1(\Omega).$$

If $\alpha(g_1, g_2) = \beta(g_1, g_2)$, then $(\widetilde{u}, \widetilde{v}) = (u_0, v_0)$.

(ii) *Assume that*

$$\alpha(g_1, g_2) = \beta(g_1, g_2), \tag{37}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega \left(1-|u_\varepsilon|^2\right)^2\mathrm{d}x \leq \gamma_3, \tag{38}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega \left(1-|v_\varepsilon|^2\right)^2\mathrm{d}x \leq \gamma_4. \tag{39}$$

If there is a constant C_6 such that for ε small enough we have

$$\frac{1}{2}\int_\Omega \left(|\nabla u_\varepsilon|^2+|\nabla v_\varepsilon|^2\right)\mathrm{d}x \geq C_6 > \frac{1}{2}\int_\Omega \left(|\nabla u_0|^2+|\nabla v_0|^2\right)\mathrm{d}x + \sqrt{\gamma_1\gamma_3}+\sqrt{\gamma_1\gamma_4}, \tag{40}$$

then

either $|u_\varepsilon|$ or $|v_\varepsilon|$ does not converges uniformly to 1 on $\overline{\Omega}$.

We will prove Theorems 3 and 4 in Section 2. The proofs of Theorems 7 and 8 are given in Sections 3 and 4, respectively.

2. Proofs of Theorems 3 and 4

Throughout this section, we assume (10) and prove Theorems 3 and 4. Then, we can write

$$g = e^{i\varphi_0} \quad \text{where } \varphi_0 : \partial\Omega \to \mathbb{R}.$$

Moreover, the function u_0 is lifted by a harmonic function φ such that

$$\begin{cases} \Delta\varphi = 0 \quad \text{in } \Omega & \text{and} \quad \varphi = \varphi_0 \quad \text{on } \partial\Omega, \\ u_0 = e^{i\varphi} & \text{and} \quad \displaystyle\int_\Omega |\nabla u_0|^2\,\mathrm{d}x = \int_\Omega |\nabla\varphi|^2\,\mathrm{d}x. \end{cases}$$

Proof of Theorem 3(i). Suppose that (11). Since $\|u_\varepsilon\|_\infty \leq 1$, up to a subsequence, we have $u_\varepsilon \rightharpoonup \widetilde{u}$ in $H^1_g(\Omega)$ for some $\widetilde{u} \in H^1_g(\Omega)$. By (9), $|\widetilde{u}| = 1$ a.e. on Ω and consequently $\widetilde{u} \in H^1_g(\Omega; S^1)$. Since u_0 is a minimizer of J_g, we are led to

$$\frac{1}{2}\int_\Omega |\nabla u_0|^2\,\mathrm{d}x \leq \frac{1}{2}\int_\Omega |\nabla\widetilde{u}|^2\,\mathrm{d}x \leq \liminf_{\varepsilon\to 0}\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x \leq C_1 \leq \frac{1}{2}\int_\Omega |\nabla u_0|^2\,\mathrm{d}x. \tag{41}$$

Thus, (12) is true. Since $u_\varepsilon \to u_0$ weakly in $H^1(\Omega)$, we deduce that

$$\int_\Omega |\nabla u_\varepsilon - \nabla u_0|^2\,\mathrm{d}x = \int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x + \int_\Omega |\nabla u_0|^2\,\mathrm{d}x - 2\int_\Omega \nabla u_\varepsilon \cdot \nabla u_0\,\mathrm{d}x \longrightarrow 0. \tag{42}$$

Hence, $u_\varepsilon \to u_0$ in $H^1_g(\Omega)$. Thus, Theorem 1 holds true by Theorem 2. □

In the above proof, we prove the following corollary.

Corollary 9. *If u_ε is any solution for* (2), *then*

$$\liminf_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x \geq \int_\Omega |\nabla u_0|^2\,\mathrm{d}x.$$

Proof. If we assume the contrary, up to a subsequence, we may assume that

$$\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x \leq C_1 < \frac{1}{2}\int_\Omega |\nabla u_0|^2\,\mathrm{d}x$$

for some C_1. Then, we get a contradiction by arguing as in (41). □

To prove Theorem 3(ii), we need two lemmas.

Lemma 10. *Let u_ε be a solution for* (2). *If $|u_\varepsilon| \to 1$ uniformly on $\overline{\Omega}$, then*

$$\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x \leq 2\int_\Omega |\nabla u_0|^2\,\mathrm{d}x. \tag{43}$$

Proof. Since $|u_\varepsilon| \to 1$ uniformly on $\overline{\Omega}$, we may assume that

$$|u_\varepsilon| \geq \frac{1}{2} \text{ on } \Omega \text{ for } \varepsilon > 0 \text{ small enough.} \tag{44}$$

Then, $u_\varepsilon/|u_\varepsilon|$ can be lifted by a smooth function ζ_ε such that

$$\frac{u_\varepsilon}{|u_\varepsilon|} = e^{i\zeta_\varepsilon} \quad \text{on } \Omega.$$

Hence, we can write

$$u_\varepsilon = \rho_\varepsilon e^{i\zeta_\varepsilon} \quad \text{with } \rho_\varepsilon = |u_\varepsilon|.$$

Then, $\zeta_\varepsilon = \varphi_0$ on $\partial\Omega$ and

$$|\nabla u_\varepsilon|^2 = |\nabla \rho_\varepsilon|^2 + \rho_\varepsilon^2 |\nabla \zeta_\varepsilon|^2 \tag{45}$$

and the equation (2) is transformed into a system of ρ_ε and ζ_ε:

$$\operatorname{div}(\rho_\varepsilon^2 \nabla \zeta_\varepsilon) = 0 \qquad \text{in } \Omega, \tag{46}$$

$$-\Delta\rho_\varepsilon + \rho_\varepsilon |\nabla\zeta_\varepsilon|^2 = \frac{1}{\varepsilon^2}\rho_\varepsilon(1-\rho_\varepsilon^2) \qquad \text{in } \Omega. \tag{47}$$

Multiplying (46) by $\rho_\varepsilon - 1$, we obtain

$$\int_\Omega |\nabla\rho_\varepsilon|^2\,\mathrm{d}x + \int_\Omega \rho_\varepsilon^2 |\nabla\zeta_\varepsilon|^2\,\mathrm{d}x - \int_\Omega \rho_\varepsilon |\nabla\zeta_\varepsilon|^2\,\mathrm{d}x = \frac{1}{\varepsilon^2}\int_\Omega \rho_\varepsilon(\rho_\varepsilon - 1)(1-\rho_\varepsilon^2)\,\mathrm{d}x \leq 0. \tag{48}$$

Hence, it comes from (44), (45) and (48) that

$$\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x \leq \int_\Omega \rho_\varepsilon |\nabla\zeta_\varepsilon|^2\,\mathrm{d}x \leq 2\int_\Omega \rho_\varepsilon^2 |\nabla\zeta_\varepsilon|^2\,\mathrm{d}x.$$

On the other hand, multiplying (46) by $\zeta_\varepsilon - \varphi$, we have

$$\int_\Omega \rho_\varepsilon^2 |\nabla\zeta_\varepsilon|^2\,\mathrm{d}x = \int_\Omega \rho_\varepsilon^2 \nabla\zeta_\varepsilon \cdot \nabla\varphi\,\mathrm{d}x \leq \left(\int_\Omega \rho_\varepsilon^2 |\nabla\zeta_\varepsilon|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega \rho_\varepsilon^2 |\nabla\varphi|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

In this integration, we used the fact $u_0 = u_\varepsilon = g$ on $\partial\Omega$, i.e., $\varphi = \zeta_\varepsilon = \varphi_0$ on $\partial\Omega$. Hence, we conclude that

$$\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x \leq 2\int_\Omega \rho_\varepsilon^2 |\nabla\varphi|^2\,\mathrm{d}x \leq 2\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x.$$ □

In fact, we can obtain a better, sharp estimate.

Lemma 11. *Let u_ε be a solution for* (2)*. If $|u_\varepsilon| \to 1$ uniformly on $\overline{\Omega}$, then*

$$\limsup_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x \le \int_\Omega |\nabla u_0|^2\,\mathrm{d}x. \tag{49}$$

Proof. Let us assume the contrary. Then, there exists a constant $C_2 > 0$ and a subsequence, still denoted by u_ε, such that

$$\frac{1}{2}\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x = \frac{1}{2}\int_\Omega |\nabla u_0|^2\,\mathrm{d}x < C_2 \le \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x. \tag{50}$$

Since $|u_\varepsilon| \to 1$ uniformly on $\overline{\Omega}$, we may keep the notations in the proof of Lemma 10. Given $\delta \in \left(0, \frac{1}{4}\right)$, if ε is small enough, then

$$\frac{1}{2} < \rho_\varepsilon < \rho_\varepsilon^2 + \delta \quad \text{i.e.,} \quad \frac{1+\sqrt{1-4\delta}}{2} < \rho_\varepsilon < 1. \tag{51}$$

Let

$$\psi_\varepsilon = \zeta_\varepsilon - \varphi.$$

Then, by (45) and (50),

$$C_2 \le \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x = \frac{1}{2}\int_\Omega |\nabla \rho_\varepsilon|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \rho_\varepsilon^2\big|\nabla(\varphi+\psi_\varepsilon)\big|^2\,\mathrm{d}x. \tag{52}$$

We rewrite (46) and (47) as

$$\mathrm{div}\big(\rho_\varepsilon^2\nabla(\varphi+\psi_\varepsilon)\big) = 0 \qquad \text{in } \Omega, \tag{53}$$

$$-\Delta\rho_\varepsilon + \rho_\varepsilon\big|\nabla(\varphi+\psi_\varepsilon)\big|^2 = \frac{1}{\varepsilon^2}\rho_\varepsilon(1-\rho_\varepsilon^2) \qquad \text{in } \Omega. \tag{54}$$

Multiplying (54) by $\rho_\varepsilon - 1$ and integrating it over Ω, and using the boundary condition $\rho_\varepsilon = 1$ on $\partial\Omega$, we obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla\rho_\varepsilon|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \rho_\varepsilon^2\big|\nabla(\varphi+\psi_\varepsilon)\big|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega \rho_\varepsilon\big|\nabla(\varphi+\psi_\varepsilon)\big|^2\,\mathrm{d}x &\\ = \frac{1}{2\varepsilon^2}\int_\Omega \rho_\varepsilon(\rho_\varepsilon-1)(1-\rho_\varepsilon^2)\,\mathrm{d}x &\\ \le 0.&\end{aligned} \tag{55}$$

Then, from (51), (52) and (55), it follows that

$$\begin{aligned}C_2 &\le \frac{1}{2}\int_\Omega \rho_\varepsilon\big|\nabla(\varphi+\psi_\varepsilon)\big|^2\,\mathrm{d}x\\ &\le \frac{1}{2}\int_\Omega (\rho_\varepsilon^2+\delta)\big|\nabla(\varphi+\psi_\varepsilon)\big|^2\,\mathrm{d}x\\ &\le \frac{1}{2}\int_\Omega \rho_\varepsilon^2\big(|\nabla\varphi|^2 + 2\nabla\varphi\cdot\psi_\varepsilon + |\nabla\psi_\varepsilon|^2\big)\,\mathrm{d}x + \frac{1}{2}\delta\|\nabla\zeta_\varepsilon\|_2^2\\ &\le \frac{1}{2}\int_\Omega |\nabla\varphi|^2 + \frac{1}{2}\int_\Omega \rho_\varepsilon^2\big(2\nabla\varphi\cdot\psi_\varepsilon + |\nabla\psi_\varepsilon|^2\big)\,\mathrm{d}x + \frac{1}{2}\delta\|\nabla\zeta_\varepsilon\|_2^2.\end{aligned} \tag{56}$$

Multiplying (53) by ψ_ε, integrating it over Ω, and using the boundary condition $\psi_\varepsilon = 0$ on $\partial\Omega$, we obtain

$$\int_\Omega \rho_\varepsilon^2|\nabla\psi_\varepsilon|^2\,\mathrm{d}x + \int_\Omega \rho_\varepsilon^2\nabla\varphi\cdot\nabla\psi_\varepsilon\,\mathrm{d}x = 0. \tag{57}$$

Furthermore, by (43) and (51),

$$\|\nabla\zeta_\varepsilon\|_2^2 \le 4\int_\Omega \rho_\varepsilon^2|\nabla\zeta_\varepsilon|^2\,\mathrm{d}x \le 8\int_\Omega |\nabla u_0|^2\,\mathrm{d}x. \tag{58}$$

Hence, by (56), (57) and (58), we are led to

$$0 < C_2 - \frac{1}{2}\int_\Omega |\nabla\varphi|^2 \le -\int_\Omega \rho_\varepsilon^2|\nabla\psi_\varepsilon|^2\,\mathrm{d}x + 4\delta\|\nabla u_0\|_2^2 \le 4\delta\|\nabla u_0\|_2^2.$$

Letting $\delta \to 0$, we arrive at a contradiction. □

Lemma 12. *Let u_ε be a solution for* (2) *that satisfies* (15)*. Then,* $|u_\varepsilon| \to 1$ *uniformly on* $\overline{\Omega}$*.*

Proof. See [1, Step A.1, B.2]. □

Lemma 13. *Let u_ε be a solution for* (2)*. If $u \to u_0$ in $H^1(\Omega)$, then $|u_\varepsilon| \to 1$ uniformly on* $\overline{\Omega}$*.*

Proof. By multiplying (7) by $1-|u_\varepsilon|^2$, we obtain

$$2\int_\Omega |\nabla u_\varepsilon|^2\big(1-|u_\varepsilon|^2\big)\,\mathrm{d}x = \frac{2}{\varepsilon^2}\int_\Omega |u_\varepsilon|^2\big(1-|u_\varepsilon|^2\big)^2\,\mathrm{d}x + \int_\Omega \big|\nabla\big(1-|u_\varepsilon|^2\big)\big|^2\,\mathrm{d}x. \tag{59}$$

Given $\delta \in \big(0,\frac{1}{4}\big)$, let

$$\Omega_\varepsilon^\delta = \big\{x\in\Omega : 1-|u_\varepsilon|^2 > \delta\big\}.$$

By (9),

$$\gamma_0 \ge \frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon^\delta}\big(1-|u_\varepsilon|^2\big)^2\,\mathrm{d}x \ge \frac{(1-\delta)^2}{\varepsilon^2}|\Omega_\varepsilon^\delta|.$$

Hence, for all $\delta \in \big(0,\frac{1}{4}\big)$, $|\Omega_\varepsilon^\delta| \to 0$ as $\varepsilon\to 0$. Since $u\to u_0$ in $H^1(\Omega)$, it follows that for each fixed $\delta \in \big(0,\frac{1}{4}\big)$,

$$\int_{\Omega_\varepsilon^\delta}|\nabla u_\varepsilon|^2\,\mathrm{d}x \le 2\int_{\Omega_\varepsilon^\delta}|\nabla u_\varepsilon-\nabla u_0|^2\,\mathrm{d}x + 2\int_{\Omega_\varepsilon^\delta}|\nabla u_0|^2\,\mathrm{d}x \longrightarrow 0$$

as $\varepsilon\to 0$. Since $u\to u_0$ in $H^1(\Omega)$, we have $\|\nabla u_\varepsilon\|_2^2 \le C$ for some C. Now, we see that as $\varepsilon \to 0$,

$$\int_\Omega |\nabla u_\varepsilon|^2\big(1-|u_\varepsilon|^2\big)\,\mathrm{d}x \le \delta\int_{\Omega\setminus\Omega_\varepsilon^\delta}|\nabla u_\varepsilon|^2\,\mathrm{d}x + \int_{\Omega_\varepsilon^\delta}|\nabla u_\varepsilon|^2\,\mathrm{d}x \le C\delta + o(1).$$

So, we deduce from (59) that for all $\delta\in\big(0,\frac{1}{4}\big)$,

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega |u_\varepsilon|^2\big(1-|u_\varepsilon|^2\big)^2\,\mathrm{d}x + \limsup_{\varepsilon\to 0}\int_\Omega\big|\nabla\big(1-|u_\varepsilon|^2\big)\big|^2\,\mathrm{d}x \le C\delta.$$

Letting $\delta\to 0$, we obtain that

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega |u_\varepsilon|^2\big(1-|u_\varepsilon|^2\big)^2\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega \big(1-|u_\varepsilon|^2\big)^2\,\mathrm{d}x - \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega\big(1-|u_\varepsilon|^2\big)^3\,\mathrm{d}x. \end{aligned} \tag{60}$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega\big|\nabla\big(1-|u_\varepsilon|^2\big)\big|^2\,\mathrm{d}x = 0. \tag{61}$$

By using (9), (61) and the Gagliardo–Nirenberg inequality

$$\|u\|_3^3 \le C\|u\|_2^2\|\nabla u\|_2 \quad \text{for } u\in H_0^1(\Omega),$$

we are led to

$$\begin{aligned} \frac{1}{\varepsilon^2}\int_\Omega\big(1-|u_\varepsilon|^2\big)^3\,\mathrm{d}x &\le \frac{C}{\varepsilon^2}\left(\int_\Omega\big(1-|u_\varepsilon|^2\big)^2\,\mathrm{d}x\right)\left(\int_\Omega\big|\nabla\big(1-|u_\varepsilon|^2\big)\big|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &\le C\gamma_0\left(\int_\Omega\big|\nabla\big(1-|u_\varepsilon|^2\big)\big|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \longrightarrow 0. \end{aligned}$$

In the sequel, we conclude from (60) that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega\big(1-|u_\varepsilon|^2\big)^2\,\mathrm{d}x = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega\big(1-|u_\varepsilon|^2\big)^3\,\mathrm{d}x = 0, \tag{62}$$

which implies by Lemma 12 that $|u_\varepsilon|\to 1$ uniformly on $\overline{\Omega}$. This finishes the proof. □

Proof of Theorem 3(ii). Let us assume the contrary. Then, $|u_\varepsilon|\to 1$ uniformly on $\overline{\Omega}$. Hence, (49) holds by Lemma 11 which contradicts (13). □

Proof of Theorem 4. Suppose that (15) holds. Then, $|u_\varepsilon| \to 1$ uniformly on Ω by Lemma 12. Moreover, by Corollary 9 and Lemma 11, we have

$$\lim_{\varepsilon\to\infty}\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x = \int_\Omega |\nabla u_0|^2\,\mathrm{d}x.$$

Since $u_\varepsilon \to u_0$ weakly in $H^1(\Omega)$, we deduce from (42) that $u_\varepsilon \to u_0$ in $H^1_g(\Omega)$.

Conversely, suppose that (16) is true. Since $|u_\varepsilon| \to 1$ uniformly on $\overline{\Omega}$ by Lemma 13, we may assume that $|u_\varepsilon|^2 \ge 1/2$ and use notations in Lemmas 10 and 11. Multiplying (54) by $\rho_\varepsilon - 1$, we obtain

$$\begin{aligned}\int_\Omega |\nabla\rho_\varepsilon|^2\,\mathrm{d}x + \frac{1}{\varepsilon^2}\int_\Omega \rho_\varepsilon(1-\rho_\varepsilon)(1-\rho_\varepsilon^2)\,\mathrm{d}x &= \int_\Omega (\rho_\varepsilon-\rho_\varepsilon^2)\left|\nabla(\varphi+\psi_\varepsilon)\right|^2\mathrm{d}x\\ &\le \|1-\rho_\varepsilon\|_\infty \int_\Omega \left|\nabla(\varphi+\psi_\varepsilon)\right|^2\mathrm{d}x \longrightarrow 0.\end{aligned}$$

Here, we used the fact that $u_\varepsilon \to u_0$ in $H^1(\Omega)$ such that $\left\|\nabla(\varphi+\psi_\varepsilon)\right\|_2$ is bounded as $\varepsilon \to 0$. As a consequence,

$$\begin{aligned}0 &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega \rho_\varepsilon(1-\rho_\varepsilon)(1-\rho_\varepsilon^2)\,\mathrm{d}x\\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega \frac{\rho_\varepsilon}{1+\rho_\varepsilon}(1-\rho_\varepsilon^2)^2\,\mathrm{d}x\\ &= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon^2}\int_\Omega (1-\rho_\varepsilon^2)^2\,\mathrm{d}x\\ &= 0\end{aligned}$$

and the proof is complete. □

3. Proof of Theorem 7

Throughout this section, we assume (30) and prove Theorem 7. We also assume that Ω is star-shaped. We can write

$$g_1 = e^{i\varphi_0} \quad\text{and}\quad g_2 = e^{i\psi_0} \quad \text{where } \varphi_0, \psi_0 \colon \partial\Omega \to \mathbb{R}.$$

The functions u_0 and v_0 are lifted by harmonic functions φ and ψ respectively such that

$$\begin{cases}\Delta\varphi = 0 \quad \text{in } \Omega & \text{and} \quad \varphi = \varphi_0 \quad \text{on } \partial\Omega,\\ u_0 = e^{i\varphi} & \text{and} \quad \displaystyle\int_\Omega |\nabla u_0|^2\,\mathrm{d}x = \int_\Omega |\nabla\varphi|^2\,\mathrm{d}x,\end{cases} \tag{63}$$

and

$$\begin{cases}\Delta\psi = 0 \quad \text{in } \Omega & \text{and} \quad \psi = \psi_0 \quad \text{on } \partial\Omega,\\ v_0 = e^{i\psi} & \text{and} \quad \displaystyle\int_\Omega |\nabla v_0|^2\,\mathrm{d}x = \int_\Omega |\nabla\psi|^2\,\mathrm{d}x.\end{cases} \tag{64}$$

Proof of Theorem 7(i). Suppose that (31) is valid. Since $\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty \le 3$ by Lemma 5(ii), up to a subsequence, we have $(u_\varepsilon, v_\varepsilon) \rightharpoonup (\widetilde{u}, \widetilde{v})$ in $H^1(\Omega)\times H^1(\Omega)$ for some $(\widetilde{u}, \widetilde{v}) \in H^1_{g_1}(\Omega)\times H^1_{g_2}(\Omega)$. By (25), $|\widetilde{u}| = 1$ and $|\widetilde{v}| = 1$ a.e. on Ω and consequently $\widetilde{u} \in H^1_{g_1}(\Omega; S^1)$ and $\widetilde{v} \in H^1_{g_2}(\Omega; S^1)$. Since (u_0, v_0) is a unique minimizer of $I_{(g_1,g_2)}$ on $\mathscr{Y}(g_1, g_2)$, we are led to

$$\begin{aligned}\frac12\int_\Omega \left(|\nabla u_0|^2 + |\nabla v_0|^2\right)\mathrm{d}x &\le \frac12\int_\Omega \left(|\nabla \widetilde{u}|^2 + |\nabla \widetilde{v}|^2\right)\mathrm{d}x\\ &\le \liminf_{\varepsilon\to 0}\frac12\int_\Omega \left(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2\right)\mathrm{d}x\\ &\le C_3\\ &\le \frac12\int_\Omega \left(|\nabla u_0|^2 + |\nabla v_0|^2\right)\mathrm{d}x.\end{aligned}$$

Thus, (32) is true. Moreover, $u_\varepsilon \to u_0$ in $H^1_{g_1}(\Omega)$ and $v_\varepsilon \to v_0$ in $H^1_{g_2}(\Omega)$ as in the proof of Theorem 3(i). □

Proof of Theorem 7(ii). Let us assume the contrary so that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Omega\Big[\big(2-|u_\varepsilon|^2-|v_\varepsilon|^2\big)^2+\big(1-|u_\varepsilon|^2\big)^2\Big]\,\mathrm{d}x=0. \tag{65}$$

If (65) is valid, then it follows from [5, Lemma 2.5] that $|u_\varepsilon|\to 1$ and $|v_\varepsilon|\to 1$ uniformly on $\overline{\Omega}$. So, we may assume that $|u_\varepsilon|^2\ge 1/2$ and $|v_\varepsilon|^2\ge 1/2$ on Ω. We can write

$$u_\varepsilon=\rho_\varepsilon e^{i\zeta_\varepsilon}\qquad\text{and}\qquad v_\varepsilon=\sigma_\varepsilon e^{i\xi_\varepsilon}, \tag{66}$$

where $\rho_\varepsilon=|u_\varepsilon|$ and $\sigma_\varepsilon=|v_\varepsilon|$. Set

$$\eta_\varepsilon=\zeta_\varepsilon-\varphi\qquad\text{and}\qquad\chi_\varepsilon=\xi_\varepsilon-\psi. \tag{67}$$

Then, (20) is written as

$$\operatorname{div}\big(\rho_\varepsilon^2\nabla(\varphi+\eta_\varepsilon)\big)=0, \tag{68}$$

$$-\Delta\rho_\varepsilon+\rho_\varepsilon|\nabla\varphi+\nabla\eta_\varepsilon|^2=\frac{1}{\varepsilon^2}\rho_\varepsilon(2-\rho_\varepsilon^2-\sigma_\varepsilon^2)+\frac{1}{\varepsilon^2}\rho_\varepsilon(1-\rho_\varepsilon^2), \tag{69}$$

$$\operatorname{div}\big(\sigma_\varepsilon^2\nabla(\psi+\chi_\varepsilon)\big)=0, \tag{70}$$

$$-\Delta\sigma_\varepsilon+\sigma_\varepsilon|\nabla\psi+\chi_\varepsilon|^2=\frac{1}{\varepsilon^2}\sigma_\varepsilon(2-\rho_\varepsilon^2-\sigma_\varepsilon^2). \tag{71}$$

By multiplying (69) by $\rho_\varepsilon-1$ and (71) by $\sigma_\varepsilon-1$, we obtain from (33)

$$\begin{aligned}C_4&\le\frac{1}{2}\int_\Omega\big(|\nabla u_\varepsilon|^2+|\nabla v_\varepsilon|^2\big)\,\mathrm{d}x\\&=\frac{1}{2}\int_\Omega\big(|\nabla\rho_\varepsilon|^2+|\nabla\sigma_\varepsilon|^2+\rho_\varepsilon^2|\nabla\varphi+\nabla\eta_\varepsilon|^2+\sigma_\varepsilon^2|\nabla\psi+\nabla\chi_\varepsilon|^2\big)\\&=\frac{1}{2}\int_\Omega\big(\rho_\varepsilon|\nabla\varphi+\nabla\eta_\varepsilon|^2+\sigma_\varepsilon|\nabla\psi+\nabla\chi_\varepsilon|^2\big)\,\mathrm{d}x+D_1+D_2+D_3,\end{aligned}$$

where

$$\begin{cases}D_1=\dfrac{1}{\varepsilon^2}\displaystyle\int\rho_\varepsilon(\rho_\varepsilon-1)(2-\rho_\varepsilon^2-\sigma_\varepsilon^2),\\ D_2=\dfrac{1}{\varepsilon^2}\displaystyle\int\sigma_\varepsilon(\sigma_\varepsilon-1)(2-\rho_\varepsilon^2-\sigma_\varepsilon^2),\\ D_3=\dfrac{1}{\varepsilon^2}\displaystyle\int\rho_\varepsilon(\rho_\varepsilon-1)(1-\rho_\varepsilon^2).\end{cases} \tag{72}$$

Then, $D_j\to 0$ for each $j=1,2,3$ as $\varepsilon\to 0$. Indeed, by Hölder's inequality and the condition (65), we can show that $D_1\to 0$ and $D_3\to 0$ as $\varepsilon\to 0$. Moreover, as $\varepsilon\to 0$, we have

$$\begin{aligned}o(1)&=\frac{1}{\varepsilon^2}\int_\Omega(2-\rho_\varepsilon^2-\sigma_\varepsilon^2)^2\,\mathrm{d}x\\&=\frac{1}{\varepsilon^2}\int_\Omega(1-\rho_\varepsilon^2)^2\,\mathrm{d}x+\frac{2}{\varepsilon^2}\int_\Omega(1-\rho_\varepsilon^2)(1-\sigma_\varepsilon^2)\,\mathrm{d}x+\frac{1}{\varepsilon^2}\int_\Omega(1-\sigma_\varepsilon^2)^2\,\mathrm{d}x\\&=o(1)+\frac{2}{\varepsilon^2}\int_\Omega(1-\rho_\varepsilon^2)(1-\sigma_\varepsilon^2)\,\mathrm{d}x+\frac{1}{\varepsilon^2}\int_\Omega(1-\sigma_\varepsilon^2)^2\,\mathrm{d}x.\end{aligned}$$

Hence, by Hölder's inequality, we obtain

$$\frac{1}{\varepsilon^2}\int_\Omega(1-\sigma_\varepsilon^2)^2\,\mathrm{d}x\le o(1)+2\left[\frac{1}{\varepsilon^2}\int_\Omega(1-\rho_\varepsilon^2)^2\,\mathrm{d}x\right]^{\frac{1}{2}}\left[\frac{1}{\varepsilon^2}\int_\Omega(1-\sigma_\varepsilon^2)^2\,\mathrm{d}x\right]^{\frac{1}{2}}.$$

Thus, $\|1-\sigma_\varepsilon^2\|_2\to 0$ and then Hölder's inequality implies that $D_2\to 0$.

We have shown that as $\varepsilon \to 0$,

$$\begin{aligned} C_4 &\le o(1) + \frac{1}{2}\int_\Omega \rho_\varepsilon |\nabla\varphi + \nabla\eta_\varepsilon|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega \sigma_\varepsilon |\nabla\psi + \nabla\chi_\varepsilon|^2 \,\mathrm{d}x \\ &=: o(1) + A_1 + A_2. \end{aligned} \tag{73}$$

Let $\delta \in \left(0, \frac{1}{4}\right)$ be given and we may assume (51). So, we have

$$\begin{aligned} A_1 &\le \frac{1}{2}\int_\Omega \rho_\varepsilon^2 |\nabla\varphi + \nabla\eta_\varepsilon|^2 \,\mathrm{d}x + \frac{\delta}{2}\int_\Omega |\nabla\varphi + \nabla\eta_\varepsilon|^2 \,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega \rho_\varepsilon^2 |\nabla\varphi|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega \rho_\varepsilon^2 \left(2\nabla\varphi\cdot\nabla\eta_\varepsilon + |\nabla\eta_\varepsilon|^2\right) \mathrm{d}x + \frac{\delta}{2}\int_\Omega |\nabla\varphi + \nabla\eta_\varepsilon|^2 \,\mathrm{d}x. \end{aligned}$$

By multiplying (68) by ψ_ε, we obtain

$$\int_\Omega \rho_\varepsilon^2 |\nabla\eta_\varepsilon|^2 \,\mathrm{d}x + \int_\Omega \rho_\varepsilon^2 \nabla\varphi\cdot\nabla\eta_\varepsilon \,\mathrm{d}x = 0. \tag{74}$$

So,

$$A_1 \le \frac{1}{2}\int_\Omega \rho_\varepsilon^2 |\nabla\varphi|^2 \,\mathrm{d}x - \frac{1}{2}\int_\Omega \rho_\varepsilon^2 |\nabla\eta_\varepsilon|^2 \,\mathrm{d}x + \frac{\delta}{2}\int_\Omega |\nabla\varphi + \nabla\eta_\varepsilon|^2 \,\mathrm{d}x. \tag{75}$$

On the other hand, (74) implies that

$$\begin{aligned} \int_\Omega \rho_\varepsilon^2 |\nabla\varphi + \nabla\eta_\varepsilon|^2 \,\mathrm{d}x &= \int_\Omega \rho_\varepsilon^2 (\nabla\varphi + \nabla\eta_\varepsilon)\cdot\nabla\varphi \,\mathrm{d}x \\ &\le \left(\int_\Omega \rho_\varepsilon^2 |\nabla\varphi + \nabla\eta_\varepsilon|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega \rho_\varepsilon^2 |\nabla\varphi|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}. \end{aligned}$$

Since $\rho_\varepsilon^2 \ge 1/2$, this inequality implies that

$$\frac{1}{2}\int_\Omega |\nabla\varphi + \nabla\eta_\varepsilon|^2 \,\mathrm{d}x \le \int_\Omega \rho_\varepsilon^2 |\nabla\varphi + \nabla\eta_\varepsilon|^2 \,\mathrm{d}x \le \int_\Omega \rho_\varepsilon^2 |\nabla\varphi|^2 \,\mathrm{d}x.$$

Hence, we can rewrite (75) as

$$A_1 \le \frac{1}{2}\int_\Omega \rho_\varepsilon^2 |\nabla\varphi|^2 \,\mathrm{d}x + \delta\int_\Omega \rho_\varepsilon^2 |\nabla\varphi|^2 \,\mathrm{d}x.$$

By a similar argument, we also obtain

$$A_2 \le \frac{1}{2}\int_\Omega \sigma_\varepsilon^2 |\nabla\psi|^2 \,\mathrm{d}x + \delta\int_\Omega \sigma_\varepsilon^2 |\nabla\psi|^2 \,\mathrm{d}x.$$

In the sequel, we deduce from (73) that

$$C_4 \le o(1) + \frac{1}{2}\int_\Omega \left(\rho_\varepsilon^2 |\nabla\varphi|^2 + \sigma_\varepsilon^2 |\nabla\psi|^2\right) \mathrm{d}x + \delta\int_\Omega \left(\rho_\varepsilon^2 |\nabla\varphi|^2 + \sigma_\varepsilon^2 |\nabla\psi|^2\right) \mathrm{d}x.$$

Letting $\varepsilon \to 0$, we are led to

$$C_4 \le \frac{1}{2}\int_\Omega \left(|\nabla\varphi|^2 + |\nabla\psi|^2\right) \mathrm{d}x + \frac{\delta}{2}\int_\Omega \left(|\nabla\varphi|^2 + |\nabla\psi|^2\right) \mathrm{d}x.$$

Finally, by taking the limit $\delta \to 0$, we get a contradiction from the assumption (33). □

4. Proof of Theorem 8

This section is devoted to the proof of Theorem 8. Throughout this section, we assume that (30) holds and Ω is star-shaped.

Proof of Theorem 8(i). Suppose that (31) is valid. Since $\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty \le 2$ by Lemma 5(i), up to a subsequence, we have $(u_\varepsilon, v_\varepsilon) \rightharpoonup (\widetilde{u}, \widetilde{v})$ in $H^1(\Omega)\times H^1(\Omega)$ for some $(\widetilde{u}, \widetilde{v}) \in H^1_{g_1}(\Omega)\times H^1_{g_2}(\Omega)$. By (24), $|\widetilde{u}|^2 + |\widetilde{v}|^2 = 2$ a.e. on Ω and thus $(\widetilde{u}, \widetilde{v}) \in \mathscr{X}(g_1, g_2)$. Since (u_*, v_*) is a minimizer of $I_{(g_1,g_2)}$ on $\mathscr{X}(g_1, g_2)$, we are led to

$$\begin{aligned}\frac{1}{2}\int_\Omega \left(|\nabla u_*|^2 + |\nabla v_*|^2\right)\mathrm{d}x &\le \frac{1}{2}\int_\Omega \left(|\nabla \widetilde{u}|^2 + |\nabla \widetilde{v}|^2\right)\mathrm{d}x\\ &\le \liminf_{\varepsilon\to 0}\frac{1}{2}\int_\Omega \left(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2\right)\mathrm{d}x\\ &\le C_5\\ &\le \frac{1}{2}\int_\Omega \left(|\nabla u_*|^2 + |\nabla v_*|^2\right)\mathrm{d}x.\end{aligned}$$

Thus, (36) is obtained. As in the proof of Theorem 3(i), it also holds that $u_\varepsilon \to \widetilde{u}$ in $H^1_{g_1}(\Omega)$ and $v_\varepsilon \to \widetilde{v}$ in $H^1_{g_2}(\Omega)$. Furthermore, if $\alpha(g_1, g_2) = \beta(g_1, g_2)$, then it is easy to see that $(u_*, v_*) = (\widetilde{u}, \widetilde{v}) = (u_0, v_0)$. This completes the proof. □

Remark 14. We do not know the uniqueness of solution to the problem (28). If this problem has a unique solution, then we obtain $(u_*, v_*) = (\widetilde{u}, \widetilde{v})$ in the proof of Theorem 8(i).

Proof of Theorem 8(ii). Let us assume the contrary so that $|u_\varepsilon| \to 1$ and $|v_\varepsilon| \to 1$ uniformly on $\overline{\Omega}$. Then, $|u_*| = 1$ and $|v_*| = 1$. Since $\alpha(g_1, g_2) = \beta(g_1, g_2)$ by (37), it follows that $(u_*, v_*) = (u_0, v_0)$. So, we can use the notations (63) and (64). Moreover, we may assume that $|u_\varepsilon|^2 \ge 1/2$ and $|v_\varepsilon|^2 \ge 1/2$ on Ω, and take the notations (66) and (67). We can rewrite (19) as

$$\operatorname{div}\left(\rho_\varepsilon^2 \nabla(\varphi + \eta_\varepsilon)\right) = 0, \tag{76}$$

$$-\Delta\rho_\varepsilon + \rho_\varepsilon|\nabla\varphi + \nabla\eta_\varepsilon|^2 = \frac{1}{\varepsilon^2}\rho_\varepsilon(2 - \rho_\varepsilon^2 - \sigma_\varepsilon^2), \tag{77}$$

$$\operatorname{div}\left(\sigma_\varepsilon^2 \nabla(\psi + \chi_\varepsilon)\right) = 0, \tag{78}$$

$$-\Delta\sigma_\varepsilon + \sigma_\varepsilon|\nabla\psi + \chi_\varepsilon|^2 = \frac{1}{\varepsilon^2}\sigma_\varepsilon(2 - \rho_\varepsilon^2 - \sigma_\varepsilon^2). \tag{79}$$

By proceeding as in the proof of Theorem 7(ii), we obtain

$$C_6 \le \frac{1}{2}\int_\Omega \left(\rho_\varepsilon|\nabla\varphi + \nabla\eta_\varepsilon|^2 + \sigma_\varepsilon|\nabla\psi + \nabla\chi_\varepsilon|^2\right)\mathrm{d}x + D_1 + D_2,$$

where D_1 and D_2 are defined by (72). By (24), (38), (39) and Hölder's inequality, we obtain

$$\begin{aligned} D_1 &= \frac{1}{\varepsilon^2}\int_\Omega \frac{\rho_\varepsilon}{\rho_\varepsilon + 1}(\rho_\varepsilon^2 - 1)(2 - \rho_\varepsilon^2 - \sigma_\varepsilon^2) \le \sqrt{\gamma_1\gamma_3},\\ D_2 &= \frac{1}{\varepsilon^2}\int_\Omega \frac{\sigma_\varepsilon}{\sigma_\varepsilon + 1}(\sigma_\varepsilon^2 - 1)(2 - \rho_\varepsilon^2 - \sigma_\varepsilon^2) \le \sqrt{\gamma_1\gamma_4}.\end{aligned}$$

So,

$$C_6 \le \frac{1}{2}\int_\Omega \rho_\varepsilon|\nabla\varphi + \nabla\eta_\varepsilon|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \sigma_\varepsilon|\nabla\psi + \nabla\chi_\varepsilon|^2\,\mathrm{d}x + \sqrt{\gamma_1\gamma_3} + \sqrt{\gamma_1\gamma_4}.$$

Furthermore, by arguing as in the proof of Theorem 7(ii), we are led to

$$C_6 \le \frac{1}{2}\int_\Omega \left(|\nabla\varphi|^2 + |\nabla\psi|^2\right)\mathrm{d}x + \sqrt{\gamma_1\gamma_3} + \sqrt{\gamma_1\gamma_4},$$

which contradicts the assumption (40). □

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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