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Coxeter-type quotients of surface braid groups

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Abstract. Let M be a closed surface, $q \geq 2$ and $n \geq 2$. In this paper, we analyze the Coxeter-type quotient group $B_n(M)(q)$ of the surface braid group $B_n(M)$ by the normal closure of the element σ_1^q, where σ_1 is the standard Artin generator of the braid group B_n. Also, we study the Coxeter-type quotient groups obtained by taking the quotient of $B_n(M)$ by the commutator subgroup of the respective pure braid group $[P_n(M), P_n(M)]$ and adding the relation $\sigma_1^q = 1$, when M is a closed orientable surface or the disk.

Keywords. Artin braid group, surface braid group, finite group.

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1. Introduction

The braid groups of the 2-disk, or Artin braid groups, were introduced by Artin in 1925 and further studied in 1947 [2,3]. Surface braid groups were initially studied by Zariski [24], and were later generalized by Fox and Neuwirth to braid groups of arbitrary topological spaces using configuration spaces as follows [10]. Let S be a compact, connected surface, and let $n \in \mathbb{N}$. The *nth ordered configuration space of S*, denoted by $F_n(S)$, is defined by:

$$F_n(S) = \left\{(x_1, \ldots, x_n) \in S^n \mid x_i \neq x_j \text{ if } i \neq j;\ i, j = 1, \ldots, n\right\}.$$

The *n-string pure braid group* $P_n(S)$ *of* S is defined by $P_n(S) = \pi_1(F_n(S))$. The symmetric group S_n on n letters acts freely on $F_n(S)$ by permuting coordinates, and the *n-string braid group* $B_n(S)$ *of* S is defined by $B_n(S) = \pi_1(F_n(S)/S_n)$. This gives rise to the following short exact sequence:

$$1 \longrightarrow P_n(S) \longrightarrow B_n(S) \xrightarrow{\sigma} S_n \longrightarrow 1. \tag{1}$$

The map $\sigma\colon B_n(S) \to S_n$ is the standard homomorphism that associates a permutation to each element of $B_n(S)$. We note the following.

(1) When $M = D^2$ is the disk, then $B_n(D^2)$ (resp. $P_n(D^2)$) is the classical Artin braid group denoted by B_n (resp. the classical pure Artin braid group denoted by P_n).
(2) It follows from the definition that $F_1(S) = S$ for any surface S, and consequently that the groups $P_1(S)$ and $B_1(S)$ are both isomorphic to $\pi_1(S)$. For this reason, braid groups over a surface S may be regarded as natural generalizations of the fundamental group of S.

For more information on general aspects of surface braid groups we recommend [17] and also the survey [16], in particular its Section 2 where equivalent definitions of these groups are given, showing different viewpoints of them. We recall that the Artin braid group B_n admits the following presentation [2]:

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j|>1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for } |i-j|=1 \end{array} \right\rangle. \tag{2}$$

It is well known that the symmetric group S_n admits the following presentation:

$$S_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } 1 \le i \le n-2 \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \ge 2 \\ \sigma_1^2 = 1 & \end{array} \right\rangle.$$

Let $\langle\langle g \rangle\rangle$ denote the normal closure of an element g in a group G. Hence, it follows from (2) that the quotient $B_n/\langle\langle \sigma_1^2 \rangle\rangle$ is isomorphic to the symmetric group S_n.

Let $B_n(2) = B_n/\langle\langle \sigma_1^2 \rangle\rangle$. Note that $B_n(2)$ is a finite group, whereas the braid group B_n is infinite and torsion-free. A natural question that arises is whether the groups $B_n(k) = B_n/\langle\langle \sigma_1^k \rangle\rangle$ are finite for every $k \ge 3$. This problem was answered by Coxeter [8] using classical geometric methods, revealing an unexpected connection between braid groups and Platonic solids. He showed that $B_n(k)$ is finite if and only if $(k-2)(n-2) < 4$; see Theorem 4 (see also [19, Chapter 5, Proposition 2.2]).

It is worth noting that Coxeter's result was later proved by Assion [4] using the Burau representation of braid groups. Assion also obtained presentations of certain symplectic groups as quotients of braid groups, a line of work that was further developed by Wajnryb [23], who gave a braid-like presentation of the symplectic group $\mathrm{Sp}(n, p)$. More recently, in [6, Theorem 2.1], the authors studied the relationship between level-m congruence subgroups $B_n[m]$ and the normal closure of the element σ_1^m. In particular, they characterized precisely when this normal closure has finite index in $B_n[m]$, and provided explicit generating sets for the corresponding finite quotients. A closely related statement was independently formulated in [5, Remark 1.2]. However, the result in [6] is established with a complete proof and includes an explicit description of the generators.

Coxeter-type presentations also appear in recent work of Goldman on Shephard groups. In [11, Section 2.1], finite Shephard groups are characterized via presentations that generalize Coxeter groups by allowing generators of arbitrary finite order, while preserving Coxeter-like braid relations. Although the groups considered there are not braid groups over surfaces, this work further illustrates how imposing finite-order relations on standard generators leads to rigid quotient structures, a perspective that is conceptually related to the quotients studied in the present paper.

Motivated by Coxeter's work on Artin braid groups, we investigate the analogous problem for surface braid groups. From now on, let $B_n(M)(q)$ denote the quotient of the surface braid group $B_n(M)$ by the normal closure of the element σ_1^q, where σ_1 is the standard Artin generator of the braid group B_n permuting the first two strands [2].

The main purpose of this paper is to study Coxeter-type quotients of surface braid groups $B_n(M)(q)$. In contrast with the classical case of the disk, we prove that for every closed surface different from the sphere and the projective plane, the quotient group $B_n(M)(q)$ is infinite for all

$n, q \geq 3$. In Section 2.1 we prove the following result, where $H_1(M)$ is the first homology group of the surface M.

Theorem 1. *Let $q \geq 3$ and $n \geq 2$ integers. Let M be a closed surface different from the sphere and the projective plane.*

(1) *If M is orientable then the abelianization of the group $B_n(M)(q)$ is isomorphic to $\mathbb{Z}_q \oplus H_1(M)$.*

(2) *If M is non-orientable then the abelianization of the group $B_n(M)(q)$ is isomorphic to*

$$\begin{cases} H_1(M) & \text{if } q \text{ is odd,} \\ \mathbb{Z}_2 \oplus H_1(M) & \text{if } q \text{ is even.} \end{cases}$$

(3) *For any surface M different from the sphere and the projective plane, the group $B_n(M)(q)$ is infinite.*

We note that Theorem 1 is also true for $q = 2$. For instance, in [15, p. 226], the authors claimed that for closed orientable surfaces, of genus $g \geq 1$, the quotient group $B_n(M)(2)$ is isomorphic to $\pi_1(M)^n \rtimes S_n$. So, it is infinite.

In Section 2.2 we analyze the cases where M is the sphere or the projective plane. We compute the abelianization of $B_n(M)(q)$ and prove the following result for few strings for sphere braid groups.

Theorem 2. *Let $q \geq 3$.*

(1) $B_2(\mathbb{S}^2)(q) = \begin{cases} \mathbb{Z}_2 & \text{if } q \text{ is even,} \\ \{1\} & \text{if } q \text{ is odd.} \end{cases}$

(2) $B_3(\mathbb{S}^2)(q) \cong \begin{cases} B_3(\mathbb{S}^2) & \text{if } \gcd(4,q) = 4, \\ S_3 & \text{if } \gcd(4,q) = 2, \\ \{1\} & \text{if } \gcd(4,q) = 1. \end{cases}$

(3) *$B_4(\mathbb{S}^2)(q)$ is an infinite group if and only if $q \geq 6$.*

Finally, in Section 3 we show that the quotient group $B_n(M)/[P_n(M), P_n(M)]\ (q)$ is finite when M is the disk, see Theorem 11, and that it is infinite when M is a closed orientable surface M of genus $g \geq 1$, see Proposition 13, where $q \geq 3$, $n \geq 2$ and $[P_n(M), P_n(M)]$ is the commutator subgroup of the pure braid group of the surface M.

2. Quotients of surface braid groups

Our main purpose is to study the Coxeter-type quotient of surface braid groups $B_n(M)(q)$ obtained by considering $\sigma_1^q = 1$, for $q \geq 3$ and where σ_1 is the classical Artin generator, see [2]. We will use presentations of surface braid groups that have in the set of generators the Artin generators.

We start this section with the following elementary result that will be useful in this work.

Lemma 3. *Let a and b be positive integers and let g be an element in a group G. If $g^a = 1$ and $g^b = 1$ then $g^d = 1$, where $d = \gcd(a,b)$ denotes the greatest common divisor of the integers a and b.*

Proof. This result is a consequence of the Bezout's identity: if a and b are integers (not both 0), then there exist integers u and v such that $\gcd(a,b) = au + bv$, see [18, Theorem 1.7, Section 1.2]. □

We recall Coxeter's result for braid groups over the disk that strongly motivates this paper.

Theorem 4 ([8]). *Let $p \geq 3$, and let $B_n(p)$ denote the quotient of the braid group B_n by the relation $\sigma_1^p = 1$. Then $B_n(p)$ is finite if and only if*

$$(n,p) \in \{(3,3),(3,4),(3,5),(4,3),(5,3)\}.$$

Motivated by this unexpected result from Coxeter's work on the classical braid groups, we are interested in exploring these quotients for surface braid groups, as we show in the following sections.

2.1. *Braid groups over surfaces different from the sphere and the projective plane*

Let $n \geq 2$ and let $B_n(M)$ denote the braid groups over a surface M. Compared with the case of the disk (see [8]) the group $B_n(M)(q)$ is infinite for any integer $q \geq 3$, for closed surfaces different from the sphere and the projective plane. The goal of this section is to prove Theorem 1. To do it, we will use the following presentations of surface braid groups that have in the set of generators, the Artin generators.

Proposition 5 (Theorem 1.4 of [21]). *Let $M_{g,0}$ be an orientable surface of genus $g \geq 1$. The group $B_n(M_{g,0})$ admits the following presentation.*

Generators: $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ *and* ρ_{ij} *for* $1 \leq i \leq n$ *and* $1 \leq j \leq 2g$.

Relations:

(I) (i) $\sigma_i\sigma_j = \sigma_j\sigma_i$, $|i-j| \geq 2$;

(ii) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.

(II) (i) $(\rho_{i,2g}^{-1}\rho_{i,2g-1}^{-1}\rho_{i,2g}\rho_{i,2g-1})\cdots(\rho_{i,2}^{-1}\rho_{i1}^{-1}\rho_{i,2}\rho_{i,1}) = A_{i,i+1}\cdots A_{i,n}A_{i,1}^{-1}\cdots A_{i,i-1}^{-1}$;

(ii) $\rho_{ij}\rho_{kl} = \rho_{kl}\rho_{ij}$ *if* $i<k$, l *is odd and* $j<l$ *or* $i<k$, l *is even and* $j<l-1$;

(iii) (a) $\rho_{jk}^{-1}\rho_{ik}\rho_{jk} = B_{ij}^{-1}\rho_{ik}B_{ij}$ *if* $i<j$;

(b) $\rho_{jk}\rho_{ik}\rho_{jk}^{-1} = \rho_{ik}^{-1}B_{ij}\rho_{ik}B_{ij}^{-1}\rho_{ik}$ *if* $i<j$;

(c) $\rho_{jk}^{-1}\rho_{ik}\rho_{jk} = \rho_{ik}B_{ij}\rho_{ik}B_{ij}^{-1}\rho_{ik}^{-1}$ *if* $i>j$;

(d) $\rho_{jk}\rho_{ik}\rho_{jk}^{-1} = B_{ij}^{-1}\rho_{ik}B_{ij}$ *if* $i>j$;

(iv) (a) $\rho_{j,2t-1}^{-1}\rho_{j,2t}\rho_{j,2t-1} = B_{ij}^{-1}\rho_{i,2t-1}B_{ij}\rho_{i,2t-1}^{-1}\rho_{i,2t}B_{ij}$ *if* $i<j$;

(b) $\rho_{j,2t-1}\rho_{j,2t}\rho_{j,2t-1}^{-1} = \rho_{i,2t-1}^{-1}B_{ij}\rho_{i,2t-1}B_{ij}^{-1}\rho_{i,2t}\rho_{i,2t-1}^{-1}B_{ij}^{-1}\rho_{i,2t-1}$ *if* $i<j$;

(c) $\rho_{j,2t-1}^{-1}\rho_{j,2t}\rho_{j,2t-1} = \rho_{i,2t}\rho_{i,2t-1}B_{ij}^{-1}\rho_{i,2t-1}^{-1}$ *if* $i>j$;

(d) $\rho_{j,2t-1}\rho_{j,2t}\rho_{j,2t-1}^{-1} = \rho_{i,2t}B_{ij}$ *if* $i>j$;

(v) (a) $\rho_{j,2t}^{-1}\rho_{j,2t-1}\rho_{j,2t} = B_{ij}^{-1}\rho_{i,2t-1}$ *if* $i>j$;

(b) $\rho_{j,2t}\rho_{j,2t-1}\rho_{j,2t}^{-1} = \rho_{i,2t}^{-1}B_{ij}\rho_{i,2t}\rho_{i,2t-1}$ *if* $i<j$;

(c) $\rho_{j,2t}^{-1}\rho_{j,2t-1}\rho_{j,2t} = \rho_{i,2t}B_{ij}\rho_{i,2t}^{-1}\rho_{i,2t-1}BB_{ij}\rho_{i,2t}BB_{ij}^{-1}\rho_{i,2t}^{-1}$ *if* $i>j$;

(d) $\rho_{j,2t}\rho_{j,2t-1}\rho_{j,2t}^{-1} = B_{ij}^{-1}\rho_{i,2t-1}\rho_{i,2t}^{-1}B_{ij}^{-1}\rho_{i,2t}B_{ij}$ *if* $i>j$;

(vi) (a) $\rho_{kl}^{-1}\rho_{ij}\rho_{kl} = B_{ik}^{-1}\rho_{il}B_{ik}\rho_{il}^{-1}\rho_{ij}\rho_{il}B_{ik}^{-1}\rho_{il}^{-1}B_{ik}$ *if* $i<k$, $j>l$ *and* (C);

(b) $\rho_{kl}\rho_{ij}\rho_{kl}^{-1} = \rho_{il}^{-1}B_{ik}\rho_{il}B_{ik}^{-1}\rho_{ij}B_{ik}\rho_{il}^{-1}B_{ik}^{-1}\rho_{il}$ *if* $i<k$, $j>l$ *and* (C);

(c) $\rho_{kl}^{-1}\rho_{ij}\rho_{kl} = \rho_{il}B_{ik}^{-1}\rho_{il}^{-1}B_{ik}\rho_{ij}B_{ik}^{-1}\rho_{il}^{-1}B_{ik}\rho_{il}^{-1}$ *if* $i>k$, $j>l$ *and* (C);

(d) $\rho_{kl}\rho_{ij}\rho_{kl}^{-1} = B_{ik}^{-1}\rho_{il}^{-1}B_{ik}\rho_{il}\rho_{ij}\rho_{il}^{-1}B_{ik}^{-1}\rho_{il}B_{ik}$ *if* $i>k$, $j<l$ *and* (C);

where:

$$(j,l) \neq (2t,2t-1) \quad \text{and} \quad (j,l) \neq (2t-1,2t),$$
$$B_{ij} = \sigma_i\cdots\sigma_{j-2}\sigma_{j-1}^2\sigma_{j-2}^{-1}\cdots\sigma_i^{-1} \quad \text{and} \quad A_{ij} = \sigma_{j-2}^{-1}\cdots\sigma_i^{-1}\sigma_{j-1}^2\sigma_i\cdots\sigma_{j-2}. \tag{C}$$

(III) (i) $\sigma_i\rho_{jk} = \rho_{jk}\sigma_i$ *if* $j \neq i$ *or* $i+1$;

(ii) $\rho_{ik} = \sigma_i\rho_{i+1,k}\sigma_i^{-1}$.

Now we consider the non-orientable cases.

Proposition 6 (Theorem 1.2 of [21]). *Let $M_{g,0}$ be a non-orientable surface of genus $g \geq 2$. The group $B_n(M_{g,0})$ admits the following presentation.*

Generators: $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ *and* ρ_{ij} *for* $1 \leq i \leq n$ *and* $1 \leq j \leq g$.

Relations:

(I) (i) $\sigma_i \sigma_j = \sigma_j \sigma_i$, $i - j \geq 2$;

(ii) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

(II) (i) $\rho_{ig}^2 \rho_{i,g-1}^2 \cdots \rho_{i,1}^2 = A_{i,i+1} \cdots A_{i,n} A_{i,1}^{-1} \cdots A_{i,i-1}^{-1}$;

(ii) $\rho_{ij} \rho_{kl} = \rho_{kl} \rho_{ij}$ *if* $i < k$ *and* $j < l$;

(iii) (a) $\rho_{jk}^{-1} \rho_{ik} \rho_{jk} = B_{ij}^{-1} \rho_{ik}$ *if* $i < j$;

(b) $\rho_{jk} \rho_{ik} \rho_{jk}^{-1} = \rho_{ik}^{-1} B_{ij} \rho_{ik}^2$ *if* $i < j$;

(c) $\rho_{jk}^{-1} \rho_{ik} \rho_{jk} = \rho_{ik}^2 B_{ij}^{-1} \rho_{ik}^{-1}$ *if* $i > j$;

(d) $\rho_{jk} \rho_{ik} \rho_{jk}^{-1} = \rho_{ik} B_{ij}$ *if* $i > j$;

(iv) (a) $\rho_{kl}^{-1} \rho_{ij} \rho_{kl} = B_{ik}^{-1} \rho_{il} B_{ik} \rho_{il}^{-1} \rho_{ij} \rho_{il} B_{ik}^{-1} \rho_{il}^{-1} B_{ik}$ *if* $i < k$, $j > l$;

(b) $\rho_{kl} \rho_{ij} \rho_{kl}^{-1} = \rho_{il}^{-1} B_{ik} \rho_{il} B_{ik}^{-1} \rho_{ij} B_{ik} \rho_{il}^{-1} B_{ik}^{-1} \rho_{il}$ *if* $i < k$, $j > l$;

(c) $\rho_{kl}^{-1} \rho_{ij} \rho_{kl} = \rho_{il} B_{ik}^{-1} \rho_{il}^{-1} B_{ik} \rho_{ij} B_{ik}^{-1} \rho_{il} B_{ik} \rho_{il}^{-1}$ *if* $i > k$, $j > l$;

(d) $\rho_{kl} \rho_{ij} \rho_{kl}^{-1} = B_{ik}^{-1} \rho_{il}^{-1} B_{ik} \rho_{il} \rho_{ij} \rho_{il}^{-1} B_{ik}^{-1} \rho_{il} B_{ik}$ *if* $i > k$, $j < l$.

(III) (i) $\sigma_i \rho_{jk} = \rho_{jk} \sigma_i$ *if* $j \neq i$ *or* $i + 1$;

(ii) $\rho_{ik} = \sigma_i \rho_{i+1,k} \sigma_i^{-1}$, *where*

$$B_{ij} = \sigma_i \cdots \sigma_{j-2} \sigma_{j-1}^2 \sigma_{j-2}^{-1} \cdots \sigma_i^{-1}$$

and

$$A_{ij} = \sigma_{j-2}^{-1} \cdots \sigma_i^{-1} \sigma_{j-1}^2 \sigma_i \cdots \sigma_{j-2}$$

for $1 \leq i < j \leq n$.

We are now ready to prove Theorem 1. Throughout the proof, the abelianization of a group G will be denoted by G^{ab}.

Proof of Theorem 1. Let $q \geq 3$ and $n \geq 3$ integers and let M be a closed surface different from the sphere and the projective plane.

(1). The proof of this item follows using a presentation of the braid group over orientable surfaces given in Proposition 5. Since the argument is similar in both cases (orientable and non-orientable) we give more details for the non-orientable case below.

(2). Let

$$M = \underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{g \text{ projective planes}},$$

where $g \geq 2$ is the genus of the non-orientable surface M. We give a presentation of the abelianization of the group $B_n(M)(q)$. To do this, we use the presentation of $B_n(M)$ given by Scott, see Proposition 6:

Generators: $\sigma_1, \ldots, \sigma_{n-1}$ and $\rho_{i,j}$ where $1 \leq i \leq n$, $1 \leq j \leq g$.

Relations: All generators commute. From this and using the Scott's presentation, we get the following information:

(1) From Proposition 6(I-ii) it follows that $\sigma_i = \sigma_{i+1}$, for $i = 1, \ldots, n-2$.

(2) From Proposition 6(III-ii) we get $\rho_{i,k} = \rho_{i+1,k}$, for $1 \leq i \leq n-1$, $1 \leq k \leq g$.

(3) In Proposition 6(II) were defined elements $A_{i,j}$ and $B_{i,j}$, for all $1 \leq i < j \leq n$, as conjugates of σ_i^2. From Proposition 6(II-iii), (see also [21, Theorem 1.1, II(iii)]) we obtain, for all $1 \leq i < j \leq n$, $B_{i,j} = 1$ in $\left(B_n(M)(q)\right)^{\mathrm{ab}}$. So, in $\left(B_n(M)(q)\right)^{\mathrm{ab}}$ it holds that $\sigma_i^2 = 1$, for all $1 \leq i \leq n-1$, as well as $A_{i,j} = 1$, for all $1 \leq i < j \leq n$.

(4) As a consequence of the previous item and Proposition 6(II-i) (see also [21, Theorem 1.1, II(i)]) we get $\rho_{i,g}^2 \rho_{i,g-1}^2 \cdots \rho_{i,1}^2 = 1$, for all $i = 1, \ldots, n-1$.

The other relations in Proposition 6 do not contribute further information about $\big(B_n(M)(q)\big)^{\mathrm{ab}}$.

Since $\sigma_1^2 = 1$ and $\sigma_1^q = 1$, we have $\sigma_1^d = 1$, where $d = \gcd(2,q)$, by Lemma 3. Therefore, a presentation of the abelianization of $B_n(M)(q)$ is given by the following.

Generators: σ_1 and $\rho_{1,j}$ for $1 \le j \le g$.
Relations:

(1) All generators commute.
(2) $\sigma_1^2 = 1$, and $\sigma_1^q = 1$, for $q \ge 3$. By Lemma 3, this implies that $\sigma_1^d = 1$, for $q \ge 3$, where $d = \gcd(2,q)$.
(3) $\boldsymbol{\rho}_{1,g}^2\boldsymbol{\rho}_{1,g-1}^2\cdots\boldsymbol{\rho}_{1,1}^2 = \mathbf{1}$.

We recall that a presentation of the fundamental group of the non-orientable surface M of genus g is given by

$$\pi_1(M) = \big\langle \rho_1,\ldots,\rho_g \mid \rho_g^2\rho_{g-1}^2\cdots\rho_1^2 = 1\big\rangle. \tag{3}$$

Hence, from the computations above, we have proved this item

$$\big(B_n(M)(q)\big)^{\mathrm{ab}} \cong \mathbb{Z}_d \oplus H_1(M),$$

where $d = \gcd(2,q)$.

(3). Since the first homology group of the closed surfaces different from the sphere and the projective plane are infinite:

$$H_1(M) \cong \begin{cases} \mathbb{Z}^{2g} & \text{if } M \text{ is orientable of genus } g,\\ \mathbb{Z}^{g-1}\oplus\mathbb{Z}_2 & \text{if } M \text{ is non-orientable of genus } g,\end{cases}$$

then we conclude that the Coxeter-type quotient $B_n(M)(q)$ is infinite. □

2.2. *The sphere and the projective plane*

Now, we exhibit some information of $B_n(M)(q)$ when M is either the sphere or the projective plane.

From [9] we know that the sphere braid group with n strings, $B_n(\mathbb{S}^2)$, admits a presentation with generators σ_i for $i = 1,2,\ldots,n-1$ and relations as in (2) plus:

- the surface relation $\sigma_1\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1 = 1$.

Recall that a perfect group G is a group such that $G = [G,G]$.

Proposition 7. *Let $q \ge 2$ and $n \ge 3$ integers. Let $d = \gcd\big(q,\, 2(n-1)\big)$.*

(1) *The abelianization of $B_n(\mathbb{S}^2)(q)$ is isomorphic to the cyclic group $\mathbb{Z}_d$.*
(2) *If q and $2(n-1)$ are coprimes then $B_n(\mathbb{S}^2)(q)$ is perfect.*

Proof. Let $q \ge 2$ and $n \ge 3$ integers and let $d = \gcd\big(q,\, 2(n-1)\big)$. Using the presentation of $B_n(\mathbb{S}^2)$ we conclude that the abelianization of the quotient group $B_n(\mathbb{S}^2)(q)$ has the presentation

$$\big\langle \sigma_1 \mid \sigma_1^q = 1,\ \sigma_1^{2(n-1)} = 1\big\rangle,$$

where the second equality comes from the surface relation. Lemma 3 implies that the order of $\sigma_1 \in \big(B_n(\mathbb{S}^2)(q)\big)^{\mathrm{ab}}$ is equal to d, where $d = \gcd\big(q, 2(n-1)\big)$. From this, we proved the first item.

From the first item of this result and the hypothesis of the second item, we get $\sigma_1 = 1$. Since the abelianization of $B_n(\mathbb{S}^2)(q)$ is the trivial group, then we conclude that $B_n(\mathbb{S}^2)(q)$ is perfect, proving the second item. □

For the special case of few strings, in Theorem 2 we have the result for the Coxeter-type quotient of the sphere braid group, that we prove below. When analyzing the case of four strings, we use triangle groups as defined in [19, Appendix I, Section 7], see also [1].

Proof of Theorem 2. Let $q \geq 3$.

(1). Since the group $B_2(\mathbb{S}^2) = \mathbb{Z}_2$ is generated by σ_1, then the result of this item follows immediately from Lemma 3.

(2). Recall from [9, third lemma on p. 248] (see also [19, Chapter 11, Proposition 2.4]) that $B_3(\mathbb{S}^2)$ has order 12 and the elements σ_1 and σ_2 have order 4. So, from Lemma 3, in $B_3(\mathbb{S}^2)$ it holds

$$\begin{cases} \sigma_1^4 = 1, & \text{if } \gcd(4,q) = 4, \\ \sigma_1^2 = 1, & \text{if } \gcd(4,q) = 2, \\ \sigma_1 = 1, & \text{if } \gcd(4,q) = 1. \end{cases}$$

From this, it is clear that $B_3(\mathbb{S}^2)(q) \cong B_3(\mathbb{S}^2)$ if $\gcd(4,q) = 4$, and that $B_3(\mathbb{S}^2)(q)$ is the trivial group $\{1\}$ if $\gcd(4,q) = 1$. Finally, suppose that $\gcd(4,q) = 2$, then it follows from the proof of [9, third lemma on p. 248] (see also the proof of [19, Chapter 11, Proposition 2.4]) that $B_3(\mathbb{S}^2)(q) \cong S_3$ in this last case, completing the proof of this item.

(3). The group $B_4(\mathbb{S}^2)(q)$ admits the following presentation:

$$B_4(\mathbb{S}^2)(q) = \left\langle \sigma_1, \sigma_2, \sigma_3 \;\middle|\; \begin{array}{c} \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3,\ \sigma_1\sigma_3 = \sigma_3\sigma_1, \\ \sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1 = 1,\ \sigma_1^q = 1 \end{array} \right\rangle. \tag{4}$$

We used the GAP System [22] to show that $B_4(\mathbb{S}^2)(q)$ is a finite group in the following cases:

($q = 3$) The group $B_4(\mathbb{S}^2)(3)$ is isomorphic to the alternating group A_4.
($q = 4$) In this case the group $B_4(\mathbb{S}^2)(4)$ has order 192.
($q = 5$) The group $B_4(\mathbb{S}^2)(5)$ is isomorphic to the alternating group A_5.

We elucidate the routine used in the GAP computations for the case $B_4(\mathbb{S}^2)(3)$, the other cases are similar:

```
f3 := FreeGroup("a","b","c");;
gens:= GeneratorsOfGroup(f3);;
a:= gens[1];;b:= gens[2];;c:= gens[3];;
B4S23:= f3/[ a*b*a*b^-1*a^-1*b^-1, b*c*b*c^-1*b^-1*c^-1,
a*c*a^-1*c^-1, a^3, b^3, c^3, a*b*c^2*b*a ];
Order (B4S23);
StructureDescription (B4S23);
```

Now, for $q \geq 6$, we show that the group $B_4(\mathbb{S}^2)(q)$ is infinite. Let $\langle\!\langle \sigma_1\sigma_3^{-1} \rangle\!\rangle$ be the normal closure of the element $\sigma_1\sigma_3^{-1}$ in $B_4(\mathbb{S}^2)(q)$. Then

$$B_4(\mathbb{S}^2)(q) / \langle\!\langle \sigma_1\sigma_3^{-1} \rangle\!\rangle = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ (\sigma_1\sigma_2)^3 = 1,\ \sigma_1^q = 1 \rangle.$$

Taking $a = \sigma_1\sigma_2\sigma_1$ and $b = \sigma_1\sigma_2$ follows that $(ab) = \sigma_1^{-1}$ and so

$$B_4(\mathbb{S}^2)(q) / \langle\!\langle \sigma_1\sigma_3^{-1} \rangle\!\rangle = \langle a, b \mid a^2 = b^3 = (ab)^q = 1 \rangle.$$

Hence $B_4(\mathbb{S}^2)(q) / \langle\!\langle \sigma_1\sigma_3^{-1} \rangle\!\rangle$ is isomorphic to the triangular group $T(2,3,q)$ that is infinite if, and only if $q \geq 6$, see [19, Theorem 7.1, Appendix I]. □

Now we move to the case of the projective plane. Recall a presentation of the braid group of the projective plane.

Proposition 8 (Section III of [7]). *The braid group of the projective plane on n strings, $B_n(\mathbb{R}P^2)$ admits the following presentation.*

Generators: $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \rho_1, \rho_2, \ldots, \rho_n$.

Relations:

(I) $\sigma_i\sigma_j = \sigma_j\sigma_i$ *if* $|i-j| \geq 2$.
(II) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ *for* $i = 1,\ldots,n-2$.
(III) $\sigma_i\rho_j = \rho_j\sigma_i$ *for* $j \neq i, i+1$.
(IV) $\rho_i = \sigma_i\rho_{i+1}\sigma_i$ *for* $i = 1,\ldots,n-1$.
(V) $\rho_{i+1}^{-1}\rho_i^{-1}\rho_{i+1}\rho_i = \sigma_i^2$.
(VI) $\rho_1^2 = \sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2\sigma_1$.

For the case of braid groups over the projective plane we have the following.

Proposition 9. *Let $q \geq 2$ and $n \geq 2$ integers. The abelianization of the group $B_n(\mathbb{R}P^2)(q)$ is isomorphic to $\mathbb{Z}_2$ if q is odd, otherwise it is the Klein four group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

Proof. We obtain the result from Lemma 3 and the presentation of $B_n(\mathbb{R}P^2)$ given by Van Buskirk in [7] (see Proposition 8 and also [19, p. 202, Theorem 4.1]). □

Remark 10. Except for the information of Theorem 2, we do not know under which conditions on n and q the groups $B_n(M)(q)$ are finite, when M is either the sphere or the projective plane.

3. Quotients of crystallographic surface braid groups

The quotients of surface braid groups $B_n(M)$ by the commutator subgroup of the respective pure braid group $[P_n(M), P_n(M)]$ considered in this section were deeply studied in [12] for the case of the disk and in [13] for the case of closed surfaces, in both cases exploring its connection with crystallographic groups.

In what follows, we analyze the Coxeter-type quotient groups $B_n(M)/[P_n(M), P_n(M)]\,(q)$ by adding to the presentation of $B_n(M)/[P_n(M), P_n(M)]$ the relation $\sigma_1^q = 1$, for braid groups over closed orientable surfaces and also for the disk.

3.1. *Braid groups over the disk*

Unlike the case of the Coxeter quotient of the Artin braid group [8], see Theorem 4, for all $n, q \geq 3$ the Coxeter-type quotient $B_n/[P_n, P_n]\,(q)$ is finite. The following result is part of the dissertation thesis of the third author, see [20, Theorem 3.3].

Theorem 11. *Let $n, q \geq 3$ and $k \in \mathbb{N}$. For any integer number $q \geq 3$, the group $B_n/[P_n, P_n]\,(q)$ is finite.*

(a) *If $q = 2k+1$, then $B_n/[P_n, P_n]\,(q)$ is isomorphic to $\mathbb{Z}_q$.*
(b) *When $q = 2k$, then $B_n/[P_n, P_n]\,(q)$ has order $\frac{n(n-1)k}{2} \cdot n!$.*

Proof. Let $n, q \geq 3$ and suppose that $\sigma_1^q = 1$. The integer q is equal to $2k + r$, with $0 \leq r \leq 1$ and $r, k \in \mathbb{N}$.

For item (a), as a consequence of the presentation of the Artin braid group B_n given in (2) we get $\sigma_i^{-1}\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}$, for all $1 \leq i \leq n-2$, and so $\sigma_i^q = 1$, for all $1 \leq i \leq n-2$. Hence, $\sigma_i = \sigma_i^{-2k} = A_{i,i+1}^{-k}$, for all $1 \leq i \leq n-1$, where $A_{i,j}$ is an Artin generator of the pure Artin braid group. So, in the group $B_n/[P_n, P_n]\,(q)$ it holds $[\sigma_i, \sigma_j] = 1$, for all $1 \leq i < j \leq n-1$. Therefore,

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} &\iff \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_{i+1}\sigma_i \\ &\iff \sigma_i = \sigma_{i+1}, \end{aligned}$$

for all $1 \leq i \leq n-1$. Then, $B_n/[P_n, P_n]\,(q)$ is isomorphic to $\langle \sigma_1 \mid \sigma_1^p = 1 \rangle$, proving item (a).

Now we prove item (b). By hypothesis, we have $\sigma_1^{2k} = 1$. As before, we may conclude that $\sigma_i^{2k} = 1$, for all $1 \leq i \leq n$, so $A_{i,i+1}^k = 1$, for all $1 \leq i \leq n$. Recall the definition of the pure Artin generator $A_{i,j} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_i^2\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$. So, $A_{i,j}^k = 1$, for all $1 \le i < j \le n$. We recall that the group $P_n/[P_n, P_n]$ is free Abelian with a basis given by the classes of pure Artin generators $\{A_{i,j} \mid 1 \le i < j \le n\}$. Hence, in $B_n/[P_n, P_n]\,(q)$ the natural projection of the group $P_n/[P_n, P_n] \le B_n/[P_n, P_n]$ is isomorphic to

$$\underbrace{\mathbb{Z}_k \times \cdots \times \mathbb{Z}_k}_{\frac{n(n-1)}{2}}.$$

From the above we get the following short exact sequence

$$1 \longrightarrow \underbrace{\mathbb{Z}_k \times \cdots \times \mathbb{Z}_k}_{\frac{n(n-1)}{2}} \longrightarrow B_n/[P_n, P_n]\,(q) \longrightarrow S_n \longrightarrow 1.$$

Therefore the middle group $B_n/[P_n, P_n]\,(q)$ has finite order $\frac{n(n-1)k}{2} \cdot n!$ and with this we verify item (b).

From items (a) and (b) we proved that for any integer number $q \ge 3$, the group $B_n/[P_n, P_n]\,(q)$ is finite. □

3.2. *Braid groups over orientable surfaces*

Let M be a compact, orientable surface without boundary of genus $g \ge 1$, and let $n \ge 2$. We will use the presentation of $B_n(M)/[P_n(M), P_n(M)]$ given in [13].

Proposition 12 ([13, Proposition 9]). *Let M be a compact, orientable surface without boundary of genus $g \ge 1$, and let $n \ge 1$. The quotient group $B_n(M)/[P_n(M), P_n(M)]$ has the following presentation.*

Generators: $\sigma_1, \ldots, \sigma_{n-1}, a_{i,r}$, $1 \le i \le n$, $1 \le r \le 2g$.
Relations:

(a) *The Artin relations:*

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for all } 1 \le i, j \le n-1, |i-j| \ge 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for all } 1 \le i \le n-2. \end{cases}$$

(b) $\sigma_i^2 = 1$, *for all* $i = 1, \ldots, n-1$.
(c) $[a_{i,r}, a_{j,s}] = 1$, *for all* $i, j = 1, \ldots, n$ *and* $r, s = 1, \ldots, 2g$.
(d) $\sigma_i a_{j,r} \sigma_i^{-1} = a_{\tau_i(j),r}$ *for all* $1 \le i \le n-1$, $1 \le j \le n$ *and* $1 \le r \le 2g$.

In [14, Figure 9] we may see geometrically the elements $a_{i,r}$ of Proposition 12. We have the following result about Coxeter-type quotients and the quotient groups considered in [13].

Proposition 13. *Let M be a compact, orientable surface without boundary of genus $g \ge 1$, $q \ge 3$ and let $n \ge 2$. The group $B_n(M)/[P_n(M), P_n(M)]\,(q)$ is infinite.*

(1) *If q is odd, the Coxeter-type quotient $B_n(M)/[P_n(M), P_n(M)]\,(q)$ is isomorphic to a free Abelian group of rank $2g$.*
(2) *The group $B_n(M)/[P_n(M), P_n(M)]\,(q)$ is isomorphic to the crystallographic group quotient $B_n(M)/[P_n(M), P_n(M)]$ if q is even.*

Proof. Let M be a compact, orientable surface without boundary of genus $g \ge 1$, and let $n \ge 2$. We shall use the presentation of the quotient groups $B_n(M)/[P_n(M), P_n(M)]$ given in Proposition 12. From Proposition 12(b) we have that in $B_n(M)/[P_n(M), P_n(M)]\,(q)$ it holds $\sigma_i^2 = 1$, for all $1 \le i \le n-1$.

Hence, for all $1 \le i \le n-1$, it is true that $\sigma_i^2 = 1$ and $\sigma_i^q = 1$. If q is odd, then from Lemma 3 we get $\sigma_i = 1$, for all $1 \le i \le n-1$, proving item (1). In the case that q is even then, for all $1 \le i \le n-1$, $\sigma_i^2 = 1$ independently of the number q, getting item (2). □

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