Comptes Rendus

Mathématique

Juan García Escudero

Some lower bounds for the maximal number of *A*-singularities in algebraic surfaces

Research article
Algebraic geometry

Some lower bounds for the maximal number of *A*-singularities in algebraic surfaces

Juan García Escudero ®, *a*

a Madrid, Spain

E-mail: jjgemplubg@gmail.com

Abstract. We construct algebraic surfaces with a large number of type *A* singularities. Bivariate polynomials presented in previous works for the construction of nodal surfaces and certain families of Belyi polynomials are used. In some cases explicit expressions in terms of classical Jacobi polynomials are obtained.

1. Introduction

In [9] a family of degree $3m$ polynomials was presented with the aim of improving the lower bounds for the maximal number of nodes or A_1 singularities in degree $3m$ complex algebraic surfaces. Besides the classical Cayley cubic [6] and Kummer quartic [19] surfaces, previous lower bounds were obtained in [4,7,8,20,23,25] and upper bounds in [5,18,22,26] (some images of low degree surfaces with many nodes can be seen in [3]).

We denote by $\mu_{A_\nu}(d)$ ($\mu_{A_\nu}^{(\mathbb{R})}(d)$) the maximal number of A_ν complex (real) singularities for a degree d surface in the complex (real) projective space $\mathbb{P}^3(\mathbb{C})$ ($\mathbb{P}^3(\mathbb{R})$). General lower bounds for the number of nodes on degree d complex algebraic surfaces were given in [7]. The affine equations describing the surfaces consist of the sum of the classical univariate Chebyshev polynomials and the bivariate folding polynomials studied in [17,27]. By considering a different class of bivariate polynomials, which were motivated by previous studies on substitution tilings constructed with the help of certain deltoid tangents, new lower bounds for nodes were obtained in [9]. In [12] we showed that there are real surfaces with the same number of real singularities as those studied in [9]. The surfaces can also be defined over the rationals (or over the integers, after clearing the denominators) as shown in [13] and the lower bounds are

$$\mu_{A_1}^{(\mathbb{R})}(3m) \geq \frac{3m(3m-1)}{2}\left\lfloor \frac{3m}{2} \right\rfloor + \big(3m(m-1)+1\big)\left\lfloor \frac{3m-1}{2} \right\rfloor \tag{1}$$

with explicit equations for the surfaces.

With the purpose of getting surfaces with many A_ν singularities with $\nu > 1$, a certain class of Belyi polynomials (used instead of Chebyshev polynomials) together with the folding polynomials were studied in [21]. By using other types of Belyi polynomials in addition to the bivariate

polynomials presented in [9], hypersurfaces with many non-nodal singularities were also constructed in [10,11]. For cusps or A_2 singularities, we studied in [11] several special cases improving existing lower bounds and the results were extended in [10, Proposition 3.4], where we obtained:

$$\mu_{A_2}(3m) \geq \frac{3m^2(3m-1)}{2} + \big(3m(m-1)+1\big)\left\lfloor \frac{m-1}{2} \right\rfloor. \tag{2}$$

An explicit equation for $d = 9$ with 127 complex cusps is given in [11, Proposition 2].

In the cases of singularities of type A_ν, $\nu > 2$, some results for low degrees studied in [11] were also extended in [10, Proposition 3.6] giving:

$$\mu_{A_{3m+1}}\big(3(2m+1)\big) \geq 3m(10m+7)+4. \tag{3}$$

In particular we showed that there are explicit equations for real surfaces of degree 9 with 55 real A_4 singularities [11, Figure 2(c)] and degree 15 with 166 real A_7 singularities [10, Figure 3].

In this work we generalise the results given in [10,11]. In order to investigate the existence of surfaces with many singularities, we use the family of bivariate polynomials $\mathscr{J}$ defined over $\mathbb{Q}$ considered in [13] in conjunction with several families of Belyi polynomials, which we study in Section 2. In Section 3 we first recall the results about nodal hypersurfaces obtained with $\mathscr{J}$ in [13], and then we construct families of surfaces with many A_ν singularities, improving the corresponding lower bounds given in [21]. We also get explicit expressions in terms of Jacobi polynomials for some cases in Section 4.

2. Belyi polynomials

If w_0 is a zero of a polynomial $P(w)$, $w \in \mathbb{C}$, with critical value $\zeta = P(w_0)$, then the order of a zero w_0 of $\frac{dP(w)}{dw}$ is called its multiplicity ν (all the derivatives of $P(w)$ up to order ν vanish at w_0). A univariate polynomial with no more than two different critical values is called a Belyi polynomial. A graph without cycles (plane tree) with a bicoloring for the vertices is used to represent a Belyi polynomial whose critical points have the multiplicities given by the number of edges adjacent to the vertices minus one [2,21]. The degree of a vertex is the number of edges incident to it. A leaf vertex is a vertex with degree one. Black and white vertices represent critical points with critical value $\zeta = -1$ and $\zeta = 1$ respectively. Well known examples of Belyi polynomials are the degree d Chebyshev polynomials of the first kind $T_d(w)$. The tree corresponding to $T_9(w)$ is shown in Figure 1, where the two leaf vertices correspond to non-critical points and the eight degree 2 vertices represent critical points with multiplicity 1.
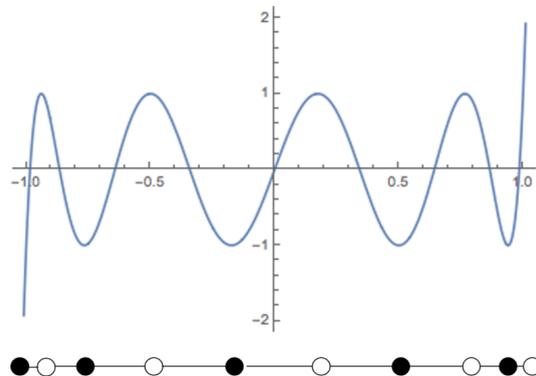

Figure 1. Chebyshev polynomial $T_9(w)$ (top) and its associated plane tree (bottom).

We use the notation $\mathscr{B}_{d,\nu,\epsilon}(w)$ for a degree d Belyi polynomial having critical points $w_1, w_2, \ldots, w_n$ of multiplicity ν with critical value $\zeta = -1$, one critical point w_0 of multiplicity ν with $\zeta = 1$, and one additional critical point w_u with multiplicity ϵ and $\zeta = -1$. The polynomials are then solutions of

$$\frac{\mathrm{d}\mathscr{B}_{d,\nu,\epsilon}(w)}{\mathrm{d}w} = (w - w_0)^{\nu}(w - w_u)^{\epsilon} \prod_{l=1}^{n}(w - w_l)^{\nu}$$

with $\mathscr{B}_{d,\nu,\epsilon}(w_l) = -1$, $l = 1, 2, \ldots, n$, $\mathscr{B}_{d,\nu,\epsilon}(w_0) = 1$ and $\mathscr{B}_{d,\nu,\epsilon}(w_u) = -1$ (the computation of some Belyi polynomials has been done in [11,21] by choosing one critical point equal to zero).

When there is no critical point with multiplicity ϵ and $\zeta = -1$ we denote the polynomial either by $\mathscr{B}_{d,\nu,0}(w)$ or $\mathscr{B}_{d,\nu}(w)$. We also write $\mathscr{B}[k_1, k_2, \ldots, k_l]$ if the polynomial can be characterised by the parameters $k_1, k_2, \ldots, k_l$ as in this section's final remarks.

Lemma 1. *There exist polynomials $\mathscr{B}_{d,\nu}^{(1)}(w)$, also denoted by $\mathscr{B}^{(1)}[n, m]$, with $k - 1$ critical points of multiplicity ν with critical value $\zeta = -1$ and one critical point of multiplicity ν with critical value $\zeta = 1$, where $k = 3m + 1$, $m \in \mathbb{Z}^{+}$ and*

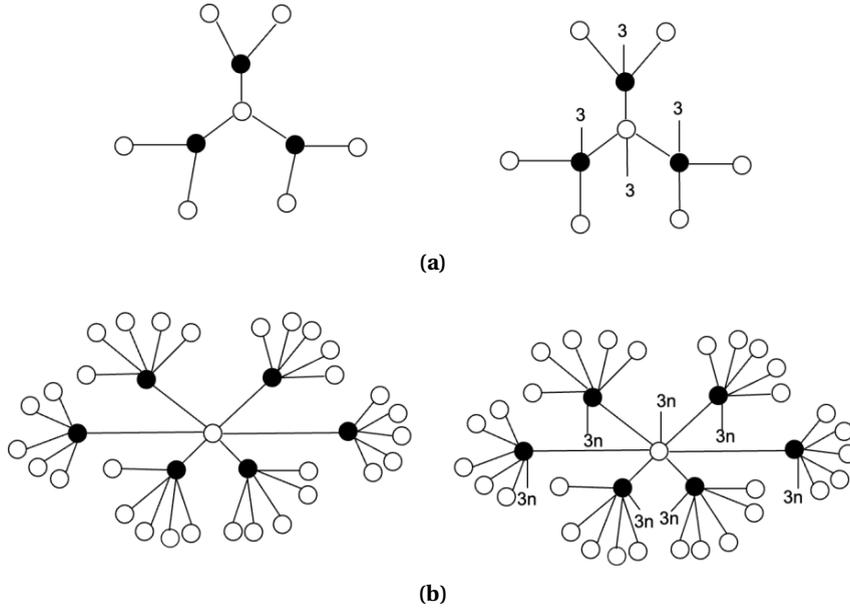$$d = k\nu + 1, \quad \nu = 3n + k - 2, \quad n \in \mathbb{Z}^{\geq 0}. \tag{4}$$

(a)

(b)

Figure 2. Plane trees for $\mathscr{B}_{d,\nu}^{(1)}(w)$. (a) $d = 12n + 9$, $\nu = 3n + 2$, $n = 0$ (left), $n = 1$ (right). (b) $d = 21n + 36$, $\nu = 3n + 5$, $n = 0$ (left), $n > 0$ (right).

Proof. The starting point, for $m = 1$, is a plane tree made up of a central white vertex, 3 black vertices and additional white vertices connected with the black ones (see Figure 2(a)-left). This corresponds to $k = 4$, $n = 0$ in (4). We then successively add $3n \in \mathbb{Z}^{+}$ edges (see the case $n = 1$ in Figure 2(a)-right) connected to vertices with opposite color (in Figure 2(a)-right we only show one edge with a number indicating the number of additional adjacent edges and omit 3 black and 9 white vertices). We get the trees associated to the series $\mathscr{B}_{12n+9, 3n+2}^{(1)}(w)$, $n \in \mathbb{Z}^{\geq 0}$.

In order to construct the initial trees for each of the remaining series, which correspond to $n = 0$ in (4), we add $k - 1 = 3m$, $m = 2, 3, 4, \ldots$ edges to the central white vertex in Figure 2(a)-left. In this way we obtain the trees for $\mathscr{B}_{(k-1)^2, k-2}^{(1)}(w)$. The case $m = 2$ is shown in Figure 2(b)-left.

We use the trees of $\mathcal{B}^{(1)}_{(k-1)^2,\,k-2}(w)$ as initial trees of the series obtained in the following way: for each value of k we add $3n$ edges to each vertex in the corresponding tree, as in the $m=1$ case, and we get the series

$$\mathcal{B}^{(1)}_{3(n+3m(n+m)),\,3(n+m)-1}(w),\, n\in\mathbb{Z}^{\geq 0}, m\in\mathbb{Z}^{+}. \tag{5}$$

In Figure 2(b)-right we represent the next step ($k=7$), which corresponds to the series $\mathcal{B}^{(1)}_{21n+36,\,3n+5}(w),\, n\in\mathbb{Z}^{+}$. $\qquad\qquad\square$

Lemma 2. *There exist polynomials* $\mathcal{B}^{(2)}_{d,v,\epsilon}(w)$ *($\mathcal{B}^{(2)}[j,n,m,l]$) with $b-1$ critical points of multiplicity v with critical value $\zeta=-1$, one critical point of multiplicity v with $\zeta=1$ and one additional critical point of multiplicity ϵ with $\zeta=-1$ where*

$$d=bv+a,\quad v=3n+b-1,\quad \epsilon=a-1,\quad n\in\mathbb{Z}^{\geq 0} \tag{6}$$

with $a=3m+j$, $b=3l+j+1$, $j\in\{0,1\}$, $m\in\mathbb{Z}^{+}$ if $j=0$, $m\in\mathbb{Z}^{\geq 0}$ if $j=1$, and $l=m,m+1,m+2,\ldots$
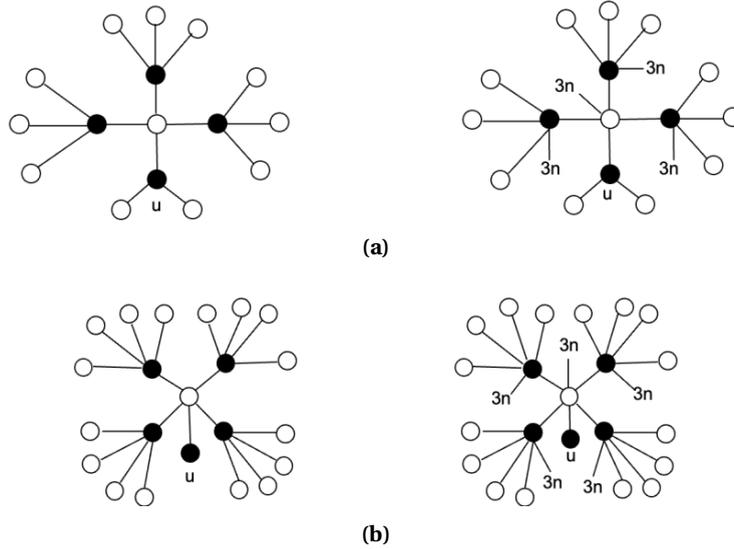
Figure 3. Plane trees for $\mathcal{B}^{(2)}_{d,v,\epsilon}(w)$. (a) $d=12n+15$, $v=3n+3$, $\epsilon=2$, $n=0$ (left) and $n>0$ (right). (b) $d=15n+21$, $v=3n+4$, $\epsilon=0$, $n=0$ (left) and $n>0$ (right).

Proof. The initial trees are generated as follows.

(1) $j=0$. For $m\in\mathbb{Z}^{+}$ and $l=m,m+1,m+2,\ldots$ the polynomials have trees made of a central white vertex with $b-1=3l$ edges connected to black vertices, each one connected to another $3l$ white vertices. In addition, the central white vertex is connected to an additional black vertex u which has $a-1=3m-1$ edges connected to white vertices. The corresponding Belyi polynomials are $\mathcal{B}^{(2)}_{3m+3l(3l+1),\,3l,\,3m-1}(w)$. The tree for $m=1$, $l=1$ is represented in Figure 3(a)-left.

(2) $j=1$. For $m\in\mathbb{Z}^{\geq 0}$, $l=m,m+1,m+2,\ldots$ there is a tree made up of a central white vertex with $b-1=3l+1$ edges connected to black vertices, and each black vertex is connected to b white vertices. The central white vertex is also connected to another black vertex u which has $a-1=3m$ edges connected to additional white vertices. The corresponding Belyi polynomials are $\mathcal{B}^{(2)}_{3m+1+(3l+1)(3l+2),\,3l+1,\,3m}(w)$. The tree for $m=0$, $l=1$ is represented in Figure 3(b)-left.

Therefore the initial trees are described by

$$d_0 = bv_0 + a, \quad v_0 = b - 1, \quad \epsilon_0 = a - 1 \tag{7}$$

where $a = 3m + j$, $b = 3l + j + 1$, $j = 0, 1$, $m \in \mathbb{Z}^+$ if $j = 0$, $m \in \mathbb{Z}^{\geq 0}$ if $j = 1$, $l = m, m+1, m+2, \ldots$

The trees in (1) and (2) are then used as initial trees for the following series. For each tree in (1) and (2) we construct a series of trees by adding $3n$ edges connected to vertices of opposite color for all the non leaf vertices with the exception of u, which is the only black vertex that remains unchanged in the whole process. The two series we obtain in this way are:

(3) $j = 0$, $d = (3n + 3l)(3l + 1) + 3m$, $v = 3n + 3l$, $\epsilon = \epsilon_0 = 3m - 1$, $n \in \mathbb{Z}^{\geq 0}$. In Figure 3(a)-right we represent the case $m = 1$, $l = 1$: $d = 12n + 15$, $v = 3n + 3$, $n \in \mathbb{Z}^+$.

(4) $j = 1$, $d = (3n + 3l + 1)(3l + 2) + 3m + 1$, $v = 3n + 3l + 1$, $\epsilon = \epsilon_0 = 3m$, $n \in \mathbb{Z}^{\geq 0}$. For $m = 0$, $l = 0$ we have the series corresponding to the bounds of (3). In Figure 3(b)-right we see the trees for the series $\mathscr{B}^{(2)}_{15n+21, 3n+4, 0}(w)$, $n \in \mathbb{Z}^+$ for $m = 0$, $l = 1$.

The series we obtain in (3) and (4) is

$$\mathscr{B}^{(2)}_{3(m+lj+(n+l)(3l+j+1))+j(j+2), 3(n+l)+j, 3m+j-1}(w) \tag{8}$$

with $j \in \{0, 1\}$, $n, m \in \mathbb{Z}^+$ if $j = 0$, $n, m \in \mathbb{Z}^{\geq 0}$ if $j = 1$, $l = m, m+1, m+2, \ldots$ ☐

Lemma 3. *There exist polynomials $\mathscr{B}^{(3)}_{d,v,\epsilon}(w)$ ($\mathscr{B}^{(3)}[x, p, n, m, l]$) with $p - 1$ critical points of multiplicity $v = x + 3n + p - 1$ with critical value $\zeta = -1$, one critical point of multiplicity $v = x + 3n + p - 1$ with critical value $\zeta = 1$, and one additional critical point with multiplicity $\epsilon = z$ and $\zeta = -1$ in the following cases:*

(a) $x = 1$:
(a1) $p = 3m - 1$, $d = 3(np + m(3m - 2) + l + 1)$, $v = 3(n + m) - 1$, $z = 3l + 1$;
(a2) $p = 3m$, $d = 3(np + 3m^2 + l + 1)$, $v = 3(n + m)$, $z = 3l + 2$;
(a3) $p = 3m + 1$, $d = 3(np + m(3m + 2) + l + 1)$, $v = 3(n + m) + 1$, $z = 3l + 1$;
(b) $x = 2$:
(b1) $p = 3m - 1$, $d = 3(np + m(3m - 1) + l + 1)$, $v = 3(n + m)$, $z = 3l + 2$;
(b2) $p = 3m$, $d = 3(np + m(3m + 1) + l + 1)$, $v = 3(n + m) + 1$, $z = 3l + 2$;
(b3) $p = 3m + 1$, $d = 3(np + 3m(m + 1) + l + 1)$, $v = 3(n + m) + 2$, $z = 3l$;
(c) $x = 3$:
(c1) $p = 3m - 1$, $d = 3(np + 3m^2 + l)$, $v = 3(n + m) + 1$, $z = 3l$;
(c2) $p = 3m$, $d = 3(np + m(3m + 2) + l + 1)$, $v = 3(n + m) + 2$, $z = 3l + 2$;
(c3) $p = 3m + 1$, $d = 3(np + (3m + 1)(m + 1) + l + 1)$, $v = 3(n + m + 1)$, $z = 3l + 2$;

where $m \in \mathbb{Z}^+$, $p \geq 4$, $n \in \mathbb{Z}^{\geq 0}$, $l = 0$ if $m = 1$ and $l = 0, 1, 2, \ldots, m - 2$ if $m \geq 2$.
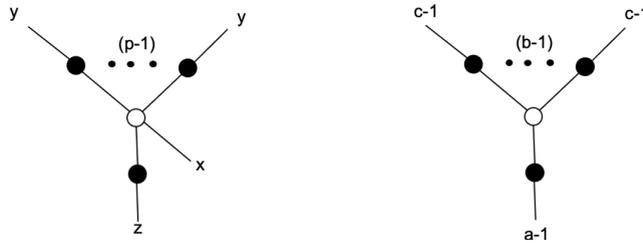

Figure 4. Plane trees for (left) Belyi polynomials in Lemma 3 with $n = 0$, (right) Belyi polynomials $\mathscr{B}_{a+c(b-1), c-1, a-1}(w)$ with explicit equations given by $G_{a,b,c}(w)$. The number of edges with label y or $c - 1$ is indicated in brackets above the suspension points.

Proof. The initial trees have the form given in Figure 4-left, which corresponds to $\mathcal{B}^{(3)}_{d_0,v_0,\epsilon_0}(w)$ with $d_0 = x + p + (p-1)y + z$, $v_0 = y$, $\epsilon_0 = z$. We look for the values of $d_0 = 3q$ and v_0 satisfying $\lfloor \frac{d_0}{v_0+1} \rfloor = \lfloor \frac{d_0-1}{v_0} \rfloor - 1 = p - 1$ (see Section 3). We find the following solutions for x and z ($p \geq 4$, $m \in \mathbb{Z}^+$, $l = 0$ if $m \leq 2$ and $l = 0,1,2,\ldots,m-2$ if $m > 2$):

(a) $x = 1$:
  (a1) $p = 3m - 1$, $q = m(3m-2) + l + 1$, $v_0 = 3m - 1$, $z = 3l + 1$;
  (a2) $p = 3m$, $q = 3m^2 + l + 1$, $v_0 = 3m$, $z = 3l + 2$;
  (a3) $p = 3m + 1$, $q = m(3m+2) + l + 1$, $v_0 = 3m + 1$, $z = 3l + 1$;
(b) $x = 2$:
  (b1) $p = 3m - 1$, $q = m(3m-1) + l + 1$, $v_0 = 3m$, $z = 3l + 2$;
  (b2) $p = 3m$, $q = m(3m+1) + l + 1$, $v_0 = 3m + 1$, $z = 3l + 2$;
  (b3) $p = 3m + 1$, $q = 3m(m+1) + l + 1$, $v_0 = 3m + 2$, $z = 3l$;
(c) $x = 3$:
  (c1) $p = 3m - 1$, $q = 3m^2 + l$, $v_0 = 3m + 1$, $z = 3l$;
  (c2) $p = 3m$, $q = m(3m+2) + l + 1$, $v_0 = 3m + 2$, $z = 3l + 2$;
  (c3) $p = 3m + 1$, $q = (3m+1)(m+1) + l + 1$, $v_0 = 3(m+1)$, $z = 3l + 2$.

We now add $3n$ edges as in Lemma 2 and we get the series for $\mathcal{B}^{(3)}_{d,v,\epsilon}(w)$ with $d = d_0 + 3np$, $v = v_0 + 3n$, $\epsilon = \epsilon_0$, $n \in \mathbb{Z}^{\geq 0}$. They satisfy $\lfloor \frac{d}{v+1} \rfloor = p - 1$, $\lfloor \frac{d-1}{v} \rfloor = p$ (see (18)). In Figure 5(a)–(b) we show two examples which correspond to $(x,z,y) = (1,1,4),(3,2,8)$ respectively: $d = 12n + 18$, $v = 3n + 4$, $n \in \mathbb{Z}^{\geq 0}$, $\epsilon = 1$ (see Figure 5(a)) and $d = 18n + 51$, $v = 3n + 8$, $n \in \mathbb{Z}^{\geq 0}$, $\epsilon = 2$ (see Figure 5(b), where we have used the notation of Figure 4).

The polynomials corresponding to this lemma are

$$\mathcal{B}^{(3)}_{3\left(m(3(m+n)+x-1)+j\left(n+2m+\lfloor\frac{x}{3}\rfloor\right)+l+1\right),\,3(n+m)+j+x-1,\,3l+2\left\lfloor 1+\left(\lfloor\frac{x-1}{2}\rfloor-\frac{1}{2}\right)j\right\rfloor+j\lfloor\frac{1}{x}\rfloor}(w) \tag{9}$$

with $j \in \{-1,0,1\}$, $x \in \{1,2,3\}$, $m \in \mathbb{Z}^+$, $n \in \mathbb{Z}^{\geq 0}$, $3m + j \geq 4$; $l = 0$ if $m = 1$ and $l = 0,1,\ldots,m-2$ if $m \geq 2$. $\qquad\qquad\square$
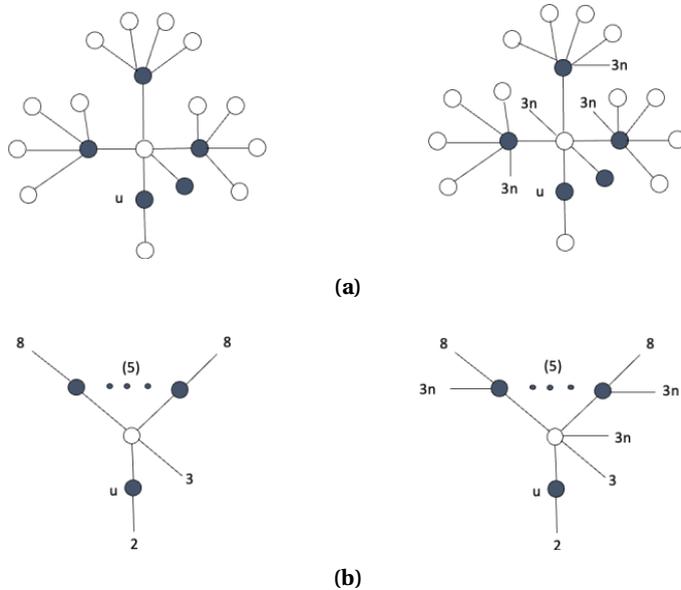

(a)

(b)

Figure 5. Plane trees for $\mathcal{B}^{(3)}_{d,v,\epsilon}(w)$. (a) $d = 12n+18$, $v = 3n+4$, $\epsilon = 1$, $n = 0$ (left) and $n > 0$ (right). (b) $d = 18n+51$, $v = 3n + 8$, $\epsilon = 2$, $n = 0$ (left) and $n > 0$ (right).

The analysis of the associated trees shows that in certain cases some polynomials given in Lemma 3 coincide with some given in Lemmas 1 and 2. We now use the notation $\mathscr{B}^{(1)}[n,m]$, $\mathscr{B}^{(2)}[j,n,m,l]$ and $\mathscr{B}^{(3)}[x,p,n,m,l]$ to compare the polynomials. For Lemma 3 ($x = 2$) and Lemma 1 we have $\mathscr{B}^{(3)}[2,3m+1,n,m,0] = \mathscr{B}^{(1)}[n+1,m]$ with $n \in \mathbb{Z}^{\geq 0}$ and $m \in \mathbb{Z}^+$. As an example we can see that the tree in Figure 2(a)-right is the same as the tree of the case $(x,z,y) = (2,0,5)$ in Lemma 3 with $d = 12n+21$, $v = 3n+5$, $\epsilon = 0$, $n = 0$, hence $\mathscr{B}^{(3)}[2,4,0,1,0] = \mathscr{B}^{(1)}[1,1]$. The cases with $l \neq 0$ in Lemma 3 are not included in Lemma 1 and the initial trees, associated to $\mathscr{B}^{(1)}[0,m]$ in Lemma 1, are not included in Lemma 3. The comparison between Lemma 3 ($x = 3$) and Lemma 2 gives $\mathscr{B}^{(3)}[3,3m-1,n,m,l] = \mathscr{B}^{(2)}[1,n+1,l,m-1]$ with $n \in \mathbb{Z}^{\geq 0}$, $m \in \mathbb{Z}^{\geq 2}$, $l = 0,1,\ldots,m-2$, and $\mathscr{B}^{(3)}[3,3m+1,n,m,l] = \mathscr{B}^{(2)}[0,n+1,l+1,m]$ for $n \in \mathbb{Z}^{\geq 0}$, $m \in \mathbb{Z}^{\geq 1}$, $l = 0$ if $m = 1$, $l = 0,1,\ldots,m-2$ if $m \geq 2$. In addition to the initial trees, that correspond to $\mathscr{B}^{(2)}[0,0,m,l]$ and $\mathscr{B}^{(2)}[1,0,m,l]$, there are other cases in Lemma 2 which are not included in Lemma 3, like $\mathscr{B}^{(2)}[1,n,m,m]$ with $n \in \mathbb{Z}^{\geq 1}$, $m \in \mathbb{Z}^{\geq 0}$ and $\mathscr{B}^{(2)}[0,n,m,m]$ for $n \in \mathbb{Z}^{\geq 1}$, $m \in \mathbb{Z}^{\geq 2}$.

3. Surfaces with many A singularities

The affine equations of the surfaces considered in this paper are obtained with the polynomials $\mathscr{B}^{(t)}$ discussed in Section 2 and polynomials related to the family

$$\widehat{J}_{d,\tau}(x,y) := \lambda_{d,\tau} \prod_{\mu} L_{d,\tau,\mu}(x,y) \tag{10}$$

where

$$L_{d,\tau,\mu}(x,y) := y + \left(\cos\left(\frac{2\pi}{d}\left(\frac{6\mu-1}{6}-\frac{\tau}{\pi}\right)\right)-x\right)\tan\left(\frac{\pi}{d}\left(\frac{6\mu-1}{6}-\frac{\tau}{\pi}\right)\right)+\sin\left(\frac{2\pi}{d}\left(\frac{6\mu-1}{6}-\frac{\tau}{\pi}\right)\right)$$

with $\mu = -\lfloor\frac{d-2}{2}\rfloor, -\lfloor\frac{d-2}{2}\rfloor+1, -\lfloor\frac{d-2}{2}\rfloor+2,\ldots,\lfloor\frac{d+1}{2}\rfloor$, $(x,y) \in \mathbb{R}^2$ and $\tau \in \mathbb{R}$. The parameters in (10) are $\lambda_{d,(6m-3d-1)\frac{\pi}{6}} = (-1)^m 2d$ and $\lambda_{d,\tau} = 2\cos\left(\tau+\frac{d\pi}{2}+\frac{2\pi}{3}\right)$ if $\tau \neq (6m-3d-1)\frac{\pi}{6}$, $m \in \mathbb{Z}$ (the line $L_{d,\tau,\mu}(x,y) = 0$ parallel to the y-axis is interpreted as the line $x+1 = 0$).

The following result will be necessary in what follows [12, Lemma 1].

Lemma 4. *The real polynomial $\widehat{J}_{d,0}(x,y)$ has $\binom{d}{2}$ critical points with critical value 0. The number of points with critical value 8 is $\frac{d(d-3)}{6}$ if $d = 0$ mod 3, and $\frac{(d-1)(d-2)}{6}$ otherwise. The number of critical points with critical value -1 is $\frac{d^2}{3} - d + 1$ for $d = 0$ mod 3, and $\frac{(d-1)(d-2)}{3}$ otherwise.*

In [13, Lemma 4.2] we have shown that

$$\mathscr{J}_d(x,y) := \widehat{J}_{d,0}\left(x,\frac{y}{\sqrt{3}}\right) \tag{11}$$

are defined over $\mathbb{Q}$, and also that we can get the Chebyshev polynomials from

$$\mathscr{J}_d(z,0) = -2T_d\left(\frac{z-1}{2}\right) + 1. \tag{12}$$

There are families of degree d nodal hypersurfaces, defined over $\mathbb{Q}$, with affine equations

$$\mathscr{J}_d(x,y) + \frac{1}{4}\left(3 - \mathscr{J}_d(2z+1,0)\right) = 0 \tag{13}$$

with $(x,y,z) \in \mathbb{R}^3$ and

$$\mathscr{J}_d(x,y) - \mathscr{J}_d(z,w) = 0 \tag{14}$$

with $(x,y,z,w) \in \mathbb{R}^4$ having many singularities as shown in [13, Theorem 4.3].

Nodal hypersurfaces obtained by generalising (13) and (14) have been studied from the point of view of their invariants and projective rigidity in [14]. Related to $\mathscr{J}_d$ are also certain maximising curves presented in [15]. The proof of [13, Theorem 4.3] is based on the characterisation of the critical points of $\mathscr{J}_d$. This analysis, given in Lemma 4, together with the results in Section 2 leads to the following.

Proposition 5. *The degree $d = 3q$ surfaces with affine equations*

$$\mathcal{J}_d(u,v) + \frac{\mathcal{B}_{d,v,\epsilon}^{(t)}(w) + 1}{2} = 0, \tag{15}$$

where $(u,v,w) \in \mathbb{C}^3$ and $s = k, b, p$ for $t = 1, 2, 3$ respectively, have

$$\frac{d(d-1)(s-1)}{2} + \frac{d(d-3)}{3} + 1 \tag{16}$$

singularities of type A_v.

Proof. We denote by $N_\zeta(\mathcal{J})$ the number of critical points of $\mathcal{J}_d(u,v)$ with critical value ζ and by $N_\zeta(\mathcal{B}, \rho)$ the number of critical points of $\mathcal{B}_{d,v,\epsilon}^{(t)}(w)$ with critical value ζ and multiplicity ρ. Then the number of singularities of type A_v of the surfaces described by (15) is $N_0(\mathcal{J})N_{-1}(\mathcal{B}, v) + N_{-1}(\mathcal{J})N_1(\mathcal{B}, v)$.

According to Lemma 4, for $d = 3q + \alpha$, $\alpha \in \{0, 1, 2\}$, $p \in \mathbb{Z}^+$, the polynomial $\mathcal{J}_d(u,v)$ has $N_0(\mathcal{J}) = \binom{d}{2}$ critical points with critical value $\zeta = 0$, $N_8(\mathcal{J}) = (d^2 - 3d + 2\lceil \frac{\alpha}{2} \rceil)/6$ critical points with $\zeta = 8$, and $N_{-1}(\mathcal{J}) = (d^2 - \lceil \frac{\alpha}{2} \rceil)/3 - d + 1$ critical points with $\zeta = -1$ [13].

The derivatives of $\mathcal{B}_{d,v,\epsilon}^{(t)}(w)$ ($s = k, b, p$ for $t = 1, 2, 3$ respectively) are

$$\frac{d\mathcal{B}_{d,v,\epsilon}^{(t)}(w)}{dw} = (w - w_0)^v (w - w_u)^\epsilon \prod_{l=1}^{s-1} (w - w_l)^v \tag{17}$$

with $\mathcal{B}_{d,v,\epsilon}^{(t)}(w_l) = -1$, $l = 1, 2, \ldots, s-1$, $\mathcal{B}_{d,v,\epsilon}^{(t)}(w_0) = 1$ and $\mathcal{B}_{d,v,\epsilon}^{(t)}(w_u) = -1$ if $\epsilon \neq 0$. Considering the results of Lemmas 1, 2 and 3 we have $N_{-1}(\mathcal{B}, v) = s - 1$ for $s = k, b, p$ and $N_1(\mathcal{B}, v) = 1$. On the other hand, the number of critical points of $\mathcal{J}_d$ for $d = 3q$ is $N_0(\mathcal{J}) = \frac{d(d-1)}{2}$, $N_8(\mathcal{J}) = \frac{d(d-3)}{6}$, $N_{-1}(\mathcal{J}) = \frac{d(d-3)}{3} + 1$, hence, we see that the number of type A_v singularities of the surfaces with affine equations $\mathcal{J}_d(u,v) + (\mathcal{B}_{d,v,\epsilon}^{(t)}(w) + 1)/2 = 0$ is given by (16). In addition, when $\epsilon \neq 0$, they also have $\frac{d(d-1)}{2}$ singularities of type A_ϵ, because $N_{-1}(\mathcal{B}, \epsilon) = 1$ and $N_1(\mathcal{B}, \epsilon) = 0$. $\qquad\square$

The lower bounds for the maximal number of A_v singularities given in [21] are improved slightly. In that paper it is shown that there exists a polynomial $M_d^j(w)$ of degree d with $\lfloor \frac{d}{j+1} \rfloor$ critical points of multiplicity j with critical value -1 and $\lfloor \frac{d-1}{j} \rfloor - \lfloor \frac{d}{j+1} \rfloor$ critical points with critical value $+1$. If $F_d^{\mathbf{A}_2}(u,v)$ denotes the folding polynomial associated to the root lattice $\mathbf{A}_2$ obtained in [17,27], then the surfaces with affine equations $F_d^{\mathbf{A}_2}(u,v) + M_d^j(w) = 0$ have $\frac{d(d-1)}{2} \lfloor \frac{d}{j+1} \rfloor + \frac{d(d-3)}{3}(\lfloor \frac{d-1}{j} \rfloor - \lfloor \frac{d}{j+1} \rfloor)$ singularities of type A_j when $d \equiv 0 \pmod 3$.

The polynomials $\mathcal{B}_{d,v,\epsilon}^{(t)}(w)$, $t = 1, 2, 3$, have degrees d and multiplicities v satisfying

$$\left\lfloor \frac{d}{v+1} \right\rfloor = s - 1, \qquad \left\lfloor \frac{d-1}{v} \right\rfloor = s \tag{18}$$

for $s = k, b, p$, hence the surfaces $\mathcal{J}_d(u,v) + (\mathcal{B}_{d,v,\epsilon}^{(t)}(w) + 1)/2$ have one more A_v singularity than $F_d^{\mathbf{A}_2}(u,v) + M_d^v(w)$.

4. Some explicit expressions in terms of classical Jacobi polynomials

In general Belyi polynomials can be computed only for low degree by using Groebner basis (see [10,11] for some examples by using *Singular* computing tool [16]). In this section we analyse some cases that can be obtained in terms of classical Jacobi polynomials $P_k^{l,m}(z)$ [1,24]. We extend the results in [11] by considering the three-parameter family of degree $d = a + (b-1)c$ polynomials

$$G_{a,b,c}(w) = w^a P_{b-1}^{a/c,-b}(1 - 2w)^c. \tag{19}$$

One of the zeros of $\frac{dG_{a,b,c}}{dw}$ with critical value $\zeta = 0$ has order $v = a-1$, there are also $b-1$ zeros with critical value $\zeta = 0$ and $v = c-1$, while the unique remaining root $w = 1$ has $v = b-1$ with critical value $\zeta = 1$. The general form of the trees for G can be seen in Figure 4-right. The polynomials used in this work have $c = b$. We now show that some polynomials in Lemmas 1 and 2 can be described in terms of $G_{a,b,c}(w)$. No explicit expressions have been found for $\mathscr{B}_{d,v,\epsilon}^{(3)}(w)$ in terms of classical Jacobi polynomials (see Figure 4).

For the polynomials in Lemma 1 corresponding to $n = 0$ in (4) we have that

$$\left(\mathscr{B}_{9m^2,\,3m-1}^{(1)}(w)+1\right)\big/2 = G_{3m,\,3m,\,3m}(w) = w^{3m}P_{3m-1}^{1,-3m}(1-2w)^{3m} \tag{20}$$

has $w = 0$ with $\zeta = 0,(v = 3m-1)$, $w = 1$ with $\zeta = 1,(v = 3m-1)$ and $3m-1$ points with $\zeta = 0,(v = 3m-1)$.

Example 6. $G_{3,3,3}(w)$ was already considered in [11, Proposition 2] for constructing a nonic surface with 127 cusps, but the surface is not defined over the rationals, by contrast with $\mathscr{J}_9(u,v) + \left(\mathscr{B}_{9,2}^{(1)}(w)+1\right)\big/2$. The polynomial

$$\left(\mathscr{B}_{36,5}^{(1)}(w)+1\right)\big/2 = G_{6,6,6}(w) = w^6(6-15w+20w^2-15w^3+6w^4-w^5)^6$$

has critical points $w = 0,(v = 5)$, $w = 2,(v = 5)$, $w = \frac{1\pm i\sqrt{3}}{2},(v = 5)$, $w = \frac{3\pm i\sqrt{3}}{2},(v = 5)$, with critical value $\zeta = 0$ and $w = 1,(v = 5)$ with $\zeta = 1$. The surface $\mathscr{J}_{36}(u,v) + \left(\mathscr{B}_{36,5}^{(1)}(w)+1\right)\big/2$ is also defined over $\mathbb{Q}$ and has 4177 complex singularities A_5.

The polynomials corresponding to the initial trees in Lemma 2 can be expressed as

$$\left(\mathscr{B}_{a+b(b-1),\,b-1,\,a-1}^{(2)}(w)+1\right)\big/2 = G_{a,b,b}(w) = w^a P_{b-1}^{a/b,-b}(1-2w)^b \tag{21}$$

for $a = 1,3,4,6,7,9,10,\ldots,3m,3m+1,\ldots$ and $b > a$. In particular:

(1) $\left(\mathscr{B}_{3m+3l(3l-1),\,3l-1,\,3m-1}^{(2)}(w)+1\right)\big/2 = G_{3m,\,3l+1,\,3l+1}(w)$, $m \in \mathbb{Z}^+$, $l = m, m+1, m+2,\ldots$;

(2) $\left(\mathscr{B}_{3m+1+(3l+2)(3l+1),\,3l+1,\,3m}^{(2)}(w)+1\right)\big/2 = G_{3m+1,\,3l+2,\,3l+2}(w)$, $m \in \mathbb{Z}^{\geq 0}$, $l = m, m+1, m+2,\ldots$
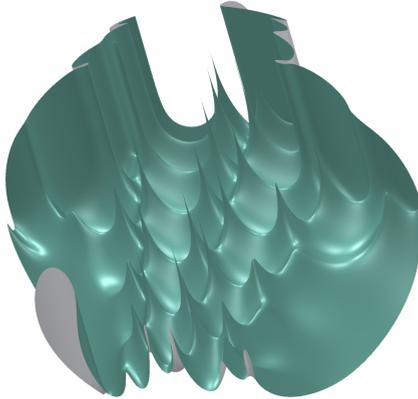

Figure 6. A fragment of the real part of a degree 21 surface with A_4 singularities.

Example 7.

(1) The lower bound $\mu_{A_3}(15) \geq 376$ was obtained in [11]. An explicit expression of $\left(\mathscr{B}_{15,3,2}^{(2)}(w)+1\right)\big/2 = G_{3,4,4}(w)$ is given in the proof of [11, Proposition 4]. The surface is defined over $\mathbb{C}$, whereas $\mathscr{J}_{15}(u,v) + \left(\mathscr{B}_{15,3,2}^{(2)}(w)+1\right)\big/2$ is defined over $\mathbb{Q}$, and both have 376 A_3 and 105 A_2 singularities.

(2) $\left(\mathscr{B}_{3,1,0}^{(2)}(w) + 1\right)/2 = G_{1,2,2}(w)$ is associated with the starting tree of the series with $d = 6n + 3$, $v = 3n + 1$, and the associated surface has the same number of nodes as the Cayley cubic surface. This case is studied in [10, Proposition 3.6], where we derived the lower bound given by (3). Explicit expressions for $\mathscr{B}_{d,1,0}^{(2)}(w)$, $d = 9, 15$, were presented in [10,11] by using Groebner basis, and the surfaces obtained give $\mu_{A_4}^{(\mathbb{R})}(9) \geq 55$ and $\mu_{A_7}^{(\mathbb{R})}(15) \geq 166$. Another particular case is

$$\left(\mathscr{B}_{21,4,0}^{(2)}(w) + 1\right)/2 = G_{1,5,5}(w) = \frac{w}{5^{20}}(924 - 616\,w + 504\,w^2 - 231\,w^3 + 44\,w^4)^5$$

which gives the degree 21 surface $\mathscr{J}_{21}(u, v) + \left(\mathscr{B}_{21,4,0}^{(2)}(w) + 1\right)/2$, with 757 complex singularities A_4. A representation of a fragment of this surface's real part with the *Surfer* visualisation tool can be seen in Figure 6.

Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, vol. 55, U.S. Government Printing Office, 1964.

[2] N. Adrianov and A. Zvonkin, "Composition of plane trees", *Acta Appl. Math.* **52** (1998), no. 1–3, pp. 239–245.

[3] J. Baez, "Visual insight", in *AMS Blogs*. Online at https://blogs.ams.org/visualinsight/ (accessed on February 17, 2026).

[4] W. P. Barth, "Two projective surfaces with many nodes, admitting the symmetries of the icosahedron", *J. Algebr. Geom.* **5** (1996), no. 1, pp. 173–186.

[5] A. Beauville, "Sur le nombre maximum de points doubles d'une surface dans $\mathbf{P}^3$ ($\mu(5) = 31$)", in *Journées de Géometrie Algébrique d'Angers, Juillet 1979* (A. Beauville, ed.), Sijthoof & Noordhoof, 1980, pp. 207–215.

[6] A. Cayley, "A memoir on cubic surfaces", *Philos. Trans. R. Soc. Lond.* **159** (1869), pp. 231–326.

[7] S. V. Chmutov, "Examples of projective surfaces with many singularities", *J. Algebr. Geom.* **1** (1992), no. 2, pp. 191–196.

[8] S. Endrass, "A projective surface of degree eight with 168 nodes", *J. Algebr. Geom.* **6** (1997), no. 2, pp. 325–334.

[9] J. G. Escudero, "On a family of complex algebraic surfaces of degree $3n$", *Comptes Rendus. Mathématique* **351** (2013), no. 17–18, pp. 699–702.

[10] J. G. Escudero, "Arrangements of real lines and surfaces with A and D singularities", *Exp. Math.* **23** (2014), no. 4, pp. 482–491.

[11] J. G. Escudero, "Hypersurfaces with many A_j-singularities: explicit constructions", *J. Comput. Appl. Math.* **259** (2014), pp. 87–94.

[12] J. G. Escudero, "A construction of algebraic surfaces with many real nodes", *Ann. Mat. Pura Appl. (4)* **195** (2016), no. 2, pp. 571–583.

[13] J. G. Escudero, "Singular and Calabi–Yau varieties linked with billiard trajectories and diffusion operators", *Ann. Mat. Pura Appl. (4)* **197** (2018), no. 5, pp. 1613–1629.

[14] J. G. Escudero, "Projective rigidity and Alexander polynomials of certain nodal hypersurfaces", *J. Algebra Appl.* **23** (2024), no. 4, article no. 2450075 (17 pages).

[15] J. G. Escudero, "Special curves and Hamiltonian systems from polynomial eigenfunctions of an elliptic operator", *Rev. Roum. Math. Pures Appl.* **71** (2026), no. 1–2.

[16] G.-M. Greuel and G. Pfister, *A Singular introduction to commutative algebra*, Springer, 2008.

[17] M. E. Hoffman and W. D. Withers, "Generalized Chebyshev polynomials associated with affine Weyl groups", *Trans. Am. Math. Soc.* **308** (1988), no. 1, pp. 91–104.

[18] D. B. Jaffe and D. Ruberman, "A sextic surface cannot have 66 nodes", *J. Algebr. Geom.* **6** (1997), no. 1, pp. 151–168.

[19] E. E. Kummer, "Über die Flächen vierten Grades mit sechzehn singulären Punkten", *Ber. Akad. Wiss. Berlin* **1864** (1864), pp. 246–260.

[20] O. Labs, "A septic with 99 real nodes", *Rend. Semin. Mat. Univ. Padova* **116** (2006), pp. 299–313.

[21] O. Labs, "Dessins d'enfants and hypersurfaces with many A_j-singularities", *J. Lond. Math. Soc. (2)* **74** (2006), no. 3, pp. 607–622.

[22] Y. Miyaoka, "The maximal number of quotient singularities on surfaces with given numerical invariants", *Math. Ann.* **268** (1984), no. 2, pp. 159–171.

[23] A. Sarti, "Pencils of symmetric surfaces in $\mathbb{P}_3$", *J. Algebra* **246** (2001), no. 1, pp. 429–452.

[24] G. Szegő, *Orthogonal polynomials*, 4th edition, Colloquium Publications, vol. XXIII, American Mathematical Society, 1975.

[25] E. G. Togliatti, "Una notevole superficie di 5° ordine con soli punti doppi isolati", *Vierteljschr. Naturforsch. Ges. Zürich* **85** (1940), pp. 127–132.

[26] A. N. Varchenko, "On semicontinuity of the spectrum and an upper estimate for the number of singular points of a projective hypersurface", *Sov. Math., Dokl.* **27** (1983), pp. 735–739.

[27] W. D. Withers, "Folding polynomials and their dynamics", *Am. Math. Mon.* **95** (1988), no. 5, pp. 399–413.