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Research article
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Minimizing solutions of degenerate vector Allen–Cahn equations with three wells in $\mathbb{R}^2$

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Abstract. We characterize all minimizers of the vector-valued Allen–Cahn equation in $\mathbb{R}^2$ under the assumption that the potential W has three wells and that the associated degenerate metric does not satisfy the usual strict triangle inequality. These minimizers depend on one variable only in a suitable coordinate system. In particular, we show that no minimizing solution to $\Delta u = \nabla W(u)$ on $\mathbb{R}^2$ can approach the three distinct values of the potential wells.

Keywords. Elliptic systems, phase transitions, optimal partitions, entire solutions.

2020 Mathematics Subject Classification. 35J47, 35B08.

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1. Introduction

The study of entire solutions $u\colon \mathbb{R}^2 \to \mathbb{R}^2$ to the vector Allen–Cahn system

$$\Delta u = \nabla W(u)$$

with a triple well potential $W\colon \mathbb{R}^2 \to \mathbb{R}$ started with a first important result in [8] and includes the more recent works [5,9,11,17] allowing for general, in particular, non-symmetric W. There is also a growing collection of works in the more general setting of $u\colon \mathbb{R}^n \to \mathbb{R}^m$ and multi-well potentials $W\colon \mathbb{R}^m \to \mathbb{R}$, see [4] and the references therein. A standard assumption in these works is that one has a strict triangle inequality between the potential wells for the degenerate metric $d(p,q)$ defined below in (3). Instead, here we pursue the degenerate case of equality as in (4), and under a set of further generic assumptions on W presented below, we establish a rigidity result on the possible entire, minimizing solutions.

Let us state our assumptions on the potential.

For $W \in C^3\big(\mathbb{R}^2;[0,\infty)\big)$, we assume that

$$\big\{p \in \mathbb{R}^2 : W(p)=0\big\} = P := \{p_1,p_2,p_3\},$$

and we assume non-degeneracy of the potential wells in the sense that

$$\mathrm{D}^2 W(p_\ell) \text{ has two } \textit{distinct, positive} \text{ eigenvalues for } \ell = 1,2,3. \tag{1}$$

Additionally, we assume that for some $M>0$,

$$p \cdot \nabla W(p) \geqslant 0 \quad \text{for } |p| \geqslant M. \tag{2}$$

Next, we introduce the degenerate metric on $\mathbb{R}^2$ which arises in many studies of vector Allen–Cahn:

$$d(p,q) := \inf\left\{\sqrt{2}\int_0^1 W^{1/2}\big(\gamma(t)\big)\big|\gamma'(t)\big|\,\mathrm{d}t : \gamma \in C^1\big([0,1],\mathbb{R}^2\big),\ \gamma(0)=p,\ \gamma(1)=q\right\}. \tag{3}$$

We will assume that, contrary to the usual assumption of a strict triangle inequality, the following *equality* holds:

$$d(p_1,p_3) = d(p_1,p_2) + d(p_2,p_3). \tag{4}$$

This condition comes naturally in models such as tri-block co-polymers and leads to an interesting geometrical phenomenon, see [3]. We will make the generic assumption that the geodesics from p_1 to p_2 and from p_2 to p_3 are *unique*. For the case when there is nonuniqueness of geodesics, see for example the work of [2]. We also assume that the *only* length-minimizing geodesic between p_1 and p_3 goes through p_2.

The reason we make this last assumption is that, although we do not attempt to prove it, we believe that coupled with (4), it gives a set of conditions that are generic, i.e. remain true under small perturbations of W in the space of three-well potentials. On the other hand, if there were another geodesic between p_1 and p_3 that does not go through p_2, we believe that an arbitrarily small perturbation of W could break the equality in (4) into a strict triangle inequality.

Remark 1. The conditions above are clearly satisfied by any potential of the form

$$W(u,v) = w(u) + v^2, \tag{5}$$

where $w\colon \mathbb{R} \to [0,\infty)$ is a smooth function with three nondegenerate zeros $a_1 < a_2 < a_3$. Of course in this case, the problem reduces to a one-dimensional one since minimizing solutions will be $\mathbb{R}$-valued. However, less trivial examples could be imagined, either by perturbing (5), or taking W of the form $W(re^{i\theta}) = w(e^{i\theta}) + (R^2 - r^2)^2$ with R suitably large and w a three-well potential on the unit circle.

We also note that an equivalent variational description of the distances $d(p_i,p_j)$ is given by

$$d(p_i,p_j) = \inf\big\{E(f,\mathbb{R}) : f \in H^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^2),\ f(-\infty)=p_i,\ f(\infty)=p_j\big\}, \tag{6}$$

where

$$E(f,I) := \int_I \left(\frac{1}{2}\big|f'(t)\big|^2 + W\big(f(t)\big)\right)\mathrm{d}t, \quad I \subset \mathbb{R}. \tag{7}$$

Then a minimizer ζ_{ij}, when it exists, is the parametrization of a geodesic joining p_i and p_j. It satisfies the system of ODE's

$$\zeta''_{ij}(t) = \nabla W\big(\zeta_{ij}(t)\big) \quad \text{for } -\infty < t < \infty,\ \zeta_{ij}(-\infty) = p_i,\ \zeta_{ij}(\infty) = p_j. \tag{8}$$

From the perspective of ODE's, these parametrized geodesics ζ_{ij} represent heteroclinic connections between the potential wells.

We have the following result.

Lemma 2. *Under our assumptions on W, there exists a length-minimizing heteroclinic connection between p_1 and p_2 as well as between p_2 and p_3, with respect to the metric d. However, there is no length-minimizing connection between p_1 and p_3.*

Proof. Since (4) implies the strict triangle inequalities

$$d(p_1,p_2) < d(p_1,p_3) + d(p_3,p_2) \qquad \text{and} \qquad d(p_2,p_3) < d(p_2,p_1) + d(p_1,p_3),$$

the existence of a heteroclinic connection between p_1 and p_2 and between p_2 and p_3 is by now standard, with numerous proofs; see, for example, [15, Theorem 3]. The fact that a length-minimizing heteroclinic connection between p_1 and p_3 does not exist follows from our assumption that the only geodesic connecting p_1 to p_3 passes through p_2. Indeed, were such a heteroclinic connection, say ζ, to exist with $\zeta(t^*) = p_2$ for some t^* then integrating (8) one would find that $\frac{1}{2}\left|\zeta'(t^*)\right|^2 = W\big(\zeta(t^*)\big) = W(p_2) = 0$, and so by uniqueness of solutions to (8), necessarily $\zeta \equiv p_2$. □

We make the further assumptions that:

- the heteroclinic connections between p_1 and p_2 and between p_2 and p_3 are unique up to translation of the parameter;
- the only nontrivial $H^1(\mathbb{R})$ solution to the linearized equations $v''(t) = \mathrm{D}^2W(\zeta_{ij})v$ is $v = \zeta'_{ij}$, for $ij = 12$ or 23).

This last hypothesis is formulated in a paper by M. Schatzman [18] and proved there to be generic. We will crucially use [18, Lemma 4.5] in our proof of Lemma 10. We will also need the following assumption, which is also certainly generic, although we will not try to prove this here.

Assumption 3. *As $t \to +\infty$, we have $\zeta'_{12}/|\zeta'_{12}| \to e$ and $\zeta'_{32}/|\zeta'_{32}| \to -e$, where e is an eigenvector of $\mathrm{D}^2W(p_2)$ corresponding to the smallest eigenvalue.*

We remark that from the assumption (1), it follows that each ζ_{ij} approaches its end-states p_i and p_j at an exponential rate, i.e.

$$\left|\zeta_{ij}(s) - p_j\right| \leqslant Ce^{-cs} \quad \text{as } s \to \infty \tag{9}$$

for some constant c depending on W, with a similar estimate holding as $s \to -\infty$. (See Section 2 for details.)

Next, for any bounded open $\Omega \subset \mathbb{R}^2$ and any $u \in H^1(\Omega,\mathbb{R}^2)$, we define

$$E(u,\Omega) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + W(u)\right)\mathrm{d}x\,\mathrm{d}y. \tag{10}$$

We say that $u\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a minimizer of E if it minimizes E with respect to its own boundary conditions on any bounded domain $\Omega \subset \mathbb{R}^2$. We remark that such an entire solution is alternatively referred to as a local minimizer (in the sense of De Giorgi) in much of the literature.

Our main result is the following.

Theorem 4. *Assume $u\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a nonconstant minimizer of E. Then, possibly after rotating and translating the coordinates, either $u(x,y) \equiv \zeta_{12}(x)$ or $u(x,y) \equiv \zeta_{23}(x)$.*

Our article is organized as follows: in Section 2, we present some preliminary results, in particular we show that, up to a translation and/or rotation, the only possible blowdown limits of a nonconstant minimizer u are the piecewise constant functions H_{12}, H_{23} and H_{13}, where

$$H_{ij}(x,y) := p_i\mathbf{1}_{\{x<0\}}(x,y) + p_j\mathbf{1}_{\{x>0\}}(x,y). \tag{11}$$

A result of [17] shows that if, up to translation and rotation, the blowdown limits of u coincide with those of either ζ_{12} or ζ_{23}, namely H_{12} or H_{23} respectively, then u itself must be equal to ζ_{12} or ζ_{23} (up to the invariances of the problem). The proof then consists of ruling out the possibility that the blowdown limit is H_{13}.

The crucial fact we use is that under our hypothesis on W, for $u\colon \mathbb{R} \to \mathbb{R}^2$ to transition optimally from p_1 to p_3, there must be two transitions: one from p_1 to p_2 and one from p_2 to p_3, but these two transitions must also be as far apart as possible. We make this statement precise in Section 3, where we study the structure of almost minimizers in 1D. In Section 4, we obtain a precise upper bound for the energy of minimizers on the ball B_R for $R \gg 1$, which exhibit nearly optimal p_1-p_3 transitions on the boundary. Finally, in Section 5, we prove a lower bound for the energy on B_R of a minimizer whose blowdown is H_{13}, and observe that this lower bound is incompatible with the upper bound we computed before, and hence conclude the proof of the theorem.

2. Preliminaries and background

We begin with some preliminary results on the behavior of a geodesic γ connecting a point a to a well $p \in P$. Though, of course, a distance-minimizing geodesic in the plane satisfies a second order system of ODE's, one can alternatively express its dynamics as a gradient flow with respect to the distance d given by (3). Thus, we have

$$\gamma(0) = a, \qquad \gamma(+\infty) = p, \qquad \text{and} \qquad \gamma'(t) = -\nabla d_p\big(\gamma(t)\big), \tag{12}$$

where $d_p(\cdot) = d(p,\cdot)$. In the case where W is a quadratic in a neighborhood of p, that is, where

$$W(q) \equiv \lambda_1 (q^{(1)} - p^{(1)})^2 + \lambda_2 (q^{(2)} - p^{(2)})^2$$

in an orthonormal coordinate system given by the eigenvectors of $D^2 W(p)$, it is shown in [1, Section 2.1] that d_p is given by

$$d_p(q) = \frac{1}{2}\big(\sqrt{\lambda_1}(q^{(1)} - p^{(1)})^2 + \sqrt{\lambda_2}(q^{(2)} - p^{(2)})^2\big),$$

so that in the purely quadratic case, with say $p = 0$, one has

$$(\gamma^{(1)}, \gamma^{(2)})' = -\big(\sqrt{\lambda_1}\gamma^{(1)}, \sqrt{\lambda_2}\gamma^{(2)}\big). \tag{13}$$

Therefore if p is a rest point of the ODE (12), then the linearization of the ODE at p is necessarily (13), and consequently, it is a contraction. It then follows from [13] that for some $\eta > 0$, there exists a C^1 diffeomorphism h_p from $B_\eta(p)$ to a neighborhood of $0 \in \mathbb{R}^2$ such that if $\gamma' = -\nabla d_p(\gamma)$, with γ taking values in $B_\eta(p)$, then $X = h \circ \gamma$ solves $X' = AX$ with $A := -\mathrm{diag}\big[\sqrt{\lambda_1}, \sqrt{\lambda_2}\big]$.

Note that this result does not hold in $\mathbb{R}^n$ for $n > 2$ (see [13]), however in that case there still exists a homeomorphism which is differentiable at p, while its inverse is differentiable at 0. This result is proved in [21] for discrete dynamical systems but, as mentioned in [13], using [19, Lemma 4], this carries over to smooth dynamical systems.

We thus have the following result.

Lemma 5. *Assume that $W : \mathbb{R}^2 \to \mathbb{R}_+$ is C^2 and that $\nabla W(p) = 0$, $D^2 W(p) > 0$. Then there exist $\eta > 0$ and a C^1-diffeomorphism h from $U = B_\eta(p)$ to a neighborhood of $0 \in \mathbb{R}^2$ such that $Dh(p) = \mathrm{Id}$ and such that if $\gamma : \mathbb{R}_+ \to U$ is a geodesic with the property that $\gamma(t) \to p$ as $t \to +\infty$, then $h \circ u = X$, where X solves $X' = -\mathrm{diag}\big[\sqrt{\lambda_1}, \sqrt{\lambda_2}\big]X$.*

In particular, choosing orthonormal coordinates (x, y) in which $p = 0$ and $\frac{1}{2} D^2 W(p) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, and assuming $0 < \lambda_1 < \lambda_2$, it follows that if $\gamma(0) = (x_0, y_0)$, then $\gamma(t) \approx (x_0 e^{-\sqrt{\lambda_1} t}, y_0 e^{-\sqrt{\lambda_2} t})$ as $t \to +\infty$.

The property (9) also follows from this lemma.

Next, we note that even though minimizing (7) fixes a parametrization of the geodesic, there is still translation invariance. We may remove this invariance and then unambiguously define $\zeta_{ij} : \mathbb{R} \to \mathbb{R}^2$ by requiring that

$$d\big(p_i, \zeta_{ij}(0)\big) = d\big(p_j, \zeta_{ij}(0)\big). \tag{14}$$

From now on we will write

$$d_{ij} = d(p_i, p_j). \tag{15}$$

We now summarize the well-known Gamma-convergence properties of the rescaled energy $E_R(u,\Omega)$ on a bounded smooth domain Ω where

$$E_R(u,\Omega) := \int_\Omega \left(\frac{1}{2R}|\nabla u|^2 + RW(u)\right) \mathrm{d}x\,\mathrm{d}y. \tag{16}$$

Then the L^1 Gamma-limit of the sequence $\{E_R\}$ is the functional E_0 given by

$$E_0(u,\Omega) := \sum_{1\leqslant i<j\leqslant 3} d_{ij}\mathscr{H}^1\big(\partial^* S_i \cap \partial^* S_j \cap \Omega\big), \tag{17}$$

defined on $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^2,P)$, where $P=\{p_1,p_2,p_3\}$, where $S_j := u^{-1}(p_j)$ for $j=1,2,3$, and $\partial^* S$ refers to the reduced boundary of a set S of finite perimeter, cf. [12]. This follows from [6] in the absence of boundary conditions, and from [10] for the case of Dirichlet boundary conditions. Note that if u is a minimizer of E_0, and in the case were the triangle inequality is strict for the distance between the wells, the interface between the sets S_j consists of line segments that can meet at triple junctions, the angles they make are determined through stationarity by the d_{ij}'s and are nonzero.

Next, for any $R>0$, and any $u\colon \mathbb{R}^2\to\mathbb{R}^2$, we introduce the notation $u_R = u(R\,\cdot)$ to denote the *blowdown* of the function u. One can then readily check that the relation $E(u,R\Omega) = RE_R(u_R,\Omega)$ holds. We will make use of the following well-known fact in the proof of the proposition to follow.

Remark 6. If $u\colon \mathbb{R}^2\to\mathbb{R}^2$ is a minimizer of E, and if $u_R\to u_0$ in L^1_{loc}, where $u_0\in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^2,P)$, then u_R converges locally uniformly to p_j in the interior of each $S_j := u_0^{-1}(p_j)$. See e.g. [17, Proposition 4.2].

Recalling the definition of H_{ij} given in (11), we have the following characterization of the possible limit of blowdowns.

Proposition 7. *Assume $u\colon \mathbb{R}^2\to\mathbb{R}^2$ is a non-constant minimizer of E, then for any sequence $\{R_n\}$ of radii tending to $+\infty$, there exists a subsequence, still denoted $\{R_n\}$, such that, possibly after rotating the coordinates, $u(R_n\,\cdot)$ converges to H_{ij} in L^1_{loc}, for some $1\le i\neq j\le 3$.*

Moreover, for any bounded open $\Omega\subset\mathbb{R}^2$, it holds that

$$E_{R_n}\big(u(R_n\,\cdot),\Omega\big) = E_0(H_{ij},\Omega) + o(1).$$

Proof. Applying [17, Proposition 3.1], which in particular does not rely on a strict triangle inequality, we have that for any sequence $\{R_n\}\to\infty$, there exists a subsequence, still denoted by $\{R_n\}$, and a function $u_0\in \mathrm{BV}_{\mathrm{loc}}\big(\mathbb{R}^2;P\big)$ such that the blowdowns $\{u_{R_n}\}$ of u satisfy

$$u_{R_n} \longrightarrow u_0 \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2). \tag{18}$$

Furthermore, u_0 is a minimizer of E_0 and in light of the Gamma-convergence of E_R to E_0, we have that

$$E_{R_n}\big(u(R_n\,\cdot),\Omega\big) = E_0(u_0,\Omega) + o(1) \tag{19}$$

for any bounded open $\Omega\subset\mathbb{R}^2$.

We first eliminate the possibility that $u_0 = p_j$ for some $j\in\{1,2,3\}$. Suppose, by way of contradiction, that say $u_0 = p_1$. Then by (19), we have that $E_{R_n}(u_{R_n},B_1) = o(1)$ as $n\to\infty$, or equivalently, $E(u,B_{R_n}) = o(R)$. Hence, an application of the Mean Value Theorem leads to a sequence of radii, say $\rho_n\to\infty$, such that $E(u,\partial B_{\rho_n})\to 0$. Also, through an appeal to [17, Proposition 4.2], we have that

$$u_{\rho_n} \longrightarrow p_1 \quad \text{locally uniformly on } \mathbb{R}^2,$$

which implies, in particular, that

$$\sup_{(x,y)\in\partial B_{\rho_n}} \big|u(x,y) - p_1\big| = o(1). \tag{20}$$

With (19) and (20) in hand, we may construct a low-energy competitor on the ball B_{ρ_n} by linearly interpolating between the values of u on ∂B_{ρ_n} and p_1 on ∂B_{ρ_n-1}. A straightforward estimation reveals that the cost on this annulus is $o(1)$. (One can view the estimates in [17, pp. 4179–4181] for a similar estimation in a more complicated situation.) Then filling in the ball B_{ρ_n-1} with the value p_1 we have shown that the minimizer u satisfies $E(u, B_{\rho_n}) = o(1)$. Hence, in fact $E(u, \mathbb{R}^2) = 0$ and necessarily $u \equiv p_1$, contradicting the assumption that u is nonconstant.

Next we appeal to the regularity result for minimizing partitions of the plane. Appealing to the result [20, Theorem 3], it is argued in [16, Theorem 4.3] that even in the absence of a strict triangle inequality for, say, the cost $d(p_1, p_3)$, the phase boundaries of any minimizing partition of the plane consist of a union of line segments joined at triple junctions. What is more, in light of (4), there can be no junctions at all since stationarity at such a junction would require the phase corresponding to p_3 to be enclosed with an angle of size 0. It follows that the only possibility for a minimizing partition would be a collection of parallel strips — that is, phase boundaries consisting of parallel lines.

We may assume without loss of generality that these lines are vertical, so that $u_0(x, y) = u_0(x)$. It is then not difficult to argue by constructing competitors that u_0, as a function of one variable, must be a minimizer for the one-dimensional transition energy. Then a simple examination of this problem shows that if u_0 has more than one transition, then the only possibility is that it transitions from p_1 to p_2 and then from p_2 to p_3, either from left-to-right or from right-to-left.

Also, going back to the 2-dimensional picture, if u_0 is not a constant, then one of the transition lines must go through the origin, for otherwise we would have $u_0 \equiv p_i$ in $B(0, \eta)$, which would imply, from the uniform convergence of u_{R_n} to u_0 on $B(0, \eta)$, that $u \equiv p_i$.

Thus, with no loss of generality, suppose the phase boundaries are vertical and that the y-axis separates the p_1-phase from the p_2-phase. Then there are numbers $a < 0$ and $b > 0$ such that $u_0 = p_1$ for $a < x < 0$ and $u_0 = p_2$ for $0 < x < b$ with $i \neq k$. Let $\{\eta_j\} \to 0$ be an arbitrary sequence of positive numbers. Then for each j there exists a value R_{n_j} (which we denote simply by R_j) such that $\eta_j^{1/3} R_j \to \infty$ and such that

$$\| u_{R_j} - p_1 \|_{L^1([a/2, -\eta_j] \times [-1/\eta_j, 1/\eta_j])} < \eta_j \tag{21}$$

and

$$\| u_{R_j} - p_2 \|_{L^1([\eta_j, b/2] \times [-1/\eta_j, 1/\eta_j])} < \eta_j \tag{22}$$

But letting $\tilde{x} = x/\eta_j^{1/3}$ and $\tilde{y} = y/\eta_j^{1/3}$ these two conditions are equivalent to the conditions

$$\| u_{\eta_j^{1/3} R_j} - p_1 \|_{L^1([a/2\eta_j^{1/3}, -\eta_j^{2/3}] \times [-1/\eta_j^{4/3}, 1/\eta_j^{4/3}])} < \eta_j^{1/3} \tag{23}$$

and

$$\| u_{\eta_j^{1/3} R_j} - p_2 \|_{L^1([\eta_j^{2/3}, b/2\eta_j^{1/3}] \times [-1/\eta_j^{4/3}, 1/\eta_j^{4/3}])} < \eta_j^{1/3}. \tag{24}$$

In other words, along the sequence $\{\eta_j^{1/3} R_j\} \to \infty$ the blowdowns converge in L^1_{loc} to H_{12}. Consequently, we may apply [17, Theorem 3.1] to conclude that $u \equiv \zeta_{12}$ up to a translation. But this contradicts the assumption that the blowdown consists of more than one phase boundary. Of course, the same argument would apply if phases p_2 and p_3 were adjacent. □

Ultimately, we will show that the blowdown limits can only be H_{12} or H_{23}, not H_{13} and then that the only minimizers are ζ_{12} and ζ_{23}, up to the invariances of E, thus proving Theorem 4.

3. Structure of almost minimizers in 1D

Lemma 8. *Assume* 0 *is a nondegenerate minimizer of* $W: \mathbb{R}^2 \to \mathbb{R}_+$. *Then there exist* $\eta > 0$ *and* $R_0, C, c > 0$ *such that if* $R > R_0$, *the following holds: for any* $u: [0,R] \to \mathbb{R}^2$ *such that* $u(0) = a$, *with* a *such that* $d(0,a) \le \eta$ *it holds that*

$$E(u) \ge d(0,a) - Ce^{-cR}.$$

The proof uses a strategy suggested to us by A. Monteil [14], which is used again in the next lemma.

Proof. First we note that we may assume that u minimizes the energy on $[0,R]$ with respect to the initial condition $u(0) = a$. In particular, choosing $\eta > 0$ sufficiently small so that W is convex on the set of points p satisfying $d(0,p) \le \eta$, we may assume that $d\big(u(t),a\big) \le \eta$ for any t. Let $\gamma_a: [0,+\infty) \to \mathbb{R}^2$ be the distance minimizing geodesic from a to 0, parametrized so that $E(\gamma_a) = d(a,0)$. Let also $p = u(R)$ and $q = \gamma_a(R)$.

We define $\tilde u: \mathbb{R}_+ \to \mathbb{R}^2$ by $\tilde u(t) = u(t)$ for $t \in [0,R]$ and $\tilde u(t) = \gamma_p(t-R)$ for $t \ge R$, where $\gamma_p: [0,+\infty) \to \mathbb{R}^2$ is the distance minimizing geodesic from p to 0 such that $E(\gamma_p) = d(p,0)$.

The maps $\tilde u$ and γ_a have the same boundary conditions on $[0,+\infty]$, therefore, letting Q be the quadratic form associated to $\frac12 \mathrm{D}^2 W(0)$, we have

$$\begin{aligned} E(\tilde u) - E(\gamma_a) &= \int_0^{+\infty} \left(\frac12 |\tilde u'|^2 + W(\tilde u)\right) - \left(\frac12 |\gamma_a'|^2 + W(\gamma_a)\right) \mathrm{d}t \\ &\ge \int_0^{+\infty} \frac12 \big(|\tilde u'|^2 - |\gamma_a'|^2\big) + \mathrm{D}W(\gamma_a)(\tilde u - \gamma_a) + (1 - C\eta) Q(\tilde u - \gamma_a)\,\mathrm{d}t \\ &= \int_0^{+\infty} \gamma_a' \cdot (\tilde u - \gamma_a)' + \frac12 \big|(\tilde u - \gamma_a)'\big|^2 + \mathrm{D}W(\gamma_a)(\tilde u - \gamma_a) \\ &\quad + (1 - C\eta) Q(\tilde u - \gamma_a)\,\mathrm{d}t \\ &= \int_0^{+\infty} \frac12 \big|(\tilde u - \gamma_a)'\big|^2 + (1 - C\eta) Q(\tilde u - \gamma_a)\,\mathrm{d}t, \end{aligned} \tag{25}$$

where we have used the fact that $\gamma_a'' = \mathrm{D}W(\gamma_a)$. Therefore (and recalling that C denotes a constant that may change from line to line),

$$E(\tilde u) - d(a,0) \ge \sqrt{1 - C\eta}\,\sqrt{2} \int_0^{+\infty} \sqrt{Q(\tilde u - \gamma_a)}\,\big|(\tilde u - \gamma_a)'\big|\,\mathrm{d}t.$$

We now introduce the distance associated to the form Q, which we denote by d_Q, defined by replacing W by Q in (3). Then the inequality above can be phrased as

$$E(\tilde u) - d(a,0) \ge 2\sqrt{1 - C\eta}\, d_Q(0, p - q),$$

where the factor of 2 emerges since the path $\tilde u - \gamma_a$ goes from 0 to $u(R) - \gamma_a(R) = p - q$ and then returns. Since $E(\tilde u) = E(u) + d(p,0)$ and $d(a,0) = d(a,q) + d(q,0)$ we deduce that

$$E(u) \ge d(a,q) + d(q,0) - d(p,0) + 2\sqrt{1 - C\eta}\, d_Q(0, p - q). \tag{26}$$

Now (see [1, Remark 2.4]) we have that $x \to \sqrt{d_Q(x,0)}$ is a norm on $\mathbb{R}^2$, that we denote $\|x\|_Q$, therefore

$$d_Q(0, p - q) = \|p - q\|_Q^2 \ge d_Q(0,p) + d_Q(0,q) - 2\|p\|_Q \|q\|_Q \ge d_Q(p,q) - 2\|p\|_Q \|q\|_Q.$$

Together with (26), and using the fact that $d_Q \ge (1 - C\eta) d$ this implies that

$$E(u) \ge d(a,q) + d(q,0) - d(p,0) + d(p,q) + (1 - C\eta)\|p - q\|_Q^2 - 4\|p\|_Q \|q\|_Q.$$

and therefore,

$$E(u) \ge d(a,q) + (1 - C\eta)\|p - q\|_Q^2 - 4\|p\|_Q \|q\|_Q. \tag{27}$$

Now consider η chosen sufficiently small such that $C\eta < 1/2$. Then we claim that (27) implies the inequality

$$E(u) \geq d(a,q) - C\|q\|_Q^2.$$

Indeed if, say $\|p\|_Q > M\|q\|_Q$ for some large enough M, then in fact (27) would imply that $E(u) \geq d(a,q)$ since the middle term on the right-hand side of (27) would then dominate the last term. On the other hand, if $\|p\|_Q \leqslant M\|q\|_Q$, then one has the claim with $C = 4M$ by ignoring the middle term on the right-hand side of (27).

Thus,

$$E(u) \geq d(a,q) - Cd(0,q) = d(0,a) - d(0,q) - C\|q\|_Q^2 = d(0,a) - Cd(0,q).$$

Since $\gamma_a(t)$ converges to 0 exponentially fast as $t \to +\infty$, we find that $d(0,q) = d\big(0,\gamma_a(R)\big) \leq Ce^{-cR}$, and so

$$E(u) \geq d(0,a) - Ce^{-cR}. \qquad \square$$

Lemma 9. *Assume* 0 *is a nondegenerate minimizer of* $W\colon \mathbb{R}^2 \to \mathbb{R}_+$ *and choose coordinates* (x,y) *such that* $D^2W(0) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ *with* $0 < \lambda_1 < \lambda_2$ *in these coordinates. Let d be the distance associated to W as in* (3)*. For* $\eta > 0$ *and* $\varepsilon > 0$ *sufficiently small depending on W, if* $f \in H^1\big([-R,R],\mathbb{R}^2\big)$ *satisfies* $f(-R) = a$ *and* $f(R) = b$ *with* $a = (a_1,a_2)$ *and* $b = (b_1,b_2)$*, where* $d(a,0) = \eta = d(b,0)$, $a_1 < 0$, $b_1 > 0$, *and* $|a_2|,|b_2| \leq \varepsilon|a_1|,\varepsilon|b_1|$*, then one has*

$$E\big(f,[-R,R]\big) \geq 2\eta + c\Big(\min_{t\in[-R,R]} \big|f(t) - p_2\big|^2 + e^{-CR}\Big), \tag{28}$$

for some $c,C > 0$ *independent of* R.

Proof. Let $f \in H^1\big([-R,R],\mathbb{R}^2\big)$ be a minimizer of E from (7) with $I = [-R,R]$ under the boundary conditions $f(-R) = a$ and $f(R) = b$. Then for η sufficiently small depending on W, we have $d\big(f(t),0\big) \leq \eta$ for any $t \in (-R,R)$, and furthermore there exists $t_0 \in (-R,R)$ such that $f(t_0) = q := (0,q_2)$, since $a_1b_1 < 0$.

We denote by $\gamma_p\colon [0,+\infty) \to \mathbb{R}^2$ the distance minimizing geodesic from any point p to 0, parametrized so that $E\big(\gamma_p,[0,+\infty)\big) = d(p,0)$. We define $f_a\colon [-R,+\infty) \to \mathbb{R}^2$ by requiring that

$$f_a(t) = \begin{cases} f(t) & \text{if } t \in [-R,t_0], \\ \gamma_q(t-t_0) & \text{if } t \in [t_0,+\infty). \end{cases}$$

The maps f_a and $\widetilde{\gamma}_a := \gamma_a(\cdot + R)$ have the same boundary conditions on $[-R,+\infty]$, therefore

$$\begin{aligned} &E\big(f_a,[-R,\infty)\big) - E\big(\gamma_a(\cdot+R),[-R,\infty)\big) \\ &\qquad = \int_{-R}^{+\infty} \Big(\frac{1}{2}|f_a'|^2 + W(f_a)\Big) - \Big(\frac{1}{2}|\widetilde{\gamma}_a'|^2 + W(\widetilde{\gamma}_a)\Big)\,\mathrm{d}t \\ &\qquad \geq \int_{-R}^{+\infty} \frac{1}{2}\big(|f_a'|^2 - |\widetilde{\gamma}_a'|^2\big) + \mathrm{D}W(\widetilde{\gamma}_a)(f_a - \widetilde{\gamma}_a) + (1-C\eta)Q(f_a - \widetilde{\gamma}_a)\,\mathrm{d}t \\ &\qquad = \int_{-R}^{+\infty} \widetilde{\gamma}_a' \cdot (f_a - \widetilde{\gamma}_a)' + \frac{1}{2}\big|(f_a - \widetilde{\gamma}_a)'\big|^2 + \mathrm{D}W(\widetilde{\gamma}_a)(f_a - \widetilde{\gamma}_a) \\ &\qquad\quad + (1-C\eta)Q(f_a - \widetilde{\gamma}_a)\,\mathrm{d}t \\ &\qquad = \int_{-R}^{+\infty} \frac{1}{2}\big|(f_a - \widetilde{\gamma}_a)'\big|^2 + (1-C\eta)Q(f_a - \widetilde{\gamma}_a)\,\mathrm{d}t, \end{aligned} \tag{29}$$

where Q is the quadratic form associated to $\frac{1}{2}\mathrm{D}^2W(0)$, and where we have used the fact that $\widetilde{\gamma}_a'' = \mathrm{D}W(\widetilde{\gamma}_a)$.

Therefore

$$E\big(f_a,[-R,\infty)\big) - d(a,0) \geq \sqrt{2}(1-C\eta)\int_{-R}^{+\infty} \sqrt{Q(f_a - \widetilde{\gamma}_a)}\big|(f_a - \widetilde{\gamma}_a)'\big|\,\mathrm{d}t.$$

We now recall the distance associated to the form Q, which we denote by d_Q, analogously to (3). Then the inequality above can be phrased as

$$E(f_a) - d(a,0) \geq 2(1-C\eta)d_Q\big(0, q-\tilde{\gamma}_a(t_0)\big), \tag{30}$$

where the factor of 2 emerges since the path $f_a - \tilde{\gamma}_a$ goes from 0 to $q - \tilde{\gamma}_a(t_0)$ and then returns. Denoting the coordinates of $\tilde{\gamma}_a(t_0)$ by (α_1, α_2), one has that

$$d_Q\big(0, q-\tilde{\gamma}_a(t_0)\big) = \frac{1}{\sqrt{2}}\big(\sqrt{\lambda_1}\alpha_1^2 + \sqrt{\lambda_2}(q_2-\alpha_2)^2\big) \quad \text{while} \quad d_Q(0,q) = \frac{1}{\sqrt{2}}\sqrt{\lambda_2}\, q_2^2 \tag{31}$$

(again, see [1, Remark 2.4]). Moreover, from Lemma 5 and in view of the hypothesis $|a_2| \leq \varepsilon|a_1|$, it holds that $|\alpha_2| \leq C\varepsilon|\alpha_1|$. Since $0 < \lambda_1 < \lambda_2$ and since $d(0,q) \leq (1+C\eta)d_Q(0,q)$, one can then use the inequalities (30) and (31) to find that for η and ε chosen small enough

$$2(1-C\eta)d_Q\big(0, q-\tilde{\gamma}_a(t_0)\big) - d(0,q) \geq c(\alpha_1^2 + q_2^2) \geq c\big(d\big(0,\tilde{\gamma}_a(t_0)\big) + \big|f(t_0)-p\big|^2\big),$$

by considering separately the case where $|q_2|$ is negligible compared with $|\alpha_2|$ and where it is not, and using that $|\alpha_2| \leq C\varepsilon|\alpha_1|$ in the second inequality. Here c is a positive constant depending on W.

Inserting this inequality into (30) we obtain

$$E(f_a) = E\big(f,[-R,t_0]\big) + d(0,q) \geq d(0,a) + d(0,q) + c\big(d\big(0,\tilde{\gamma}_a(t_0)\big) + \big|f(t_0)-0\big|^2\big). \tag{32}$$

Since $t_0 + R \leq 2R$, we have, using Lemma 5, that $d\big(0,\tilde{\gamma}_a(t_0)\big) \geq ce^{-CR}$ and therefore,

$$E\big(f,[-R,t_0]\big) \geq d(0,a) + c\big(\big|f(t_0)-p\big|^2 + e^{-CR}\big).$$

By a similar argument we have the inequality

$$E\big(f,[t_0,R]\big) \geq d(0,b) + c\big(\big|f(t_0)-p\big|^2 + e^{-CR}\big).$$

Adding these two inequalities proves the lemma. □

Lemma 10. *There exist $\eta > 0$ and $C > 0$ such that for any $\gamma > 0$ small enough and any $R > 0$ large enough the following holds.*

Assume $f\colon [-R,R] \to \mathbb{R}^2$ is such that:

- *for every $t \in [-R,-R/2]$, $\big|f(t)-p_1\big| \leq \eta$;*
- *for every $t \in [R/2,R]$, $\big|f(t)-p_3\big| \leq \eta$;*
- *$E\big(f,[-R,R]\big) \leq d_{13} + \gamma$.*

Then, there exist a, b and T, with $-R/2 < a < T < b < R/2$ such that:

- *$\big\|f(a+\cdot)-\zeta_{12}(\cdot)\big\|^2_{H^1([-R/2,T])} \leq C(\gamma + e^{-cR})$;*
- *$\big\|f(b+\cdot)-\zeta_{23}(\cdot)\big\|^2_{H^1([T,R/2])} \leq C(\gamma + e^{-cR})$;*
- *$\big|f(T)-p_2\big|^2 < C(\gamma + e^{-cR})$.*

Proof. Let $\eta > 0$ be such that W is strictly convex in $B(p_i, 3\eta)$ for $i = 1,2,3$. It suffices to show that if R_n converges to $+\infty$ and γ_n converges to 0, and if f_n satisfies the above hypotheses, then for any n large enough there exist a_n, b_n and T_n with $-R_n/2 < a_n < T_n < b_n < R_n/2$ such that the conclusion of the proposition holds for some $C > 0$ independent of n. Throughout the proof, C denotes a positive large constant independent of n, and c denotes a positive small constant independent of n.

Let A_n be the set of $t \in [-R_n, R_n]$ such that $d\big(f_n(t), P\big) > \eta$. Then A_n is a union of disjoint open intervals included in $(-R_n/2, R_n/2)$. We denote $\mathscr{I} = \{I_1, \ldots, I_l\}$ the family of those intervals in which there exists t such that $d\big(f_n(t), P\big) > 4\eta$, and call them the transition intervals. If I is a transition interval, then $E(f_n, I) \geq 3\eta$, using the trivial lower bound $E\big(f_n,[t,t']\big) \geq d\big(f_n(t), f_n(t')\big)$ which results from the definition of d.

Because of the energy bound, the number of transition intervals is thus bounded independently of n. It is easy to check that the energy bound also implies that their size is bounded above and below independently of n.

The complement of $\bigcup_j I_j$ is a union of the sequence of closed disjoint intervals $\mathscr{J} = \{J_0, \dots, J_k\}$. On each J_i, with $0 \le i \le k$, the map f_n is close to one of the wells that we denote q_i. For $0 \le i \le k$ we let $J_i = (x_i, y_i)$, so that $x_0 = -R_n$ and $y_k = R_n$, and so that $d\big(f_n(x_i), q_i\big) \le \eta$ and $d\big(f_n(y_i), q_i\big) \le \eta$. We say that I_i is a transition interval from the well q_{i-1} to the well q_i. Note that $q_0 = p_1$ and $q_k = p_3$.

From Lemma 8, we have $E(f_n, J_0) \ge d\big(p_1, f_n(y_1)\big) - ce^{-CR_n}$ and $E(f_n, J_k) \ge d\big(f_n(x_k), p_3\big) - ce^{-CR_n}$.

We claim that, in the sequence $q_0, \dots, q_k$, no point in P can appear twice. Indeed, if we had $q_i = q_j = p \in P$ with $i < j$, then we would have $d\big(f_n(y_i), p\big) \le \eta$ and $d\big(f_n(x_j), p\big) \le \eta$. By using the trivial lower bound $E\big(f, [t, t']\big) \ge d\big(f(t), f(t')\big)$ on each interval in $\mathscr{I}$ or $\mathscr{J}$ to the left of y_i, including J_i we find, denoting I_{left} their union,

$$E(f_n, I_{\text{left}}) \ge d\big(p_1, f_n(y_i)\big) - ce^{-CR_n}.$$

Similarly, denoting I_{right} the union of the intervals to the right of x_j, including J_j, we have

$$E(f_n, I_{\text{right}}) \ge d\big(f_n(x_j), p_3\big) - ce^{-CR_n}.$$

But between y_i and x_j, there is at least a transition interval, on which the energy is bounded below (as we have seen) by 3η. Thus

$$E\big(f_n, [-R_n, R_n]\big) \ge d\big(p_1, f_n(y_i)\big) + d\big(f_n(x_j), p_3\big) + 3\eta - ce^{-CR_n}.$$

Since $\eta \ge d\big(f_n(y_i), p\big)$ and $\eta \ge d\big(p, f_n(x_j)\big)$ we deduce

$$E\big(f_n, [-R_n, R_n]\big) \ge d_{13} + \eta - ce^{-CR_n},$$

which contradicts the energy upper bound if n is large enough, hence proving the claim.

Thus we have shown that there is either a single transition interval from $q_0 = p_1$ to $q_1 = p_3$ or two transition intervals: one from $q_0 = p_1$ to $q_1 = p_2$ and the other from $q_1 = p_2$ to $q_2 = p_3$.

We now exclude the first possibility. If there was a single transition, then by choosing $a_n \in I_1$, and going to a subsequence if necessary, we would have $f_n(a_n + \cdot) \to f$ weakly in $H^1_{\text{loc}}(\mathbb{R})$ and locally uniformly by compact Sobolev embedding, with $E(f, \mathbb{R}) \le d_{13}$ and $d\big(f(t), p_1\big) \le 4\eta$ for t small enough while $d\big(f(t), p_3\big) \le 4\eta$ for t large enough. This and the finiteness of $E(f, \mathbb{R})$ easily implies that $f(-\infty) = p_1$ and $f(+\infty) = p_3$, and thus that f is a heteroclinic joining p_1 to p_3, contradicting our assumptions on W.

We thus have two transition intervals $I_1 = [y_0, x_1]$ and $I_2 = [y_1, x_2]$ whose complement are J_0, J_1 and J_2. From now on we denote by $I_1 = [\alpha_n, \beta_n]$ and $I_2 = [\alpha'_n, \beta'_n]$ and we recall that $\alpha_n \ge \frac{-R_n}{2}$, while $\beta'_n \le \frac{R_n}{2}$. Moreover $\alpha'_n - \beta_n$ must converge to $+\infty$ as $n \to +\infty$ as the opposite would imply, as in the previous paragraph, the existence of a heteroclinic joining p_1 to p_3. Then, choosing $a_n \in I_1$ and $b_n \in I_2$ and going to a subsequence if necessary, we deduce that $f_n(a_n + \cdot) \to \widetilde{f}$ and $f_n(b_n + \cdot) \to \widetilde{g}$, where $d\big(\widetilde{f}(t), p_1\big) \le 4\eta$ for t small enough and $d\big(\widetilde{f}(t), p_2\big) \le 4\eta$ for t large enough while $d\big(\widetilde{g}(t), p_2\big) \le 4\eta$ for t small enough and $d\big(\widetilde{g}(t), p_3\big) \le 4\eta$ for t large enough.

Moreover,

$$d_{13} \ge \lim_{n \to +\infty} E(f_n, \mathbb{R}) \ge E(\widetilde{f}, \mathbb{R}) + E(\widetilde{g}, \mathbb{R}).$$

As above we deduce that

$$\widetilde{f}(-\infty) = p_1, \qquad \widetilde{f}(+\infty) = p_2, \qquad \widetilde{g}(-\infty) = p_2, \qquad \widetilde{g}(+\infty) = p_3,$$

which implies that $E(\widetilde{f}, \mathbb{R}) \ge d_{12}$ and $E(\widetilde{g}, \mathbb{R}) = d_{23}$.

Thus, in view of the upper bound $d_{13} \geq E(\widetilde{f},\mathbb{R}) + E(\widetilde{g},\mathbb{R})$, we find that $\widetilde{f}$ and $\widetilde{g}$ are heteroclinics connecting, respectively, p_1 to p_2 and p_2 to p_3. Shifting $\{a_n\}_n$ and $\{b_n\}_n$ if necessary, we may thus assume that

$$f_n(a_n + \cdot) \longrightarrow \zeta_{12}, \qquad f_n(b_n + \cdot) \longrightarrow \zeta_{23},$$

in the weak H^1_{loc} topology and locally uniformly. Indeed, since $\lim E(f_n) = E(\widetilde{f}) + E(\widetilde{g})$, it follows that the convergence is, in fact, strong in H^1_{loc}.

Moreover, from the energy upper bound we deduce that for any fixed $R > 0$ and n large enough depending on R,

$$E\Big(f_n, \mathbb{R} \setminus \big([a_n - R, a_n + R] \cup [b_n - R, b_n + R]\big)\Big) \leq \gamma_n + Ce^{-cR}.$$

It remains to prove the quantitative estimates. To this end, we recall [18, Lemma 4.5] which states that there exist α and β depending only on W such that if $h\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$\|h - \zeta_{12}\|_{H^1(\mathbb{R})} \leq \beta \tag{33}$$

then there exists $t \in \mathbb{R}$ such that

$$\big\|h - \zeta_{12}(\cdot + t)\big\|^2_{H^1(\mathbb{R})} \leqslant \alpha\big(E(h) - E(\zeta_{12})\big), \tag{34}$$

with a similar statement pertaining to ζ_{23}. We note that the hypotheses on W in [18] differ from ours globally; for example there one assumes the existence of two heteroclinic connections between a pair of wells. However, in a neighborhood of a particular heteroclinic, our situation is identical to that of [18] and so the same conclusions hold.

In order to apply this result, we consider extensions of f_n to $\mathbb{R}$ as follows. First, using Lemma 9, we show that there exists $T_n \in J_1$ such that

$$\big\|f_n(T_n) - p_2\big\|^2 \leq C(\gamma_n + e^{-cR_n}).$$

Indeed, Lemma 9 with $I = J_1$ yields

$$E(f_n, J_1) \geq 2\eta + c\Big(\min_{t\in[-R_n/2, R_n/2]} \big|f(t) - p_2\big|^2 + e^{-CR_n}\Big).$$

Further on $J_i, i = 0,2$, we apply Lemma 8 to f_n to obtain, since at one endpoint of J_0 (resp J_2) f_n is at distance η from p_1 (resp. p_3), that $E(f_n, J_i) \geq \eta - ce^{-CR_n}$, $i = 0,2$. For the remaining intervals I_1, I_2, we note that on I_1, f_n connects values from the geodesic balls centered at p_1 to the one centered at p_2, while on I_2, f_n connects values from the geodesic balls centered at p_2 to the one centered at p_3, so that $E(f_n, I_1) \geq d_{12} - 2\eta, E(f_n, I_2) \geq d_{23} - 2\eta$. Finally using the upper bound $E\big(f_n, [-R_n, R_n]\big) \leq d_{13} + \gamma_n$, we obtain the desired $T_n \in J_1$ with $\big\|f_n(T_n) - p_2\big\|^2 \leq C(\gamma_n + e^{-cR_n})$.

Next we define $h_n\colon \mathbb{R} \to \mathbb{R}$ and $h'_n\colon \mathbb{R} \to \mathbb{R}$ as follows. Their respective restriction to $[-R_n/2, T_n]$ and to $[T_n, R_n/2]$ agree with f_n. Outside these intervals, we extend h_n to the right of T_n via the energy minimizer connecting $f_n(T_n)$ to p_2 and to the left of $-R_n/2$ via the energy minimizer connecting $f_n(-R_n/2)$ to p_1. Similarly we extend h'_n to the right of $R_n/2$ via the energy minimizer connecting $f_n(R_n/2)$ to p_3 and to the left of T_n via the energy minimizer connecting $f_n(T_n)$ to p_2.

We now verify that h_n satisfies condition (33) if n is large enough. First we note that, recalling $I_1 = [\alpha_n, \beta_n]$ and $I_2 = [\alpha'_n, \beta'_n]$,

$$\big\|h_n - \zeta_{12}(\cdot - a_n)\big\|_{H^1([\alpha_n,\beta_n])} \longrightarrow 0 \quad \text{as } n \to \infty. \tag{35}$$

On the other hand, since

$$d\big(f_n(\alpha_n), p_1\big) \leq \eta, \qquad d\big(f_n(\beta_n), p_2\big) \leq \eta, \qquad d\big(f_n(\alpha'_n), p_2\big) \leq \eta, \qquad d\big(f_n(\beta'_n), p_3\big) \leq \eta,$$

we have $E\big(f_n, [\alpha_n, \beta_n]\big) \geq d_{12} - 2\eta$ and $E\big(f_n, [\alpha'_n, \beta'_n]\big) \geq d_{23} - 2\eta$, so that

$$E\Big(f_n, [-R_n, R_n] \setminus \big([\alpha_n, \beta_n] \cup [\alpha'_n, \beta'_n]\big)\Big) \leq 4\eta + \gamma_n.$$

However, on $(-\infty,\alpha_n]$, both $\zeta_{12}(\cdot-a_n)$ and h_n take values in the geodesic ball centered at p_1 with radius 4η. Therefore their energy is comparable to the square of their H^1 distance to p_1 on this interval. Thus, in particular,

$$\left\| h_n-\zeta_{12}(\cdot-a_n)\right\|^2_{H^1(-\infty,\alpha_n)}<C\eta, \tag{36}$$

for n large enough. Similarly,

$$\| h_n-p_2\|^2_{H^1([\beta_n,+\infty])}<C\eta. \tag{37}$$

It follows from (35), (36) and (37) that (33) holds for h_n if η is small enough, depending only on W, and for n large enough. The same is true of the H^1 distance between h_n' and $\zeta_{23}(\cdot-b_n)$.

In order to exploit the conclusion (34), we will now prove that

$$E(h_n)-E(\zeta_{12})<C(\gamma_n+e^{-cR_n})\qquad\text{and}\qquad E(h_n')-E(\zeta_{23})<C(\gamma_n+e^{-cR_n}). \tag{38}$$

We first note that

$$E(h_n,\mathbb{R})=E\big(f_n,[-R_n/2,T_n]\big)+d\big(f_n(-R_n/2),p_1\big)+d\big(f_n(T_n),p_2\big)$$

and

$$E(h_n',\mathbb{R})=E\big(f_n,[T_n,R_n/2]\big)+d\big(f_n(R_n/2),p_3\big)+d\big(f_n(T_n),p_2\big).$$

Next, we have that $d\big(f_n(-R_n/2),p_1\big)\le E\big(f_n,[-R_n,-R_n/2]\big)+Ce^{-cR_n}$ and $d\big(f_n(T_n),p_2\big)\le C(\gamma_n+e^{-cR_n})$. Also, we have that $d\big(f_n(R_n/2),p_3\big)\le E\big(f_n,[R_n/2,R_n]\big)+Ce^{-cR_n}$. Thus, we deduce that

$$\begin{aligned}E(h_n,\mathbb{R})&\le E\big(f_n,[-R_n,T_n]\big)+C(\gamma_n+e^{-cR_n}),\\ E(h_n',\mathbb{R})&\le E\big(f_n,[T_n,R_n]\big)+C(\gamma_n+e^{-cR_n}).\end{aligned}$$

Consequently,

$$E(h_n,\mathbb{R})+E(h_n',\mathbb{R})\le E\big(f_n,[-R_n,R_n]\big)+C(\gamma_n+e^{-cR_n}).$$

Moreover, since $E\big(f_n,[-R_n,R_n]\big)\le d_{12}+d_{23}+\gamma_n$ and $E(h_n,\mathbb{R})\ge d_{12}$ and $E(h_n',\mathbb{R})\ge d_{23}$, we deduce that

$$E(h_n,\mathbb{R})-E(\zeta_{12})\le C(\gamma_n+e^{-cR_n})\qquad\text{and}\qquad E(h_n',\mathbb{R})-E(\zeta_{23})\le C(\gamma_n+e^{-cR_n}). \tag{39}$$

Applying (34), we find that for some $\widetilde{a}_n$ and $\widetilde{b}_n$ one has

$$\left\| f_n(\widetilde{a}_n+\cdot)-\zeta_{12}(\cdot)\right\|^2_{H^1([-R_n/2,T_n])}\le C(\gamma_n+e^{-cR_n})$$

and

$$\left\| f_n(\widetilde{b}_n+\cdot)-\zeta_{23}(\cdot)\right\|^2_{H^1([T_n,R_n/2])}\le C(\gamma_n+e^{-cR_n}).$$

This concludes the proof. □

The following corollary is a direct consequence of Lemma 10 and the fact that both $\left|f(T)-p_2\right|^2$ and $\left|f(T)-\zeta_{12}(T-a)\right|^2$ are bounded by $C(\gamma+e^{-cR})$, and hence

$$\left|p_2-\zeta_{12}(T-a)\right|^2\le C(\gamma+e^{-cR}).$$

Corollary 11. *The numbers a and b in the conclusion of Lemma 10 satisfy*

$$T-a\ge c\min\big(|\log\gamma|,R\big),\qquad b-T\ge c\min\big(|\log\gamma|,R\big),$$

for some $c>0$ *independent of* γ *and* R.

In addition, Lemma 10 also allows us to easily obtain the following proposition, illustrated in Figure 1. Here we denote by s the arclength variable on the circle ∂B_R such that $s(Re^{i\theta})=R\theta$ for $\theta\in(-\pi,\pi]$.

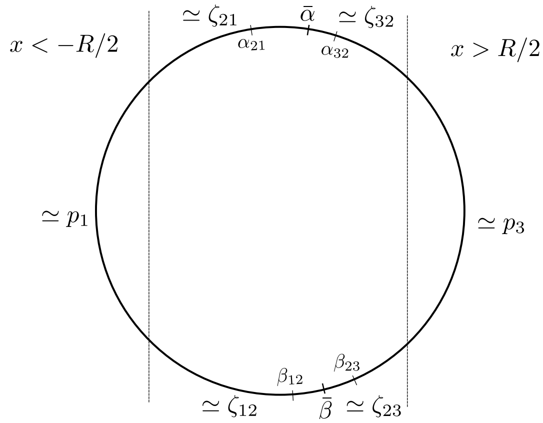

Figure 1. The map $u\big|_{\partial B_R}$ and the transition angles according to Proposition 12.

Proposition 12. *There exist constants* $C, \eta > 0$ *and* $R_0, \gamma_0 > 0$ *such that for any* $R > R_0$ *and any* $0 < \gamma < \gamma_0$, *if* $u\colon \partial B_R \to \mathbb{R}^2$ *is such that*

$$\big|u(x,y) - p_1\big| < \eta \text{ for any } (x,y) \in \partial B_R \text{ with } x < -R/2, \tag{40}$$

$$\big|u(x,y) - p_3\big| < \eta \text{ for any } (x,y) \in \partial B_R \text{ with } x > R/2, \tag{41}$$

$$E(u, \partial B_R) \le 2d_{13} + \gamma, \tag{42}$$

then there exist angles $\pi/4 < \alpha_{32} < \overline{\alpha} < \alpha_{21} < 3\pi/4$ *and* $-3\pi/4 < \beta_{12} < \overline{\beta} < \beta_{23} < -\pi/4$ *such that the following holds: writing* $I_{21} = [\overline{\alpha}, 3\pi/4]$, $I_{32} = [\pi/4, \overline{\alpha}]$ *and* $I_{12} = [-3\pi/4, \overline{\beta}]$, $I_{23} = [\overline{\beta}, -\pi/4]$, *we have*

$$\begin{aligned} &\big\| u(Re^{i(\alpha_{21}+\cdot/R)}) - \zeta_{21}(\cdot)\big\|^2_{H^1(I_{21})} < C\gamma, \qquad \big\| u(Re^{i(\alpha_{32}+\cdot/R)}) - \zeta_{32}(\cdot)\big\|^2_{H^1(I_{32})} < C\gamma, \\ &\big\| u(Re^{i(\beta_{12}+\cdot/R)}) - \zeta_{12}(\cdot)\big\|^2_{H^1(I_{12})} < C\gamma, \qquad \big\| u(Re^{i(\beta_{23}+\cdot/R)}) - \zeta_{23}(\cdot)\big\|^2_{H^1(I_{23})} < C\gamma. \end{aligned} \tag{43}$$

and such that for any $s \in [-R\pi, R\pi] \setminus \bigcup_{jk} I_{jk}$,

$$d\big(u(Re^{is/R}), P\big) < \eta. \tag{44}$$

Finally,

$$R\min\big(\overline{\alpha} - \alpha_{32}, \alpha_{21} - \overline{\alpha}, \beta_{23} - \overline{\beta}, \overline{\beta} - \beta_{12}\big) \ge c\min\big(|\log\gamma|, R\big). \tag{45}$$

4. Upper bound

Here we establish a key upper bound.

Proposition 13. *There exists* $C > 0$ *such that for any* $R > C$ *and* $1/C > \eta > 0$, $1/C > \gamma > 0$ *the following holds.*

If $u\colon \partial B_R \to \mathbb{R}^2$ *satisfies the hypotheses of Proposition 12 and is such that* $u(x,y)$ *is* η*-close to* p_1 *if* $x < -\eta R$ *and* η*-close to* p_3 *if* $x > \eta R$, *then for any* $C \le \sigma$, $\rho \le R/C$ *it holds that*

$$E(u, B_R) \le d_{12}L_1 + d_{23}L_3 + C\left(\frac{\rho^2}{\sigma} + \frac{\sigma}{\rho}e^{-\ell} + Re^{-(\ell+\rho)} + \eta\ell e^{-c\ell} + e^{-c\ell} + \gamma + \eta^2\right), \tag{46}$$

where, using the angles $\alpha_{32}, \alpha_{21}, \beta_{12}, \beta_{23}, \overline{\alpha}, \overline{\beta}$ *from the conclusion of Proposition 12, we have used the notation*

$$L_1 = R|e^{i\alpha_{21}} - e^{i\beta_{12}}|, \quad L_3 = R|e^{i\alpha_{32}} - e^{i\beta_{23}}|, \quad \ell = \frac{1}{2}R\,\alpha_{\min} \tag{47}$$

and

$$\alpha_{\min} := \min\big(\overline{\alpha} - \alpha_{32}, \alpha_{21} - \overline{\alpha}, \beta_{23} - \overline{\beta}, \overline{\beta} - \beta_{12}\big).$$

Proof of Proposition 13. The proposition is proved by constructing a comparison map $v\colon B_R \to \mathbb{R}^2$ coinciding with u on ∂B_R.

Step 1: Annulus. We begin by defining a competitor in the annulus $A = B_R \setminus B_{R-1}$. To this end, it will be convenient to define the following modification of the heteroclinics ζ_{ij} using the parameter ℓ from (47). We set

$$\zeta_{ij}^{\ell}(s) = \begin{cases} p_i & \text{if } s \le -\ell, \\ \zeta_{ij}(s) & \text{if } |s| < \ell - 1, \\ p_j & \text{if } s \ge \ell, \end{cases} \tag{48}$$

and we take ζ_{ij}^{ℓ} to be a linear interpolation on the intermediate intervals. It is easy to check that

$$E\big(\zeta_{ij}^{\ell}, [-\ell, \ell]\big) \le d(p_i, p_j) + e^{-c\ell}, \qquad \|\zeta_{ij}^{\ell} - \zeta_{ij}\|_{L^2(\mathbb{R})}^2 \le e^{-c\ell}. \tag{49}$$

Adopting the notation from Proposition 12, we note that the line through $e^{i\bar{\alpha}}$ and $e^{i\bar{\beta}}$, which we denote by L, separates B_R into two regions, one, say B_R^{left}, containing the line segment joining $Re^{i\alpha_{21}}$ and $Re^{i\beta_{12}}$ and B_R^{right} containing the line segment joining $Re^{i\alpha_{32}}$ and $Re^{i\beta_{23}}$.

Denoting by S_{12} the first of these two segments and by S_{23} the second, we define a competitor, say v, in A as the linear interpolation in the radial direction between the values of u on ∂B_R and the function, say $V\colon \partial B_{R-1} \to \mathbb{R}^2$ given by

$$V(x, y) := \begin{cases} \zeta_{12}^{\ell}\,\mathrm{dist}\big((x, y), S_{12}\big) & \text{for } (x, y) \in \partial B_{R-1} \cap B_R^{\text{left}}, \\ \zeta_{23}^{\ell}\,\mathrm{dist}\big((x, y), S_{23}\big) & \text{for } (x, y) \in \partial B_{R-1} \cap B_R^{\text{right}}. \end{cases} \tag{50}$$

Here $\mathrm{dist}\big((x, y), S_{12}\big)$ denotes the signed distance function to S_{12}, taken to be negative to the left of S_{12} and $\mathrm{dist}\big((x, y), \ell_{23}\big)$ is the signed distance function to S_{23}, taken to be negative to the left of S_{23}. We note that, so defined, the function V is continuous taking value p_2 at the two points $\partial B_{R-1} \cap \partial B_R^{\text{left}} \cap \partial B_R^{\text{right}}$. We also note that in light of the definition of ℓ, cf. (47), the segments S_{12} and S_{23} are both more than ℓ distance from the line L.

Estimating the energetic cost of v in the annulus A is entirely analogous to a similar calculation carried out in [17, step 3 of the proof of Theorem 1.3]. Therefore, we omit the details. One finds that with two transitions from p_1 to p_2 and two transitions from p_2 to p_3, one has

$$E(v, A) \le 2(d_{12} + d_{23}) + C(e^{-c\ell} + \gamma + \eta^2),$$

through an appeal to (49) and (43). Here we have used the fact that the direction of the segments S_{12} and S_{23} and the radial direction at their endpoints make angles bounded by $C\eta$ resulting in the $O(\eta^2)$ term above.

Step 2: Defining v in "triangles" inside B_{R-1} bordering ∂B_{R-1}. The line L passing through $e^{i\bar{\alpha}}$ and $e^{i\bar{\beta}}$ separates B_{R-1} into two parts: one to the left, which we denote by D, containing the segment $S_{12} \cap B_{R-1}$, and the other one containing the segment $S_{23} \cap B_{R-1}$. We then introduce a coordinate system (x', y') to define the competitor v in the region D by taking the segment S_{12} to coincide with the y' axis, with $S_{23} \cap B_{R-1}$ extending in y' coordinates from say $y' = y_-$ to $y' = y_+$. We also take the x'-axis to go through the middle of $S_{23} \cap B_{R-1}$ so that $y_- = -y_+$. Before proceeding, we remark that in these coordinates, the function V defined in (50) is simply a function of x' for $(x, y) \in \partial B_{R-1} \cap B_R^{\text{left}}$.

In these coordinates, consider the strip $\big\{(x', y') \bigm| |x'| < \ell\big\}$. Its intersection with B_{R-1} may be decomposed as a rectangle $\mathcal{R}$, defined as the largest rectangle of the form $\big\{(x', y') : |x'| < \ell,\ y' \in [y_-, y+]\big\}$ inscribed in B_{R-1}, and two "triangles" C_+ and C_-, each having one side which is in fact an arc in ∂B_{R-1}. The other two sides are parallel to the x' and y' axis, respectively. Since $Re^{i\alpha_{21}}$ and $Re^{i\beta_{12}}$ both must lie to the right of the line $x = -\eta R$ by assumption, it follows that the ends of the segment $S_{12} \cap B_{R-1}$ are at a distance of at most ηR from the y-axis, and so the ratio of the length of the y'-side to the x'-side for both of the aforementioned "triangles" is bounded by $C\eta$. Consequently, the length of the y'-side of these triangles is $O(\eta\ell)$.

On C_+ and C_-, we define v to be independent of y so that

$$v(x',y_+) = v(x',-y_+) = \zeta_{12}^{\ell}(x'),$$

and

$$\begin{aligned} E(v, C_+ \cup C_-) &\le E(\zeta_{12}^{\ell}) \times \left(\left|(R-1)e^{i\alpha_{21}} - (R-1)e^{i\beta_{12}}\right| - 2y_+\right) \\ &\le d_{12}\left(\left|(R-1)e^{i\alpha_{21}} - (R-1)e^{i\beta_{12}}\right| - 2y_+\right) + C\eta\ell e^{-c\ell}. \end{aligned} \tag{51}$$

Step 3: Dilation. Though tempting, it is too costly energetically to simply take our competitor v to be $\zeta_{12}^{\ell}(x')$ throughout D. Therefore, we must further separate the center of this modified heteroclinic from the line L. To this end, we proceed to define v first on

$$D_+ := D \cap \mathbb{R} \times [y_+ - \sigma, y_+], \qquad D_- := D \cap \mathbb{R} \times [-y_+, \sigma - y_+].$$

By enforcing the condition $v(x',y') = v(x',-y')$ on $D_+ \cup D_-$, in fact, we only need to work on D_+.

The idea is to interpolate between $\zeta_{12}^{\ell}(x')$ and $\zeta_{12}^{\ell+\rho}(x'+\rho)$, in the y' variable for a parameter ρ to be chosen judiciously later. That is, for $(x',y') \in D_+$ we take our competitor v to be given by

$$v(x',y') = \zeta_{12}^{\ell + \frac{\rho}{\sigma}(y_+ - y')}\left(x' + \frac{\rho}{\sigma}(y_+ - y')\right). \tag{52}$$

In light of (49), the energy due to the x' derivative and the potential term in D_+ is bounded by

$$\int_0^{\sigma} d_{12} + e^{-c(\ell + t\rho/\sigma)}\,\mathrm{d}t \le \sigma d_{12} + C\frac{\sigma}{\rho}e^{-c\ell}. \tag{53}$$

From (52) and the definition of ζ_{12}^{ℓ}, it is straightforward to deduce that

$$\frac{1}{2}\int_{D_+}\left|\frac{\partial v}{\partial y'}\right|^2 \le \left(\frac{\rho}{\sigma}\right)^2 \int_0^{\sigma} d_{12} + e^{-c(\ell + t\rho/\sigma)}\,\mathrm{d}t \leqslant C\frac{\rho^2}{\sigma}. \tag{54}$$

As noted at the beginning of this step, the same bounds hold for D_-.

Step 4: Remainder and conclusion. It remains to define v in $D \cap \mathbb{R} \times [y_- + \sigma, y_+ - \sigma]$. We simply require v to be independent of y' there, so that $v = v(x') = \zeta_{12}^{\ell+\rho}(x'+\rho)$. Hence, in this region the energy bound takes the form

$$E\big(v, D \cap \mathbb{R} \times [y_- + \sigma, y_+ - \sigma]\big) \le (y_+ - y_- - 2\sigma)(d_{12} + e^{-c(\ell+\rho)}). \tag{55}$$

Adding the upper bounds (51), (53), (54) and (55), recalling that $y_- = -y_+$, and noting that $y_+ < R$, we obtain

$$\begin{aligned} E(v,D) \le (2y_+ - 2\sigma)(d_{12} + e^{-c(\ell+\rho)}) + C\frac{\rho^2}{\sigma} + 2\sigma d_{12} \\ + C\frac{\sigma}{\rho}e^{-c\ell} + d_{12}\left(\left|(R-1)e^{i\alpha_{21}} - (R-1)e^{i\beta_{12}}\right| - 2y_+\right) + C\eta\ell e^{-c\ell}. \end{aligned}$$

Therefore,

$$E(v,D) \le d_{12}\left|(R-1)e^{i\alpha_{21}} - (R-1)e^{i\beta_{12}}\right| + C\left(Re^{-c(\ell+\rho)} + \frac{\rho^2}{\sigma} + \frac{\sigma}{\rho}e^{-c\ell} + \eta\ell e^{-c\ell}\right).$$

Carrying out an analogous construction in the portion of B_{R-1} lying to the right of the line L, that is, with ζ_{12}^{ℓ} replaced by ζ_{23}^{ℓ}, etc, we deduce that

$$\begin{aligned} E(v,B_{R-1}) \le d_{12}\left|(R-1)e^{i\alpha_{21}} - (R-1)e^{i\beta_{12}}\right| + d_{23}\left|(R-1)e^{i\alpha_{32}} - (R-1)e^{i\beta_{23}}\right| \\ + C\left(Re^{-c(\ell+\rho)} + \frac{\rho^2}{\sigma} + \frac{\sigma}{\rho}e^{-c\ell} + \eta\ell e^{-c\ell}\right). \end{aligned}$$

Then because all of the transition angles are within $C\eta$ of being $\pm\pi/2$ due to the hypothesis, we have

$$\left|Re^{i\alpha_{21}} - Re^{i\beta_{12}}\right| = \left|(R-1)e^{i\alpha_{21}} - (R-1)e^{i\beta_{12}}\right| + 2 + O(\eta^2),$$

with a similar statement holding if we replace α_{21} and β_{12} by α_{32} and β_{23}. Thus, once we add in the cost of the annulus interpolation $E(v,A)$ from Step 1, we finally arrive at the upper bound

$$E(v,B_R) \le d_{12}L_{12} + d_{23}L_{23} + C\left(Re^{-c(\ell+\rho)} + \frac{\rho^2}{\sigma} + \frac{\sigma}{\rho}e^{-c\ell} + \eta\ell e^{-c\ell} + e^{-c\ell} + \gamma + \eta^2\right). \qquad \square$$

Lemma 14. *Assume $u\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a minimizer of E and that there exists a sequence $R_n \to +\infty$ such that, possibly after rotating the coordinates, $u(R_n\cdot) \to H_{13}$ in L^1. Then* any *sequence $R_n \to +\infty$ is such that, possibly after rotating the coordinates, $u(R_n\cdot) \to H_{13}$.*

Proof. Assume by contradiction that no subsequence of $u(R_n\cdot)$ converges to H_{13} modulo a rotation, then a subsequence must converge to H_{12} or H_{23} modulo a rotation, which implies, from [17, Theorem 1.3], that, modulo a rotation, $u(x,y) = \zeta_{12}(x)$ or $u(x,y) = \zeta_{23}(x)$, which contradicts the assumptions. $\square$

Lemma 15. *Under the same hypothesis, there exist $R_0 > 0$ such that for any $R > R_0$,*

$$E(u,\partial B_R) \ge 2d_{13} - e^{-CR}.$$

Proof. It suffices to prove that for any sequence $R_n \to +\infty$ the bound is satisfied with a constant C independent of n. We choose $\eta > 0$ small enough depending on W, to be determined below.

Since $u(R_n\cdot)$ converges in L^1 to H_{13} modulo a rotation, for any R large enough in the sequence there exist α, α' and β, β' with

$$\pi/2 - \eta < \alpha < \pi/2 < \alpha' < \pi/2 + \eta, \qquad -\pi/2 - \eta < \beta < -\pi/2 < \beta' < -\pi/2 + \eta$$

such that $u(Re^{i\alpha})$, $u(Re^{i\beta})$ are η-close to p_1 and such that $u(Re^{i\alpha'})$, $u(Re^{i\beta'})$ are η-close to p_3.

To bound from below the energy of u on $\partial B_R^+ = \left\{Re^{i\theta} \mid \theta \in [0,\pi]\right\}$ we let $v(s) = u(Re^{is/R})$, so that $E(u,\partial B_R^+) = E\big(v,[0,\pi R]\big)$. Then

$$E\big(v,[0,\pi R]\big) = E\big(v,[0,R\alpha']\big) + E\big(v,[R\alpha',R\alpha]\big) + E\big(v,[R\alpha,R\pi]\big).$$

We have $E\big(v,[R\alpha',R\alpha]\big) \ge d\big(v(R\alpha'),v(R\alpha)\big)$ and, using Lemma 8, we find that $E\big(v,[0,R\alpha']\big) \ge d\big(v(R\alpha'),p_3\big) - Ce^{-cR}$ while $E\big(v,[R\alpha,R\pi]\big) \ge d\big(v(R\alpha),p_1\big) - Ce^{-cR}$. Therefore

$$E(u,\partial B_R^+) \ge d_{13} - e^{-cR}.$$

The same holds on ∂B_R^-, proving the result. $\square$

We are now ready to prove the following important upper bound.

Proposition 16. *Assume $u\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a minimizer of E such that $u(R\cdot) \to H_{13}$, possibly after taking subsequences and modulo a rotation of the coordinates.*

Then there exist sequences of positive numbers $R_n \to +\infty$ and $\eta_n \to 0$, $\gamma_n \to 0$, such that the restriction of u to ∂B_{R_n} satisfies the hypotheses of Proposition 12 with the parameters η_n, γ_n. Moreover, denoting α_{21}, α_{32}, β_{12}, β_{23} the transition angles (which depend on n) given by Proposition 12 we have, as $n \to +\infty$,

$$E(u,B_{R_n}) \le R_n\big(|e^{i\alpha_{21}} - e^{i\beta_{12}}|d_{12} + |e^{i\alpha 32} - e^{i\beta_{23}}|d_{23}\big) + o(1).$$

Proof of Proposition 16. From Lemma 14, any sequence $R_n \to +\infty$ has a subsequence such that, modulo a rotation, $u(R_n\cdot)$ converges to H_{13}. It follows that, given η, $\gamma > 0$, if n is large enough then (40), (41) are satisfied for every $R \in [\eta R_n, R_n]$. Moreover, from Lemma 15, for $R \in [\eta R_n, R_n]$ we have

$$E(u,\partial B_R) \ge 2d_{13} - e^{-cR}.$$

On the other hand Proposition 7 implies, after rescaling, that as $n \to +\infty$,

$$E(u, B_{R_n}) \le 2R_n d_{13} + o(R_n),$$

since the diameter of B_{R_n} is $2R_n$.

Combining the two, we find that (42) is satisfied with $\gamma_n = o(1)$, for most $R \in [\eta R_n, R_n]$. Hence (40), (41) and (42) are satisfied for some sequence $\{R_n\}$ converging to $+\infty$ with $\gamma_n = o(1)$.

We then apply Proposition 13 with $\sigma = \rho = R^\theta$ for some fixed $0 < \theta < 1$, for any R in our sequence. We find that all the error terms may be absorbed in CR^θ so that

$$E(u, B_R) \le d_{12} L_1 + d_{23} L_3 + CR^\theta,$$

Moreover, from Lemma 15, we have for any $r > R_0$ that

$$E(u, \partial B_r) \ge 2d_{13} - e^{-Cr}.$$

It follows that

$$\begin{aligned}\int_{R/2}^{R} E(u,\partial B_r) - 2d_{13}\,\mathrm{d}r &= E(u,B_R) - E(u,B_{R/2}) - Rd_{13}\\ &\le E(u,B_R) - \int_{R_0}^{R/2} 2d_{13} - e^{-Cr}\,\mathrm{d}r - Rd_{13}\\ &\le E(u,B_R) - 2Rd_{13} + C\\ &\le L_1 d_{12} + L_3 d_{23} + CR^\theta - 2Rd_{13}\\ &\le CR^\theta,\end{aligned} \tag{56}$$

since L_1, $L_3 \le 2R$. Hence there must exist $R' \in [R/2, R]$ such that

$$E(u, \partial B_{R'}) \le 2d_{13} + CR^{\theta-1}.$$

We denote $\{R'_n\}$ a subsequence of $\{R'\}$ such that $u(R'_n \cdot) \to H_{13}$ modulo a rotation. We may then apply Proposition 12 on $\partial B_{R'}$, with η_n tending to 0 and $\gamma = C(R'_n)^{\theta-1}$, which also tends to 0 as $n \to +\infty$. It yields transition angles α_{ij}, β_{ij} such that

$$c\log\big((R'_n)^\theta\big) \le \ell := \frac{R'_n}{2}\min\big(|e^{i\alpha_{21}} - e^{i\alpha_{32}}|, |e^{i\beta_{12}} - e^{i\beta_{23}}|\big).$$

Therefore, Proposition 13 applied with $\rho = (R'_n)^\varepsilon$ and $\sigma = (R'_n)^{3\varepsilon}$, with $\varepsilon > 0$ chosen small enough (taking $\varepsilon < c\theta$ works), yields

$$E(u, B_{R'_n}) \le L_1 d_{12} + L_3 d_{23} + o(1). \qquad \square$$

5. Proof of Theorem 4

We now proceed to prove our main result, by contradiction. It is proved in [17] that if u is a minimizer of E and $\{R_n\}$ is a sequence of radii tending to $+\infty$ such that $u(R_n \cdot)$ converges to H_{12} in L^1, then $u(x, y) = \zeta_{12}(x)$. The same holds if $\{1,2\}$ is replaced by $\{2,3\}$. In light of Proposition 7, it thus remains to obtain a contradiction under the assumption that, for any $\{R_n\}$ converging to $+\infty$, there exists a subsequence such that, possibly after a rotation of the coordinates, $\|u(R_n \cdot) - H_{13}(R_n \cdot)\|_{L^1}$ converges to 0 as $n \to +\infty$, which we assume henceforth.

The contradiction will be obtained by defining $\{R_n\}$, $\{\eta_n\}$, $\{\gamma_n\}$ to be as in the conclusion of Proposition 16. Our aim is to derive a lower bound for $E(u, B_{R_n})$ that will contradict the upper bound of Proposition 16, for large enough n.

The map $u(R_n \cdot)$ converges to H_{13} in L^1, and therefore locally uniformly outside the y-axis, see Remark 6. Therefore, for any $\eta > 0$ we may choose n_0 large enough so that $d\big(u(x, y), p_1\big) < \eta$ for any $(x,y)\in B_{R_{n_0}}$ such that $x<-\eta R_{n_0}$, while $d\big(u(x,y),p_3\big)<\eta$ if $(x,y)\in B_{R_{n_0}}$ is such that $x>\eta R_{n_0}$. Relabeling the sequence we let $n_0=0$ and $R_{n_0}=R_0$. We thus have:

$$\begin{aligned} &d\big(u(x,y),p_1\big)<\eta \quad \text{in } \mathscr{R}_{\text{left}}=B_{R_0}\cap\{x<-\eta R_0\}\\ \text{and}\quad &d\big(u(x,y),p_3\big)<\eta \quad \text{in } \mathscr{R}_{\text{right}}=B_{R_0}\cap\{x>\eta R_0\}. \end{aligned} \tag{57}$$

We now proceed to prove the desired lower bound for $E(u,B_{R_n})$. First we extend u to $\tilde u$ defined on B_{R_n+1}, which will allow us to clean up the boundary data. We claim that we may define $\tilde u$ on the annulus $A_n=B_{R_n+1}\setminus B_{R_n}$ so that, as $n\to+\infty$

$$E(\tilde u,A_n)\le 2(d_{12}+d_{23})+o(1), \tag{58}$$

and so that $\tilde u$ restricted to ∂B_{R_n+1} is equal to ζ_{ij}^{ℓ}, defined by (48), on an arc of length ℓ centered at the angle α_{ij} or β_{ij} and $\tilde u$ is constant equal to one of the wells on each of the connected components of the complement of these arcs. Here the angles α_{ij}, β_{ij} are the transition angles (which depend on n) given by Proposition 12. Note that $\tilde u$ depends on n as well, but only on the annulus A_n. The construction of $\tilde u$ is by radial interpolation, as in the proof of Proposition 13, Step 1, and the estimate (58) follows from the same arguments, hence is omitted.

Again using the fact that u converges to H_{13} locally uniformly to p_1 on the half-plane $x<0$ and to p_3 on the half-plane $x>0$, we have that transition angles tend to $\pm\pi/2$ as $n\to+\infty$. Therefore

$$\big|(R_n+1)e^{i\alpha_{21}}-(R_n+1)e^{i\beta_{12}}\big|=\big|R_ne^{i\alpha_{21}}-R_ne^{i\beta_{12}}\big|+2+o(1),$$

and a similar statement holds if we replace α_{21} and β_{12} by α_{32} and β_{23}. Thus, from (58) and the upper bound of Proposition 16 we deduce that

$$E(\tilde u,B_{R_n+1})\le(R_n+1)\big(|e^{i\alpha_{21}}-e^{i\beta_{12}}|d_{12}+|e^{i\alpha_{32}}-e^{i\beta_{23}}|d_{23}\big)+o(1).$$

From now on we drop the tilde and denote u the extension of u to B_{R_n+1}, and write R instead of R_n+1 for simplicity. The above upper bound thus becomes

$$E(u,B_R)\le R\big(|e^{i\alpha_{21}}-e^{i\beta_{12}}|d_{12}+|e^{i\alpha_{32}}-e^{i\beta_{23}}|d_{23}\big)+o(1). \tag{59}$$

We proceed to prove a lower bound contradicting (59) for large enough n.

Let

$$a_1=Re^{i\alpha_{21}},\qquad b_1=Re^{i\beta_{12}},\qquad a_3=Re^{i\alpha_{32}},\qquad b_3=Re^{i\beta_{23}}.$$

We define $\zeta\colon B_R\to\mathbb{R}$ such that

$$\|\nabla\zeta\|_\infty\le 1,\qquad \zeta(a_1)-\zeta(b_1)=|a_1-b_1|,\qquad \zeta(a_3)-\zeta(b_3)=|a_3-b_3|. \tag{60}$$

Such a ζ exists: indeed, since

$$\pi>\alpha_{21}>\alpha_{32}>\beta_{23}>\beta_{12}>-\pi,$$

the minimal connection joining the points a_1, a_3 to the points b_1, b_3 is exactly the union of the segments $[a_1,b_1]$ and $[a_3,b_3]$, see [7]. An explicit formula for ζ is

$$\zeta=\max(\zeta_1,\zeta_3),\qquad \zeta_1(x)=x\cdot u_1-\lambda_1,\qquad \zeta_3(x)=x\cdot u_3-\lambda_3, \tag{61}$$

where $u_1=(a_1-b_1)/|a_1-b_1|$ and $u_3=(a_3-b_3)/|a_3-b_3|$, and λ_1, λ_3 are suitably chosen real numbers such that ζ satisfies the required properties.

Since ζ_1 and ζ_3 are linear functions each of whose level sets (see Figure 2) necessarily consist of a line segment with endpoints on ∂B_R, it follows that each nonempty level set of ζ consists of one or two segments which can be consecutive or disjoint, hence intersect ∂B_R at either 2, 3 or 4 points. The desired lower bound will be obtained by integrating w.r.t. the variable t the lower bound we next compute on the level set $\Gamma_t=\{\zeta=t\}$.

The essential parameter in the lower bound is the boundary condition at the points of $\Gamma_t\cap\partial B_R$. The domain $\overline{B}_R\setminus\big([a_1,b_1]\cup[a_3,b_3]\big)$ has three connected components. We denote by U_1, U_2, U_3 the intersections of these three components with ∂B_R, numbered so that u restricted to U_i is closest

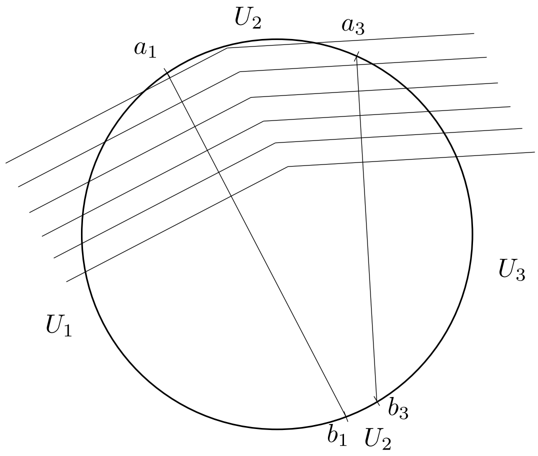

Figure 2. The disk B_R and some level-sets of the function ζ.

to the well p_i, in a way that we will quantify below. We now examine to which U_i's the points in $\Gamma_t \cap \partial B_R$ belong.

To this end we note that on $[a_1, b_1]$, the function ζ is one-to-one, since $\zeta(a_1) - \zeta(b_1) = |a_1 - b_1|$ and $\|\nabla\zeta\|_\infty \le 1$. For the same reason ζ restricted to $[a_3, b_3]$ is one-to-one, and therefore Γ_t intersects either of these segments at most once. Four cases may thus occur, depending on the value of t:

(a) $t \in \big[\zeta(a_1), \zeta(b_1)\big]$ and $t \notin \big[\zeta(a_3), \zeta(b_3)\big]$, then Γ_t intersects $[a_1, b_1]$ only: thus Γ_t contains a segment or broken line which connects U_1 to U_2;

(b) $t \notin \big[\zeta(a_1), \zeta(b_1)\big]$ and $t \in \big[\zeta(a_3), \zeta(b_3)\big]$, then Γ_t contains a segment or broken line which connects U_2 to U_3;

(c) $t \in \big[\zeta(a_i), \zeta(b_i)\big]$, for both $i = 1, 3$, then Γ_t contains either a segment or broken line connecting U_1 to U_3, or two segments: one connecting U_1 to U_2 and the other connecting U_2 to U_3;

(d) $t \notin \big[\zeta(a_1), \zeta(b_1)\big]$ and $t \notin \big[\zeta(a_3), \zeta(b_3)\big]$, then every connected component of Γ_t starts and ends in the same U_i. We will use the trivial lower bound $E(u, \Gamma_t) \ge 0$ in this case.

We now quantify the distance of $u(q)$, where $q \in \Gamma_t \cap U_i$, to the well p_i. Let $t = \zeta_1(a_1) - \delta$. Then $q = a_1 e^{i\alpha_1}$ for some α_1 and, if we denote $z = a_1 - \delta u_1$, then $q - z$ is orthogonal to u_1. Therefore, writing $a_1 = R u_1 e^{i\theta_1}$ and recalling that the inner product is given by $\Re(\bar{v}w)$, we have $Re^{i(\alpha_1+\theta_1)} - Re^{-i\theta_1} + \delta \in i\mathbb{R}$, i.e.

$$R\cos(\alpha_1 + \theta_1) - R\cos\theta_1 + \delta = 0. \tag{62}$$

From (62) we deduce that either $|\alpha_1| > R^{-1/2}$ or, using the Mean Value Theorem, that $\big(|\theta_1| + R^{-1/2}\big)|\alpha_1| \ge \delta/R$, so that

$$|\alpha_1| \ge \min\left(R^{-1/2}, \frac{\delta}{\varepsilon R + \sqrt{R}}\right),$$

where

$$\varepsilon = \max\big(|\alpha_{21} - \pi/2|, |\beta_{12} + \pi/2|, |\alpha_{32} - \pi/2|, |\beta_{23} + \pi/2|\big). \tag{63}$$

(Note that $\varepsilon = o(1)$ as $n \to +\infty$ and $|\theta_1| \le 2\varepsilon$.)

Therefore, since $|q - a_1| = R|e^{i\alpha_1} - 1|$, and using Taylor's expansion of the sine function,

$$|q - a_1| \ge \frac{1}{2}\min\left(\sqrt{R}, \frac{\delta}{\varepsilon + R^{-1/2}}\right).$$

Similar estimates hold for b_1, and for a_3, b_3 using the function ζ_3. From this we deduce that if

$$\delta(t) := \min\big(\big|t-\zeta(a_1)\big|, \big|t-\zeta(b_1)\big|, \big|t-\zeta(a_3)\big|, \big|t-\zeta(b_3)\big|\big), \tag{64}$$

then for any $q \in \Gamma_t \cap \partial B_R$ we have

$$\min\big(|q-a_1|, |q-b_1|, |q-a_3|, |q-b_3|\big) \geq \frac{1}{2}\min\left(\sqrt{R}, \frac{\delta(t)}{\varepsilon + R^{-1/2}}\right).$$

Therefore, using the particular boundary data we have on ∂B_R, if $q \in \Gamma_t \cap U_i$, then

$$d\big(p_i, u(q)\big) \leq \exp\left(-c\min\left(\sqrt{R}, \frac{\delta(t)}{\varepsilon + R^{-1/2}}\right)\right),$$

In view of Lemma 8, we deduce that

$$E(u, \Gamma_t) \geq d(p, p') - \exp\left(-c\min\left(\sqrt{R}, \frac{\delta(t)}{\varepsilon + R^{-1/2}}\right)\right), \tag{65}$$

where $p = p_1$ and $p' = p_2$ in case (a), where $p = p_2$ and $p' = p_3$ in case (b), and where $p = p_1$ and $p' = p_3$ in case (c).

If we integrate (65) with respect to t, in view of (60) and (64), we find

$$E(u, B_R) \geq |a_1 - b_1| d_{12} + |a_3 - b_3| d_{23} - C\big(\varepsilon + R^{-1/2} + Re^{-c\sqrt{R}}\big).$$

This is not enough to contradict (59), but we may now use (57) to improve the lower bound. First, using again the fact that the transition angles tend to $\pm\pi/2$ as $n \to +\infty$, we note that u_1 and u_3 tend to $(0,1)$ as $n \to +\infty$. This implies that, for n large enough,

$$|\mathscr{T}| \geq 2R_0 - \eta, \quad \text{where } \mathscr{T} = \big\{t \in \mathbb{R} \,\big|\, \Gamma_t \cap \mathscr{R}_{\text{left}} \neq \emptyset \text{ and } \Gamma_t \cap \mathscr{R}_{\text{right}} \neq \emptyset\big\}. \tag{66}$$

For each $t \in \mathscr{T}$, Γ_t is a broken line of length $2R + o(1)$, and from the definition of $\mathscr{R}_{\text{left}}$ and $\mathscr{R}_{\text{right}}$ there are points $s_1, s_3 \in \Gamma_t$ such that

$$d\big(u(s_1), p_1\big) < \eta, \quad d\big(u(s_3), p_3\big) < \eta, \quad s_1,\ s_3 \in B_{R_0}. \tag{67}$$

These two points divide Γ_t in three portions I_{left}, I_0 and I_{right}. The energy of u on I_{left} and I_{right} is bounded below using Lemma 8:

$$E(u, I_{\text{left}}) \geq d\big(p_1, u(s_1)\big) - Ce^{-cR}, \qquad E(u, I_{\text{right}}) \geq d\big(u(s_3), p_3\big) - Ce^{-cR}.$$

Then, to bound I_0 from below, we note that in light of (67) and our assumption of uniqueness of the geodesic joining p_1 and p_3, we know that the geodesic joining the points $u(s_1)$ and $u(s_3)$ must be close to $\zeta_{12} \cup \zeta_{23}$. Therefore, in light of Assumption 3, we may invoke Lemma 9 to find that

$$E(u, I_0) \geq d\big(u(s_1), u(s_3)\big) + ce^{-CR_0}.$$

Adding these inequalities we obtain

$$E(u, \Gamma_t) \geq d_{13} + ce^{-CR_0} - Ce^{-cR}. \tag{68}$$

We now integrate our lower bounds with respect to t, using either (65) or (68) according to whether $t \in \mathscr{T}$ or not. In view of (66) we find that

$$E(u, B_R) \geq R\big(|e^{i\alpha_{21}} - e^{i\beta_{12}}| d_{12} + |e^{i\alpha_{32}} - e^{i\beta_{23}}| d_{23}\big) + cR_0 e^{-cR_0} - \Delta,$$

where

$$\Delta = C\big(\varepsilon + R^{-1/2} + Re^{-c\sqrt{R}} + Re^{-cR\eta}\big).$$

Since $\Delta \to 0$ as $n \to +\infty$, we obtain a contradiction with (59) if n is large enough, proving Theorem 4.

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