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Algebra

# Representations of a noncommutative and noncocommutative bialgebra: quantum version

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**Abstract.** In this article we classify all simple modules over a noncommutative and noncocommutative bialgebra $M(p,q)$ assuming $q$ is a root of unity.

**Keywords.** Simple modules, polynomial identity algebra, Smith normal form, eigenvalue analysis, noncommutative and noncocommutative bialgebra.

**2020 Mathematics Subject Classification.** 16D60, 16D70, 16S85.

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**Notation.**

- The set of nonnegative integers is denoted by $\mathbb{Z}^{\geq 0}$.
- $\mathbb{K}$ is an algebraically closed field of characteristic 0.
- $q$ is a nonzero element in $\mathbb{K}$ and a primititve $m$-th root of unity.
- All modules in the article are right modules.
- $(k)_q = \frac{q^k-1}{q-1}$.

## 1. Introduction

Quantum groups are mathematical objects that emerged from the study of the quantum inverse scattering method, particularly the Yang–Baxter equation. They are noncommutative and noncocommutative Hopf algebras.

In [11] Chen first constructed a noncommutative and noncocommutative bialgebra $M(p,q)$ for two parameters $p$ and $q$ which is generated as an algebra by four elements $a$, $b$, $c$ and $d$ satisfying relations similar to those of $U_q(\mathfrak{sl}_2)$. Then he localized at the two group-like generators to get a Hopf algebra $H(p,q)$ containing $M(p,q)$. If $q=1$, $H(p,q)$ was shown to be braided. He also showed that if $q$ is a root of the $n$-th cyclotomic polynomial ($n \geq 2$) then $H(p,q)$ has a finite-dimensional quotient Hopf algebra $H_n(p,q)$ which is a double cross product. If $q$ is a primitive $n$-th root of unity and $p \neq 0$, $H_n(p,q)$ is quasi-triangular and is isomorphic to a Drinfeld double [12]. The methods used involve skew pairings and double cross products.

The algebra $M(p,q)$ is generated as an algebra by four elements $a$, $b$, $c$ and $d$ satisfying relations

$$ba = qab, \qquad db = qbd, \qquad ca = qac, \qquad dc = qcd, \qquad bc = cb, \qquad da - qad = p(1-bc),$$

where $p$ and $q$ are elements in $\mathbb{K}$ with $q \neq 0$. $\{a^l b^s c^t d^r : l,s,t,r \in \mathbb{Z}^{\geq 0}\}$ is a basis for the algebra $M(p,q)$ [11, Proposition 2.3]. Also the algebra $M(p,q)$ is Noetherian and can be written as an iterated skew polynomial ring

$$M(p,q) = \mathbb{K}[b,c][a;\sigma_1][d;\sigma_2,\delta_2]$$

where $\sigma_1$ is an automorphism of $\mathbb{K}[b,c]$ with $\sigma_1(b) = q^{-1}b, \sigma_1(c) = q^{-1}c$ and $\sigma_2$ is an automorphism of $\mathbb{K}[b,c][a;\sigma_1]$ with $\sigma_2(b) = qb$, $\sigma_2(c) = qc$ and $\sigma_2(a) = qa$. Finally $\delta_2$ is a $\sigma_2$-derivation of $\mathbb{K}[b,c][a;\sigma_1]$ with $\delta_2(b) = \delta_2(c) = 0$ and $\delta_2(a) = p(1-bc)$. Also $M(p,q)$ is a noncommutative and noncocommutative bialgebra [11, Theorem 2.4].

The main purpose of this article is to classify the simple modules over the bialgebra $M(p,q)$ assuming $p \neq 0$. Note that if $p = 0$ then $M(p,q)$ turns into a quantum affine space whose simple modules have already been studied in [16]. We classify all simple modules over $M(p,q)$ on which the actions of $b$ and $c$ are invertible (i.e. $b$ and $c$-torsionfree simple $M(p,q)$-module). As both $b$ and $c$ are normal elements in $M(p,q)$ their action on any simple module is either 0 or invertible. If $b$ (respectively $c$) acts as 0 on any simple $M(p,q)$-module then it becomes a simple module over a quantum affine space $M(p,q)/\langle b\rangle$ (respectively $M(p,q)/\langle c\rangle$). We first show that the algebra is a polynomial identity algebra and compute its PI-degree. This guarantees us that all simple $M(p,q)$-modules are finite dimensional and then we proceed to classify all the simple $M(p,q)$-modules.

After a two-decade hiatus, there has been a renewed interest in classifying various natural classes of simple or indecomposable modules over both established and newly developed infinite-dimensional algebras. Much of the recent progress has focused on the classification of simple modules over generalized Weyl algebras (see [1–3,5–9]) and Ore extensions with Dedekind rings as coefficient rings (see [4]). Additionally, Jordan has provided a classification of finite-dimensional simple $R$-modules for a specific category of iterated skew polynomial rings, specifically $R = A[y;\alpha][x;\alpha^{-1},\delta]$, where $A$ is an affine, commutative domain over an algebraically closed field (see [14]).

**Arrangement.** The article is organized as follows. In Section 2, we present some preliminary facts about the algebra and prove that it is a PI algebra (see Theorem 2). We then focus on computing the PI-degree using the De Concini–Procesi algorithm [13, Proposition 7.1], and ultimately determine the PI-degree (see Theorem 3). In Section 3, we construct three nonisomorphic simple modules over the algebra (see Remark 8). In Section 4, we identify some central elements of the algebra, which, by Schur's Lemma, must act as scalars. Based on whether these scalars are zero or nonzero, we classify the simple modules over the algebra, ultimately proving that the three nonisomorphic simple modules we previously constructed are the entire set of simple modules (Theorem 9). Finally, Section 5 provides the necessary and sufficient conditions for two simple modules within the same class to be isomorphic, completing the classification problem.

## 2. Preliminaries

In this section, our primary goal is to demonstrate that $M(p,q)$ is a polynomial identity algebra. This ensures that all simple modules of $M(p,q)$ are finite dimensional, with their dimensions bounded above by the PI-degree of the algebra [10, Theorem I.13.5]. The following lemma will be useful to prove the main results.

**Lemma 1.** *The following identities hold in $M(p,q)$:*

(1) $d^k a = q^k a d^k + p(k)_q(1-q^{k-1}bc)d^{k-1}$;

(2) $a^k d = q^{-k} d a^k - p q^{-k}(k)_q(1-q^{-(k-1)}bc)a^{k-1}$.

Now we are ready to state the main result.

**Theorem 2.** *$M(p,q)$ is a PI algebra.*

**Proof.** It is clear from Lemma 1 the subalgebra $A$ generated by $a^m, b^m, c^m, d^m$ is a central subalgebra of $M(p,q)$. Note $M(p,q)$ is a finitely generated module over $A$. Hence the result follows from [15, Corollary 13.1.13]. □

**Theorem 3.** PI-deg$\big(M(p,q)\big) = m$.

**Proof.** From [10, Corollary I.14.1] we have

$$\text{PI-deg}\big(M(p,q)\big) = \text{PI-deg}(Q)$$

where the antisymmetric matrix $H$ associated to $Q$ is a $4 \times 4$ matrix

$$\begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Now we can easily verify that $H$ is similar to the integral matrix

$$H' = \operatorname{diag}\Big(\big(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\big), 0_2\Big).$$

Hence they share the same invariant factors. Hence by [17, Lemma 5.7], PI-deg$\big(M(p,q)\big) =$ ord$(q) = m$. □

**Remark 4.** From the above discussion, it is quite clear that each simple $M(p,q)$-module is finite dimensional and can have dimension at most PI-deg $M(p,q)$. Therefore the calculation of PI-degree for $M(p,q)$ is of substantial importance.

## 3. Construction of simple modules

In this section we construct three nonisomorphic simple modules over $M(p,q)$.

**Simple modules of type I.** For $(\alpha,\beta,\gamma,\delta) \in (\mathbb{K}^*)^3 \times \mathbb{K}$, let us consider the $\mathbb{K}$-vector space $\mathcal{V}_1(\alpha,\beta,\gamma,\delta)$ with basis $\{v_k : 0 \le k \le m-1\}$. Define an $M(p,q)$-module structure on the $\mathbb{K}$-space $\mathcal{V}_1(\alpha,\beta,\gamma,\delta)$ as follows:

$$v_k a = \begin{cases} \big(\delta q^k + p(k)_q(1-q^{k-1}\beta\gamma)\big)v_{k-1}, & k \neq 0, \\ \alpha^{-1}\delta v_{m-1}, & k = 0, \end{cases}$$

$$v_k b = \beta q^k v_k,$$

$$v_k c = \gamma q^k v_k$$

$$v_k d = \begin{cases} v_{k+1}, & k \neq m-1, \\ \alpha v_0, & k = m-1. \end{cases}$$

These actions indeed define a $M(p,q)$-module structure on $\mathcal{V}_1(\alpha,\beta,\gamma,\delta)$. Note that for $k \neq 0, m-1$:

$$\begin{aligned} v_k(da - qad) &= v_{k+1}a - q\big(\delta q^k + p(k)_q(1-q^{k-1}\beta\gamma)\big)v_{k-1}d \\ &= \big(\delta q^{k+1} + p(k+1)_q(1-q^k\beta\gamma)\big)v_k - \big(\delta q^{k+1} + pq(k)_q(1-q^{k-1}\beta\gamma)\big)v_k \\ &= (p - pq^{2k}\beta\gamma)v_k \\ &= v_k\big(p(1-bc)\big). \end{aligned}$$

The other relations are easy to check. Now we have the following theorem.

**Theorem 5.** *$\mathcal{V}_1(\alpha,\beta,\gamma,\delta)$ is a simple $M(p,q)$-module of dimension $m$.*

**Proof.** Let $W$ be a nonzero submodule of $\mathcal{V}_1(\alpha,\beta,\gamma,\delta)$. Then we claim that $W$ contains a basis vector of $\mathcal{V}_1(\alpha,\beta,\gamma,\delta)$. Indeed, let $w=\sum_{i=0}^{m-1}\lambda_i v_i$, with $\lambda_i\in\mathbb{K}$, be a nonzero element of $W$. Suppose that there are two nonzero scalars, say $\lambda_k,\lambda_s$. Since $W$ is a submodule, the vector $w\cdot b$ is in $W$, where $w\cdot b=\sum_{i=0}^{m-1}\beta q^i\lambda_i v_i$. Then we consider the vector

$$w\cdot b-\beta q^k w=\sum_{i=0,\,i\neq k}^{m-1}(q^i-q^k)\beta\lambda_i v_i \tag{1}$$

in $N$. If $w\cdot b-\beta q^k w=0$, then from (1) we obtain that $q^s=q^k$. This implies that $m$ divides $(s-k)$, which contradicts the assumption that $0\le k\neq s\le m-1$. Thus $w\cdot b-\beta q^k w$ is a nonzero element in $N$ of smaller length than $w$. Hence by induction, there must exist some $0\le k\le m-1$ such that $v_k\in N$. Since the action of $d$ permutes all the basis vectors, we have $W=\mathcal{V}_1(\alpha,\beta,\gamma,\delta)$. This completes the proof. □

**Simple modules of type II.** For $(\alpha,\beta,\gamma)\in(\mathbb{K}^*)^3$, let us consider the $\mathbb{K}$-vector space $\mathcal{V}_2(\alpha,\beta,\gamma)$ with basis $\{v_k:0\le k\le m-1\}$. Define an $M(p,q)$-module structure on the $\mathbb{K}$-space $\mathcal{V}_2(\alpha,\beta,\gamma)$ as follows:

$$\begin{aligned}
v_k a &= \begin{cases} v_{k+1}, & k\neq m-1,\\ \alpha v_0, & k=m-1,\end{cases}\\
v_k b &= \beta q^{-k} v_k,\\
v_k c &= \gamma q^{-k} v_k,\\
v_k d &= \begin{cases} -pq^{-k}(k)_q(1-q^{-(k-1)}\beta\gamma)v_{k-1}, & k\neq 0,\\ 0, & k=0.\end{cases}
\end{aligned}$$

Using similar arguments as in Theorem 5, we have the following theorem.

**Theorem 6.** *$\mathcal{V}_2(\alpha,\beta,\gamma)$ is a simple $M(p,q)$-module of dimension $m$.*

**Simple modules of type III.** For $(\alpha,\beta)\in(\mathbb{K}^*)^2$ and depending on the parameters $\alpha$ and $\beta$ define

$$s:=\text{solution of the equation } (r)_q(\alpha\beta-q^{r-1})=0. \tag{2}$$

let us consider the $\mathbb{K}$-vector space $\mathcal{V}_3(\alpha,\beta,s)$ with basis $\{v_k:0\le k\le s-1\}$. Define an $M(p,q)$-module structure on the $\mathbb{K}$-space $\mathcal{V}_3(\alpha,\beta,s)$ as follows:

$$\begin{aligned}
v_k a &= \begin{cases} v_{k+1}, & k\neq s-1,\\ 0, & k=s-1,\end{cases}\\
v_k b &= \alpha q^{-k} v_k,\\
v_k c &= \beta q^{-k} v_k,\\
v_k d &= \begin{cases} -pq^{-k}(k)_q(1-q^{-(k-1)}\alpha\beta)(v_{k-1}), & k\neq 0,\\ 0, & k=0.\end{cases}
\end{aligned}$$

Again using similar arguments as in Theorem 5, we have the following theorem.

**Theorem 7.** *$\mathcal{V}_3(\alpha,\beta,s)$ is a simple $M(p,q)$-module of dimension $s$.*

**Remark 8.** We have the following observations:

(1) $d^m$ does not annihilate the simple $M(p,q)$-module $\mathcal{V}_1(\alpha,\beta,\gamma,\delta)$;
(2) $d^m$ annihilates the simple $\mathcal{V}_2(\alpha,\beta,\gamma)$-module $\mathcal{V}_2(\alpha,\beta,\gamma)$, but $a^m$ does not;

(3) both $d^m$ and $a^m$ annihilate the simple $M(p,q)$-module $\mathcal{V}_3(\alpha,\beta,s)$.

Thus the above three types of simple $M(p,q)$-modules are nonisomorphic.

## 4. Classification of simple modules

Let $N$ be a simple $M(p,q)$-module. Since $d^m$ and $a^m$ are central element in $M(p,q)$, by Schur's lemma they act as scalars on $N$, say $\mu_1$ and $\mu_2$ respectively. Depending on the scalars $\mu_1$ and $\mu_2$ we have the following three cases.

**Case I.** Let $\mu_1 \neq 0$. Note that the elements $ad$, $b$ and $c$ commute in $M(p,q)$. Since $N$ is finite dimensional they have a common eigenvector say $v$. Let

$$vad = \lambda_1 v, \qquad vb = \lambda_2 v, \qquad vc = \lambda_3 v$$

where $\lambda_1 \in \mathbb{K}$ and $\lambda_2, \lambda_3 \in \mathbb{K}^*$. Let us choose $\alpha := \mu_1$, $\beta := \lambda_2$, $\gamma := \lambda_3$ and $\delta := \lambda_1$ so that $(\alpha,\beta,\gamma,\delta) \in (\mathbb{K}^*)^3 \times \mathbb{K}$. Now define a $\mathbb{K}$-linear map $\Phi_1 : \mathcal{V}_1(\alpha,\beta,\gamma,\delta) \to N$ by specifying the image of the basis vectors of $\mathcal{V}_1(\alpha,\beta,\gamma,\delta)$ as follows:

$$\Phi_1(v_k) := vd^k, \quad 0 \le k \le m-1.$$

One can easily verify that $\Phi_1$ is a nonzero $M(p,q)$-module homomorphism. In this verification the following computations in $N$ will be very useful:

- for $k \neq 0$,

$$\begin{aligned}(vd^k)a &= v\big(q^k ad^k + p(k)_q(1-q^{k-1}bc)d^{k-1}\big)\\ &= \big(\lambda_1 q^k + p(k)_q(1-q^{k-1}\lambda_2\lambda_3)\big)vd^{k-1};\end{aligned}$$

- for $k = 0$,

$$(v)a = \alpha^{-1}(vd^m)a = \alpha^{-1}(vad^m) = \alpha^{-1}\lambda_1(vd^{m-1}).$$

Thus by Schur's lemma, $\Phi_1$ is an isomorphism because $\mathcal{V}_1(\alpha,\beta,\gamma,\delta)$ and $N$ are both simple $M(p,q)$-modules.

**Case II.** Let $\mu_1 = 0$ and $\mu_2 \neq 0$, i.e. $d$ is a nilpotent operator on $N$. Note that the commuting operators $b$ and $c$ keep $\ker(d)$ invariant. Hence we can choose the common eigenvector $u \in \ker d$. Assume

$$ub = \lambda_1 b, \qquad uc = \lambda_2 c$$

where $\lambda_1, \lambda_2 \in \mathbb{K}^*$. Let us choose $\alpha := \mu_2$, $\beta := \lambda_1$, $\gamma := \lambda_2$ so that $(\alpha,\beta,\gamma) \in (\mathbb{K}^*)^3$. Now define a $\mathbb{K}$-linear map $\Phi_2 : \mathcal{V}_2(\alpha,\beta,\gamma) \to N$ by specifying the image of the basis vectors of $\mathcal{V}_2(\alpha,\beta,\gamma)$ as follows:

$$\Phi_2(v_k) := ua^k, \quad 0 \le k \le m-1.$$

One can easily verify that $\Phi_2$ is a nonzero $M(p,q)$-module homomorphism. Thus again by Schur's lemma, $\Phi_2$ is an isomorphism because $\mathcal{V}_2(\alpha,\beta,\gamma)$ and $N$ are both simple $M(p,q)$-modules.

**Case III.** Let $\mu_1 = \mu_2 = 0$, i.e. both $a$ and $d$ are nilpotent operators on $N$. Again as in Case II we can choose the common eigenvector of the operators $b$ and $c$ in $\ker d$. Assume

$$ub = \lambda_1 u, \qquad uc = \lambda_2 u$$

where $\lambda_1, \lambda_2 \in \mathbb{K}^*$. Since $\mu_2 = 0$, let $r$ be the minimum index such that $ua^{r-1} \neq 0$ and $ua^r = 0$. Then

$$\begin{aligned}0 &= (ua^r)d\\ &= u\big(q^{-r}da^r - pq^{-r}(r)_q(1-q^{-(r-1)}bc)a^{r-1}\big)\\ &= -pq^{-r}(r)_q(1-q^{-(r-1)}\lambda_1\lambda_2)(ua^{r-1}).\end{aligned}$$

This implies $(r)_q(1-q^{-(r-1)}\lambda_1\lambda_2)=0$. Therefore we set

$$s := \text{solution of the equation } (r)_q(\alpha\beta - q^{r-1}) = 0$$

and $\alpha := \lambda_1$, $\beta := \lambda_2$. Define a $\mathbb{K}$-linear map $\Phi_3\colon \mathcal{V}_3(\alpha,\beta,s)\to N$ by specifying the image of the basis vectors of $\mathcal{V}_3(\alpha,\beta,s)$ as follows:

$$\Phi_3(v_k) := ua^k, \quad 0\le k\le s-1.$$

One can easily verify that $\Phi_3$ is a nonzero $M(p,q)$-module homomorphism. Thus again by Schur's lemma, $\Phi_3$ is an isomorphism because $\mathcal{V}_3(\alpha,\beta,s)$ and $N$ are both simple $M(p,q)$-modules.

The preceding discussion leads to one of the key results of this section, offering a framework for classifying simple $M(p,q)$-modules in terms of scalar parameters.

**Theorem 9.** *Suppose $\mathcal{N}$ is a simple $M(p,q)$-module on which $b$ and $c$ act invertibly. Then $\mathcal{N}$ is isomorphic to one of the following simple $M(p,q)$-modules:*

(1) *$\mathcal{V}_1(\alpha,\beta,\gamma,\delta)$ for some $(\alpha,\beta,\gamma,\delta)\in(\mathbb{K}^*)^3\times\mathbb{K}$ if $\mathcal{N}$ is $d$-torsionfree;*
(2) *$\mathcal{V}_2(\alpha,\beta,\gamma)$ for some $(\alpha,\beta,\gamma)\in(\mathbb{K}^*)^3$ if $\mathcal{N}$ is $d$-torsion and $a$-torsionfree;*
(3) *$\mathcal{V}_3(\alpha,\beta,s)$ for some $(\alpha,\beta)\in(\mathbb{K}^*)^2$ and $s :=$ solution of the equation $(r)_q(\alpha\beta-q^{r-1})=0$ if $\mathcal{N}$ is $a$-torsion and $d$-torsion.*

## 5. Isomorphism between simple $M(p,q)$-modules

In this section, we investigate the conditions under which two modules, as classified in Theorem 9, are isomorphic. Remark 8 implies that modules from different types, as described in the theorem, cannot be isomorphic to one another. However, it remains possible for two distinct modules of the same type to be isomorphic.

**Theorem 10.** *Let $(\alpha_1,\beta_1,\gamma_1,\delta_1)$ and $(\alpha_2,\beta_2,\gamma_2,\delta_2)$ belong to $(\mathbb{K}^*)^3\times\mathbb{K}$. Then $\mathcal{V}_1(\alpha_1,\beta_1,\gamma_1,\delta_1)\cong\mathcal{V}_1(\alpha_2,\beta_2,\gamma_2,\delta_2)$ if and only if $\alpha_1=\alpha_2$ and there exists some $k$ with $0\le k\le m-1$ such that*

$$\beta_1 = q^k\beta_2,\ \gamma_1=q^k\gamma_2 \qquad \text{and} \qquad \delta_1=\delta_2 q^k + p(k)_q(1-q^{k-1}\beta_2\gamma_2).$$

**Proof.** Suppose $\phi\colon \mathcal{V}_1(\alpha_1,\beta_1,\gamma_1,\delta_1)\to\mathcal{V}_1(\alpha_2,\beta_2,\gamma_2,\delta_2)$ is a module isomorphism. The central element $d^m$ acts on $\mathcal{V}_1(\alpha_1,\beta_1,\gamma_1,\delta_1)$ and $\mathcal{V}_1(\alpha_2,\beta_2,\gamma_2,\delta_2)$ as multiplication by $\alpha_1$ and $\alpha_2$ respectively. This implies $\alpha_1=\alpha_2$. Note that $v_k=v_0d^k$ holds in $\mathcal{V}_1(\alpha_1,\beta_1,\gamma_1,\delta_1)$. Therefore $\phi$ can be uniquely determined by $\phi(v_0)$. Suppose

$$\phi(v_0)=\sum_{p\in I}\lambda_p v_p$$

where $I$ is a nonempty subset of $\{0,\ldots,m-1\}$ with $\lambda_p\neq 0$ for all $p\in I$. Now the equality $\phi(v_0b)=\phi(v_0)b$ gives $\beta_1-\beta_2q^p=0$ for all $p\in I$. This implies the index set $I$ must be a singleton. Therefore $\phi(v_0)=\lambda_k v_k$ for some $k$ with $0\le k\le m-1$ and $\lambda_k\neq 0$. Then the equalities $\phi(v_0b)=\phi(v_0)b$ and $\phi(v_0b)=\phi(v_0)b$ give $\beta_1=q^k\beta_2$ and $\gamma_1=q^k\gamma_2$ respectively. Similarly the equality $\phi(v_0ad)=\phi(v_0)ad$ gives

$$\delta_1=\delta_2q^k+p(k)_q(1-q^{k-1}\beta_2\gamma_2).$$

Conversely assume the relations between $(\alpha_1,\beta_1,\gamma_1,\delta_1)$ and $(\alpha_2,\beta_2,\gamma_2,\delta_2)$ hold. Define a $\mathbb{K}$-linear map $\psi\colon\mathcal{V}_1(\alpha_1,\beta_1,\gamma_1,\delta_1)\to\mathcal{V}_1(\alpha_2,\beta_2,\gamma_2,\delta_2)$ by $\psi(v_i)=\lambda v_{k+i}$ for some nonzero scalar $\lambda$, with indices taken modulo $m$. It is straightforward to verify that $\psi$ is an $M(p,q)$-module isomorphism. □

Similarly we can state the following theorems.

**Theorem 11.** *Let* $(\alpha_1,\beta_1,\gamma_1)$ *and* $(\alpha_2,\beta_2,\gamma_2)$ *belong to* $(\mathbb{K}^*)^3$. *Then* $\mathcal{V}_2(\alpha_1,\beta_1,\gamma_1)\cong\mathcal{V}_2(\alpha_2,\beta_2,\gamma_2)$ *if and only if* $\alpha_1=\alpha_2$ *and there exists some* $k$ *with* $0\le k\le m-1$ *such that*

$$\beta_1=q^{-k}\beta_2 \qquad \text{and} \qquad \gamma_1=q^{-k}\gamma_2.$$

**Theorem 12.** *Let* $(\alpha_1,\beta_1)$ *and* $(\alpha_2,\beta_2)$ *belong to* $(\mathbb{K}^*)^2$. *Then* $\mathcal{V}_3(\alpha_1,\beta_1,s_1)\cong\mathcal{V}_3(\alpha_2,\beta_2,s_2)$ *if and only if* $s_1=s_2$ *and there exists some* $k$ *with* $0\le k\le s_1-1$ *such that*

$$\alpha_1=q^{-k}\alpha_2 \qquad \text{and} \qquad \beta_1=q^{-k}\beta_2.$$

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