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Remarks on Gauduchon Kähler-like manifolds

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Abstract. We introduce conditions weaker than Gauduchon Kähler-like and use them to derive rigidity results. These results extend those obtained in [15,16] under the stronger Gauduchon Kähler-like assumption.

1. Introduction

Let X be a complex manifold with $\dim_{\mathbb{C}} X = n \geq 2$. Let

$$\omega = \sqrt{-1} g_{i\bar{j}} \mathrm{d} z^i \wedge \mathrm{d} \bar{z}^j$$

be a Hermitian metric on X.

A metric ω is called Gauduchon if $\partial \bar{\partial} \omega^{n-1} = 0$, balanced if $\mathrm{d} \omega^{n-1} = 0$, and Kähler if $\mathrm{d} \omega = 0$. While there are obstructions to the existence of balanced (or Kähler) metrics, a classical result due to Gauduchon [5] asserts the following: on any compact complex manifold X with $n \geq 2$, there exists a unique (up to a positive constant) Gauduchon metric in the conformal class of every Hermitian metric.

Let ∇ and ∇^B be the Chern and Bismut connections of a Hermitian metric ω. For any $t \in \mathbb{R}$, the corresponding Gauduchon connection ∇^t is defined by [6]

$$\nabla^t = (1 - t)\nabla + t \nabla^B.$$

If ω is non-Kähler, the Gauduchon connections ∇^t are all distinct. In particular, the Lichnerowicz connection $\nabla^{\frac{1}{2}}$ coincides with the restriction of the complexified Levi-Civita connection to the holomorphic tangent bundle $T^{1,0}X$. For the geometry of these connections, see [4,10,12] and references therein.

The connection ∇^t is called Kähler-like if its curvature F satisfies all the symmetries of a Kähler metric's curvature; namely, its $(2,0)$- and $(1,1)$-parts satisfy $F^{(2,0)} = 0$ and $F^{(1,1)}$ is symmetric (see (11)). For background on Kähler-like manifolds, we refer to [1,9,13,15,16] and the references therein.

Let $F^{t,1}_{i\bar{j}}$, $F^{t,2}_{i\bar{j}}$, $F^{t,3}_{i\bar{j}}$ and $F^{t,4}_{i\bar{j}}$ denote the four Ricci curvatures of $F^{(1,1)}$ (see Definition 9). If ∇^t is Kähler-like, then these four Ricci curvatures coincide.

Our starting point is the following.

Proposition 1. *There exists a Hermitian metric ω on the Hopf surface $S^3 \times S^1$, whose curvatures of the Bismut connection ∇^B satisfy*

$$F^{(2,0)} = 0, \qquad F^{1,1}_{i\bar{j}} = F^{1,2}_{i\bar{j}} = F^{1,3}_{i\bar{j}} = F^{1,4}_{i\bar{j}}.$$

While ∇^B is not Kähler-like.

Our main results are the following.

Theorem 2. *Let (X, ω) be a compact Hermitian manifold with $n \geq 3$. If the curvature of the Gauduchon connection ∇^t satisfies*

$$F^{t,1}_{i\bar{j}} = F^{t,3}_{i\bar{j}}$$

for some $t \in \big((-\infty, 2 - \sqrt{3}] \cup [2 + \sqrt{3}, +\infty)\big) \setminus \{0\}$, then ω is Kähler. While for $t = 0$, ω is balanced.

As an instance of Theorem 2 with $t = 0$, consider the Iwasawa threefold with its natural metric ω. It is well known that ω is balanced and has a flat Chern connection ∇. Furthermore, for the special case of compact complex surfaces, the conclusion of Theorem 2 can be strengthened under an additional hypothesis.

Theorem 3. *Let (X, ω) be a compact Hermitian surface. If the curvature of the Gauduchon connection ∇^t satisfies*

$$F^{t,1}_{i\bar{j}} = F^{t,2}_{i\bar{j}} = F^{t,3}_{i\bar{j}}$$

for some $t \neq 1$, then ω is Kähler.

During the proof of Theorem 3, we note that the Lichnerowicz connection $\nabla^{\frac{1}{2}}$ has a special property valid for all dimensions n (see (18)). This leads us to study Gauduchon connections ∇^t satisfying

$$F^{(2,0)} = 0, \qquad F^{t,1}_{i\bar{j}} = F^{t,2}_{i\bar{j}} = F^{t,3}_{i\bar{j}}. \tag{1}$$

If ∇^t is Kähler-like, then ∇^t satisfies (1).

Proposition 4. *Let (X, ω) be a compact Gauduchon manifold with $n \geq 3$. If the curvature of the Lichnerowicz connection $\nabla^{\frac{1}{2}}$ satisfies (1), then ω is Kähler.*

The compactness condition can be replaced by introducing an additional connection ∇^s.

Proposition 5. *Let (X, ω) be a Hermitian surface. Let s and t be two real numbers with $s \neq t$. If both the curvatures of Gauduchon connections ∇^s and ∇^t satisfy (1), then ω is Kähler.*

Proposition 6. *Let (X, ω) be a Hermitian manifold with $n \geq 3$. Let s and t be two real numbers with $(s - t)\big(t - \frac{1}{2}\big) \neq 0$. If both the curvatures of Gauduchon connections ∇^s and ∇^t satisfy (1), then ω is Kähler.*

Theorems 1 and 2 and Proposition 1 of [15] were proved under the assumption that ∇^t is Kähler-like. Our present results (Theorems 2 and 3, and Propositions 4, 5 and 6) extend those in [15] to weaker conditions.

The following results can be proved by arguments analogous to those for Theorems 2, 3 and Proposition 4; their proofs are therefore omitted.

Theorem 7. *Let (X, ω) be a compact Hermitian manifold with $n \geq 3$. If the curvature of the Bismut connection ∇^B satisfies (1), then ω is a Gauduchon metric satisfying*

$$\Lambda_\omega \partial \bar{\partial} \omega = 0, \qquad \nabla^B \tau = 0,$$

where $\tau = \Lambda_\omega \partial \omega$.

Theorem 8. *Let (X, ω) be a Hermitian surface. Then the following statements are equivalent:*

  (1) *the Bismut connection ∇^B satisfies (1);*
  (2) $\nabla^B T = 0$;
  (3) *ω is Vaisman.*

Here T is the torsion of the Chern connection ∇ of ω.

Theorem 7 can be seen as an analogue, to some extent, of [16, Theorem 1]. The cited theorem characterizes when the Bismut connection ∇^B of a (not necessarily compact) Hermitian manifold (X, ω) is Kähler-like by the conditions $\partial\bar{\partial}\omega = 0$ and $\nabla^B T = 0$. (We remark that the condition $\Lambda_\omega \partial\bar{\partial}\omega = 0$ appears in works such as [3,14].) In a similar vein, Theorem 8 corresponds to [16, Theorem 2].

Section 2 sets up the notation and main definitions, and contains the proof of Proposition 1. Theorems 2, 3 and Proposition 4 are proved in Sections 3 and 4.

The proofs of Propositions 5 and 6 are omitted, as they follow directly from the arguments for [15, Theorem 2].

2. Preliminaries

Let X be a complex manifold with $\dim_{\mathbb{C}} X = n \geq 2$. Let

$$\omega = \sqrt{-1} g_{i\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$$

be a Hermitian metric on X. Denote

$$\Gamma^p_{ik} = g^{p\bar{j}} \partial_k g_{i\bar{j}}.$$

The Chern connection ∇ of ω is the unique connection which is compatible with the complex structure of X and the Hermitian metric ω. Locally, we have

$$\nabla_k \left(\frac{\partial}{\partial z^i} \right) = \Gamma^p_{ik} \frac{\partial}{\partial z^p}, \qquad \nabla_{\bar{l}} \left(\frac{\partial}{\partial z^i} \right) = 0,$$

where $\nabla_k = \nabla_{\frac{\partial}{\partial z^k}}$ and $\nabla_{\bar{l}} = \nabla_{\frac{\partial}{\partial \bar{z}^l}}$ denote the covariant derivatives of ∇ along the vector fields $\frac{\partial}{\partial z^k}$ and $\frac{\partial}{\partial \bar{z}^l}$. Hence, for $\theta = \theta_{\bar{l}} \mathrm{d}\bar{z}^l$, we have

$$\nabla_k \theta_{\bar{l}} = \partial_k \theta_{\bar{l}}, \qquad \nabla_{\bar{l}} \bar{\theta}_{\bar{k}} = \partial_{\bar{l}} \bar{\theta}_{\bar{k}}.$$

These terms may be used interchangeably.

The Chern connection ∇ has torsion tensor T whose components are

$$T_{ki\bar{j}} = \partial_k g_{i\bar{j}} - \partial_i g_{k\bar{j}}.$$

Then, we have $T_{ki\bar{j}} = -T_{ik\bar{j}}$. Denote

$$T^p_{ik} := g^{p\bar{j}} T_{ik\bar{j}} = \Gamma^p_{ki} - \Gamma^p_{ik}$$

and

$$T_i := g^{k\bar{l}} T_{ik\bar{l}}.$$

Then the torsion $(1,0)$-form τ of ∇ is defined by

$$\tau := \Lambda \partial \omega = T_i \mathrm{d}z^i.$$

Since each Hermitian metric ω satisfies

$$\partial \omega^{n-1} = \tau \wedge \omega^{n-1}, \qquad (2)$$

it is direct to verify that ω is balanced if and only if

$$\tau = 0,$$

and that ω is Gauduchon if and only if

$$\partial^* \tau = 0.$$

Moreover, for a Hermitian metric ω, we calculate

$$-\partial^* \tau = g^{k\bar{l}} \nabla_{\bar{l}} T_k + |\tau|^2, \tag{3}$$

where

$$g^{k\bar{l}} \nabla_{\bar{l}} T_k = g^{k\bar{l}} \nabla_k \overline{T}_l, \qquad |\tau|^2 = g^{k\bar{l}} T_k \overline{T}_l.$$

Integration by parts will be used frequently. Let (X, ω) be a compact Hermitian manifold. For $\theta = \theta_{\bar{l}} d\bar{z}^l$, we obtain from (2) that

$$\begin{aligned}
0 &= n \int_X \sqrt{-1} \, d\big((\theta - \bar{\theta}) \wedge \omega^{n-1}\big) \\
&= n \int_X \sqrt{-1} (\partial\theta - \bar{\partial}\bar{\theta} + \tau \wedge \theta + \bar{\theta} \wedge \bar{\tau}) \wedge \omega^{n-1} \\
&= \int_X 2 \operatorname{Re}\big(g^{k\bar{l}} (\nabla_k \theta_{\bar{l}} + T_k \theta_{\bar{l}})\big) \omega^n.
\end{aligned} \tag{4}$$

Hereafter we may omit the volume form $\frac{\omega^n}{n!}$.

The curvature R_ω of ∇ is

$$\nabla \circ \nabla \left(\frac{\partial}{\partial z^i}\right) = R^p_{ik\bar{l}} \, dz^k \wedge d\bar{z}^l \otimes \frac{\partial}{\partial z^p},$$

where (e.g., [8, Chapter 1])

$$R^p_{ik\bar{l}} = g^{p\bar{j}}(-\partial_{\bar{l}} \partial_k g_{i\bar{j}} + g^{m\bar{n}} \partial_{\bar{l}} g_{m\bar{j}} \partial_k g_{i\bar{n}}). \tag{5}$$

Denote

$$R_{i\bar{j}k\bar{l}} = g_{p\bar{j}} R^p_{ik\bar{l}} = -\partial_{\bar{l}} \partial_k g_{i\bar{j}} + g^{m\bar{n}} \partial_{\bar{l}} g_{m\bar{j}} \partial_k g_{i\bar{n}}.$$

Then we have

$$R_{i\bar{j}k\bar{l}} = \overline{R}_{j\bar{i}l\bar{k}}. \tag{6}$$

The components of the four Ricci curvatures are

$$R_{i\bar{j}} = g^{k\bar{l}} R_{k\bar{l}i\bar{j}}, \qquad K_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}}, \qquad R^3_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{l}k\bar{j}}, \qquad R^4_{i\bar{j}} = g^{k\bar{l}} R_{k\bar{j}i\bar{l}}.$$

Then we obtain from (6) that

$$R_{i\bar{j}} = \overline{R}_{j\bar{i}}, \qquad K_{i\bar{j}} = \overline{K}_{j\bar{i}}, \qquad R^3_{i\bar{j}} = \overline{R^4}_{j\bar{i}}.$$

The Bianchi identities are well-known (e.g., [11, (2.6)]):

$$\begin{aligned}
R_{i\bar{j}k\bar{l}} - R_{k\bar{j}i\bar{l}} &= \nabla_{\bar{l}} T_{ik\bar{j}}, \qquad R_{i\bar{j}k\bar{l}} - R_{i\bar{l}k\bar{j}} = \nabla_k \overline{T}_{jl\bar{i}}, \\
R_{i\bar{j}k\bar{l}} - R_{k\bar{l}i\bar{j}} &= \nabla_{\bar{l}} T_{ik\bar{j}} + \nabla_i \overline{T}_{jl\bar{k}} = \nabla_{\bar{j}} T_{ik\bar{l}} + \nabla_k \overline{T}_{jl\bar{i}}.
\end{aligned} \tag{7}$$

In the local expression (5), we follow the notations in [8]. So the first two indices and the last two indices of the curvature tensor in [11, (2.4)] have been swapped. We obtain from (7) that

$$\begin{aligned}
R^3_{i\bar{j}} &= R_{i\bar{j}} + \nabla_{\bar{j}} T_i, \\
K_{i\bar{j}} - R_{i\bar{j}} &= \frac{1}{2}(\nabla_{\bar{j}} T_i + \nabla_i \overline{T}_j) - \frac{1}{2} g^{k\bar{l}}(\nabla_{\bar{l}} T_{ki\bar{j}} + \nabla_k \overline{T}_{lj\bar{i}}).
\end{aligned} \tag{8}$$

The Bismut connection ∇^B of ω is defined by

$$\nabla^B_k \left(\frac{\partial}{\partial z^i}\right) = \Gamma^p_{ki} \frac{\partial}{\partial z^p}, \qquad \nabla^B_{\bar{l}} \left(\frac{\partial}{\partial z^i}\right) = g^{p\bar{q}} \overline{T}_{lq\bar{i}} \frac{\partial}{\partial z^p}.$$

Then for every $t \in \mathbb{R}$, the Gauduchon connection ∇^t is defined by [6]

$$\nabla_k \left(\frac{\partial}{\partial z^i}\right) = \big((1-t)\Gamma^p_{ik} + t\Gamma^p_{ki}\big) \frac{\partial}{\partial z^p}, \qquad \nabla_{\bar{l}} \left(\frac{\partial}{\partial z^i}\right) = t g^{p\bar{q}} \overline{T}_{lq\bar{i}} \frac{\partial}{\partial z^p}. \tag{9}$$

The curvature of the Gauduchon connection ∇^t is

$$\nabla^t \circ \nabla^t \left(\frac{\partial}{\partial z^i}\right) = \left(\frac{1}{2} F^p_{imk}\, \mathrm{d}z^m \wedge \mathrm{d}z^k + F^p_{ik\bar{l}}\, \mathrm{d}z^k \wedge \mathrm{d}\bar{z}^l + \frac{1}{2} F^p_{i\bar{n}\bar{l}}\, \mathrm{d}\bar{z}^n \wedge \mathrm{d}\bar{z}^l\right) \otimes \frac{\partial}{\partial z^p},$$

where (e.g., [4, Proposition 4.2])

$$F^p_{imk} = t g^{p\bar{j}}\left(\nabla_i T_{mk\bar{j}} + (1-t)(T_{kr\bar{j}} T^r_{mi} + T_{mr\bar{j}} T^r_{ik})\right) = -g^{p\bar{j}} \overline{F}^q_{j\bar{m}\bar{k}} g_{i\bar{q}}$$

and

$$F^t_{i\bar{j}k\bar{l}} = g_{p\bar{j}} F^p_{ik\bar{l}} = R_{i\bar{j}k\bar{l}} + t(\nabla_{\bar{l}} T_{ki\bar{j}} + \nabla_k \overline{T}_{lj\bar{i}}) + t^2 g^{p\bar{q}}(T_{ki\bar{q}} \overline{T}_{lj\bar{p}} - T_{pk\bar{j}} \overline{T}_{ql\bar{i}}).$$

By (6), we have

$$F^t_{i\bar{j}k\bar{l}} = \overline{F}^t_{j\bar{i}l\bar{k}}. \tag{10}$$

Hence, ∇^t is Kähler-like if and only if (e.g., [15, Section 2])

$$F^p_{imk} = 0, \qquad F^t_{i\bar{j}k\bar{l}} = F^t_{k\bar{j}i\bar{l}}. \tag{11}$$

Indeed, we obtain from (10) and the second identity that in (11) that $F^t_{k\bar{l}i\bar{j}} = F^t_{i\bar{j}k\bar{l}}$.

Denote

$$Q^1_{i\bar{j}} = g^{k\bar{l}} g^{p\bar{q}} T_{ik\bar{q}} \overline{T}_{jl\bar{p}}, \qquad Q^2_{i\bar{j}} = g^{k\bar{l}} g^{p\bar{q}} T_{pk\bar{j}} \overline{T}_{ql\bar{i}}$$

and

$$U_{i\bar{j}} = g^{p\bar{q}} \overline{T}_q T_{pi\bar{j}}, \qquad V_{i\bar{j}} = g^{k\bar{l}} \nabla_{\bar{l}} T_{ki\bar{j}}.$$

Then we have

$$Q^1_{i\bar{j}} = \overline{Q}^1_{j\bar{i}}, \qquad Q^2_{i\bar{j}} = \overline{Q}^2_{j\bar{i}}.$$

Definition 9. *Let (X, ω) be a Hermitian manifold. The four Ricci curvatures of the Gauduchon connection ∇^t are*

$$F^{t,1}_{i\bar{j}} = g^{k\bar{l}} F^t_{k\bar{l}i\bar{j}} = R_{i\bar{j}} + t(\nabla_{\bar{j}} T_i + \nabla_i \overline{T}_j),$$

$$F^{t,2}_{i\bar{j}} = g^{k\bar{l}} F^t_{i\bar{j}k\bar{l}} = K_{i\bar{j}} + t(V_{i\bar{j}} + \overline{V}_{j\bar{i}}) + t^2(Q^1_{i\bar{j}} - Q^2_{i\bar{j}}),$$

$$F^{t,3}_{i\bar{j}} = g^{k\bar{l}} F^t_{i\bar{l}k\bar{j}} = R^3_{i\bar{j}} - t(\nabla_{\bar{j}} T_i + \overline{V}_{j\bar{i}}) - t^2(Q^1_{i\bar{j}} + \overline{U}_{j\bar{i}}),$$

$$F^{t,4}_{i\bar{j}} = g^{k\bar{l}} F^t_{k\bar{j}i\bar{l}} = R^4_{i\bar{j}} - t(\nabla_i \overline{T}_j + V_{i\bar{j}}) - t^2(Q^1_{i\bar{j}} + U_{i\bar{j}}).$$

Then we have (by (10))

$$F^{t,1}_{i\bar{j}} = \overline{F}^{t,1}_{j\bar{i}}, \qquad F^{t,2}_{i\bar{j}} = \overline{F}^{t,2}_{j\bar{i}}, \qquad F^{t,3}_{i\bar{j}} = \overline{F}^{t,4}_{j\bar{i}}.$$

Remark 10. Let (X, ω) be a compact Hermitian manifold.

If $V_{i\bar{j}} = 0$, then

$$g^{i\bar{j}} \nabla_{\bar{j}} T_i = g^{i\bar{j}} V_{i\bar{j}} = 0.$$

Integration by parts (see (4)) yields

$$0 = -\int_X \partial^* \tau = \int_X \left(g^{i\bar{j}} \nabla_{\bar{j}} T_i + |\tau|^2\right) = \int_X |\tau|^2,$$

which implies ω is balanced.

Hence, for $t = 0$, we obtain from (8) and Definition 9 that

$$F^{0,1}_{i\bar{j}} = F^{0,2}_{i\bar{j}} = F^{0,3}_{i\bar{j}} = F^{0,4}_{i\bar{j}} = R_{i\bar{j}}.$$

Moreover, if ω is of pointwise constant (Chern) holomorphic sectional curvature $u \in \mathscr{A}^0_{\mathbb{R}}(X) \setminus \{0\}$, then ω is Kähler and u is a (non-zero) constant. The proof is similar to [7, Theorem 1.1], so we omit it.

We close this section by a proof of Proposition 1.

Suppose $n \geq 2$ and $(\delta_1, \ldots, \delta_n) \in \mathbb{C}^n \setminus \{0\}$ with $|\delta_1| = \cdots = |\delta_n| \neq 1$. Let $S^{2n-1} \times S^1 = (\mathbb{C}^n \setminus \{0\})/\sim$ be the Hopf manifold, where

$$(z^1, \ldots, z^n) \sim (\delta_1 z^1, \ldots, \delta_n z^n).$$

For $\alpha > 0$ and $\beta > -\alpha$, we consider the natural metric $\omega = \sqrt{-1}\, g_{i\bar{j}}\, \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$ on $S^{2n-1} \times S^1$, where

$$g_{i\bar{j}} = \alpha \frac{\delta_{ij}}{|z|^2} + \beta \frac{\bar{z}^i z^j}{|z|^4}$$

for $|z|^2 = \sum_{l=1}^n |z^l|^2$. Denote $\gamma = \frac{\beta}{\alpha} > -1$.

Proof of Proposition 1. Direct calculation yields that the $(1,1)$-part of the curvature of ∇^t of ω is (e.g., [2, Section 3] for $t = 1$)

$$
\begin{aligned}
F^t_{i\bar{j}k\bar{l}} = &\left(\left((1+\gamma)(2-t)t - \gamma\right)\delta_{il}\delta_{kj} + \left(1 - 2(1+\gamma)t\right)\delta_{ij}\delta_{kl}\right)\frac{\alpha}{|z|^4} \\
&+ \left(\left((1+\gamma t)(\gamma - t) - (\gamma t)^2\right)(\bar{z}^i z^l \delta_{kj} + \bar{z}^k z^j \delta_{il})\right. \\
&+ \left(\left((1+\gamma)t - \gamma\right)^2 + 2\gamma\right)\bar{z}^i z^j \delta_{kl} + \left(2(1+\gamma)t - 1\right)\bar{z}^k z^l \delta_{ij}\right)\frac{\alpha}{|z|^6} \\
&+ \alpha\gamma\left((1+\gamma)t^2 - (3+\gamma)\right)\frac{\bar{z}^i z^j \bar{z}^k z^l}{|z|^8},
\end{aligned}
$$

which implies

$$F^{t,1}_{i\bar{j}} = \left(\delta_{ij} - \frac{\bar{z}^i z^j}{|z|^2}\right)\frac{n - 2(n-1)(1+\gamma)t}{|z|^2}$$

and

$$F^{t,2}_{i\bar{j}} = \left(n - \left(t^2 + 2(n-2)t + 1\right)(1+\gamma)\right)\frac{\delta_{ij}}{|z|^2} + \left(\left(2n - 1 + (n-1)\gamma\right)\gamma - 2(1+\gamma)\left(1 + (n-1)\gamma\right)t\right.$$
$$\left. + \left((n-1)\gamma + n\right)(1+\gamma)t^2\right)\frac{\bar{z}^i z^j}{|z|^4}$$

and

$$F^{t,3}_{i\bar{j}} = \left(-n(1+\gamma)t^2 + (2n-3)(1+\gamma)t + 1 - (n-1)\gamma\right)\frac{\delta_{ij}}{|z|^2}$$
$$+ \left((n-1)\gamma - 1 + (1+\gamma)\left((n-1)\gamma - (n-2)\right)t + \left(1 - (n-1)\gamma\right)(1+\gamma)t^2\right)\frac{\bar{z}^i z^j}{|z|^4}$$
$$= F^{t,4}_{i\bar{j}}.$$

A necessary condition for $F^{t,1}_{i\bar{j}} = F^{t,2}_{i\bar{j}}$ is

$$n - 2(n-1)(1+\gamma)t = n - \left(t^2 + 2(n-2)t + 1\right)(1+\gamma),$$

i.e.,

$$(1+\gamma)(t-1)^2 = 0.$$

The only choice is $t = 1$ since $1 + \gamma > 0$.

Then we have

$$F^{1,1}_{i\bar{j}} = F^{1,2}_{i\bar{j}} = \left(\delta_{ij} - \frac{\bar{z}^i z^j}{|z|^2}\right)\frac{n - 2(n-1)(1+\gamma)}{|z|^2}$$

and

$$F^{1,3}_{i\bar{j}} = F^{1,4}_{i\bar{j}} = \left(\delta_{ij} - \frac{\bar{z}^i z^j}{|z|^2}\right)\frac{n - 2(1+\gamma)}{|z|^2}.$$

Hence $F_{i\bar{j}}^{1,1} = F_{i\bar{j}}^{1,3}$ if and only if

$$(n-2)(1+\gamma) = 0,$$

i.e., $n = 2$ since $1 + \gamma > 0$.

Direct calculation yields

$$\Gamma_{ik}^p = \left(\gamma \bar{z}^i \delta_{pk} - \bar{z}^k \delta_{pi} - \gamma \frac{\bar{z}^i \bar{z}^k z^p}{|z|^2}\right) \frac{1}{|z|^2} \quad \text{and} \quad T_{ki\bar{j}} = (\bar{z}^i \delta_{kj} - \bar{z}^k \delta_{ij}) \frac{\alpha(1+\gamma)}{|z|^4},$$

which implies

$$\nabla_p T_{ki\bar{j}} = \partial_p T_{ki\bar{j}} - T_{mi\bar{j}} \Gamma_{kp}^m - T_{km\bar{j}} \Gamma_{ip}^m = 0.$$

Then the $(2, 0)$-part of the curvature of ∇^t of ω is $F_{imk}^p = \frac{1}{2} g^{p\bar{j}} \nabla_i T_{mk\bar{j}} = 0$.

It is direct to check the Bismut connection ∇^B of ω is not Kähler-like. $\qquad\square$

3. Proofs of Theorems 2 and 3

Let (X, ω) be a compact Hermitian manifold with $n \geq 2$.

Proof of Theorem 2. Since $F_{i\bar{j}}^{t,3} = \overline{F}_{j\bar{i}}^{t,4}$ for every Hermitian metric, we obtain from $F_{i\bar{j}}^{t,1} = F_{i\bar{j}}^{t,3}$ that

$$F_{i\bar{j}}^{t,3} = F_{i\bar{j}}^{t,4}.$$

By (8) and Definition 9, we get

$$F_{i\bar{j}}^{t,3} = R_{i\bar{j}} + \frac{1-t}{2}(\nabla_{\bar{j}} T_i + \nabla_i \overline{T}_j) - \frac{t}{2}(V_{i\bar{j}} + \overline{V}_{j\bar{i}}) - \frac{t^2}{2}(U_{i\bar{j}} + \overline{U}_{j\bar{i}}) - t^2 Q_{i\bar{j}}^1,$$

which implies

$$(1 - 3t)(\nabla_{\bar{j}} T_i + \nabla_i \overline{T}_j) = t(V_{i\bar{j}} + \overline{V}_{j\bar{i}}) + t^2(U_{i\bar{j}} + \overline{U}_{j\bar{i}}) + 2t^2 Q_{i\bar{j}}^1. \qquad (12)$$

Since

$$g^{i\bar{j}} V_{i\bar{j}} = g^{i\bar{j}} \nabla_{\bar{j}} T_i = g^{i\bar{j}} \nabla_i \overline{T}_j = g^{i\bar{j}} \overline{V}_{j\bar{i}},$$

$$g^{i\bar{j}} U_{i\bar{j}} = |\tau|^2 = g^{i\bar{j}} \overline{U}_{j\bar{i}},$$

$$g^{i\bar{j}} Q_{i\bar{j}}^1 = g^{i\bar{j}} Q_{i\bar{j}}^2 = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} T_{pk\bar{j}} \overline{T}_{ql\bar{i}} = |T|^2,$$

we obtain from (12) that

$$(1 - 4t) g^{i\bar{j}} \nabla_{\bar{j}} T_i = t^2(|\tau|^2 + |T|^2).$$

Integration by parts yields (see (3) and (4))

$$0 = (t^2 - 4t + 1) \int_X |\tau|^2 + t^2 \int_X |T|^2. \qquad (13)$$

Let $n \geq 3$. If $t = 0$, the identity (13) is

$$0 = \int_X |\tau|^2,$$

which implies ω is balanced. If $t \in \left((-\infty, 2 - \sqrt{3}] \cup [2 + \sqrt{3}, +\infty)\right) \setminus \{0\}$, the right-hand side of (13) is non-negative, which implies ω is Kähler. $\qquad\square$

Remark 11. In Theorem 2, if we replace the condition $F_{i\bar{j}}^{t,1} = F_{i\bar{j}}^{t,3}$ by one of the following conditions:

(1) $F_{i\bar{j}}^{t,2} = F_{i\bar{j}}^{t,3}$;

(2) $F_{i\bar{j}}^{t,1} + F_{i\bar{j}}^{t,2} = F_{i\bar{j}}^{t,3} + F_{i\bar{j}}^{t,4}$;

we still have the identity (13), while other combinations of the four Ricci curvatures fail to give interesting identities like (13).

To prove Theorem 3, we need the following.

Lemma 12. *Let (X, ω) be a compact Hermitian surface. If*

$$F_{i\bar{j}}^{\frac{1}{3},1} = F_{i\bar{j}}^{\frac{1}{3},2} = F_{i\bar{j}}^{\frac{1}{3},3} = F_{i\bar{j}}^{\frac{1}{3},4},$$

then ω is Kähler.

With Lemma 12 in hand, we can prove Theorem 3.

Proof of Theorem 3. Suppose $F_{i\bar{j}}^{t,1} = F_{i\bar{j}}^{t,3}$. Since (see (2))

$$|T|^2 = 2|\tau|^2,$$

we obtain from (13) that

$$0 = (t-1)(3t-1)\int_X |\tau|^2.$$

By Lemma 12, for $t \neq 1$, we have $\tau = 0$, which implies ω is Kähler by (2). $\qquad\square$

We have to prove Lemma 12. We calculate for general $n \geq 2$ and $t \in \mathbb{R}$.

Let (X, ω) be a Hermitian manifold whose Gauduchon connection ∇^t satisfies

$$F_{i\bar{j}}^{t,1} = F_{i\bar{j}}^{t,2} = F_{i\bar{j}}^{t,3} = F_{i\bar{j}}^{t,4}$$

for some $t \in \mathbb{R}$.

By the Bianchi identities (7), we get

$$\nabla_{\bar{j}} T_{ik\bar{l}} + \nabla_{\bar{l}} T_{ki\bar{j}} = \nabla_i \overline{T}_{jl\bar{k}} + \nabla_k \overline{T}_{lj\bar{i}},$$

which implies

$$\nabla_{\bar{j}} T_i + V_{i\bar{j}} = \nabla_i \overline{T}_j + \overline{V}_{j\bar{i}}. \tag{14}$$

Since $F_{i\bar{j}}^{t,1} = F_{i\bar{j}}^{t,2}$, we obtain from (8) and Definition 9 that

$$(2t-1)(\nabla_{\bar{j}} T_i + \nabla_i \overline{T}_j) = (2t-1)(V_{i\bar{j}} + \overline{V}_{j\bar{i}}) + 2t^2(Q_{i\bar{j}}^1 - Q_{i\bar{j}}^2). \tag{15}$$

Combining (12) with (15) yields

$$V_{i\bar{j}} + \overline{V}_{j\bar{i}} = (3-8t)(\nabla_{\bar{j}} T_i + \nabla_i \overline{T}_j) - 2t^2(U_{i\bar{j}} + \overline{U}_{j\bar{i}} + Q_{i\bar{j}}^1 + Q_{i\bar{j}}^2)$$

Inserting the above identity into (15) yields

$$(2t-1)(4t-1)(\nabla_{\bar{j}} T_i + \nabla_i \overline{T}_j) = t^2(1-2t)(U_{i\bar{j}} + \overline{U}_{j\bar{i}}) + 2t^2(1-t)Q_{i\bar{j}}^1 - 2t^3 Q_{i\bar{j}}^2. \tag{16}$$

Since $F_{i\bar{j}}^{t,3} = F_{i\bar{j}}^{t,4}$, we obtain from Definition 9 that

$$(t-1)(\nabla_{\bar{j}} T_i - \nabla_i \overline{T}_j) = t(V_{i\bar{j}} - \overline{V}_{j\bar{i}}) + t^2(U_{i\bar{j}} - \overline{U}_{j\bar{i}}).$$

Inserting (14) into the above identity yields

$$(2t-1)(\nabla_{\bar{j}} T_i - \nabla_i \overline{T}_j) = t^2(U_{i\bar{j}} - \overline{U}_{j\bar{i}}). \tag{17}$$

If $t = \frac{1}{4}$, then ω is already Kähler by (13). Thus for $t \neq \frac{1}{4}$, we obtain from (16) and (17) that

$$(2t-1)(4t-1)\nabla_{\bar{j}} T_i = t^3 U_{i\bar{j}} + t^2(1-3t)\overline{U}_{j\bar{i}} + t^2(1-t)Q_{i\bar{j}}^1 - t^3 Q_{i\bar{j}}^2. \tag{18}$$

Take $n = 2$. Then we have (see (2))

$$U_{i\bar{j}} = |\tau|^2 g_{i\bar{j}} - T_i T_{\bar{j}}, \qquad V_{i\bar{j}} = g^{k\bar{l}} \nabla_{\bar{l}} T_k g_{i\bar{j}} - \nabla_{\bar{j}} T_i$$

and

$$Q_{i\bar{j}}^1 = |\tau|^2 g_{i\bar{j}}, \qquad Q_{i\bar{j}}^2 = 2|\tau|^2 g_{i\bar{j}} - 2T_i T_{\bar{j}}.$$

Inserting these into (18) for $t = \frac{1}{3}$ yields

$$-3\nabla_{\bar{j}} T_i = |\tau|^2 g_{i\bar{j}} + T_i T_{\bar{j}}, \tag{19}$$

which implies

$$-g^{i\bar{j}}\nabla_{\bar{j}} T_i = |\tau|^2.$$

We obtain from (3) that ω is Gauduchon.

Proof of Lemma 12. Let (X, ω) be a compact Hermitian surface.

By (2), we calculate

$$[\nabla_{\bar{j}}, \nabla_{\bar{l}}] T_i = \nabla_{\bar{q}} T_i \overline{T}^q_{lj} = \overline{T}_l \nabla_{\bar{j}} T_i - \overline{T}_j \nabla_{\bar{l}} T_i.$$

which implies

$$-3 \cdot 2 \operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} T_k \nabla_{\bar{j}} \nabla_{\bar{l}} T_i) = -3 \cdot 2 \operatorname{Re}\big(g^{k\bar{l}} T_k \nabla_{\bar{l}}(g^{i\bar{j}} \nabla_{\bar{j}} T_i)\big) - 6|\tau|^2 g^{k\bar{l}} \nabla_{\bar{l}} T_k$$
$$+ 3 \cdot 2 \operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} T_k \overline{T}_j \nabla_{\bar{l}} T_i).$$

Since ω is Gauduchon, integration by parts (see (4)) for the right-hand side yields

$$-3 \int_X 2 \operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} T_k \nabla_{\bar{j}} \nabla_{\bar{l}} T_i) = 6 \int_X |\tau|^4 + 3 \int_X 2 \operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} T_k \overline{T}_j \nabla_{\bar{l}} T_i). \tag{20}$$

On the other hand, we obtain from (19) that

$$-3 \cdot 2 \operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} T_k \nabla_{\bar{j}} \nabla_{\bar{l}} T_i) = 2 \operatorname{Re}\big(g^{k\bar{l}} T_k \nabla_{\bar{l}} |\tau|^2\big) + 2 \operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} T_k T_i \nabla_{\bar{j}} \overline{T}_j) + 2|\tau|^2 g^{k\bar{l}} \nabla_{\bar{l}} T_k.$$

Integration by parts (see (4)) yields

$$-3 \int_X 2 \operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} T_k \nabla_{\bar{j}} \nabla_{\bar{l}} T_i) = 2 \int_X 2 \operatorname{Re}\big(g^{k\bar{l}} T_k \nabla_{\bar{l}} |\tau|^2\big) - \int_X 2 \operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} T_k \overline{T}_j \nabla_{\bar{l}} T_i) - 2 \int_X |\tau|^4$$
$$= -\int_X 2 \operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} T_k \overline{T}_j \nabla_{\bar{l}} T_i) - 2 \int_X |\tau|^4.$$

Combining the above identity with (20) yields

$$2 \int_X |\tau|^4 = -\int_X 2 \operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} T_k \overline{T}_j \nabla_{\bar{l}} T_i).$$

Inserting (19) into the right-hand side yields

$$6 \int_X |\tau|^4 = -3 \int_X 2 \operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} T_k \overline{T}_j \nabla_{\bar{l}} T_i) = 4 \int_X |\tau|^4.$$

We have $\tau = 0$, which implies ω is Kähler by (2). □

4. Proof of Proposition 4

Let (X, ω) be a compact Hermitian manifold with $n \geq 3$. We consider the Lichnerowicz connection $\nabla^{\frac{1}{2}}$ of ω satisfying

$$F^p_{imk} = 0, \qquad F^{\frac{1}{2},1}_{i\bar{j}} = F^{\frac{1}{2},2}_{i\bar{j}} = F^{\frac{1}{2},3}_{i\bar{j}}. \tag{21}$$

By the calculation in Section 2 and the first identity in (21), we have

$$\nabla_k T_i = -\frac{1}{2}(T_p T^p_{ki} + T^p_{km} T^m_{ip}). \tag{22}$$

With the second identity in (21) in hand, we obtain from (15) that

$$Q^1_{i\bar{j}} = Q^2_{i\bar{j}},$$

and from (17) that

$$U_{i\bar{j}} = \overline{U}_{j\bar{i}},$$

and from (12) and (14) that

$$0 = 2(\nabla_{\bar{j}} T_i + V_{i\bar{j}}) + U_{i\bar{j}} + Q^1_{i\bar{j}}. \tag{23}$$

Lemma 13. *Let (X, ω) be a compact Hermitian manifold with $n \geq 3$. If $\nabla^{\frac{1}{2}}$ satisfies (21), then*
$$d(\tau + \bar{\tau}) = 0.$$

Proof. By (22), we have
$$\nabla_k T_i - \nabla_i T_k = T_p T_{ik}^p.$$
While by the definition of the Chern connection,
$$\nabla_k T_i - \nabla_i T_k = \partial_k T_i - \partial_i T_k + T_p T_{ik}^p.$$
Then we have
$$0 = \frac{1}{2}(\partial_k T_i - \partial_i T_k)\, dz^k \wedge dz^i = \partial \tau.$$
We only need to show
$$\bar{\partial}\tau + \partial\bar{\tau} = 0.$$

Step 1. Direct calculation yields
$$[\nabla_{\bar{l}}, \nabla_{\bar{j}}] T_i = \nabla_{\bar{q}} T_i \overline{T}_{jl}^q,$$
which implies
$$2\operatorname{Re}\big(g^{i\bar{j}} g^{k\bar{l}} T_k [\nabla_{\bar{l}}, \nabla_{\bar{j}}] T_i\big) = -2\operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} \nabla_{\bar{l}} T_i U_{k\bar{j}}). \qquad (24)$$
Integration by parts (see (4)) for the left-hand side in (24) yields (by (3))
$$\int_X 2\operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} T_k \nabla_{\bar{l}} \nabla_{\bar{j}} T_i) = \int_X 2\operatorname{Re}\big(g^{k\bar{l}} T_k \nabla_{\bar{l}}(g^{i\bar{j}} \nabla_{\bar{j}} T_i)\big)$$
$$= 2\int_X (\partial^* \tau) g^{i\bar{j}} \nabla_{\bar{j}} T_i$$
and
$$-\int_X 2\operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} T_k \nabla_{\bar{j}} \nabla_{\bar{l}} T_i) = \int_X 2\operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} \nabla_{\bar{j}} T_k \nabla_{\bar{l}} T_i) + \int_X 2\operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} \nabla_{\bar{l}} T_i T_k \overline{T}_j).$$
Integration by parts (see (4)) for the right-hand side in (24) yields
$$-\int_X 2\operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} \nabla_{\bar{l}} T_i U_{k\bar{j}}) = \int_X 2\operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} T_i \nabla_{\bar{l}} U_{k\bar{j}}) + \int_X 2\operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} U_{k\bar{j}} \overline{T}_l T_i)$$
$$= \int_X 2\operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} \nabla_{\bar{j}} \overline{T}_l T_p T_{ki}^p) - \int_X 2\operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} V_{i\bar{l}} T_k \overline{T}_j),$$
where the second equality follows from the definition of $U_{i\bar{j}}$ and $V_{i\bar{j}}$ and
$$2\operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} U_{i\bar{l}} T_k \overline{T}_j) = 2\operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} T_p T_{ki}^p \overline{T}_l \overline{T}_j) = 0, \qquad (25)$$
since $T_{ki}^p = -T_{ik}^p$.

Moreover, we obtain from (22) that
$$2\operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} \nabla_{\bar{j}} \overline{T}_l T_p T_{ki}^p) = -\frac{1}{2} 2\operatorname{Re}\big(g^{i\bar{j}} g^{k\bar{l}} (\overline{T}_q \overline{T}_{jl}^q + \overline{T}_{jn}^q \overline{T}_{lq}^n) T_p T_{ki}^p\big)$$
$$= -\frac{1}{2} 2\operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} \overline{T}_q \overline{T}_{jl}^q T_p T_{ki}^p)$$
$$= \frac{1}{2} 2\operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} Q_{i\bar{l}}^1 T_k \overline{T}_j)$$
$$= g^{i\bar{j}} g^{k\bar{l}} U_{i\bar{l}} U_{k\bar{j}} = |U|^2,$$
where we have used the definition of $Q_{i\bar{j}}^1$ and
$$Q_{i\bar{j}}^1 = Q_{i\bar{j}}^2, \qquad U_{i\bar{j}} = \overline{U}_{j\bar{i}}.$$
Then we obtain from (24) that
$$2\int_X (\partial^* \tau) g^{i\bar{j}} \nabla_{\bar{j}} T_i + \int_X 2\operatorname{Re}(g^{i\bar{j}} g^{k\bar{l}} \nabla_{\bar{j}} T_k \nabla_{\bar{l}} T_i) = \int_X |U|^2 - \int_X 2\operatorname{Re}\big(g^{i\bar{j}} g^{k\bar{l}} (\nabla_{\bar{l}} T_i + V_{i\bar{l}}) T_k \overline{T}_j\big),$$

where (by (23) and (25))

$$-\int_X 2\operatorname{Re}\big(g^{i\bar{j}}g^{k\bar{l}}(\nabla_{\bar{l}}T_i + V_{i\bar{l}})T_k\overline{T}_j\big) = \frac{1}{2}\int_X 2\operatorname{Re}\big(g^{i\bar{j}}g^{k\bar{l}}(U_{i\bar{l}} + Q^1_{i\bar{l}})T_k\overline{T}_j\big)$$
$$= \frac{1}{2}\int_X 2\operatorname{Re}\big(g^{i\bar{j}}g^{k\bar{l}}Q^1_{i\bar{l}}T_k\overline{T}_j\big)$$
$$= \int_X |U|^2.$$

Thus we get

$$\int_X 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{j}}T_k\nabla_{\bar{l}}T_i) = 2\int_X |U|^2 - 2\int_X (\partial^*\tau)g^{i\bar{j}}\nabla_{\bar{j}}T_i. \tag{26}$$

Denote $\Delta = \sqrt{-1}\Lambda_\omega\partial\bar{\partial}$. We have

$$2\Delta|\tau|^2 = 2\operatorname{Re}\big(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{l}}(\nabla_k T_i\overline{T}_j + T_i\nabla_k\overline{T}_j)\big).$$

Integration by parts (see (4)) yields (by (3))

$$2\int_X (\partial^*\tau)|\tau|^2 = \int_X 2\operatorname{Re}\big(g^{i\bar{j}}g^{k\bar{l}}\overline{T}_l(\nabla_k T_i\overline{T}_j + T_i\nabla_k\overline{T}_j)\big),$$

where we obtain from (22) that

$$2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_k T_i\overline{T}_l\overline{T}_j) = -\frac{1}{2}2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}T^p_{km}T^m_{ip}\overline{T}_l\overline{T}_j) = -|U|^2.$$

Then we get

$$\int_X 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_k\overline{T}_j\overline{T}_l T_i) = \int_X |U|^2 + 2\int_X (\partial^*\tau)|\tau|^2. \tag{27}$$

By (24), (26) and (27), we get

$$\int_X 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{l}}T_i U_{k\bar{j}}) = -3\int_X |U|^2 - 2\int_X (\partial^*\tau)|\tau|^2. \tag{28}$$

Step 2. Motivated by [9, Lemma 3.7], we calculate

$$2\operatorname{Re}\big(g^{i\bar{j}}g^{k\bar{l}}\big(\overline{T}_j[\nabla^{\frac{1}{2}}_{\bar{l}},\nabla^{\frac{1}{2}}_k]T_i - \overline{T}_l[\nabla^{\frac{1}{2}}_{\bar{j}},\nabla^{\frac{1}{2}}_k]T_i\big)\big)$$
$$= 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\overline{T}_l\nabla^{\frac{1}{2}}_k\nabla^{\frac{1}{2}}_{\bar{j}}T_i) - 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\overline{T}_j\nabla^{\frac{1}{2}}_k\nabla^{\frac{1}{2}}_{\bar{l}}T_i), \tag{29}$$

where we have used (by (9) and (22))

$$\nabla^{\frac{1}{2}}_k T_i = \nabla_k T_i + \frac{1}{2}T_p T^p_{ki} = -\frac{1}{2}T^p_{km}T^m_{ip} = \nabla^{\frac{1}{2}}_i T_k. \tag{30}$$

Since

$$2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\overline{T}_l\nabla^{\frac{1}{2}}_k\nabla^{\frac{1}{2}}_{\bar{j}}T_i) = 2\operatorname{Re}\big(g^{k\bar{l}}\overline{T}_l\nabla_k(g^{i\bar{j}}\nabla^{\frac{1}{2}}_{\bar{j}}T_i)\big)$$

and

$$\nabla^{\frac{1}{2}}_{\bar{j}}T_i = \nabla_{\bar{j}}T_i - \frac{1}{2}g^{p\bar{q}}T_p\overline{T}_{jq\bar{i}} = \nabla_{\bar{j}}T_i + \frac{1}{2}\overline{U}_{j\bar{i}} = \nabla_{\bar{j}}T_i + \frac{1}{2}U_{i\bar{j}}, \tag{31}$$

integration by parts yields (see (3) and (4))

$$\int_X 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\overline{T}_l\nabla^{\frac{1}{2}}_k\nabla^{\frac{1}{2}}_{\bar{j}}T_i) = 2\int_X (\partial^*\tau)g^{i\bar{j}}\Big(\nabla_{\bar{j}}T_i + \frac{1}{2}U_{i\bar{j}}\Big)$$
$$= 2\int_X (\partial^*\tau)g^{i\bar{j}}\nabla_{\bar{j}}T_i + \int_X (\partial^*\tau)|\tau|^2. \tag{32}$$

By (9), we get

$$\nabla^{\frac{1}{2}}_k\nabla^{\frac{1}{2}}_{\bar{l}}T_i = \partial_k\nabla^{\frac{1}{2}}_{\bar{l}}T_i - \nabla^{\frac{1}{2}}_{\bar{l}}T_p\Gamma^p_{ik} - \frac{1}{2}g^{p\bar{q}}(\nabla^{\frac{1}{2}}_{\bar{l}}T_p T_{ik\bar{q}} + \nabla^{\frac{1}{2}}_{\bar{q}}T_i T_{kp\bar{l}}).$$

Then we have

$$-\int_X 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\overline{T}_j\nabla_k^{\frac{1}{2}}\nabla_{\bar{l}}^{\frac{1}{2}}T_i) = \frac{1}{2}\int_X 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{l}}^{\frac{1}{2}}T_i U_{k\bar{j}}) + \frac{1}{2}\int_X 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{l}}^{\frac{1}{2}}T_i T_k\overline{T}_j)$$
$$+\int_X 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{l}}^{\frac{1}{2}}T_i\nabla_k\overline{T}_j). \quad (33)$$

We obtain from (25), (27), (28) and (31) that

$$\frac{1}{2}\int_X 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{l}}^{\frac{1}{2}}T_i U_{k\bar{j}}) = \frac{1}{2}\int_X 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{l}}T_i U_{k\bar{j}}) + \frac{1}{2}\int_X |U|^2$$
$$= -\int_X |U|^2 - \int_X (\partial^*\tau)|\tau|^2,$$

$$\frac{1}{2}\int_X 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{l}}^{\frac{1}{2}}T_i T_k\overline{T}_j) = \frac{1}{2}\int_X 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{l}}T_i T_k\overline{T}_j)$$
$$= \frac{1}{2}\int_X |U|^2 + \int_X (\partial^*\tau)|\tau|^2,$$

$$\int_X 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{l}}^{\frac{1}{2}}T_i\nabla_k\overline{T}_j) = 2\int_X |\overline{\nabla}\tau|^2 + \frac{1}{2}\int_X 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}U_{i\bar{l}}\nabla_k\overline{T}_j)$$
$$= 2\int_X |\overline{\nabla}\tau|^2 - \frac{3}{2}\int_X |U|^2 - \int_X (\partial^*\tau)|\tau|^2,$$

where

$$|\overline{\nabla}\tau|^2 = g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{l}}T_i\nabla_k\overline{T}_j.$$

Inserting the three identities into (33) yields

$$-\int_X 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\overline{T}_j\nabla_k^{\frac{1}{2}}\nabla_{\bar{l}}^{\frac{1}{2}}T_i) = 2\int_X |\overline{\nabla}\tau|^2 - 2\int_X |U|^2 - \int_X (\partial^*\tau)|\tau|^2.$$

Combining the above identity with (26), (29) and (32), we get

$$\int_X 2\operatorname{Re}\big(g^{i\bar{j}}g^{k\bar{l}}(\overline{T}_j[\nabla_{\bar{l}}^{\frac{1}{2}},\nabla_k^{\frac{1}{2}}]T_i - \overline{T}_l[\nabla_{\bar{j}}^{\frac{1}{2}},\nabla_k^{\frac{1}{2}}]T_i)\big) = 2\int_X |\overline{\nabla}\tau|^2 - 2\int_X |U|^2 + 2\int_X (\partial^*\tau)g^{i\bar{j}}\nabla_{\bar{j}}T_i$$
$$= 2\int_X |\overline{\nabla}\tau|^2 - \int_X 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{j}}T_k\nabla_{\bar{l}}T_i). \quad (34)$$

We deal with the left-hand side of (34). By definition (see (9)), we get (e.g., [9, Corollary 2.5])

$$[\nabla_{\bar{l}}^t,\nabla_k^t]T_i = T_p F_{ik\bar{l}}^p + t g^{p\bar{q}}(\nabla_{\bar{q}}^t T_i T_{kp\bar{l}} - \nabla_p^t T_i \overline{T}_{lq\bar{k}}).$$

For $t = \frac{1}{2}$, we get

$$2\operatorname{Re}\big(g^{i\bar{j}}g^{k\bar{l}}(\overline{T}_j[\nabla_{\bar{l}}^{\frac{1}{2}},\nabla_k^{\frac{1}{2}}]T_i - \overline{T}_l[\nabla_{\bar{j}}^{\frac{1}{2}},\nabla_k^{\frac{1}{2}}]T_i)\big)$$
$$= 2\operatorname{Re}\big(g^{i\bar{j}}g^{k\bar{l}}(F_{i\bar{l}}^{\frac{1}{2},2} - F_{i\bar{l}}^{\frac{1}{2},4})T_k\overline{T}_j\big) + \frac{1}{2}2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_k^{\frac{1}{2}}T_i\overline{T}_q\overline{T}_{jl}^q)$$
$$+ \frac{1}{2}2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_k^{\frac{1}{2}}T_i\overline{T}_l\overline{T}_j) - \frac{1}{2}2\operatorname{Re}\big(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{l}}^{\frac{1}{2}}T_i(U_{k\bar{j}} + T_k\overline{T}_j)\big),$$

where the first and second terms on the right-hand side vanish due to (21) and (30), respectively. For the third term, we obtain from (30) again that

$$\frac{1}{2}2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_k^{\frac{1}{2}}T_i\overline{T}_l\overline{T}_j) = -\frac{1}{2}|U|^2.$$

For the last term, we obtain from (25), (27), (28) and (31) that

$$-\frac{1}{2}\int_X 2\operatorname{Re}\big(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{l}}^{\frac{1}{2}}T_i(U_{k\bar{j}} + T_k\overline{T}_j)\big) = -\frac{1}{2}\int_X 2\operatorname{Re}\big(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{l}}T_i(U_{k\bar{j}} + T_k\overline{T}_j)\big) - \frac{1}{2}\int_X |U|^2$$
$$= \frac{1}{2}\int_X |U|^2.$$

Summing up yields

$$\int_X 2\operatorname{Re}\big(g^{i\bar{j}}g^{k\bar{l}}\big(\overline{T}_j[\nabla^{\frac{1}{2}}_{\bar{l}},\nabla^{\frac{1}{2}}_k]T_i - \overline{T}_l[\nabla^{\frac{1}{2}}_{\bar{j}},\nabla^{\frac{1}{2}}_k]T_i\big)\big) = 0.$$

Inserting it into (34) yields

$$0 = 2\int_X |\overline{\nabla}\tau|^2 - \int_X 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{j}}T_k\nabla_{\bar{l}}T_i). \tag{35}$$

We use the completed square

$$0 \le g^{i\bar{j}}g^{k\bar{l}}(\nabla_{\bar{l}}T_i - \nabla_i\overline{T}_l)(\nabla_k\overline{T}_j - \nabla_{\bar{j}}T_k) = 2|\overline{\nabla}\tau|^2 - 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{j}}T_k\nabla_{\bar{l}}T_i),$$

where the equality holds if and only if

$$\nabla_{\bar{l}}T_i = \nabla_i\overline{T}_l.$$

Then we have $\bar{\partial}\tau + \partial\bar{\tau} = 0$.

The conclusion follows from $\partial\tau = 0$ and $\bar{\partial}\tau + \partial\bar{\tau} = 0$. $\qquad\square$

Proof of Proposition 4. Let (X,ω) be a compact Hermitian manifold with $n \ge 3$.

We obtain from (31) that

$$2|\overline{\nabla}^{\frac{1}{2}}\tau|^2 = 2g^{i\bar{j}}g^{k\bar{l}}\nabla^{\frac{1}{2}}_{\bar{l}}T_i\nabla^{\frac{1}{2}}_k\overline{T}_j = 2|\overline{\nabla}\tau|^2 + 2\operatorname{Re}(g^{i\bar{j}}g^{k\bar{l}}\nabla_{\bar{l}}T_iU_{k\bar{j}}) + \frac{1}{2}|U|^2.$$

By (3), (26), (28) and (35), we have

$$\int_X (\partial^*\tau)^2 = \int_X |\overline{\nabla}^{\frac{1}{2}}\tau|^2 + \frac{1}{4}\int_X |U|^2.$$

Suppose ω is Gauduchon. By (3), we have

$$0 = \int_X |\overline{\nabla}^{\frac{1}{2}}\tau|^2 + \frac{1}{4}\int_X |U|^2,$$

which implies

$$U_{i\bar{j}} = 0.$$

Then we have

$$|\tau|^2 = g^{i\bar{j}}U_{i\bar{j}} = 0.$$

Taking $t = \frac{1}{2}$ in (13) yields

$$0 = 3\int_X |\tau|^2 = \int_X |T|^2,$$

which implies $T = 0$, i.e., ω is Kähler. $\qquad\square$

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Declaration of interests

References

[1]   D. Angella, A. Otal, L. Ugarte and R. Villacampa, "On Gauduchon connections with Kähler-like curvature", *Commun. Anal. Geom.* **30** (2022), no. 5, pp. 961–1006.

[2]   G. Barbaro, "Griffiths positivity for Bismut curvature and its behaviour along Hermitian curvature flows", *J. Geom. Phys.* **169** (2021), article no. 104323 (17 pages).

[3]   J. Fu, Z. Wang and D. Wu, "Semilinear equations, the γ_k function, and generalized Gauduchon metrics", *J. Eur. Math. Soc.* **15** (2013), no. 2, pp. 659–680.

[4]   J. Fu and X. Zhou, "Scalar curvatures in almost Hermitian geometry and some applications", *Sci. China, Math.* **65** (2022), no. 12, pp. 2583–2600.

[5]   P. Gauduchon, "La 1-forme de torsion d'une variété hermitienne compacte", *Math. Ann.* **267** (1984), no. 4, pp. 495–518.

[6]   P. Gauduchon, "Hermitian connections and Dirac operators", *Boll. Unione Mat. Ital., VII. Ser., B* **11** (1997), no. 2, pp. 257–288.

[7]   T. Kai, "Holomorphic sectional curvature and Kähler-like metric", *Sci. Sin., Math.* **51** (2021), no. 12, pp. 2013–2024.

[8]   S. Kobayashi, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, vol. 15, Princeton University Press, 1987.

[9]   R. A. Lafuente and J. Stanfield, "Hermitian manifolds with flat Gauduchon connections", *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)* **25** (2024), no. 4, pp. 2241–2258.

[10] K. Liu and X. Yang, "Ricci curvatures on Hermitian manifolds", *Trans. Am. Math. Soc.* **369** (2017), no. 7, pp. 5157–5196.

[11] M. Sherman and B. Weinkove, "Local Calabi and curvature estimates for the Chern–Ricci flow", *New York J. Math.* **19** (2013), pp. 565–582.

[12] J. Wang and X. Yang, "Curvatures of real connections on Hermitian manifolds", *Pac. J. Math.* **337** (2025), no. 2, pp. 365–391.

[13] B. Yang and F. Zheng, "On curvature tensors of Hermitian manifolds", *Commun. Anal. Geom.* **26** (2018), no. 5, pp. 1195–1222.

[14] X. Yang, "Manifolds with non-positive second Chern–Ricci curvature", *Math. Z.* **310** (2025), no. 4, article no. 72 (14 pages).

[15] Q. Zhao and F. Zheng, "On Gauduchon Kähler-like manifolds", *J. Geom. Anal.* **32** (2022), no. 4, article no. 110 (27 pages).

[16] Q. Zhao and F. Zheng, "Strominger connection and pluriclosed metrics", *J. Reine Angew. Math.* **796** (2023), pp. 245–267.