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On separation of variables for symmetric spaces of rank 1

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Abstract. We study existence and nonexistence of diagonal and separating coordinates for Riemannian symmetric spaces of rank 1. We generalize the results of Gauduchon and Moroianu (2020) by showing that a symmetric space of rank 1 has diagonal coordinates if and only if it has constant sectional curvature. This implies that orthogonal separation of variables on a symmetric space of rank 1 is possible only in the constant sectional curvature case. We show that on the complex projective space $\mathbb{C}P^n$ and on complex hyperbolic space $\mathbb{C}H^n$, with $n \geq 2$, separating coordinates necessarily have precisely n ignorable coordinates. In view of results of Boyer et al. (1983, 1985) and later results of Winternitz et al. (1994), this completes the description of separation of variables on $\mathbb{C}P^n$ for all n and on $\mathbb{C}H^n$ for $n = 2, 3$.

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1. Introduction

All geodesics on Riemannian compact rank 1 symmetric spaces are closed and have the same length. The three families of such manifolds are the standard sphere S^n , the complex projective space $\mathbb{C}P^n$, and the quaternionic projective space $\mathbb{H}P^n$ corresponding to the three associative

division algebras \mathbb{R} , \mathbb{C} , and Hamilton’s quaternions \mathbb{H} . The final isolated example is $\mathbb{O}P^2$ related to non-associative octonions. As symmetric spaces they are given by the following quotients

$$\begin{aligned} SO(n+1)/SO(n) &= S^n, \\ SU(n+1)/S(U(n) \times U(1)) &= \mathbb{C}P^n, \\ Sp(n+1)/(Sp(n) \times Sp(1)) &= \mathbb{H}P^n, \\ F_4/Spin(9) &= \mathbb{O}P^2. \end{aligned}$$

These spaces have noncompact twins, which are rank 1 Riemannian symmetric spaces of negative curvature:

$$\begin{aligned} SO(n,1)/SO(n) &= H^n && \text{(real) hyperbolic space,} \\ SU(n,1)/S(U(n) \times U(1)) &= \mathbb{C}H^n && \text{complex hyperbolic space,} \\ Sp(n,1)/(Sp(n) \times Sp(1)) &= \mathbb{H}H^n && \text{quaternionic hyperbolic space,} \\ F_4^-/Spin(9) &= \mathbb{O}H^2 && \text{Cayley hyperbolic plane.} \end{aligned}$$

The results of this paper can be presented in Table 1, where “diagonal” means that there exist (or do not exist) local coordinates, in which the metric tensor is diagonal, and “separable” means that there exist (or do not exist) local coordinates, in which the geodesic equation and the Laplacian are separable, see definitions in Section 2. In Table 1, we assume that $n \geq 2$.

Table 1.

	diagonal	separable
S^n	yes	orthogonal [19, Theorem 3.3]
H^n	yes	orthogonal [19, Theorem 5.1]
$\mathbb{C}P^n$	no [15, Propositions 3.1 and 3.2]	non-orthogonal, n ignorable variables (Theorem 8)
$\mathbb{C}H^n$	no (Theorem 5)	non-orthogonal, n ignorable variables (Theorem 8)
$\mathbb{H}P^n$	no [15, Proposition 4.1]	no (Theorem 6)
$\mathbb{H}H^n, \mathbb{O}P^2, \mathbb{O}H^2$	no (Theorem 5)	no (Theorem 6)

In more detail, we study $\mathbb{C}P^n$, $\mathbb{C}H^n$, $\mathbb{H}P^n$, $\mathbb{H}H^n$, $\mathbb{O}P^2$ and $\mathbb{O}H^2$ from the viewpoint of local existence of diagonal and separating coordinates. The nonexistence of diagonal coordinate systems on $\mathbb{C}P^n$ and $\mathbb{H}P^n$ was established in [15]. We generalize these results to $\mathbb{C}H^n$, $\mathbb{H}H^n$, $\mathbb{O}P^2$ and $\mathbb{O}H^2$ in Theorem 5. The proof for $\mathbb{C}H^n$ and $\mathbb{H}H^n$ follows the same ideas as in [15].

We next proceed to the study of separation of variables on these spaces; see Section 2 for definitions. The nonexistence of orthogonal separating coordinates follows from the nonexistence of diagonal coordinates. On S^n and H^n non-orthogonal separation coordinates exist, but they can always be transformed into orthogonal ones. We show in Theorem 6 that for $n \geq 2$, the spaces $\mathbb{H}P^n$, $\mathbb{H}H^n$, $\mathbb{O}P^2$ and $\mathbb{O}H^2$ admit no separation of variables.

Non-orthogonal separating variables exist on $\mathbb{C}P^n$ and on $\mathbb{C}H^n$, see [8,9,29,35] where several families of examples have been discussed. All these examples have precisely n ignorable coordinates. On the other hand, all separating coordinate systems on $\mathbb{C}P^n$ with n ignorable coordinates have been classified in [9]. Moreover, it was shown in [9] that on $\mathbb{C}P^2$, any separation of variables has two ignorable coordinates, which completes the classification of separations of variables for this space.

Non-orthogonal separating coordinates on $\mathbb{C}H^2$ with two ignorable coordinates were explicitly constructed in [8]. Non-orthogonal separating coordinates on $\mathbb{C}H^3$ having three ignorable coordinates were explicitly constructed in [35]. In general, the picture for the space $\mathbb{C}H^n$ is substantially more complicated than that for its compact dual $\mathbb{C}P^n$, since the isometry algebra $\mathfrak{su}(n, 1)$ contains $n+2$ pairwise non-conjugate abelian subalgebras of dimension n (not just a Cartan subalgebra!), see [28, Theorem 5.1]. To describe separating coordinate systems with n ignorable coordinates, one has to study whether a given n -dimensional abelian subalgebra of $\mathfrak{su}(n, 1)$ indeed generates ignorable coordinates in a non-orthogonal separation of variables, and if so, which ones. This is quite a nontrivial task which has been completed only for $n = 2, 3$ [35]. Two out of $n+2$ pairwise non-conjugate n -dimensional abelian subalgebras of $\mathfrak{su}(n, 1)$ are Cartan and are relatively easy to handle, for all $n \geq 2$. However, the remaining n have been treated in full detail only for $n = 2, 3$, and the complete description for $n = 3$ obtained in [35] is already quite involved.

We show in Theorem 8 that every separating coordinate system on $\mathbb{C}P^n$ and on $\mathbb{C}H^n$ has precisely n ignorable coordinates (as we mentioned above, for $\mathbb{C}P^2$, this fact is established in [9, Section 6B]). This implies that the classifications of separations of variables on $\mathbb{C}P^n$ and on $\mathbb{C}H^2$ and $\mathbb{C}H^3$ presented in [9,29,35] are complete.

Let us now comment on why we have chosen symmetric spaces for our investigation. It is known since Stäckel [32,33] and Levi-Civita [25] that separating coordinates are closely related to Killing tensors of order one and two. Symmetric spaces of rank 1 have a large algebra of Killing tensors of order one (of Killing vector fields) and also a large algebra of Killing tensors of order two, and so they are in a certain sense natural candidates for the existence of separating coordinates. We also hope that the methods developed in this paper, combined with those from [15], will allow to study separating and diagonal coordinates on symmetric spaces of higher rank. We would like to emphasize that by [26] not all Killing tensors of order two on $\mathbb{H}P^n$ and $\mathbb{O}P^2$ are quadratic polynomials of the Killing vector fields, and so purely algebraic methods to study separation of variables using universal enveloping algebra will not be in general sufficient. We also note that by the results of [27], the study of Killing tensors of reducible symmetric spaces is reduced to that of irreducible components, which may substantially facilitate the study of separating coordinates.

Geodesics of rank 1 symmetric spaces are completely and explicitly described in e.g. [6], in particular in the compact case they are closed and have the same length. The interest in separation of variables goes way beyond the description of the solution of the geodesic equations. In particular, in irreducible symmetric spaces the Eisenhart–Robertson condition (see e.g., [14, Section 2]) is automatically fulfilled so that the Helmholtz equation also separates in the separating coordinate systems. In addition, it is easy to introduce potential energy in the picture, such that the corresponding Hamiltonian system and the corresponding Helmholtz equation still separate.

Separations of variables was studied and used since the middle of the 19th century. The problem of describing and classifying, up to isometries, all separations of variables in space forms has been solved by Eisenhart [14] in small dimensions and under certain nondegeneracy assumptions. The solution for Riemannian space forms was completed in [19]. However, the pseudo-Riemannian case is still open: while the orthogonal case has been completed in [7,23], the description of non-orthogonal separations for constant curvature metrics of indefinite signature is unknown. The present paper solves the natural analog of the Eisenhart problem for all rank 1 symmetric spaces except for $\mathbb{C}H^n$ with $n \geq 4$, and reduces the remaining cases to those which can be solved with computer algebra.

2. Separation of variables

In this paper, we study separation of variables for the geodesic equations and adopt the formal definition of separation of variables which has been already introduced by Levi-Civita [25]. The geodesic flow with metric g_{ij} is described through the Hamiltonian $H(x, p) = \frac{1}{2} \sum g^{ij}(x) p_i p_j$ on the cotangent bundle T^*M . By separation of variables on an N -dimensional manifold M^N we understand the existence of a function $W(x_1, \dots, x_N, c_1, \dots, c_N)$ of $2N$ variables such that the following conditions are fulfilled:

- (a) the $N \times N$ -matrix $\frac{\partial^2 W}{\partial c_i \partial x_j}$ is non-singular;
- (b) $H(x, \frac{\partial W}{\partial x}) = c_1$ (the Hamilton–Jacobi equation);
- (c) $W(x, c) = \sum_{i=1}^N W_i(x_i, c_1, \dots, c_N)$ (where the function W_i depends on x_i and c only).

It is well known, see e.g. [19, Section 1], that the existence of a function W satisfying conditions (a) and (b) (such a function is called a generating function) allows one to construct a local coordinate system $(c_1, \dots, c_N, Q_1, \dots, Q_N)$ on the cotangent bundle T^*M such that $c_1 = H(x, p)$ and such that in this coordinate system, the standard symplectic form $\sum_{i=1}^N dp_i \wedge dx_i$ has the “canonical” form $\sum_{i=1}^N dc_i \wedge dQ_i$. Indeed, consider the following two local mappings:

$$\begin{aligned} \phi: \mathbb{R}^{2N}(x, c) &\longrightarrow \mathbb{R}^{2N}(x, p), & \phi(x, c) &= \left(x, \frac{\partial W}{\partial x}\right) \\ \psi: \mathbb{R}^{2N}(x, c) &\longrightarrow \mathbb{R}^{2N}(Q, c), & \psi(x, c) &= \left(\frac{\partial W}{\partial c}, c\right). \end{aligned} \tag{1}$$

By (a), the mappings ϕ , ψ , and therefore $\psi \circ \phi^{-1}$, are local diffeomorphisms. In view of the equation

$$\begin{aligned} dW &= \sum_i \frac{\partial W}{\partial x_i} dx_i + \sum_i \frac{\partial W}{\partial c_i} dc_i \\ &= \sum_i p_i dx_i + \sum_i Q_i dc_i \quad (\text{after applying } \phi, \psi), \end{aligned}$$

the equation $0 = d(dW)$ is equivalent to $\sum_i d(p_i dx_i) = \sum_i d(c_i dQ_i)$, as claimed. As $H = c_1$ by (b), the Hamiltonian system generated by H looks extremely simple in coordinates (c, Q) and its general solution is given by $(c(t), Q(t)) = (\text{const}_1, \text{const}_2, \dots, \text{const}_N, \text{Const}_1 - t, \text{Const}_2, \dots, \text{Const}_N)$.

Unfortunately, a function W satisfying (a)–(c) exists not for many Hamiltonians. In particular, for most metrics, the required coordinates do not exist at all. Finding coordinates x_1, \dots, x_N , for a given metric for which there exists a function W satisfying (a)–(c) is a nontrivial task. Such coordinates are called *separating coordinates* or *separating variables* in our paper, and we study their existence and construction for symmetric spaces of rank 1. Probably, the only effective way to find such coordinates, which we follow, is based on the relation of the separating coordinates to Killing tensors of order one and two due to Stäckel [32,33] and Levi-Civita [25].

First observe that the coordinates c_1, \dots, c_N viewed as functions on T^*M are functionally independent and Poisson-commute. These functions are called *separation constants*. It is known that they are necessarily either linear or quadratic in momenta p_1, \dots, p_N , and so they correspond to Killing tensors of order one or two. A necessary and sufficient condition for a system of r Killing vectors and $N - r$ Killing tensors on a Riemannian manifold to correspond to a separation of variables is given in the following theorem.

Theorem 1 (follows from [5, Theorem 2.7] or [2, Section 7], [21, Theorem 4] and [1], see also [3,4]). *Let M^N be a Riemannian manifold equipped with a set of r Killing vector fields V_{N-r+1}, \dots, V_N and $N - r$ quadratic Killing tensor fields K_1, \dots, K_{N-r} . There locally exists a separating coordinate system $(x_1, \dots, x_{N-r}, t_{N-r+1}, \dots, t_N)$ on M^N such that the separation constants c_1, \dots, c_N are the linear and the quadratic in momenta functions corresponding to the given Killing vectors and Killing tensors if and only if the following properties hold:*

- (I) *the linear and the quadratic in momenta functions corresponding to the Killing vectors and the Killing tensors, respectively, Poisson commute and are functionally independent;*
- (II) *one of the Killing tensors of order two is the metric tensor;*
- (III) *locally, the quadratic Killing tensors admit $N-r$ mutually orthogonal eigenvector fields X_i , $i = 1, \dots, N-r$, which are orthogonal to all Killing vector fields V_{N-r+1}, \dots, V_N .*

Moreover, in the separating coordinate system, the Killing vector fields V_j are constant linear combinations of the vector fields ∂_{t_i} , $i = N-r+1, \dots, N$, and the separation constants c_1, \dots, c_{N-r} corresponding to the quadratic Killing tensors are given by

$$c_\ell = \sum_{\alpha, \beta=1}^{N-r} k_\ell^{\alpha\beta}(x_1, \dots, x_{N-r}) p_\alpha p_\beta + \sum_{i, j=N-r+1}^N h_\ell^{ij}(x_1, \dots, x_{N-r}) p_i p_j, \tag{2}$$

and the Hamiltonian of the geodesic flow is $c_1/2$. For any $i, j \in N-r+1, \dots, N$, the functions $\sum_{\alpha, \beta=1}^{N-r} k_\ell^{\alpha\beta} p_\alpha p_\beta + h_\ell^{ij}$, $\ell = 1, \dots, N-r$, viewed as functions on the cotangent bundle to $(N-r)$ -dimensional manifolds with local coordinates x_1, \dots, x_{N-r} , are given by the Stäckel formula.

The coordinates t_{N-r+1}, \dots, t_N from Theorem 1 are called *ignorable coordinates*.

Although it is well known, let us recall the Stäckel formula following [14,31]. Take a non-singular $(N-r) \times (N-r)$ matrix $S = (S_{ij})$ with S_{ij} being a function of the i -th variable x_i only. Next, consider the functions c_α , $\alpha = 1, \dots, N-r$, given by the following system of linear equations

$$SC = P, \tag{3}$$

where $C = (c_1, c_2, \dots, c_{N-r})^\top$ and $P = (f_1(x_1)p_1^2 + \phi_1(x_1), f_2(x_2)p_2^2 + \phi_2(x_2), \dots, f_{N-r}(x_{N-r})p_{N-r}^2 + \phi_{N-r}(x_{N-r}))^\top$. It is known that the functions c_i are in involution, and the coordinates x_i are separating for the Hamiltonian system corresponding to any of them, or even to all of them together.

Theorem 1 can be understood, in the Riemannian case, as an equivalent reformulation of the definition of separating coordinates and of their existence. Though this reformulation is sufficient for us, we give the formula for the functions W_i , $i = 1, \dots, N-r$, as

$$W_i(x_i, c_1, \dots, c_N) = \pm \int_0^{x_i} \sqrt{\frac{1}{f_i(\xi)} \left(-\phi_i(\xi) + \sum_{\alpha=1}^{N-r} S_{i\alpha}(\xi) c_\alpha \right)} d\xi,$$

and for $i = N-r+1, \dots, N$, we take $W_i = c_i$.

Remark 2. Theorem 1 is shorter and is visually different from that stated, e.g., in [2,21]. This difference is due to the assumption on the metric being positive definite, which prohibits the coordinates of the second class in the terminology of Benenti. Moreover, it uses a small improvement obtained in [1].

Remark 3. From Theorem 1 we easily obtain the following.

- If a Riemannian manifold admits separation of variables such that none of the separation constants c_1, \dots, c_N , correspond to Killing vector fields, i.e., $r = 0$ in the notation of Theorem 1, then it locally admits an orthogonal coordinate system.
- In the coordinates $(x_1, \dots, x_{N-r}, t_{N-r+1}, \dots, t_N)$, from Theorem 1, the submanifolds corresponding to the coordinates x_1, \dots, x_{N-r} are totally geodesic, and are orthogonal at every point to the pairwise commuting Killing vector fields $\frac{\partial}{\partial t_{N-r+1}}, \dots, \frac{\partial}{\partial t_N}$. The metric is given by

$$ds^2 = \sum_{\alpha, \beta=1}^{N-r} g_{\alpha\beta}(x) dx_\alpha dx_\beta + \sum_{i, j=N-r+1}^N h_{ij}(x) dt_i dt_j. \tag{4}$$

Remark 4. A lot of research related to separation of variables on $\mathbb{C}P^n$ and $\mathbb{C}H^n$ (see, e.g., [29,35]) was done in the context of superintegrability and multi-separation of variables (see, e.g., [20]).

In smaller dimensions, after symplectic reduction with respect to ignorable coordinates, the geodesic flow of the metric and the corresponding Killing tensors produce a superintegrable system on the space whose commuting integrals correspond to the first block of (2) with potentials which are essentially the components of the second block of (2). We comment on this in Section 5.

3. Nonexistence of separating coordinates on $\mathbb{H}P^n, \mathbb{H}H^n, \mathbb{O}P^2$ and $\mathbb{O}H^2$

We start with the following fact, a substantial part of which follows from the work of Gauduchon and Moroianu [15].

Theorem 5. *A rank 1 Riemannian symmetric space locally admits a coordinate system in which the metric has diagonal form if and only if it has constant curvature.*

Proof. The “if” part is well known. We only need to establish the “only if” claim.

By [15, Propositions 3.1, 3.2, 4.1], no local diagonal metric exists on either $\mathbb{C}P^n$ or $\mathbb{H}P^n$ for $n > 1$. The proofs in [15] only use the algebraic properties of the curvature tensor and the local behavior of the complex structure (for $\mathbb{C}P^2$), and so almost verbatim work for the non-compact spaces $\mathbb{C}H^n$ and $\mathbb{H}H^n$ with $n > 1$.

Suppose that there is a local orthogonal coordinate system $x^i, i = 1, \dots, 16$, on $\mathbb{O}P^2$ in which the metric has a diagonal form. Relative to such a coordinate system, the components of the curvature tensor satisfy the property $R_{ijkl} = 0$, for all pairwise non-equal i, j, k and l . At a point $o \in \mathbb{O}P^2$, define an orthonormal basis $\{e_i\}$ for $T_o\mathbb{O}P^2$ such that e_i is a multiple of $\partial/\partial x^i$, for $i = 1, \dots, 16$. Then $R(e_1, e_2, Z, W) = 0$, for all $Z, W \in T_o\mathbb{O}P^2$ such that $Z, W \perp e_1, e_2$.

We can identify $T_o\mathbb{O}P^2$ with the (\mathbb{R} -) linear space \mathbb{O}^2 via a linear isometry, so that a vector $X \in T_o\mathbb{O}P^2$ is represented as $X = (x_1, x_2)$, $x_1, x_2 \in \mathbb{O}$. Under this identification, the curvature tensor of $\mathbb{O}P^2$ of sectional curvature between 1 and 4 is given in [10, Equation 6.12]. For $X = (x_1, x_2), Y = (y_1, y_2), Z = (z_1, z_2) \in T_o\mathbb{O}P^2 = \mathbb{O}^2$ we have:

$$R(X, Y)Z = (4\langle x_1, z_1 \rangle y_1 - 4\langle y_1, z_1 \rangle x_1 - (z_1 y_2) x_2^* + (z_1 x_2) y_2^* - (x_1 y_2 - y_1 x_2) z_2^*, \\ 4\langle x_2, z_2 \rangle y_2 - 4\langle y_2, z_2 \rangle x_2 - x_1^* (y_1 z_2) + y_1^* (x_1 z_2) + z_1^* (x_1 y_2 - y_1 x_2)),$$

where $*$ is the octonion conjugation and $\langle u, v \rangle = \text{Re}(uv^*)$, for $u, v \in \mathbb{O}$.

As the isotropy group $\text{Spin}(9)$ acts transitively on the unit sphere of $T_x\mathbb{O}P^2$, we can take $e_1 = (1, 0)$, and then $e_2 = (y_1, y_2)$, with $y_1, y_2 \in \mathbb{O}$ and $y_1 \perp 1$. Then for $Z, W \perp e_1, e_2$, the condition $R(e_1, e_2, Z, W) = 0$ gives

$$\langle y_2 z_2^*, w_1 \rangle + \langle 2y_1 z_2 - z_1^* y_2, w_2 \rangle = 0, \tag{5}$$

where we used the fact that $y_1 \perp 1$, and so $y_1^* = -y_1$. Take $w_2 = 0$. Then $w_1 \perp 1, y_1$, and so (5) gives $y_2 z_2^* \in \text{Span}(1, y_1)$. Assuming $y_2 \neq 0$ we arrive at a contradiction, as the left multiplication by a nonzero octonion is injective and as z_2 can be chosen arbitrarily from the 7-dimensional space $y_2^\perp \cap \mathbb{O}$. It follows that $y_2 = 0$, and then (5) gives $\langle y_1 z_2, w_2 \rangle = 0$, again leading to a contradiction, since $y_1 \neq 0$ and as $z_2, w_2 \in \mathbb{O}$ can be chosen arbitrarily.

The same argument works for the octonion hyperbolic plane $\mathbb{O}H^2$, since under a linear isometry between the tangent spaces to $\mathbb{O}P^2$ and $\mathbb{O}H^2$, their curvature tensors differ only by the sign. □

From Remark 3 and Theorem 5 it easily follows that no rank 1 symmetric space of non-constant curvature admits an *orthogonal* separation of variables.

We next address *non-orthogonal* separation of variables and prove the following.

Theorem 6. *The Riemannian symmetric spaces $\mathbb{H}P^n$ ($n \geq 2$), $\mathbb{H}H^n$ ($n \geq 2$), $\mathbb{O}P^2$ and $\mathbb{O}H^2$ admit no local non-orthogonal separation of variables.*

Proof. By Theorem 1 (see also (4)), the local existence of non-orthogonal separation of variables on a Riemannian space M of dimension N implies the local existence of two complementary, orthogonal local foliations on M , with the first consisting of totally geodesic submanifolds of dimension $N - r > 0$ admitting a diagonal metric, and the second consisting of flat submanifolds of dimension r which are orbits of an abelian r -dimensional subgroup K of the isometry group of M . The picture here is very similar to that for the polar action of the group K on M . Recall that a proper action of a group K on a complete, connected Riemannian manifold M is called *polar*, if it admits a *section*, that is, an embedded, totally geodesic submanifold Σ which meets all the orbits of K , and which intersects all the orbits orthogonally in each of its points (for the modern state of the theory of polar actions, the reader is referred to [11,12] and the bibliographies therein). In our case, all the assumptions are local, and we do not see how we can guarantee the properness of the action of K (or, a priori, even the closedness of K). However, we can make use of some results of the theory of polar actions.

Suppose $M = G/H$ is our symmetric space, being acted upon by an abelian group $K \subset G$, locally admitting a totally geodesic section Σ of complementary dimension. Denote \mathfrak{g} and \mathfrak{k} the Lie algebras of G and K , respectively. Assume that the point $o \in M$, the projection of the identity of G , is a regular point of the action of K . Denote $\mathfrak{m} = T_o\Sigma \subset T_oM$, and let $\mathfrak{m}^\perp = T_o(Ko)$ be its orthogonal complement.

Lemma 7. *With the above assumptions and notation, the following hold:*

- (a) *the subspace $\mathfrak{m}^\perp \subset T_oM$ is a Lie triple system and hence is tangent to a totally geodesic submanifold of M passing through o ;*
- (b) *the subspace $[\mathfrak{m}, \mathfrak{m}]$ is orthogonal to \mathfrak{k} relative to the Killing form of \mathfrak{g} .*

Proof. To prove assertion (a), we compute the curvature tensor R of M in the notation of formula (4). Note that the subspaces \mathfrak{m} and \mathfrak{m}^\perp are spanned, respectively, by ∂_{x_α} , $\alpha = 1, \dots, N - r$, and by ∂_{t_i} , $i = N - r + 1, \dots, N$. A direct computation shows that $R(\mathfrak{m}^\perp, \mathfrak{m}^\perp)\mathfrak{m}^\perp \subset \mathfrak{m}^\perp$. It follows that \mathfrak{m}^\perp is a Lie triple system, and hence is tangent to a totally geodesic submanifold of M , by Cartan’s Theorem [18, Theorem 7.2].

The proof of assertion (b) is verbatim the proof of assertion (ii) of the Proposition in [16, p. 195] — it only requires local arguments. □

Note that the totally geodesic submanifold in Lemma 7(a) is *not*, in general, the orbit of K , a leaf of the $(N - r)$ -dimensional flat complementary foliation.

Assume there is a local non-orthogonal separation of variables on the space $M = \mathbb{H}P^n$ or $M = \mathbb{H}H^n$, $n > 1$. The arguments for both spaces are similar, so let us assume that $M = \mathbb{H}P^n$. By Lemma 7(a) and Theorem 5, the tangent space to M at a regular point of the action of K is the orthogonal sum of two Lie triple systems, one of which is tangent to a totally geodesic submanifold of constant curvature. By [36, Theorem 1], the maximal totally geodesic submanifolds of $\mathbb{H}P^n$ are $\mathbb{H}P^k$, $k < n$, and $\mathbb{C}P^n$, and so the only possible case is that one of the totally geodesic submanifolds is $\Sigma = \mathbb{H}P^1 = S^4$ (and the other one is then $\mathbb{H}P^{n-1}$). This means that $m = 4$, and through every regular point x there passes a flat submanifold $L = Kx \subset M$ orthogonal to the totally geodesic $\mathbb{H}P^1$. But then the tangent space to L at each point is invariant relative to the quaternionic structure, and hence $L \subset M$ is a quaternionic submanifold. By [17, Theorem 5], L must be totally geodesic, which contradicts the fact that it is flat.

The fact that the octonionic projective plane $\mathbb{O}P^2 = F_4 / \text{Spin}(9)$ admits no non-orthogonal separation of variables follows from the dimension count. By [36, Theorem 1] the maximal dimension of a proper, totally geodesic submanifold of $\mathbb{O}P^2$ is 8, while the maximal dimension of an abelian subspace of the algebra \mathfrak{f}_4 is 4, as any such subspace lies in a Cartan subalgebra of \mathfrak{f}_4 .

The same simple argument does not, unfortunately, work for the octonionic hyperbolic plane $\mathbb{O}H^2 = F_4^- / \text{Spin}(9)$, as the noncompact Lie algebra \mathfrak{f}_4^- admits abelian subalgebras not lying in any

Cartan subalgebra, and having the dimension greater than the rank (up to at least 8; to see this, we note that a solvable group whose Lie algebra is a 1-dimensional extension of a 2-step nilpotent 15-dimensional algebra $\mathfrak{v} \oplus \mathfrak{z}$ with the center \mathfrak{z} of dimension 7 acts simply transitively on $\mathbb{O}H^2$; a required 8-dimensional abelian subalgebra of \mathfrak{f}_4^- can be taken as the direct sum of \mathfrak{z} and a line in \mathfrak{v}).

By Lemma 7(a) and Theorem 5, the tangent space to $\mathbb{O}H^2$ at a regular point is the orthogonal sum of two Lie triple systems, one of which is tangent to a totally geodesic submanifold of constant curvature. From [36, Theorem 1], the maximal totally geodesic submanifolds of $\mathbb{O}H^2$ are $\mathbb{H}H^2$ and the real hyperbolic space H^8 , and so the only possible case is that one of the totally geodesic submanifolds is $\Sigma = H^8$ (and the other one is also H^8).

We use the presentation given in [24, Section 4]. We have the decomposition $\mathfrak{f}_4^- = \mathfrak{so}(8) \oplus \mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$ into linear subspaces orthogonal relative to the Killing form, and $T_o\mathbb{O}H^2 = \{0\} \oplus \{0\} \oplus \mathbb{O} \oplus \mathbb{O}$. The Lie bracket on \mathfrak{f}_4^- is given by

$$[(A, u, v, w), (B, x, y, z)] = (C, r, s, t),$$

where $u, v, w, x, y, z, r, s, t \in \mathbb{O}$ and $A, B, C \in \mathfrak{so}(8)$ such that

$$\begin{aligned} C &= AB - BA - 4u \wedge x + 4\lambda^2(v \wedge y) + 4\lambda(w \wedge z), \\ r &= Ax - Bu - (vz)^* + (yw)^*, \\ s &= \lambda(A)y - \lambda(B)v + (wx)^* - (zu)^*, \\ t &= \lambda^2(A)z - \lambda^2(B)w + (uy)^* - (xv)^*, \end{aligned} \tag{6}$$

where $a \wedge b = ab^\top - ba^\top$ for $a, b \in \mathbb{O}$, and where λ and λ^2 are the automorphisms of $\mathfrak{so}(8)$ defined by $\lambda(a \wedge b) = \frac{1}{2}L_b^*L_a^*$ and $\lambda^2(a \wedge b) = \frac{1}{2}R_b^*R_a^*$, for $a, b \in \mathbb{O}$, $a \perp 1$, with L_c and R_c being the left and the right multiplications by $c \in \mathbb{O}$, respectively.

As the isotropy group $\text{Spin}(9)$ acts transitively on the set of the Lie triple systems in $T_o\mathbb{O}H^2$ tangent to totally geodesic hyperbolic spaces $H^8 \subset \mathbb{O}H^2$ (by the uniqueness part of [36, Theorem 1]), we can take $\mathfrak{m} = \{(0, 0, v, 0) \mid v \in \mathbb{O}\}$ and $\mathfrak{m}^\perp = \{(0, 0, 0, w) \mid w \in \mathbb{O}\}$. Then $[\mathfrak{m}, \mathfrak{m}] = \{(\lambda^2(v \wedge y), 0, 0, 0) \mid v, y \in \mathbb{O}\} = \mathfrak{so}(8) \oplus \{0\} \oplus \{0\} \oplus \{0\}$, as λ is an automorphism. We now need an abelian 8-dimensional subalgebra $\mathfrak{k} \subset \mathfrak{f}_4^-$ whose projection to $T_o\mathbb{O}H^2$ is \mathfrak{m}^\perp and which, according to Lemma 7(b), is orthogonal to $[\mathfrak{m}, \mathfrak{m}]$. Then $\mathfrak{k} \subset \{(0, u, 0, w) \mid u, w \in \mathbb{O}\}$. As $\dim \mathfrak{k} = 8$, we can take $U = (0, u, 0, 1) \in \mathfrak{k}$ and then for any $z \in \mathbb{O}$, $z \perp 1$, there exists $X = (0, x, 0, z) \in \mathfrak{k}$. We have $[U, X] = 0$, and so by (6) we obtain $x = zu$ and $\lambda(1 \wedge z) = u \wedge x = u \wedge (zu)$. The latter equation gives $\frac{1}{2}L_z = u \wedge (zu)$, and so the element $u \in \mathbb{O}$ has the following property: for any $a \in \mathbb{O}$ and any $z \in \mathbb{O}$, $z \perp 1$, we have $\frac{1}{2}za = \langle zu, a \rangle u - \langle u, a \rangle zu$. Taking a nonzero $z \perp 1$ we obtain $a = 2\langle zu, a \rangle z^{-1}u - 2\langle u, a \rangle u$, which is clearly a contradiction, as we can choose $a \in \mathbb{O}$ which does not lie in the real span of u and $z^{-1}u$. \square

4. Number of ignorable coordinates in separating coordinates on $\mathbb{C}P^n$ and $\mathbb{C}H^n$

As we already know, the spaces $\mathbb{C}P^n$ and $\mathbb{C}H^n$ admit no orthogonal separating coordinates ([15] and Theorem 5). On the other hand, [9,29,35] provide examples of non-orthogonal separating coordinates. See also [30] for another construction of the separation of variables on $\mathbb{C}P^n$ and $\mathbb{C}H^n$. In these examples, relative to the separating coordinates, the metric has the form (4), with $\sum_{\alpha, \beta} g_{\alpha\beta}(x) dx_\alpha dx_\beta$ being the metric of constant curvature given in ellipsoidal coordinates, and the number of ignorable variables is exactly n . We show that these properties hold for any separation of variables on $\mathbb{C}P^n$ and on $\mathbb{C}H^n$.

Theorem 8. *Let x_1, \dots, x_{2n} , with $n \geq 2$, be local separating coordinates on $\mathbb{C}P^n$ or on $\mathbb{C}H^n$.*

Then precisely n of them are ignorable, that is, in the notation of Theorem 1, we have $r = n$; we denote $x_{n+1} = t_1, \dots, x_{2n} = t_n$. The corresponding vector fields $\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n}$ are Killing and form a

maximal abelian subalgebra of $\mathfrak{su}(n+1)$ in the case of $\mathbb{C}P^n$, and of $\mathfrak{su}(n,1)$ in the case of $\mathbb{C}H^n$. The coordinates x_1, \dots, x_n are separating coordinates for the metric g of constant positive curvature on the totally geodesic submanifold $\mathbb{R}P^n$ in the case of $\mathbb{C}P^n$, and of constant negative curvature on the totally geodesic submanifold H^n in the case of $\mathbb{C}H^n$.

Note that [7,19,22], in which separating coordinates for the sphere and for the hyperbolic spaces have been constructed, provide explicit formulas for the metric g .

Proof. For non-orthogonal separating coordinate system on $\mathbb{C}P^n$ or on $\mathbb{C}H^n$, the metric tensor has the form given in (4), with the submanifolds Σ given by $t_i = \text{const}_i$ being totally geodesic, and of constant curvature, by Theorem 5. Moreover, by Lemma 7(a), at a regular point, the orthogonal complement to the tangent space of Σ must be a Lie triple system. By [36, Theorem 1], any totally geodesic submanifold of $\mathbb{C}P^n$ is congruent to either a standard $\mathbb{C}P^k$ or a standard $\mathbb{R}P^k$, with $k \leq n$. Similarly, any totally geodesic submanifold of $\mathbb{C}H^n$ is congruent to either a standard $\mathbb{C}H^k$ or a standard H^k , with $k \leq n$. Moreover, the dimension of a maximal abelian subalgebra of $\mathfrak{su}(n+1)$ is n (the rank of $\mathfrak{su}(n+1)$), and the dimension of a maximal abelian subalgebra of $\mathfrak{su}(n,1)$ is also n , by [28, Theorem 5.1]. This leaves only two possibilities in the case of $\mathbb{C}P^n$: either Σ is a totally real $\mathbb{R}P^n \subset \mathbb{C}P^n$, or $n = 2$ and $\Sigma = \mathbb{C}P^1 \subset \mathbb{C}P^2$. Similarly, for $\mathbb{C}H^n$, either Σ is a totally real $H^n \subset \mathbb{C}H^n$, or $n = 2$ and $\Sigma = \mathbb{C}H^1 \subset \mathbb{C}H^2$.

We show that both for $\mathbb{C}P^n$ and for $\mathbb{C}H^n$, only the first alternative is possible. We give a proof in the case of $\mathbb{C}P^n$; for $\mathbb{C}H^n$ it is identical, up to obvious changes.

Suppose that $n = 2$, and that Σ is congruent to $\mathbb{C}P^1$. We take

$$\mathfrak{su}(3) = \left\{ \begin{pmatrix} a_1 i & z_1 & z_2 \\ -\bar{z}_1 & a_2 i & z_3 \\ -\bar{z}_2 & -\bar{z}_3 & a_3 i \end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{C}, a_1, a_2, a_3 \in \mathbb{R}, a_1 + a_2 + a_3 = 0 \right\}.$$

In the notation of Lemma 7, at a regular point $o \in \mathbb{C}P^2$, the subspace $T_o\mathbb{C}P^2$ is given by $z_1 = a_1 = a_2 = a_3 = 0$, and then the subspaces $\mathfrak{m} \subset T_o\mathbb{C}P^2$ and $\mathfrak{m}^\perp \subset T_o\mathbb{C}P^2$, up to isotropy, are given by $z_3 = 0$ and by $z_2 = 0$, respectively. The 2-dimensional abelian subalgebra $\mathfrak{k} \subset \mathfrak{su}(3)$ tangent to the t -coordinates is orthogonal to $[\mathfrak{m}, \mathfrak{m}]$, by Lemma 7(b), relative to the Killing form of $\mathfrak{su}(3)$, and the projection of \mathfrak{k} to $T_o\mathbb{C}P^2$ equals \mathfrak{m}^\perp . But then \mathfrak{k} is spanned by the following two elements:

$$\begin{pmatrix} -si & u & 0 \\ -\bar{u} & 2si & 1 \\ 0 & -1 & -si \end{pmatrix}, \quad \begin{pmatrix} -qi & v & 0 \\ -\bar{v} & 2qi & i \\ 0 & i & -qi \end{pmatrix},$$

for some $u, v \in \mathbb{C}$, $s, q \in \mathbb{R}$, and a direct calculation shows that they do not commute. □

5. Conclusion

We solved (the natural analog of) the Eisenhart problem for certain compact rank 1 symmetric spaces. In particular, we have shown that $\mathbb{H}P^n$ with $n \geq 2$ and $\mathbb{O}P^2$ do not admit local separation of variables, and that on $\mathbb{C}P^n$, all separating coordinates are those constructed in [9]. We partially solved the Eisenhart problem for noncompact rank 1 symmetric spaces: we have shown that $\mathbb{H}H^n$ with $n \geq 2$ and $\mathbb{O}H^2$ do not admit separation of variables and that on $\mathbb{C}H^n$, any separating coordinates have n ignorable coordinates. In view of results of [8,28,29,35], this solves the Eisenhart problem for $\mathbb{C}H^2$ and $\mathbb{C}H^3$.

An algorithm for classifying possible separating variables on the space $\mathbb{C}H^n$, for any given n , is in essence given in Theorem 1, and includes the following steps. The classification of n -dimensional abelian subalgebras $\mathfrak{k} \subset \mathfrak{su}(n,1)$ is given in [28, Theorem 5.1]. For any such subalgebra \mathfrak{k} , one first constructs the totally geodesic submanifold Σ (which is necessarily isometric to H^n) orthogonal to the Killing vector fields corresponding to the subalgebra. Quadratic Killing tensor fields on $\mathbb{C}P^n$ are quadratic forms in the Killing vector fields [13,34]. By duality, this is also

true for $\mathbb{C}H^n$, so that the quadratic Killing tensor fields on $\mathbb{C}H^n$ are in one-to-one correspondence with the quadratic forms on $\mathfrak{su}(n, 1)$. With some aid of computer algebra, one then finds the subspace of all quadratic Killing tensors Poisson-commuting with \mathfrak{k} , and then, n -dimensional subspaces of that space consisting of pairwise Poisson-commuting quadratic Killing tensors. Next, for such n -dimensional subspaces one needs to verify if the restrictions of its quadratic Killing tensors to the submanifold Σ gives the orthogonal separation of variables on Σ . That is, one needs to check that they have common eigenspaces, and that their Haantjes torsion is zero. This can be reduced to a certain algebraic calculation, requiring working with Gröbner bases.

We note that the classification of polar actions on $\mathbb{C}H^n$ obtained in [12] could be very useful for finding separating variables.

Another natural question to address is the separation of variables on symmetric spaces of higher rank. Related geometric questions will include the study of the existence of diagonal coordinates on symmetric spaces (see [15, Section 5] for a list of related open problems), and also the study of abelian subalgebras of isometry algebras of symmetric spaces such that the orthogonal distribution to the span of the values of the corresponding Killing vector fields at a regular point is integrable (and hence, automatically totally geodesic).

Additional directions for future research are to explore the relation between separation of variables on $\mathbb{C}H^n$ and superintegrable and multiseparable systems. Recall that the functions h^{ij} from (4) can be viewed as a potential adding which to the kinetic energy corresponding to the $(N-r)$ -dimensional metric g does not destroy the integrability and the separability. In the context of separation of variables on $\mathbb{C}P^n$, the functions h^{ij} are constructed from a Cartan subalgebra of $\mathfrak{su}(n+1)$ only. In particular, they provide a superintegrable and multiseparable system. It is easy to check that this system is actually the so-called nondegenerate superintegrable system on the sphere. In the case of $\mathbb{C}H^n$, there exists $n+2$ pairwise non-conjugate abelian subalgebras of dimension n of $\mathfrak{su}(n, 1)$, and each of them gives a natural analog of nondegenerate superintegrable system on the (real) hyperbolic space. They were studied in detail for small n in [29,35], with special attention to Cartan subalgebras, and we plan to extend this study to all values of $n \geq 2$ and to all abelian subalgebras of dimension n of $\mathfrak{su}(n, 1)$.

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