



ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus

Mathématique

David Hokken

Topology of zero sets of polynomials with square discriminant

Volume 364 (2026), p. 101-106

Online since: 9 March 2026

<https://doi.org/10.5802/crmath.824>

 This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>



*The Comptes Rendus. Mathématique are a member of the
Mersenne Center for open scientific publishing*
www.centre-mersenne.org — e-ISSN : 1778-3569



Research article
Dynamical systems, Number theory

Topology of zero sets of polynomials with square discriminant

David Hokken ^a

^a Mathematisch Instituut, Universiteit Utrecht, Postbus 80.010, 3508 TA Utrecht, The Netherlands

E-mail: d.p.t.hokken@uu.nl

Abstract. Let $\mathcal{N} \neq \{0\}$ be a fixed set of integers, closed under multiplication, closed under negation, or containing $\{\pm 1\}$. We prove that any zero of a polynomial in $\mathbf{Z}[X]$ whose coefficients lie in \mathcal{N} can be approximated in \mathbf{C} to arbitrary precision by a zero of a polynomial in $\mathbf{Z}[X]$ with square discriminant whose coefficients also lie in \mathcal{N} . Hence the topology of the closure in \mathbf{C} of the set of zeros of all such polynomials is insensitive to the discriminant being a square, in contrast to the Galois groups of the polynomials.

Keywords. Zeros of polynomials, self-similar fractals, square discriminant.

2020 Mathematics Subject Classification. 30C15, 11C08, 28A80.

Funding. This research was financially supported by the Dutch Research Council (NWO), project number OCENW.M20.233.

Manuscript received 17 December 2024, accepted 9 February 2026, online since 9 March 2026.

1. Introduction

Let $\mathcal{N} \neq \{0\}$ be a subset of the integers. Denote by $P = P(\mathcal{N})$ the set of univariate polynomials f whose coefficients lie in \mathcal{N} and such that $f(0) \neq 0$. Define $W = W(\mathcal{N})$ as the countably infinite set of all complex zeros of all polynomials in P and denote its closure in \mathbf{C} by M . First investigated by Barnsley and Harrington [2] from the point of view of iterated function systems, the set M is a fractal when \mathcal{N} is finite and, as such, abounds in topological features. We mention some of these for the three best-studied cases $M_1 = M(\{0, \pm 1\})$, $M_2 = M(\{\pm 1\})$ (so-called Littlewood polynomials; see Figure 1), and $M_3 = M(\{0, 1\})$. It is a theorem of Bousch [5] that M_1 and M_2 are connected and locally connected, and of Odlyzko and Poonen [11] that M_3 is path-connected. Bousch also proved the fascinating result that $z \in M_2$ if $z^2 \in M_1$. Bandt [1] showed disconnectedness of the complement M_1^c . Subsequent work of Calegari, Koch, and Walker [7] established that M_1^c has infinitely many connected components, and also that M_2^c is disconnected. They also confirmed Bandt's conjecture that the interior of M_1 is dense away from the real line. We refer to [7] for further discussion of the related history and results.

In another, more algebraic direction, one may ask what typical Galois-theoretic properties of the polynomials in P are. For fixed \mathcal{N} of cardinality at least 2, a folklore conjecture [3,4,6,9,11] asserts that a random polynomial $f \in P$ of degree n has the symmetric group S_n as Galois group over \mathbf{Q} with probability tending to 1 as n tends to infinity. It is an outstanding challenge to rule

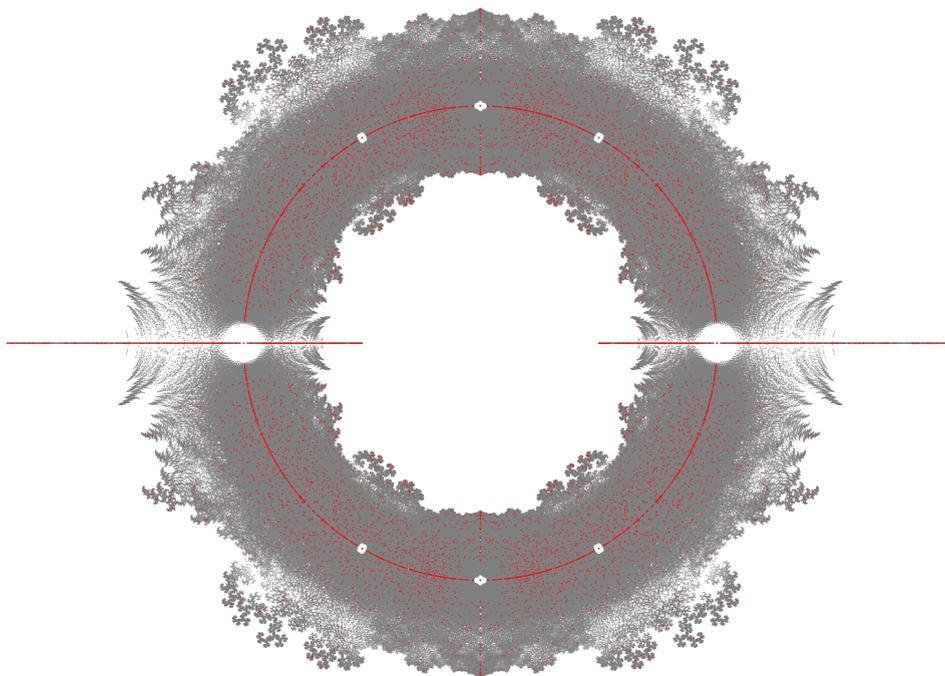


Figure 1. Approximation of M (grey) overlaid by an approximation of M^\square (red) for $\mathcal{N} = \{\pm 1\}$, where only the zeros of polynomials of degree $n \leq 16$ have been included. The figure is centered at $0 \in \mathbf{C}$, and the “circle” of red dots traces the unit circle. Theorem 1 asserts that the figure turns red as n tends to infinity.

out the alternating group A_n as likely Galois group. Assume $n > 1$, let a_n be the leading coefficient of f , and denote the zeros of f by $\alpha_1, \dots, \alpha_n \in \mathbf{C}$. Then the Galois group of f is contained in A_n if and only if the discriminant $\Delta(f) := a_n^{2n-2} \prod_{i < j} (\alpha_i - \alpha_j)^2$ of f is a nonzero integral square. Thus the discriminant affects the Galois groups of the polynomials in P , and one may ask whether the topology of the respective zero set is also sensitive to the discriminant being a square. Let $P^\square \subset P$ consist of the polynomials whose discriminant is a square (possibly zero), and define $W^\square \subset W$ and $M^\square \subset M$ in similar fashion. The result here shows that the topology of the closure of the zero set does not change if we restrict to polynomials in P with square discriminant.

Theorem 1. *Let $\mathcal{N} \neq \{0\}$ be a subset of the integers. Then $M^\square = M$ if:*

- (i) \mathcal{N} is closed under negation, that is $a \in \mathcal{N}$ if and only if $-a \in \mathcal{N}$; or
- (ii) \mathcal{N} is closed under multiplication, that is $ab \in \mathcal{N}$ if $a, b \in \mathcal{N}$; or
- (iii) $\{\pm 1\} \subset \mathcal{N}$.

In other words, for any \mathcal{N} as in Theorem 1(i), (ii) or (iii), any zero of a polynomial in $P(\mathcal{N})$ can be approximated to arbitrary precision by a zero of a polynomial in $P(\mathcal{N})$ with square discriminant. Theorem 1 covers the cases $\mathcal{N} = \{\pm 1\}, \{0, 1\}, \{0, \pm 1\}$; one may ask whether $M^\square = M$ for any $\mathcal{N} \neq \{0\}$. We cannot exclude the possibility of vanishing discriminant, except in the case $\mathcal{N} = \{\pm 1\}$ — see Corollary 5.

2. Proofs

Let $f, g \in \mathbf{Q}[X]$ be two nonzero polynomials. Denote by $\text{Res}(f, g) \in \mathbf{Q}$ their resultant. Write n and a_n for the degree and leading coefficient of f . If $\alpha_1, \dots, \alpha_n$ are the complex zeros of f , then

the discriminant of f is the rational number $\Delta(f) := a_n^{2n-2} \prod_{i < j} (\alpha_i - \alpha_j)^2$. With the convention that the empty product equals 1, we also adopt this definition when $n \in \{0, 1\}$; in particular, the discriminant $\Delta(a_0) = a_0^{-2}$ of a constant $a_0 \in \mathbf{Q}^\times$ is a square. Then $\Delta(fg) = \Delta(f)\Delta(g) \operatorname{Res}(f, g)^2$, that is

$$\Delta: \mathbf{Q}[X] \rightarrow \mathbf{Q}/(\mathbf{Q}^\times)^2 \quad \text{satisfies} \quad \Delta(fg) = \Delta(f)\Delta(g). \quad (1)$$

For nonconstant f , we also have the formula $\Delta(f) = (-1)^{n(n-1)/2} \operatorname{Res}(f, f')/a_n$.

We start with the following observation.

Lemma 2. *The sets M and M^\square are equal if and only if they coincide on the open unit disk \mathbb{D} .*

Proof. The set P comes equipped with the reversal map

$$P \ni f(X) \mapsto f_{\operatorname{rev}}(X) := X^{\deg f} f(1/X) \in P \quad (2)$$

which is discriminant-preserving by the assumption that $f(0) \neq 0$ if $f \in P$. Hence $\alpha \in W$ if and only if $1/\alpha \in W$. This means that M and M^\square are equal if and only if they coincide on the closed unit disk $\bar{\mathbb{D}}$. For the remainder, it suffices to show that M^\square contains the unit circle.

The zeros of $p_n(X) := 1 + X + \cdots + X^n$ lie dense in the unit circle as $n \equiv 1 \pmod{4}$ tends to infinity. We show that p_n has square discriminant, which implies that $ap_n \in P$ has square discriminant for any $a \in \mathcal{N} \setminus \{0\}$, proving the claim. Applying (1) to $f = p_n$ and $g = X - 1$, we find that $\Delta(p_n)$ is a square if

$$\Delta(X^{n+1} - 1) = (-1)^{n(n+1)/2} \operatorname{Res}(X^{n+1} - 1, (n+1)X^n) = (-1)^{n(n-1)/2} (n+1)^{n+1}$$

is a square, which holds by the assumption $n \equiv 1 \pmod{4}$. □

We say $f \in \mathbf{Q}[X]$ is *reciprocal* if it is of even degree and invariant under the reversal map (2).

Lemma 3. *Let $f \in \mathbf{Q}[X]$ be of even degree n . The following hold:*

- (i) *if f is reciprocal, then f has square discriminant if and only if $(-1)^{n/2} f(1)f(-1)$ is a square;*
- (ii) *the polynomial $f(X)f(X^k)$ has square discriminant for odd k .*

Proof. For (i), see e.g. [8, p. 85]. For (ii), if $f \in \mathbf{Q}$, the statement is trivial. We may assume f is monic and does not lie in \mathbf{Q} . Set $g(X) = f(X^k)$. It suffices to show that $\Delta(f)\Delta(g)$ is a square. Note that g has degree kn and $g(\beta) = 0$ if and only if $f(\beta^k) = 0$. Hence, with the notation $a_0 = g(0)$,

$$\Delta(g) = (-1)^{kn(kn-1)/2} \prod_{\beta: g(\beta)=0} g'(\beta) = (-1)^{kn(kn-1)/2} k^{kn} a_0^{k-1} \prod_{\beta: g(\beta)=0} f'(\beta^k)$$

where we used $g'(\beta) = f'(\beta^k)k\beta^{k-1}$ in the second equality. On the other hand,

$$\Delta(f)^k = (-1)^{kn(n-1)/2} \prod_{\alpha: f(\alpha)=0} f'(\alpha)^k = (-1)^{kn(n-1)/2} \prod_{\beta: g(\beta)=0} f'(\beta^k).$$

The parity assumptions on n and k imply

$$\Delta(f)\Delta(g) = (-1)^{n^2 k(k-1)/2} k^{kn} a_0^{k-1} \Delta(f)^{k+1}$$

is a square. □

Suppose $z \in \mathbb{D}$. If F is a power series whose coefficients are bounded in absolute value by H , then

$$|F(z)| \leq H(1 + |z| + |z|^2 + \cdots) = \frac{H}{1 - |z|}.$$

In particular, the radius of convergence of F is at least 1 and the set of zeros of F in \mathbb{D} is well-defined.

Lemma 4. *Let $f, g \in \mathbb{Z}[[x]]$ be two power series whose coefficients are bounded in absolute value. Fix $\epsilon > 0$ and a zero $\alpha \in \mathbb{D}$ of f . Then there is an integer N such that g has a zero $\beta \in \mathbb{D}$ with $|\alpha - \beta| < \epsilon$ if $X^N \mid (f - g)$.*

Proof. This is an application of Rouché's theorem. Consider an open disk D of radius $R \leq \epsilon$ centered at α . Make R small enough such that the closure \bar{D} is contained in the open unit disk and such that α is the only zero of f in \bar{D} ; this is possible as otherwise the set of zeros contains an accumulation point in the disk of convergence of f , contradicting the identity theorem for analytic functions, see [10, Theorem II.3.2]. By compactness, $|f|$ attains some positive minimum m on the boundary of D . Since the coefficients of f and g are bounded, there is an integer B such that the coefficients of the difference $f - g$ are at most B in absolute value. If the N initial coefficients of f and g coincide, then

$$|f(z) - g(z)| \leq |z|^N B \sum_{j \geq 0} |z|^j \leq \frac{B|z|^N}{1-|z|}.$$

Thus choosing N large enough ensures that $|f(z) - g(z)| < m \leq |f(z)|$ for any z lying on the boundary of D . By Rouché's theorem [10, Theorem VI.1.6], the power series f and g have the same number of zeros within D (counting multiplicity), so g has a zero β satisfying the conditions. \square

Proof of Theorem 1. If \mathcal{N} contains only one element, then the proof of Lemma 2 shows that $M = M^\square$ is the unit circle. So we may assume the cardinality of \mathcal{N} is at least 2. Also by Lemma 2, it suffices to show that for any $\epsilon > 0$ and any complex number $\alpha \in \mathbb{D}$ that is a zero of some polynomial $f = a_0 + a_1X + \dots + a_nX^n \in P$, there is a $\beta \in W^\square$ with $|\alpha - \beta| < \epsilon$. Observe that the power series $F(X) := f(X)/(1 - X^{n+1})$ also vanishes at α . Write $F(X) = b_0 + b_1X + \dots$ for its power series expansion. For a positive integer k to be selected later, consider the truncation $g(X) = b_0 + \dots + b_{k-1}X^{k-1}$ of F at the k -th term. In case $b_{k-1} = 0$, replace b_{k-1} by any nonzero element of \mathcal{N} . Then $g \in P$.

(i) If \mathcal{N} is closed under negation, take k even, set $h(X) = g(X) + X^k g(-X)$, and define

$$f_k(X) = h(X) + aX^{2k} + X^{2k+1}h_{\text{rev}}(X) \in P$$

where $a \in \mathcal{N}$ can be chosen freely. Then f_k is reciprocal and of degree $4k$ and

$$f_k(1) = f_k(-1) = g(1) + g(-1) + a - g_{\text{rev}}(-1) + g_{\text{rev}}(1),$$

so $f_k(1)f_k(-1)$ is a square. By Lemma 3(i), the polynomial f_k has square discriminant. Furthermore, $F - f_k$ is divisible by X^{k-1} . By making k sufficiently large, Lemma 4 ensures that f_k has a zero β with $|\alpha - \beta| < \epsilon$.

(ii) If \mathcal{N} is closed under multiplication, one can take k odd and $f_k(X) = g(X)g(X^k)$, which lies in P and has square discriminant by Lemma 3(ii). Furthermore, f_k vanishes at all the zeros of g , and $F - g$ is divisible by X^{k-1} . By making k sufficiently large, Lemma 4 ensures that g , and hence f_k , has a zero β with $|\alpha - \beta| < \epsilon$.

(iii) When $\{\pm 1\} \subset \mathcal{N}$, take k odd, and replace if necessary b_{k-1} by 1 or -1 so as to make $g(1)$ nonzero. Write $\ell = 4g(1) - u$ where $u = -1$ if $g(1) > 0$ and $u = 1$ if $g(1) < 0$. Note $u|\ell| = \ell$. Consider the reciprocal polynomial $f_k(X) = (1 + X^k)g(X) + uX^{2k}(1 + \dots + X^{|\ell|-1}) + X^{2k+|\ell|}(1 + X^k)g_{\text{rev}}(X)$. Then the degree of f_k is divisible by 4 and $f_k(1) = f_k(-1) = u$. The same reasoning as in (i) shows that f_k has square discriminant and a zero β with $|\alpha - \beta| < \epsilon$ for sufficiently large k .

This finishes the proof. \square

From the Galois-theoretic point of view, it would be more interesting to simultaneously control the discriminant and the factorisation of the polynomials f_k constructed in the proof

of Theorem 1. In (ii), the polynomial f_k is reducible, but in (i) and (iii) one would typically expect f_k to be irreducible. In each construction, one would similarly expect (thanks to the flexibility in choosing g) the polynomial f_k to have nonvanishing discriminant, but we cannot ascertain this in general. However, the following result does demonstrate this for the set $\mathcal{N} = \{\pm 1\}$. To state the result, denote by $M^{\square \neq 0} = M^{\square \neq 0}(\mathcal{N})$ the closure of the set of zeros of polynomials in P^{\square} with nonvanishing discriminant, and by $C_2 \wr S_{2k}$ the wreath product of the cyclic group C_2 with the symmetric group S_{2k} .

Corollary 5. *For $\mathcal{N} = \{\pm 1\}$, we have $M = M^{\square \neq 0}$. More precisely, for any $\epsilon > 0$ and zero α of a polynomial with coefficients in $\{\pm 1\}$, there exists a zero β of a polynomial with coefficients in $\{\pm 1\}$ and Galois group in $(C_2 \wr S_{4\ell}) \cap A_{8\ell}$ for some $\ell > 0$ with the property that $|\alpha - \beta| < \epsilon$.*

Proof. In the proof of Theorem 1(i), the polynomial f_k , with $k > 0$ even, is of degree $4k$ and has square discriminant. Over \mathbf{F}_2 , the polynomial f_k is a factor of the squarefree polynomial $X^{4k+1} - 1$, since all its coefficients are 1 mod 2. Hence f_k is squarefree in $\mathbf{Q}[X]$, which implies that $\Delta(f_k)$ does not vanish, proving that the Galois group G_k of f_k lies in A_{4k} ; since f_k is reciprocal, in addition $G_k \leq C_2 \wr S_{2k}$. \square

Acknowledgments

I am grateful to Gunther Cornelissen for discussions on this topic and comments on an earlier version of the paper.

Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

References

- [1] C. Bandt, “On the Mandelbrot set for pairs of linear maps”, *Nonlinearity* **15** (2002), no. 4, pp. 1127–1147.
- [2] M. F. Barnsley and A. N. Harrington, “A Mandelbrot set for pairs of linear maps”, *Phys. D: Nonlinear Phenom.* **15** (1985), no. 3, pp. 421–432.
- [3] L. Bary-Soroker, D. Hokken, G. Kozma and B. Poonen, “Irreducibility of Littlewood polynomials of special degrees”, *Int. Math. Res. Not.* (2025), no. 21, article no. rnaf326 (5 pages).
- [4] L. Bary-Soroker, D. Koukoulopoulos and G. Kozma, “Irreducibility of random polynomials: general measures”, *Invent. Math.* **233** (2023), no. 3, pp. 1041–1120.
- [5] T. Bousch, “Paires de similitudes”, 1988. Online at https://www.imo.universite-paris-saclay.fr/~bousch/preprints/paires_sim.pdf.
- [6] E. Breuillard and P. P. Varjú, “Irreducibility of random polynomials of large degree”, *Acta Math.* **223** (2019), no. 2, pp. 195–249.
- [7] D. Calegari, S. Koch and A. Walker, “Roots, Schottky semigroups, and a proof of Bandt’s conjecture”, *Ergodic Theory Dyn. Syst.* **37** (2017), no. 8, pp. 2487–2555.
- [8] A. Dubickas, “Salem numbers as Mahler measures of nonreciprocal units”, *Acta Arith.* **176** (2016), no. 1, pp. 81–88.
- [9] S. V. Konyagin, “On the number of irreducible polynomials with 0,1 coefficients”, *Acta Arith.* **88** (1999), no. 4, pp. 333–350.

- [10] S. Lang, *Complex analysis*, 4th edition, Graduate Texts in Mathematics, vol. 103, Springer, 1999.
- [11] A. M. Odlyzko and B. Poonen, “Zeros of polynomials with 0, 1 coefficients”, *Enseign. Math.* (2) **39** (1993), no. 3–4, pp. 317–348.