ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus
Mathématique

Luca Rossi

On the criticality and the principal eigenvalue of almost periodic elliptic operators

Volume 364 (2026), p. 321-331

Online since: 18 May 2026

https://doi.org/10.5802/crmath.829

This article is licensed under the
Creative Commons Attribution 4.0 International License.
http://creativecommons.org/licenses/by/4.0/

The Comptes Rendus. Mathématique *are a member of the*
Mersenne Center for open scientific publishing
www.centre-mersenne.org — e-ISSN : 1778-3569

ACADÉMIE DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus. Mathématique
2026, Vol. 364, p. 321-331
https://doi.org/10.5802/crmath.829

Research article
Partial differential equations

On the criticality and the principal eigenvalue of almost periodic elliptic operators

Luca Rossi [a]

[a] Istituto G. Castelnuovo, Sapienza Università di Roma, Rome, Italy
E-mail: l.rossi@uniroma1.it

Abstract. We review the notion and the properties of the generalized principal eigenvalue for elliptic operators in unbounded domains, and we relate it to the criticality theory. We focus on operators with almost periodic coefficients. We present a Liouville-type result in dimension $N \leq 2$. Next, we show with a counterexample that criticality is not equivalent to the existence of an almost periodic principal eigenfunction, even for self-adjoint operators. Finally, we exhibit an almost periodic operator which is subcritical but which admits a critical limit operator. This is a manifestation of the instability character of the criticality property in the almost periodic setting.

Keywords. Generalized principal eigenvalue, elliptic operators, almost periodic coefficients, criticality theory, Liouville-type theorem.

2020 Mathematics Subject Classification. 35B15, 35B09, 35J10, 35B53, 35P05.

Note. Article submitted by invitation.

Manuscript received 30 July 2025, revised 29 January 2026, accepted 12 March 2026, online since 18 May 2026.

In memory of Haïm Brezis, whose work is an endless source of inspiration

1. Introduction

The *principal eigenvalue* encodes several fundamental properties of an elliptic operator, such as the validity of the maximum principle and the existence, uniqueness and stability of solutions. The principal eigenvalue is classically defined as the eigenvalue which admits a positive eigenfunction. For a uniformly elliptic operator with smooth coefficients, defined in a bounded smooth domain under different types of boundary conditions (Dirichlet, Neumann, Robin), the principal eigenvalue exists, it is unique and simple. These facts can be shown using the Krein–Rutman theory, the extension of the Perron–Frobenius theory for operators on infinite dimensional spaces. The crucial conditions required by the Krein–Rutman theory are the positivity and the compactness of the resolvent of the operator, which are both consequences of the ellipticity as well as the compactness of the spatial domain. Indeed, the same theory can be applied in the case of periodic operators, that is, operators defined on the torus, and provides one with the *periodic principal eigenvalue*. In order to understand what happens beyond the periodic case, a first framework to investigate is that of almost periodic (a.p. in the sequel) functions, see the precise definition in Section 2. A.p. functions share the following property with continuous periodic functions: the set of their translations is relatively compact in L^∞. Despite this fact, one cannot

cast the problem for operators with a.p. coefficients in a compact framework, unlike the periodic case.

In order to study the existence of an *a.p. principal eigenvalue* (i.e. an eigenvalue with an a.p. positive eigenfunction) one should consider the eigenproblem in the whole space and prove a posteriori that the eigenfunction is a.p. However, in such a framework the resolvent of the operator is not compact, making the Krein–Rutman theory unavailable. As a matter of fact, both uniqueness and simplicity of the principal eigenvalue fail in general for operators in unbounded domains. These questions have been addressed in [5,6,9] by introducing several distinct notions of *generalized principal eigenvalue*.

The differences between the periodic and the almost periodic settings are not only technical. Indeed, the existence of the a.p. principal eigenvalue may fail for an operator with a.p. coefficients. A counterexample can be exhibited for an operator of the form $\Delta - c(x)$ where $c(x) = K\big(\cos(2\pi x) + \cos(2\pi\alpha x)\big)$, $\alpha \notin \mathbb{Q}$ and K is large enough, which is a *quasiperiodic* function, a subclass of a.p. functions. This follows from a result of [10] (see also [29]) — proved using large deviation techniques and the avalanche principle of [16] — and from Ruelle–Oseledec's theorem. Quite surprisingly, if K is small enough and α satisfies a certain diophantine condition, then the above operator admits an a.p. principal eigenvalue, as shown by Kozlov [17]. In Counterexample 5 below we present another example of a self-adjoint a.p. operator in dimension 1 which does not admit an a.p. principal eigenvalue. This counterexample does not make use of probabilistic tools, but exploits an elementary construction performed in our previous paper [27], that we recall in Section 4. The counterexample further shows that the non-existence of the a.p. principal eigenvalue can also occur for a *critical* operator, in the sense of Pinchover [24]. The criticality theory is related, but not equivalent, to that of the generalized principal eigenvalue, see Section 2 below. Criticality implies in particular the uniqueness of positive solutions, up to a scalar multiple.

It is not hard to see that for a self-adjoint operator in spatial dimension $N = 1$, the a.p. principal eigenvalue is necessarily simple, if it exists, see e.g. [21, Proposition 1.7]. The same is true in dimension $N = 2$ owing to a Liouville-type result by Nordmann [22]. Combining the latter with a result by Pinchover [25], one can prove a stronger property: if an operator admits an a.p. positive solution then it is critical if and only if the spatial dimension is $N \le 2$, see Corollary 4 below.

We conclude this work with another counterexample to the existence of the a.p. principal eigenvalue, which in addition shows that an operator with a.p. coefficients can be *subcritical* but can have a critical *limit operator*, cf. Counterexample 7 below.

2. Precise definitions and statements of the results

We consider elliptic operators $\mathscr{L}$ of the form

$$\mathscr{L}u = \nabla\cdot\big(A(x)\nabla u\big) + b(x)\cdot\nabla u + c(x)u, \quad x \in \mathbb{R}^N,$$

where A is a matrix field, b is a vector field and c is a scalar function. We always assume that the entries of A, b and the function c are real valued, measurable and bounded, and, in addition, that A is smooth and it is symmetric and uniformly elliptic.

One cannot define the principal eigenvalue of $-\mathscr{L}$ in a classical way through the Krein–Rutman theory, because the domain $\mathbb{R}^N$ is unbounded. In collaboration with H. Berestycki and F. Hamel, motivated by the study of reaction-diffusion equations in heterogeneous media, we considered in [6] the following notion of *generalized principal eigenvalue*:

$$\lambda_1 := \sup\big\{\lambda \in \mathbb{R} : \exists\, \phi > 0,\ -\mathscr{L}\phi \ge \lambda\phi \text{ on } \mathbb{R}^N\big\}.$$

The functions ϕ in the above formula (and in general the functions involved in differential equalities/inequalities) are understood to belong to $W^{2,N}_{\mathrm{loc}}(\mathbb{R}^N)$ (and to satisfy the inequality

a.e.). We point out that no global integrability condition is required on ϕ. The above definition coincides with the one introduced by Berestycki–Nirenberg–Varadhan [8] in the case of linear elliptic operators defined on bounded non-smooth domains. The authors showed that, up to giving a relaxed meaning to the Dirichlet boundary condition, such notion enjoys all the fundamental properties of the classical one. Some related notions and partial results had been previously given in [23]. Instead, in the whole space (or more in general in unbounded domains), the quantity λ_1 is not sufficient to encode all the classical properties of the principal eigenvalue. In fact, another notion, denoted by λ_1', has been employed in [5,6] to give a complete description of solutions of the elliptic problem, and a third quantity, λ_1'', has been introduced in [9]. The point is that the eigenvalue admitting a positive eigenfunction is not unique, as in the case of a bounded domain, but actually the whole half line $(-\infty, \lambda_1]$ does, see [9, Theorem 1.4]. For instance, for a periodic operator $-\mathscr{L}$ which is not self-adjoint, the periodic principal eigenvalue is typically smaller than λ_1, and actually coincides with λ_1' and λ_1'', see Section 3 below.

We are concerned in particular with operators with almost periodic coefficients.

Definition 1. *A function $f\colon \mathbb{R}^N \to \mathbb{R}$ is said to be* almost periodic *(a.p.) if from any sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ one can extract a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\big(f(x+x_{n_k})\big)_{k\in\mathbb{N}}$ converges uniformly in $x \in \mathbb{R}^N$.*

We say that the elliptic operator $\mathscr{L}$ is a.p. if its coefficients, i.e. the components of the matrix field A, of the vector field b and the function c, are all a.p. functions.

The *criticality theory* has been developed by Pinchover [24] in order to study the existence, uniqueness and stability of positive solutions of elliptic equations beyond the case of operators with compact resolvent, and also by [20] in the case of Schrödinger operators.

Definition 2. *An elliptic operator $\mathscr{L}$ is:*

- supercritical *if the set of positive solutions of $\mathscr{L}u = 0$ is empty;*
- critical *if the set of positive functions satisfying $\mathscr{L}u \le 0$ is one-dimensional;*
- subcritical *in the other cases.*

The above definition differs from the original one of Pinchover [24], which is expressed in terms of the existence a positive Green function or of a ground state, in the sense of Agmon [1]. For a comprehensive treatment of the subject, from both the PDE and the probability point of view, we refer to the monograph by Pinsky [26]. The above equivalent definition is given in [25], cf. also [1,13] or [26, Theorems 3.4 and 3.9]. A previous equivalent definition in the case of Schrödinger operators had been given by Simon [28]. Criticality can be related to the generalized principal eigenvalue:

$$\begin{aligned} \lambda_1 < 0 &\iff \mathscr{L} \text{ is supercritical,} \\ \lambda_1 > 0 &\implies \mathscr{L} \text{ is subcritical,} \\ \lambda_1 = 0 &\implies \mathscr{L} \text{ is either critical or subcritical.} \end{aligned}$$

Hence the criticality is not completely characterized by λ_1.

For a critical operator, the set of positive solutions of $\mathscr{L}u = 0$ coincides with the set of positive supersolutions, hence it is also one-dimensional, by definition. In dimension $N = 1$, the converse holds true, namely, an operator is critical if and only if the set of positive solutions of $\mathscr{L}u = 0$ is one-dimensional, see [26, Proposition 5.1.3] (or [20, Appendix 1] for the self-adjoint case).

It is well known that the Laplace operator in $\mathbb{R}^N$ is critical if and only if $N \le 2$. This result has been extended by Damanik–Killip–Simon [12, Theorem 5] to the operator $\mathscr{L} = \Delta + c(x)$. A related result is given by Pinsky [26, Theorem 8.2.7], proved using probabilistic techniques, that asserts that for a periodic operator $\mathscr{L}$, not necessarily self-adjoint, $\mathscr{L} + \lambda_1$ is critical if and only if $N \le 2$. Using the results of [4,22,25] one can obtain still another extension to non-periodic operators.

Proposition 3. *Let $\mathscr{L}$ be a self-adjoint operator with regular coefficients. There holds:*

(i) *if $N \le 2$ and the equation $\mathscr{L} = 0$ admits a positive bounded solution φ, then $\mathscr{L}$ is critical;*
(ii) *if $N \ge 3$ and the equation $\mathscr{L} = 0$ admits a solution φ satisfying $\inf \varphi > 0$, then $\mathscr{L}$ is subcritical.*

Proof. (i). [22, Theorem 2] implies that φ is a *generalized principal eigenfunction* of $\mathscr{L}$, i.e. $\lambda_1 = 0$. Let $\psi > 0$ satisfy $\mathscr{L}\psi \le 0$. Call $u := -\psi + k\varphi$ with $k > 0$ large enough so that $\sup u > 0$. Thus u is a subsolution of $\mathscr{L} = 0$ and it is bounded from above. It then follows again from [22, Theorem 2] that u is a multiple of φ, whence so is ψ. This shows that $\mathscr{L}$ is critical.

(ii). Assume by contradiction that $\mathscr{L}$ is critical. Then applying [25, Theorem 1.7] (with $\psi \equiv 1$) one deduces that the operator $\nabla \cdot (A\nabla)$ is critical too. But this is not the case in dimension 3 or larger, by [3,4,15]. □

An a.p. function is necessarily bounded. Moreover if it is a positive solution of an elliptic equation, then it must have a positive infimum, because otherwise it would converge locally uniformly (hence uniformly) to 0 around minimizing sequences, which is impossible. One then derives from Proposition 3 the following.

Corollary 4. *Let $\mathscr{L}$ be a self-adjoint operator with regular coefficients and assume that the equation $\mathscr{L}u = 0$ admits a positive a.p. solution φ. Then $\mathscr{L}$ is critical if and only if $N \le 2$.*

Corollary 4 extends the above-mentioned result [26, Theorem 8.2.7] in the case of self-adjoint operators, because for a self-adjoint periodic operator, the generalized principal eigenvalue λ_1 reduces to the periodic principal eigenvalue, see [6, Proposition 6.6], that is, $\mathscr{L} + \lambda_1$ admits a positive periodic solution. One can wonder whether the reciprocal of Corollary 4 holds true for an a.p. operator, namely, if in low dimension criticality is equivalent to the existence of an a.p. principal eigenvalue. The following gives a negative answer.

Counterexample 5. *There exists a self-adjoint a.p. operator $\mathscr{L}$ in dimension $N = 1$ which is critical but does not admit any positive a.p. eigenfunction.*

The proof of Counterexample 5 is given in Section 4. It makes use of a construction carried out in our previous paper [27], that we will reclaim. The operator of the counterexample possesses a ground state with an almost exponential decay, which is related to the phenomenon of *Anderson localization*, see [11].

We conclude with a last result that shows, on the one hand, that self-adjoint a.p. operators with $\lambda_1 = 0$ may not be critical, even in dimension 1, and, on the other hand, that criticality is not invariant in the passage to limit operators, which, loosely speaking, are the ω-limits of the original operator.

Definition 6. *A* limit operator *associated with $\mathscr{L}$ is an operator of the form*

$$\mathscr{L}^* u = \nabla \cdot \left(A^*(x)\nabla u\right) + b^*(x) \cdot \nabla u + c^*(x)u,$$

where $A^*(x)$, $b^*(x)$, $c^*(x)$ are the pointwise limits of $A(x + x_n)$, $b(x + x_n)$, $c(x + x_n)$ respectively, for a given sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$.

Counterexample 7. *There exists a self-adjoint a.p. operator $\mathscr{L}$ in dimension $N = 1$ which is subcritical, it has $\lambda_1 = 0$, but it admits a limit operator $\mathscr{L}^*$ which is critical.*

Remark 8. For an a.p. operator $\mathscr{L}$, the convergences of $A(x + x_n)$, $b(x + x_n)$, $c(x + x_n)$ in the definition of the limit operator $\mathscr{L}^*$ are actually uniform in $x \in \mathbb{R}^N$ (up to subsequences). It follows that $A^*(x - x_n)$, $b^*(x - x_n)$, $c^*(x - x_n)$ converge (up to subsequences) to $A(x)$, $b(x)$, $c(x)$, hence $\mathscr{L}$ is a limit operator associated with $\mathscr{L}^*$ (which is also a.p.). In particular, we can rephrase

Counterexample 7 by saying that there exists a self-adjoint a.p. operator in dimension $N = 1$ which is critical, but it admits a limit operator which is subcritical.

The operator $\mathscr{L}$ in Counterexample 7 exhibits three striking differences with respect to periodic operators:

(1) $\mathscr{L}$ does not admit an a.p. (or merely a bounded) positive eigenfunction, owing to Proposition 3;
(2) for a periodic operator $\widetilde{\mathscr{L}}$, $\widetilde{\mathscr{L}} + \lambda_1$ is always critical in dimension $N \leq 2$, by [26, Theorem 8.2.6], while $\mathscr{L}$ is not;
(3) a periodic operator shares the same criticality character with all its limit operators (which are just translations of the former), while $\mathscr{L}$ does not.

3. Other notions of generalized principal eigenvalue

Besides the generalized principal eigenvalue λ_1 introduced in [8], two other notions were employed in [5,6,9] to characterize the existence, uniqueness and the positivity of solutions of elliptic equations on unbounded domains, with application to the study of the Fisher-KPP equation in heterogeneous media. In the case where the domain is the whole space, they reduce to:

$$\lambda_1' := \inf\{\lambda \in \mathbb{R} : \exists\, \phi > 0,\ \sup \phi < +\infty,\ -\mathscr{L}\phi \leq \lambda\phi \text{ on } \mathbb{R}^N\},$$

and

$$\lambda_1'' := \sup\{\lambda \in \mathbb{R} : \exists\, \phi,\ \inf \phi > 0,\ -\mathscr{L}\phi \geq \lambda\phi \text{ on } \mathbb{R}^N\}.$$

One of the key properties of these other notions is that the operator $\mathscr{L}$ fulfils the *maximum principle* if $\lambda_1'' > 0$ and only if $\lambda_1' \geq 0$, cf. [9, Theorem 1.6].

One has that the inequalities $\lambda_1'' \leq \lambda_1' \leq \lambda_1$ hold in general, while for a self-adjoint operator it holds that $\lambda_1' = \lambda_1$, see [9, Theorem 1.7]. It is further conjectured in [9] that $\lambda_1'' = \lambda_1'$. Using a probabilistic approach connecting elliptic PDEs and branching processes, Maillard–Tough [19] have recently proved this conjecture in the case of self-adjoint operators with Hölder continuous coefficients, and disproved it with a counterexample for non-self-adjoint operators.

For a periodic operator, both quantities λ_1' and λ_1'' coincide with the periodic principal eigenvalue, that is, the unique eigenvalue admitting a positive periodic eigenfunction, while λ_1 is in general greater than the latter. Berestycki and Nadin showed that, for an operator with a.p. (and even *uniquely ergodic*), Hölder continuous coefficients, not necessarily self-adjoint, one has $\lambda_1' = \lambda_1''$. This result is derived in [7] using the homogenization techniques of Lions–Souganidis [18]. Indeed, the study of eigenproblems can be recast in the homogenization framework in terms of the existence of *approximate correctors* for a transformed operator.

4. Non-existence of an a.p. principal eigenvalue in the critical case

In [27], an a.p. function b has been constructed in order to exhibit a counterexample to a Liouville-type result for elliptic operators. We use the same function b to construct the operator $\mathscr{L}$ of Counterexample 5. Let us recall its definition: first, we set

$$\sigma(x) := \begin{cases} -1 & \text{if } x \in [-1, 0), \\ 1 & \text{if } x \in [0, 1], \end{cases}$$

then, by iteration,

$$\forall\, n \in \mathbb{N}, \quad \sigma(x) := \begin{cases} \sigma(x + 2\cdot 3^n) - \dfrac{1}{(n+1)^2} & \text{if } x \in [-3^{n+1}, -3^n), \\ \sigma(x - 2\cdot 3^n) + \dfrac{1}{(n+1)^2} & \text{if } x \in (3^n, 3^{n+1}]. \end{cases} \tag{1}$$

Finally, in order to have a regular function, we define

$$b(x) := \sigma(x)\sin^4(\pi x).$$

The function b is depicted in Figure 1. It is of class C^3, odd and *limit periodic*, that is, it is the uniform limit of a sequence of continuous periodic functions. In this case, such a sequence is given by $(b_n)_{n\in\mathbb{N}}$ with $b_n \equiv b$ on $[-3^n, 3^n]$ and then extended by periodicity. Limit periodic functions are a subclass of the a.p. functions because, as it is easily seen from Definition 1, the space of a.p. functions is closed with respect to the L^∞ norm (see e.g. [2,14]). Moreover, one can show the following inequality:

$$\forall\, x \in \mathbb{R}, \quad \int_0^x b(t)\,\mathrm{d}t \geq K\frac{|x|}{\left(\log_3|x|+1\right)^2} - K, \tag{2}$$

for some $K > 0$, see [27, Proposition 3.3].

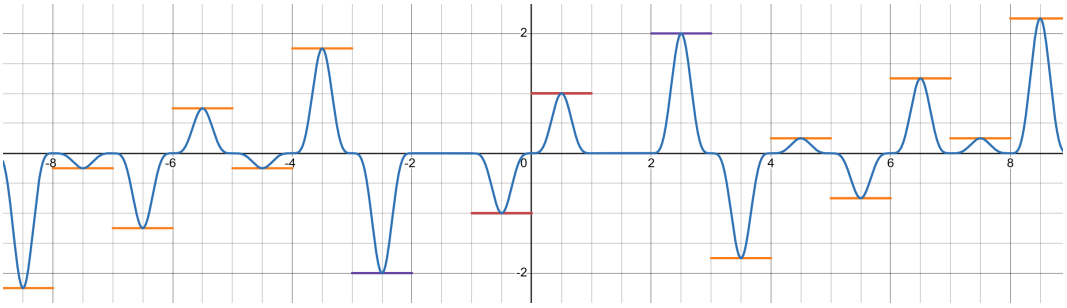

Figure 1. The graphs of σ and b.

Proof of Counterexample 5. Consider the following operator:

$$Lu := u'' - 2b(x)u', \quad x \in \mathbb{R},$$

with b defined as above. The function

$$v(x) := e^{-\int_0^x b(t)\,\mathrm{d}t}u(x)$$

satisfies $\mathscr{L}v = e^{-\int_0^x b(t)\,\mathrm{d}t}Lu$, with

$$\mathscr{L}v := v'' + (b' - b^2)v.$$

The coefficient $b' - b^2$ is a.p., because b^2 is a.p., and also b' is, being uniformly continuous, and it is known that the derivative of an a.p. function is a.p. if and only if it is uniformly continuous, see e. g. [2, Property V].

We now investigate the criticality of $\mathscr{L}$ and the sign of its principal eigenvalue λ_1. The function

$$\varphi(x) := e^{-\int_0^x b(t)\,\mathrm{d}t}$$

satisfies $\mathscr{L}\varphi = 0$. It follows from (2) that φ decays to 0 at $\pm\infty$, hence it is bounded. Recalling the definition of λ_1 and of λ_1' in Section 3 one derives $\lambda_1 \geq 0 \geq \lambda_1'$. But then, since $\lambda_1 = \lambda_1'$ by [9, Theorem 1.7(i)], one eventually deduces that $\lambda_1 = 0$. On the other hand, in dimension 1, the existence of a bounded positive eigenfunction implies that $\mathscr{L}$ is critical, as it follows from [20, Theorem A.9] or also [12, Theorem 5], [25], or even from Proposition 3 above. In turn, criticality entails that the space of positive solutions to $\mathscr{L} = 0$ has dimension 1. Therefore, $\mathscr{L}$ does not admit an a.p. positive eigenfunction, since the only candidate, φ, decays to 0 and thus it cannot be a.p.

Notice that, instead, L is critical and the unique positive eigenfunction is the constant one, which is a.p. □

5. The subcritical case

This section is devoted to the construction of Counterexample 7. We start again from the function b of [27], reclaimed in the previous section. We define

$$Lu := u'' + 2b(x)u', \quad x \in \mathbb{R}.$$

Similarly to before, we transform L into a self-adjoint operator by considering

$$v(x) = e^{\int_0^x b(t)\,\mathrm{d}t} u(x),$$

namely $\mathscr{L}v = e^{\int_0^x b(t)\,\mathrm{d}t} Lu$ with

$$\mathscr{L}v := v'' - (b' + b^2)v.$$

As noticed in Section 4, this is an a.p. operator. We first study the generalized principal eigenvalue and the criticality character of $\mathscr{L}$.

Proposition 9. *The operator $\mathscr{L}$ is subcritical and satisfies $\lambda_1 = 0$.*

Proof. Let L, $\mathscr{L}$ be the above operators. The solutions to $Lu = 0$ satisfy

$$u'(x) = Ce^{-2\int_0^x b(t)\,\mathrm{d}t},$$

for some $C \in \mathbb{R}$, hence they are bounded by (2). Then, letting u be as above with $C \neq 0$, the functions

$$u_1 \equiv 1, \qquad u_2 \equiv u + \|u\|_\infty$$

are two linearly independent positive solutions of $L = 0$. This means that L is subcritical and has $\lambda_1(-L) \geq 0$ (here and in the sequel we are stressing the operator to which the principal eigenvalue is referred to). Considering the functions $e^{\int_0^x b(t)\,\mathrm{d}t} u_{1,2}(x)$ one infers that also $\mathscr{L}$ is subcritical with $\lambda_1(-\mathscr{L}) \geq 0$.

It remains to show that $\lambda_1(-\mathscr{L}) = 0$, i.e. that $\lambda_1(-\mathscr{L}) \leq 0$. We will make use of the principal eigenvalue $\lambda_1'(-\mathscr{L})$ defined in Section 3. We indeed know from [9, Theorem 1.7(i)] that $\lambda_1(-\mathscr{L}) = \lambda_1'(-\mathscr{L})$. Notice that the solutions to $\mathscr{L} = 0$ grow like $e^{\int_0^x b}$, which is unbounded, and thus they cannot be used in the definition of $\lambda_1'(-\mathscr{L})$. However, we will show that, for any $\lambda > 0$, $(\mathscr{L} + \lambda)v = 0$ admits a positive bounded subsolution. Fix $\lambda > 0$. Define the following function:

$$\psi(x) := e^{\int_0^x [b(t) - \gamma(t)]\,\mathrm{d}t},$$

where $\gamma \in C^1(\mathbb{R})$ will be chosen later. We compute

$$\mathscr{L}\psi = \left[(b-\gamma)' + (b-\gamma)^2\right]\psi - (b' + b^2)\psi = (\gamma^2 - 2b\gamma - \gamma')\psi.$$

We choose γ small enough (in C^1 norm) so that $\gamma^2 - 2b\gamma - \gamma' \geq -\lambda$ in $\mathbb{R}$, and in addition $\gamma(-\infty) < 0 < \gamma(+\infty)$. It follows that $-\mathscr{L}\psi \leq \lambda\psi$. In order to see that ψ is bounded, we use the fact that b has zero average. We recall that for an a.p. function f, the average

$$\langle f \rangle := \lim_{h \to \pm\infty} \frac{1}{h} \int_{x_0}^{x_0+h} f(t)\,\mathrm{d}t$$

exists uniformly with respect to x_0 (and it is independent of x_0). Then, being b odd and a.p., it has zero average, since $\int_{-h}^{h} b = 0$ for all $h \in \mathbb{R}$. As a consequence,

$$\lim_{x \to \pm\infty} \frac{1}{x} \int_0^x (b - \gamma) = -\gamma(\pm\infty),$$

which immediately implies that $\psi \to 0$ as $x \to \pm\infty$. We eventually infer, from the definition, that $\lambda_1'(-\mathscr{L}) \leq \lambda$, and therefore $\lambda_1'(-\mathscr{L}) \leq 0$ by the arbitrariness of $\lambda > 0$. We have shown that $\lambda_1(-\mathscr{L}) = \lambda_1'(-\mathscr{L}) \leq 0$, as desired. □

It remains to show that the subcritical operator $\mathscr{L}$ admits a limit operator which is critical. Owing to Proposition 3, it is sufficient to exhibit a limit operator which admits a positive bounded eigenfunction with eigenvalue 0 (see also [21, Proposition 1.7] and the characterization of critical operators in [20]). We now improve these results by relaxing the boundedness of the eigenfunction, using the Harnack inequality.

Lemma 10. *Let $\mathscr{L}$ be a self-adjoint operator on $\mathbb{R}$ and assume that there exist $\lambda \in \mathbb{R}$ and a positive function φ satisfying $-\mathscr{L}\varphi = \lambda\varphi$ and*

$$\liminf_{x\to-\infty} \varphi(x) < +\infty \qquad \text{and} \qquad \liminf_{x\to+\infty} \varphi(x) < +\infty.$$

Then necessarily $\lambda = \lambda_1$ and moreover $\mathscr{L} + \lambda$ is critical.

Proof. Let ψ be another eigenfunction of $-\mathscr{L}$, not identically equal to φ, associated with the same eigenvalue λ as φ. We can assume without loss of generality that $\psi(0) \neq 0$. Let us normalize φ, ψ by $\varphi(0) = \psi(0) = 1$. Using the equation $(\mathscr{L} + \lambda) = 0$ satisfied by both φ and ψ we get, for all $x \in \mathbb{R}$,

$$\int_0^x (A\varphi')'\psi = \int_0^x (A\psi')'\varphi,$$

from which we deduce

$$A(x)(\varphi'\psi - \psi'\varphi)(x) = A(0)(\varphi' - \psi')(0),$$

or equivalently

$$(\psi/\varphi)'(x) = \frac{A(0)(\psi' - \varphi')(0)}{A(x)\varphi^2(x)}. \tag{3}$$

Since $\psi \not\equiv \varphi$, we know that $(\psi' - \varphi')(0) \neq 0$. Up to reflecting the functions φ, ψ and the coefficients of $\mathscr{L}$, it is not restrictive to assume that $(\psi' - \varphi')(0) > 0$ (notice that the reflection does not affect the hypothesis on φ). We then infer from (3) that ψ/φ is increasing on $\mathbb{R}$. Next, by hypothesis, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$x_n \longrightarrow -\infty, \qquad \lim_{n\to\infty} \varphi(x_n) \in [0, +\infty).$$

We can assume that $(x_n)_{n\in\mathbb{N}}$ satisfies (up to subsequences) $x_{n+1} \leq x_n - 1$ for $n \in \mathbb{N}$. We now use the Harnack inequality. It implies the existence of a positive constant C such that

$$\forall\ n \in \mathbb{N},\ x \in [x_n - 1, x_n + 1], \quad \varphi(x) \leq C.$$

This fact, together with (3), yields

$$\forall\ n \in \mathbb{N},\ x \in [x_n - 1, x_n + 1], \quad (\psi/\varphi)'(x) \geq \frac{A(0)(\psi' - \varphi')(0)}{C^2 \sup A},$$

from which we deduce

$$(\psi/\varphi)(x_{n+1}) \leq (\psi/\varphi)(x_n - 1) \leq (\psi/\varphi)(x_n) - \frac{A(0)(\psi' - \varphi')(0)}{C^2 \sup A} \leq \cdots \leq (\psi/\varphi)(x_1) - n\frac{A(0)(\psi' - \varphi')(0)}{C^2 \sup A}.$$

We eventually obtain that $(\psi/\varphi)(x_n) \to -\infty$ as $n \to \infty$. This shows that φ is the unique positive eigenfunction of $-\mathscr{L}$ with eigenvalue λ, up to a scalar multiple, which, according to [26, Proposition 5.1.3], implies that $\mathscr{L} + \lambda$ is critical, whence in particular $\lambda = \lambda_1$. □

In order to conclude the proof of Counterexample 7, it remains to find a limit operator associated with $\mathscr{L}$ that fulfils the hypothesis of Lemma 10. We claim that the function b (defined in Section 4) satisfies the following:

$$\forall\ k, n \in \mathbb{N},\ k \leq n, \quad \int_{3^n - 3^k}^{3^n} b(x)\,\mathrm{d}x \geq 0. \tag{4}$$

Let us first show that (4) holds in the case $n = k$, i.e. that $\int_0^{3^n} b(x)\,\mathrm{d}x \geq 0$. This inequality holds for $n = 0$. Assuming that it holds for some $n \in \mathbb{N}$, using (1) we get

$$\begin{aligned}\int_0^{3^{n+1}} b(x)\,\mathrm{d}x &= \int_0^{3^n} b(x)\,\mathrm{d}x + \int_{3^n}^{3^{n+1}} \left[\sigma(x-2\cdot 3^n) + (n+1)^{-2}\right]\sin^4(\pi x)\,\mathrm{d}x \\ &\geq \int_{-3^n}^{3^n} b(x)\,\mathrm{d}x,\end{aligned}$$

and the latter integral is 0 because b is odd. Hence (4) holds for $n = k$. We now fix $k \in \mathbb{R}$ and proceed by iteration on n. Using (1) we derive

$$\int_{3^{n+1}-3^k}^{3^{n+1}} b(x)\,\mathrm{d}x = \int_{3^{n+1}-3^k}^{3^{n+1}} \left[\sigma(x-2\cdot 3^n) + (n+1)^{-2}\right]\sin^4(\pi x)\,\mathrm{d}x \geq \int_{3^n-3^k}^{3^n} b(x)\,\mathrm{d}x,$$

and thus, if (4) holds for some n, then it holds true for $n+1$. This proves (4).

Next, using again (1), we deduce, for $k \leq n$,

$$\begin{aligned}\int_{3^n}^{3^n+3^k} b(x)\,\mathrm{d}x &= \int_{3^n}^{3^n+3^k} \left[\sigma(x-2\cdot 3^n) + (n+1)^{-2}\right]\sin^4(\pi x)\,\mathrm{d}x \\ &= \int_{-3^n}^{-3^n+3^k} b(x)\,\mathrm{d}x + \frac{C3^k}{(n+1)^2},\end{aligned}$$

where $C = \int_0^1 \sin^4(\pi x)\,\mathrm{d}x$. Then, since b is odd, we obtain

$$\int_{3^n}^{3^n+3^k} b(x)\,\mathrm{d}x = \int_{3^n}^{3^n-3^k} b(y)\,\mathrm{d}y + \frac{C3^k}{(n+1)^2},$$

and therefore, by (4), we eventually derive

$$\forall\ k, n \in \mathbb{N},\ k \leq n, \quad \int_{3^n}^{3^n+3^k} b(x)\,\mathrm{d}x \leq \frac{C3^k}{(n+1)^2}. \tag{5}$$

Let us consider the sequence $(b_n)_{n\in\mathbb{N}}$ of translations of b defined by

$$b_n(x) \coloneqq b(x+3^n).$$

By almost periodicity, $(b_n)_{n\in\mathbb{N}}$ and $(b_n')_{n\in\mathbb{N}}$ converge uniformly (up to subsequences) to some function b_∞ and its derivative b_∞', respectively. It follows from (4) that

$$\forall\ k \in \mathbb{N}, \quad \int_{-3^k}^0 b_\infty(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_{-3^k}^0 b_n(x)\,\mathrm{d}x \geq 0,$$

while, from (5),

$$\forall\ k \in \mathbb{N}, \quad \int_0^{3^k} b_\infty(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_0^{3^k} b_n(x)\,\mathrm{d}x \leq \lim_{n\to\infty} \frac{C3^k}{(n+1)^2} = 0.$$

We finally consider the limit operator

$$\mathscr{L}^* u \coloneqq u'' - (b_\infty' + b_\infty^2)u.$$

The function

$$\varphi(x) \coloneqq e^{\int_0^x b_\infty(t)\,\mathrm{d}t}$$

satisfies $\mathscr{L}^*\varphi = 0$ and, in addition,

$$\forall\ k \in \mathbb{N}, \quad \varphi(\pm 3^k) = e^{\int_0^{\pm 3^k} b_\infty(t)\,\mathrm{d}t} \leq 1.$$

We can therefore apply Lemma 10 and infer that $\mathscr{L}^*$ is critical.

Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

References

[1] S. Agmon, "On positivity and decay of solutions of second order elliptic equations on Riemannian manifolds", in *Methods of functional analysis and theory of elliptic equations (Naples, 1982)* (D. Greco, ed.), Liguori Publications, 1983, pp. 19–52.

[2] L. Amerio and G. Prouse, *Almost-periodic functions and functional equations*, Van Nostrand Reinhold Co., 1971.

[3] M. T. Barlow, "On the Liouville property for divergence form operators", *Can. J. Math.* **50** (1998), no. 3, pp. 487–496.

[4] H. Berestycki, L. Caffarelli and L. Nirenberg, "Further qualitative properties for elliptic equations in unbounded domains", *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4)* **25** (1997), no. 1–2, pp. 69–94.

[5] H. Berestycki, F. Hamel and L. Roques, "Analysis of the periodically fragmented environment model. I. Species persistence", *J. Math. Biol.* **51** (2005), no. 1, pp. 75–113.

[6] H. Berestycki, F. Hamel and L. Rossi, "Liouville-type results for semilinear elliptic equations in unbounded domains", *Ann. Mat. Pura Appl. (4)* **186** (2007), no. 3, pp. 469–507.

[7] H. Berestycki and G. Nadin, *Asymptotic spreading for general heterogeneous Fisher-KPP type equations*, Memoirs of the American Mathematical Society, vol. 1381, American Mathematical Society, 2022.

[8] H. Berestycki, L. Nirenberg and S. R. S. Varadhan, "The principal eigenvalue and maximum principle for second-order elliptic operators in general domains", *Commun. Pure Appl. Math.* **47** (1994), no. 1, pp. 47–92.

[9] H. Berestycki and L. Rossi, "Generalizations and properties of the principal eigenvalue of elliptic operators in unbounded domains", *Commun. Pure Appl. Math.* **68** (2015), no. 6, pp. 1014–1065.

[10] K. Bjerklöv, "Positive Lyapunov exponents for continuous quasiperiodic Schrödinger equations", *J. Math. Phys.* **47** (2006), no. 2, article no. 022702 (4 pages).

[11] J. Bourgain and C. E. Kenig, "On localization in the continuous Anderson–Bernoulli model in higher dimension", *Invent. Math.* **161** (2005), no. 2, pp. 389–426.

[12] D. Damanik, R. Killip and B. Simon, "Schrödinger operators with few bound states", *Commun. Math. Phys.* **258** (2005), no. 3, pp. 741–750.

[13] U. Das and Y. Pinchover, "The Landis conjecture via Liouville comparison principle and criticality theory", 2024. Online at https://arxiv.org/abs/2405.11695.

[14] A. M. Fink, *Almost periodic differential equations*, Lecture Notes in Mathematics, vol. 377, Springer, 1974.

[15] N. A. Ghoussoub and C. Gui, "On a conjecture of De Giorgi and some related problems", *Math. Ann.* **311** (1998), no. 3, pp. 481–491.

[16] M. Goldstein and W. Schlag, "Hölder continuity of the integrated density of states for quasiperiodic Schrödinger equations and averages of shifts of subharmonic functions", *Ann. Math. (2)* **154** (2001), no. 1, pp. 155–203.

[17] S. M. Kozlov, "Ground states of quasiperiodic operators", *Dokl. Akad. Nauk SSSR* **271** (1983), no. 3, pp. 532–536.

[18] P.-L. Lions and P. E. Souganidis, "Stochastic homogenization of Hamilton–Jacobi and "viscous" Hamilton–Jacobi equations with convex nonlinearities — Revisited", *Commun. Math. Sci.* **8** (2010), no. 2, pp. 627–637.

[19] P. Maillard and O. Tough, "Generalised principal eigenvalues and global survival of branching Markov processes", 2025. Online at https://arxiv.org/abs/2505.12127.

[20] M. Murata, "Structure of positive solutions to $(-\Delta+V)u=0$ in $\mathbf{R}^n$", *Duke Math. J.* **53** (1986), no. 4, pp. 869–943.

[21] G. Nadin and L. Rossi, "Generalized transition fronts for one-dimensional almost periodic Fisher-KPP equations", *Arch. Ration. Mech. Anal.* **223** (2017), no. 3, pp. 1239–1267.

[22] S. Nordmann, "Maximum principle and principal eigenvalue in unbounded domains under general boundary conditions", 2021. Online at https://arxiv.org/abs/2102.07558.

[23] R. D. Nussbaum and Y. Pinchover, "On variational principles for the generalized principal eigenvalue of second order elliptic operators and some applications", *J. Anal. Math.* **59** (1992), pp. 161–177.

[24] Y. Pinchover, "On positive solutions of second-order elliptic equations, stability results, and classification", *Duke Math. J.* **57** (1988), no. 3, pp. 955–980.

[25] Y. Pinchover, "A Liouville-type theorem for Schrödinger operators", *Commun. Math. Phys.* **272** (2007), no. 1, pp. 75–84.

[26] R. G. Pinsky, *Positive harmonic functions and diffusion*, Cambridge Studies in Advanced Mathematics, vol. 45, Cambridge University Press, 1995.

[27] L. Rossi, "Liouville type results for periodic and almost periodic linear operators", *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **26** (2009), no. 6, pp. 2481–2502.

[28] B. Simon, "Brownian motion, L^p properties of Schrödinger operators and the localization of binding", *J. Funct. Anal.* **35** (1980), no. 2, pp. 215–229.

[29] E. Sorets and T. Spencer, "Positive Lyapunov exponents for Schrödinger operators with quasi-periodic potentials", *Commun. Math. Phys.* **142** (1991), no. 3, pp. 543–566.