ACADÉMIE
DES SCIENCES
INSTITUT DE FRANCE

Comptes Rendus

Mathématique

Andrea Marrama

On the restriction of finite torsors

Volume 364 (2026), p. 431-435

Online since: 27 July 2026

https://doi.org/10.5802/crmath.836

CENTRE
MERSENNE

The Comptes Rendus. Mathématique *are a member of the*
Mersenne Center for open scientific publishing
www.centre-mersenne.org — e-ISSN : 1778-3569

ACADÉMIE DES SCIENCES
INSTITUT DE FRANCE
1666

Comptes Rendus. Mathématique
2026, Vol. 364, p. 431-435
https://doi.org/10.5802/crmath.836

Research article
Algebraic geometry

On the restriction of finite torsors

Andrea Marrama [a]

[a] Dipartimento di Matematica "Tullio Levi-Civita", Università degli Studi di Padova, Via Trieste 63, 35121 Padova, Italy

E-mail: andrea.marrama@unipd.it

Abstract. We give a detailed account of the proof of a purity theorem for finite flat torsors due to Laurent Moret-Bailly. In addition, we show that a part of this result, namely the full faithfulness, holds also under weaker assumptions.

Keywords. Torsors, finite group schemes, purity.

2020 Mathematics Subject Classification. 14L30, 14M17, 14F20.

Funding. The author was funded by the European Union, Next Generation EU, through the Ministero dell'Università e della Ricerca, project PRIN-20222B24AY.

Manuscript received 1 March 2025, revised 6 July 2025, accepted 25 April 2026, online since 27 July 2026.

Introduction

The purpose of this text is to give a more concise and convenient form to some contents of the author's Master thesis [7], together with an additional result which is surely known to the experts, but was left as an open question in loc. cit. More specifically, we give an account of the proof of a purity theorem for finite flat torsors due to Laurent Moret-Bailly (Theorem 6), which was originally published without proof in [8]. The statement is analogous to the Zariski–Nagata purity theorem for finite étale coverings (see [5, X, §3]). The proof is based on Auslander's method from [1] and relies on a crucial result from loc. cit. (Theorem 5). As hinted above, this proof follows an argument suggested by Moret-Bailly. We remark that a slightly more general theorem was recently obtained in [2, Theorem 7.1.3] using a similar strategy. The goal of this note is to provide a fully detailed proof of Moret-Bailly's result and to show that a part of it, namely the full faithfulness, holds also under weaker assumptions (Proposition 3).

Notation. For any scheme S, we denote by $\mathscr{O}_S$ its structure sheaf. Let S be a scheme, G an affine S-group-scheme. We denote by $\mathrm{Sch}_{/S}$ the category of S-schemes and by $\mathrm{Tors}(S,G)$ the category of fppf G-torsors over S. If $T \to S$ is a morphism of schemes and X is any S-scheme, we write X_T for the pullback $X \times_S T$; we will consider G_T as a T-group-scheme and, if X is an fppf G-torsor over S, we will consider X_T as a G_T-torsor over T. If X and Y are two S-schemes, we denote by $\mathrm{Hom}_S(X,Y)$ the set of homomorphisms of S-schemes from X to Y; if X and Y are fppf G-torsors over S, we write $\mathrm{Hom}_{(S,G)}(X,Y)$ for the set of homomorphisms from X to Y in $\mathrm{Tors}(S,G)$. Regular schemes are assumed to be locally Noetherian.

We first address the full faithfulness statement, which is an easy consequence of the following two lemmas; the first one is simply recalled from [3].

Lemma 1 ([3, Corollaire 6.1.14]). *Let S be a locally Noetherian normal scheme, $T \to S$ an integral morphism of schemes, $U \subseteq S$ a dense open subscheme. The restriction map*

$$\mathrm{Hom}_S(S,T) \longrightarrow \mathrm{Hom}_S(U,T)$$

is bijective.

Lemma 2. *Let S be a scheme, G an affine S-group-scheme, X and Y two fppf G-torsors over S. The following fppf sheaf of sets (with the obvious restriction maps):*

$$\begin{aligned} \underline{\mathrm{Hom}}_G(X,Y)\colon\ \mathrm{Sch}_{/S} &\longrightarrow \mathrm{Sets} \\ (T \to S) &\longmapsto \mathrm{Hom}_{(T,G_T)}(X_T, Y_T) \end{aligned}$$

is representable by an affine S-scheme, which is fppf locally isomorphic to G. In particular, if G is finite over S, then $\underline{\mathrm{Hom}}_G(X,Y)$ is representable by a finite S-scheme.

Proof. We may argue in a similar way as for the representability of fppf torsors with respect to an affine group scheme. Namely, the question is fppf local on S by effective descent of affine schemes (see e.g. [9, Tag 0245]). Thus, we are reduced to the situation where X and Y are trivial G-torsors, in which case $\underline{\mathrm{Hom}}_G(X,Y)$ is isomorphic to G as a presheaf on $\mathrm{Sch}_{/S}$ and the claim is clear. The final statement of the lemma follows because the property of being finite is fppf local on the base. □

Proposition 3. *Let S be a locally Noetherian normal scheme, $U \subseteq S$ a dense open subscheme, G a finite S-group-scheme. The pullback functor*

$$\mathrm{Tors}(S,G) \longrightarrow \mathrm{Tors}(U,G_U)$$

is fully faithful.

Proof. Let X, Y be fppf G-torsors over S. We have to show that the restriction map

$$\mathrm{Hom}_{(S,G)}(X,Y) \longrightarrow \mathrm{Hom}_{(U,G_U)}(X_U,Y_U) \tag{$*$}$$

is bijective. Let Z be the finite S-scheme representing $\underline{\mathrm{Hom}}_G(X,Y)$ by Lemma 2. The map $(*)$ may then be identified with the restriction map:

$$\mathrm{Hom}_S(S,Z) \longrightarrow \mathrm{Hom}_S(U,Z),$$

which is bijective by Lemma 1 because Z is integral over S. □

Remark 4. The proposition could also be proved bypassing Lemma 2. Indeed, the bijectivity of $(*)$ may be checked fppf locally, thus reducing the matter to the bijectivity of the restriction map $\mathrm{Hom}_S(S,G) \to \mathrm{Hom}_S(U,G)$, which holds by Lemma 1.

Regarding the essential surjectivity of the pullback functor above, we now report a result due to Laurent Moret-Bailly giving a sufficient condition. The statement is analogous to the Zariski–Nagata purity theorem for finite étale coverings (see [5, X, §3]). We provide here a proof, omitted in the original publication [8], based on a sketch which the author of loc. cit. communicated to us. The argument relies on a crucial result by Maurice Auslander, which we recall for the reader's convenience.

Theorem 5 ([1, Theorem 1.3]). *Let R be a regular local ring, M a finitely generated reflexive R-module. Suppose that the R-module of endomorphisms $\mathrm{End}_R(M)$ is isomorphic to a direct sum of copies of M. Then, M is a free R-module.*

We now come to Moret-Bailly's purity theorem for finite flat torsors.

Theorem 6 ([8, Lemme 2]). *Let S be a regular scheme, $U \subseteq S$ an open subscheme, G a finite flat S-group-scheme. Suppose that the complement of U in S has codimension at least 2. Then, the pullback functor*

$$\mathrm{Tors}(S,G) \longrightarrow \mathrm{Tors}(U,G_U)$$

is an equivalence of categories.

Proof. Full faithfulness follows from Proposition 3. To show that we have an equivalence of categories, we will exhibit, for any fppf G_U-torsor over U, an object of $\mathrm{Tors}(S,G)$ which restricts to the given one over U.

Let $j\colon U \to S$ denote the inclusion morphism, let X be an fppf G_U-torsor over U and denote by $\rho\colon G_U \times_U X \to X$ the G_U-action on X. Because $X \to U$ is finite flat, it corresponds to a coherent $\mathscr{O}_U$-algebra $\mathscr{P}$, which is locally free as an $\mathscr{O}_U$-module. Consider the quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{Q} := j_*\mathscr{P}$, which satisfies $\mathscr{Q}|_U = \mathscr{P}$. Then, $\mathscr{Q}$ corresponds to an affine S-scheme $\widetilde{X}$ such that $\widetilde{X}_U = X$. Let now $\mathscr{G}$ denote the coherent $\mathscr{O}_S$-algebra corresponding to $G \to S$. Then, $\mathscr{G}|_U$ is the coherent $\mathscr{O}_U$-algebra corresponding to $G_U \to U$ and ρ corresponds to a homomorphism of $\mathscr{O}_U$-algebras $\rho^\#\colon \mathscr{P} \to \mathscr{P} \otimes_{\mathscr{O}_U} \mathscr{G}|_U$. We set $\widetilde{\rho}^\# := j_*\rho^\#\colon \mathscr{Q} \to \mathscr{Q} \otimes_{\mathscr{O}_S} \mathscr{G}$, where $j_*(\mathscr{P} \otimes_{\mathscr{O}_U} \mathscr{G}|_U) \cong \mathscr{Q} \otimes_{\mathscr{O}_S} \mathscr{G}$ because $\mathscr{G}$ is finite locally free as an $\mathscr{O}_S$-module. We have that $\widetilde{\rho}^\#$ is a homomorphism of $\mathscr{O}_S$-algebras satisfying $\widetilde{\rho}^\#|_U = \rho^\#$, hence it corresponds to a homomorphism of S-schemes $\widetilde{\rho}\colon G \times_S \widetilde{X} \to \widetilde{X}$ whose pullback along j is ρ. By functoriality of j_* (and of the identification used in the definition of $\widetilde{\rho}^\#$), $\widetilde{\rho}$ verifies the rules of a G-action on $\widetilde{X}$. It remains to see that $\widetilde{\rho}$ makes $\widetilde{X}$ an fppf G-torsor over S.

We first observe that $\mathscr{Q}$ is a coherent $\mathscr{O}_S$-module by [4, Corollaire 5.11.4], because S is regular, $\mathscr{P}$ is finite locally free and the codimension of $S \setminus U$ in S is at least 2. In other words, $\widetilde{X}$ is a finite S-scheme. Next, we claim that $\mathscr{Q}$ is a locally free $\mathscr{O}_S$-module. In order to prove this, consider the $\mathscr{O}_S$-modules of homomorphisms (respectively endomorphisms) $\mathscr{F} := \mathscr{H}om_{\mathscr{O}_S}(\mathscr{G},\mathscr{Q})$, $\mathscr{H} := \mathscr{E}nd_{\mathscr{O}_S}(\mathscr{Q})$. We define a homomorphism of $\mathscr{O}_S$-modules $\theta\colon \mathscr{F} \to \mathscr{H}$ by setting, for $V \subseteq S$ open and $u\colon \mathscr{G}|_V \to \mathscr{Q}|_V$ a homomorphism of $\mathscr{O}_V$-modules:

$$\theta_V(u)\colon \mathscr{Q}|_V \xrightarrow{\widetilde{\rho}^\#|_V} \mathscr{Q}|_V \otimes_{\mathscr{O}_V} \mathscr{G}|_V \xrightarrow{(\mathrm{id},u)} \mathscr{Q}|_V,$$

where the last map uses the $\mathscr{O}_V$-algebra structure of $\mathscr{Q}|_V$. We argue that $\theta|_U$ is an isomorphism. Indeed, let $\Omega^\#\colon \mathscr{P} \otimes_{\mathscr{O}_U} \mathscr{P} \to \mathscr{P} \otimes_{\mathscr{O}_U} \mathscr{G}|_U$ be the homomorphism of $\mathscr{O}_U$-algebras given by the canonical map $\mathscr{P} \to \mathscr{P} \otimes_{\mathscr{O}_U} \mathscr{G}|_U$ on the first component and $\rho^\#$ on the second component. Because X is a G_U-torsor over U (with respect to ρ), we have that $\Omega^\#$ is an isomorphism. This allows to find an inverse $(\theta|_U)^{-1}\colon \mathscr{H}|_U \to\to \mathscr{F}|_U$ of $\theta|_U$, namely, for $V \subseteq U$ open and v an endomorphism of the $\mathscr{O}_V$-module $\mathscr{Q}|_V = \mathscr{P}|_V$:

$$(\theta|_U)^{-1}_V(v)\colon \mathscr{G}|_V \longrightarrow \mathscr{P}|_V \longrightarrow \otimes_{\mathscr{O}_V}\mathscr{G}|_V \xrightarrow{(\Omega^\#)|_V^{-1}} \mathscr{P}|_V \longrightarrow \otimes_{\mathscr{O}_V}\mathscr{P}|_V \xrightarrow{(\mathrm{id},v)} \mathscr{P}|_V,$$

where the first arrow is the canonical map of $\mathscr{O}_V$-algebras. Note now that, because $\mathscr{Q}$ is the pushforward of an $\mathscr{O}_U$-module along j and it is coherent, then, by [4, Théorème 5.10.5], $\mathscr{Q}$ has depth at least 2 at all points of $S \setminus U$. Therefore, the same holds for $\mathscr{F}$ and $\mathscr{H}$. But then, by loc. cit. (now used in the opposite direction than above), the fact that θ is an isomorphism over U is enough to deduce that θ is an isomorphism of $\mathscr{O}_S$-modules. In particular, for every point $s \in S$, we have an isomorphism of the stalks $\mathrm{End}_{\mathscr{O}_{S,s}}(\mathscr{Q}_s) = \mathscr{H}_s \cong \mathscr{F}_s = \mathrm{Hom}_{\mathscr{O}_{S,s}}(\mathscr{G}_s,\mathscr{Q}_s)$. Because $\mathscr{G}$ is locally free, it follows that $\mathrm{End}_{\mathscr{O}_{S,s}}(\mathscr{Q}_s)$ is isomorphic to a direct sum of copies of $\mathscr{Q}_s$. Moreover, $\mathscr{Q}_s$ is a finitely generated reflexive $\mathscr{O}_{S,s}$-module, because $\mathscr{Q}$ is coherent, locally free over U and, as observed above, it has depth at least 2 on $S \setminus U$ (then one uses the criterion [9, Tag 0AVA]). Thus, for every $s \in S$, we have that $\mathscr{Q}_s$ is a free $\mathscr{O}_{S,s}$-module by Auslander's result (Theorem 5), since S is regular. This proves that $\mathscr{Q}$ is a locally free $\mathscr{O}_S$-module, hence that $\widetilde{X}$ is a flat S-scheme. In fact, $\widetilde{X} \to S$ is an fppf covering. Indeed, it only remains to see that it is surjective, but this follows from the fact that $\widetilde{X} \to S$ is a finite, hence closed morphism and its image contains the dense subset $U \subseteq S$, as $X \to U$ is an fppf G_U-torsor, hence faithfully flat.

To conclude that $\widetilde{X}$ is an fppf G-torsor over S with respect to $\widetilde{\rho}$, it is now sufficient to show that the homomorphism of S-schemes $\widetilde{\Omega}\colon G \times_S \widetilde{X} \to \widetilde{X} \times_S \widetilde{X}$, given by $\widetilde{\rho}$ on the first component and the second projection on the second component, is an isomorphism. Equivalently, we shall prove that the homomorphism of $\mathscr{O}_S$-algebras $\widetilde{\Omega}^\#\colon \mathscr{Q} \otimes_{\mathscr{O}_S} \mathscr{Q} \to \mathscr{Q} \otimes_{\mathscr{O}_S} \mathscr{G}$, given by the canonical map $\mathscr{Q} \to \mathscr{Q} \otimes_{\mathscr{O}_S} \mathscr{G}$ on the first component and $\widetilde{\rho}^\#$ on the second component, is an isomorphism. Note that $\widetilde{\Omega}^\#|_U = \Omega^\#$ is an isomorphism of $\mathscr{O}_U$-algebras. Moreover, the coherent $\mathscr{O}_S$-modules $\mathscr{Q} \otimes_{\mathscr{O}_S} \mathscr{Q}$ and $\mathscr{Q} \otimes_{\mathscr{O}_S} \mathscr{G}$ have depth at least 2 at all points of $S \setminus U$, because $\mathscr{Q}$ and $\mathscr{G}$ are finite locally free, S is regular and $S \setminus U$ has codimension at least 2 in S. Therefore, $\widetilde{\Omega}^\#$ is an isomorphism by [4, Théorème 5.10.5] and this concludes the proof of the theorem. $\square$

A consequence of the previous theorem is that, given a regular base scheme, the obstruction to extend a finite flat torsor from a dense open subscheme lies exclusively in codimension 1. A slightly more general version of this result, based on a similar proof, is given in [2, Theorem 7.1.3]. For a non-Noetherian analogue (in the context of algebraic spaces over Prüfer bases), see [6, Theorem 6.8].

Acknowledgments

The author wishes to thank Laurent Moret-Bailly for kindly sharing a sketch of the proof of Theorem 6 and for several comments that helped improving the exposition of the subject. The author is also grateful to Gabriel Zalamansky for suggesting to work on this topic in the first place, Matteo Longo for helpful comments on an earlier version of the manuscript and the anonymous referee for useful remarks on the structure of the paper.

Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

References

[1] M. Auslander, "On the purity of the branch locus", *Am. J. Math.* **84** (1962), pp. 116–125.

[2] K. Česnavičius and P. Scholze, "Purity for flat cohomology", *Ann. Math. (2)* **199** (2024), no. 1, pp. 51–180.

[3] A. Grothendieck, "Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes", *Publ. Math., Inst. Hautes Étud. Sci.* (1961), no. 8.

[4] A. Grothendieck, "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II", *Publ. Math., Inst. Hautes Étud. Sci.* (1965), no. 24.

[5] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*, Advanced Studies in Pure Mathematics, vol. 2, North-Holland; Masson, 1968.

[6] N. Guo and F. Liu, "Purity and quasi-split torsors over Prüfer bases", *J. Éc. Polytech., Math.* **11** (2024), pp. 187–246.

[7] A. Marrama, *A purity theorem for torsors*, ALGANT Master thesis, Universiteit Leiden (The Netherlands) and Universität Duisburg-Essen (Germany), 2016. Online at https://www.math.u-bordeaux.fr/~ybilu/algant/documents/theses/Marrama.pdf.

[8] L. Moret-Bailly, “Un théorème de pureté pour les familles de courbes lisses”, *Comptes Rendus. Mathématique* **300** (1985), no. 14, pp. 489–492.
[9] The Stacks Project Authors, *Stacks Project*. Online at http://stacks.math.columbia.edu.