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Toric Schubert varieties and directed Dynkin diagrams

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Abstract. A flag variety is a smooth projective homogeneous variety G/B, where G is a simple algebraic group over the complex numbers and B is a Borel subgroup of G. A Schubert variety X_w is a subvariety of G/B indexed by an element w of the Weyl group of G. It is called toric if it is a toric variety with respect to the action of the maximal torus $T \subset B$. In this paper, we associate an edge-labeled digraph $\mathscr{G}_w$ to a toric Schubert variety X_w and classify toric Schubert varieties up to isomorphism. We also give a simple criterion for when a toric Schubert variety X_w is (weak) Fano in terms of $\mathscr{G}_w$. Finally, we discuss whether toric Schubert varieties can be distinguished by their integral cohomology rings up to isomorphism and show that this holds when G is simply-laced.

Keywords. Schubert varieties, toric varieties, Bott manifolds, cohomological rigidity.

2020 Mathematics Subject Classification. 14M25, 14M15, 57S12, 05A05.

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Introduction

A flag variety is a smooth projective homogeneous variety G/B, where G is a simple algebraic group over the complex numbers $\mathbb{C}$ and B is a Borel subgroup of G. Let T be the maximal torus of G in B and W the Weyl group of G. The Bruhat decomposition $G/B = \bigsqcup_{w \in W} BwB/B$ provides a cell decomposition of G/B. Indeed, BwB/B is isomorphic to an affine space $\mathbb{C}^{\ell(w)}$ where $\ell(w)$ denotes the Coxeter length of w. The Schubert variety X_w is the closure of the cell BwB/B in the flag variety G/B, and X_w is called toric if it is a toric variety with respect to the torus T. It is known that X_w is toric if and only if w is a product of *distinct* simple reflections (see [8,12]). This implies that a toric Schubert variety X_w is smooth; in fact, X_w is a Bott manifold that is the total space of an iterated $\mathbb{C}P^1$-bundle over a point.

In this paper, we associate to an edge-labeled digraph $\mathscr{G}_w$ a toric Schubert variety X_w and prove the following.

Theorem 1 (Theorem 18). *Let W and W' be the Weyl groups of simple algebraic groups G and G', respectively. Then toric Schubert varieties X_w ($w \in W$) and $X_{w'}$ ($w' \in W'$) are isomorphic as varieties if and only if $\mathscr{G}_w$ and $\mathscr{G}_{w'}$ are isomorphic as edge-labeled digraphs.*

The edge-labeled digraph $\mathscr{G}_w$ is defined as follows. Take a reduced expression $s_{i_1}s_{i_2}\cdots s_{i_m}$ of w as a product of simple reflections s_i in W, where $i_1, i_2, \dots, i_m$ are mutually distinct. Then the vertex set of $\mathscr{G}_w$ is $\{i_1,\dots,i_m\}$, and there is a directed edge (i_j, i_k) between two vertices in $\mathscr{G}_w$ if and only if $j > k$ and the (i_j, i_k) entry c_{i_j,i_k} in the Cartan matrix of G is nonzero. We label the directed edge by the integer $-c_{i_j,i_k}$, which is 1, 2, or 3. Therefore, the underlying graph of $\mathscr{G}_w$ is an induced subgraph of the Dynkin diagram of G, with a multiple edge (if any) replaced by a single edge. All directed edges of $\mathscr{G}_w$ are labeled 1, except when the underlying edge of the Dynkin diagram is multiple and the directed edge is oriented in the same direction as the arrow on the Dynkin diagram. In that case, the label is 2 for a double edge, and 3 for a triple edge. If the orientation is opposite to the Dynkin-diagram arrow, then its label is 1. (See, for instance, Table 4.) Here, we notice that $\mathscr{G}_w$ is independent of the choice of reduced expression of w (see Remark 16).

A smooth projective variety X is called Fano (respectively, weak Fano) if the anticanonical divisor $-K_X$ is ample (respectively, nef and big). In birational geometry, the Fano and weak Fano conditions encode the positivity of the anticanonical divisor $-K_X$. Such positivity properties play a fundamental role in the birational classification of projective varieties and single out particularly rigid and well-behaved families. The next theorem shows that the edge-labeled digraph $\mathscr{G}_w$ is also useful for determining whether the toric Schubert variety X_w is Fano or weak Fano.

Theorem 2 (Theorem 19). *A toric Schubert variety X_w is Fano (respectively, weak Fano) if and only if every vertex of $\mathscr{G}_w$ has indegree at most* 1 *(respectively,* 2*).*

An element of the Weyl group W is called a Coxeter element if it is a product of the simple reflections, each appearing exactly once. When $w \in W$ is a Coxeter element, the underlying graph of $\mathscr{G}_w$ agrees with the Dynkin diagram of G with a multiple edge (if any) replaced by a single edge. We enumerate the isomorphism classes of (weak) Fano toric Schubert varieties X_w for Coxeter elements w in each Lie type (see Table 3).

We also discuss whether a toric Schubert variety X_w is determined by its integral cohomology ring $H^*(X_w;\mathbb{Z})$, in other words, whether $\mathscr{G}_w$ can be recovered from $H^*(X_w;\mathbb{Z})$.

Theorem 3 (Theorem 32). *The edge-labeled digraph $\mathscr{G}_w$ can be recovered (up to isomorphism) from the cohomology ring $H^*(X_w;\mathbb{Z})$ if all edge labels of $\mathscr{G}_w$ are* 1.

Combining Theorem 3 with Theorem 1, we obtain the following corollary.

Corollary 4. *Let W and W' be the Weyl groups of simple algebraic groups G and G' of simply-laced type (i.e. type* A, D, *or* E*). Then toric Schubert varieties X_w ($w \in W$) and $X_{w'}$ ($w' \in W'$) are isomorphic as varieties if and only if $H^*(X_w;\mathbb{Z})$ and $H^*(X_{w'};\mathbb{Z})$ are isomorphic as graded rings.*

Remark 5. The assumption in Theorem 3 cannot be weakened as is seen in the following examples with simple reflections numbered as in Table 1.

(1) When G is of type G_2, there are two Coxeter elements w, w' and both $\mathscr{G}_w$ and $\mathscr{G}_{w'}$ consist of two vertices with a directed edge but one has label 1 while the other has label 3. However, the cohomology rings $H^*(X_w;\mathbb{Z})$ and $H^*(X_{w'};\mathbb{Z})$ are isomorphic. Indeed, if F_a denotes the Hirzebruch surface indexed by a nonnegative integer a, then X_w and $X_{w'}$ are F_1 and F_3. As is well-known, F_a and $F_{a'}$ are isomorphic if and only if $a = a'$, and $H^*(F_a;\mathbb{Z})$ and $H^*(F_{a'};\mathbb{Z})$ are isomorphic as graded rings (more strongly, F_a and $F_{a'}$ are diffeomorphic) if and only if $a \equiv a'$ (mod 2).
(2) When G is of type C_3, $\mathscr{G}_{s_1s_3}$ consists of two vertices with no edge and $\mathscr{G}_{s_2s_3}$ consists of two vertices and a directed edge with label 2. On the other hand, $X_{s_1s_3}$ and $X_{s_2s_3}$ are isomorphic to F_0 and F_2 respectively, so their integral cohomology rings are isomorphic. We obtain a similar observation when G is of type B_3 or F_4. The toric Schubert variety $X_{s_3s_2}$ in type B_3 or F_4 is isomorphic to the Hirzebruch surface F_2 while $X_{s_1s_3}$ is isomorphic to F_0.
(3) When G is of type F_4, the cohomology rings of toric Schubert varieties $X_{s_3s_2s_1s_4}$ and $X_{s_2s_1s_4s_3}$ are isomorphic but $X_{s_3s_2s_1s_4}$ and $X_{s_2s_1s_4s_3}$ are not isomorphic as varieties (see Remark 35).

As mentioned in Remark 5, not all toric Schubert varieties are distinguished as varieties by their integral cohomology rings. However, they are distinguished by their integral cohomology rings as smooth manifolds. Indeed, toric Schubert varieties are Bott manifolds, and it is known that Bott manifolds are distinguished by their integral cohomology rings up to diffeomorphism (see [4]). Related to this, it is asked and studied in [16] whether smooth (not necessarily toric) Schubert varieties are distinguished by their integral cohomology rings up to diffeomorphism or homeomorphism.

This paper is organized as follows. As mentioned above, toric Schubert varieties are Bott manifolds. In Section 1, we review fans of Bott manifolds and recall a criterion for determining when they are Fano or weak Fano. In Section 2, we associate the edge-labeled digraph $\mathscr{G}_w$ to a toric Schubert variety X_w and prove Theorems 1 and 2. Using Theorem 1, we enumerate the isomorphism classes of (Fano or weak Fano) toric Schubert varieties in Section 3. In Section 4, we prove Theorem 3 using a presentation of $H^*(X_w;\mathbb{Z})$ as a graded ring with generators and relations.

1. Preliminaries: toric Schubert varieties and Bott manifolds

In this section, we recall the classification of toric Schubert varieties and describe them in terms of Bott manifolds. To do so, we first review Bott manifolds, which are smooth projective toric varieties. Moreover, we recall criteria for the Fano and weak Fano properties of Bott manifolds.

1.1. *Toric Schubert varieties*

Let G be a simple algebraic group over $\mathbb{C}$ of rank r, let B be a Borel subgroup, and let T be a maximal torus of G contained in B. We denote by W the Weyl group of G. A *flag variety* is the homogeneous space G/B, which is a smooth projective variety. When G is of type A with rank $n-1$, the flag variety G/B is diffeomorphic to

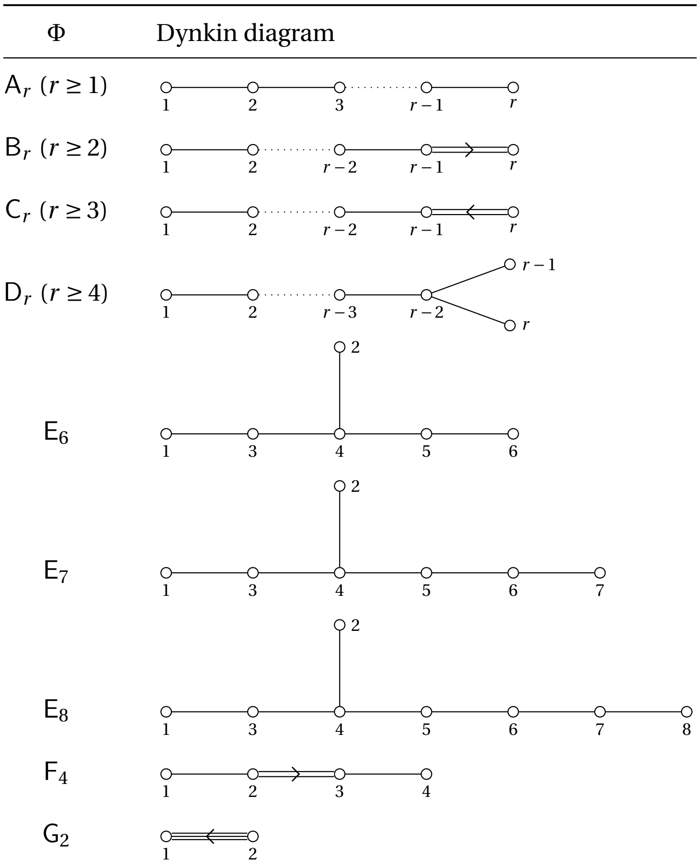
$$\mathrm{Fl}(n) := \left\{\left(\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n\right) \,\middle|\, \dim_{\mathbb{C}} V_i = i \text{ for all } i = 1,\dots,n\right\},$$

where each V_i is a linear subspace of $\mathbb{C}^n$. Moreover, the Weyl group is the symmetric group $\mathfrak{S}_n$ on the set $[n] := \{1,\dots,n\}$.

The Weyl group W of G is generated by simple reflections s_i for $i = 1,\dots,r$, so each element $w \in W$ can be expressed as a product of generators:

$$w = s_{i_1}s_{i_2}\cdots s_{i_m}.$$

Table 1. Dynkin diagrams for finite types.



If m is minimal among all such expressions for w, then m is called the (Coxeter) *length* of w and we write $\ell(w) = m$. Moreover, we call the word $s_{i_1} s_{i_2} \cdots s_{i_m}$ a *reduced decomposition* for w. An expression $s_{i_1} s_{i_2} \cdots s_{i_m}$ provides a string $(i_1, i_2, \ldots, i_m)$ in $[r]^m$, and we call it a *word*. A word $(i_1, i_2, \ldots, i_m)$ is *reduced* if the corresponding expression $s_{i_1} s_{i_2} \cdots s_{i_m}$ is reduced. In Table 1, we provide Dynkin diagrams for finite types. In this manuscript, we use the ordering on the simple roots as in the table following [10].

The left multiplication of T on G induces an action of T on G/B. Then there is a bijective correspondence between the T-fixed point set $(G/B)^T$ and the Weyl group W of G, and we denote by wB the fixed point in G/B corresponding to $w \in W$. For $w \in W$, the *Schubert variety X_w* is a subvariety of G/B defined by the (Zariski) closure of $BwB/B \subset G/B$. A Schubert variety is T-invariant and has complex dimension $\ell(w)$.

Considering a reduced decomposition for $w \in W$, one can decide whether the Schubert variety X_w is toric or not with respect to the torus action T as follows.

Theorem 6 (see [8,12]). *For $w \in W$, the following statements are equivalent:*

(1) *X_w is a toric variety (with respect to the T-action);*
(2) *X_w is a smooth toric variety (with respect to the T-action);*
(3) *some (equivalently, any) reduced expression for w consists of distinct simple reflections.*

Example 7. There are six elements in the Weyl group $\mathfrak{S}_3$ of type A_2. For each $w \in \mathfrak{S}_3$, we display whether the corresponding Schubert variety X_w is toric or not and the length $\ell(w) = \dim_{\mathbb{C}} X_w$ in the following table:

w	e	s_1	s_2	$s_1 s_2$	$s_2 s_1$	$s_1 s_2 s_1$
X_w is toric	yes	yes	yes	yes	yes	no
$\ell(w)$	0	1	1	2	2	3

Example 8. The Weyl group W_{B_2} of type B_2 is given by $W_{\mathsf{B}_2} = \langle s_1, s_2 \mid (s_1)^2 = (s_2)^2 = (s_1 s_2)^4 = e \rangle$. There are eight elements in W_{B_2}, and five of them produce toric Schubert varieties:

w	e	s_1	s_2	$s_1 s_2$	$s_2 s_1$	$s_1 s_2 s_1$	$s_2 s_1 s_2$	$s_1 s_2 s_1 s_2$
X_w is toric	yes	yes	yes	yes	yes	no	no	no
$\ell(w)$	0	1	1	2	2	3	3	4

When G is of type A_{n-1}, the fan of a toric Schubert variety X_w is the same as the normal fan of a polytope

$$\mathsf{Q}_w := \mathrm{Conv}\{(v^{-1}(1), \dots, v^{-1}(n)) \in \mathbb{R}^n \mid v \le w\}.$$

Here, we compare two elements $v, w \in W$ with respect to the *Bruhat order*, that is, if $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression, then $v \le w$ if and only if there exists a reduced expression $v = s_{i_{j_1}} s_{i_{j_2}} \cdots s_{i_{j_q}}$ with $1 \le j_1 < \cdots < j_q \le k$. To consider the fan of a toric Schubert variety X_w in general Lie types, we recall [9, Section 3.7] and [13, Section 4.3].

Theorem 9 ([13, Theorem 4.23]). *Let $w = s_{i_1} \cdots s_{i_m}$ be a reduced decomposition for $w \in W$. Assume that $i_1, \dots, i_m$ are distinct. Then the fan of the toric Schubert variety X_w is isomorphic to the fan in $\mathbb{R}^m$ whose primitive ray generators are the $2m$ column vectors of the following matrix. A subset of these column vectors spans a cone if and only if it does not contain both the k-th column of the left $m \times m$ submatrix and the k-th column of the right $m \times m$ submatrix for any $k = 1, \dots, m$:*

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & & -1 & \cdots & 0 \\ \vdots & & \ddots & & & -c_{i_j,i_k} & \ddots & \\ 0 & 0 & \cdots & 1 & & & & -1 \end{array}\right], \tag{1}$$

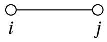
where the (j,k) entry for $m \ge j > k \ge 1$ in the right submatrix above is $-c_{i_j,i_k}$ and the integers $c_{p,q}$ are Cartan integers. Therefore, if we denote the $2m$ ray vectors in the above matrix from left to right by $\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_m}, \mathbf{w}_{i_1}, \dots, \mathbf{w}_{i_m}$, then

$$\mathbf{v}_{i_k} + \mathbf{w}_{i_k} = \sum_{j>k} (-c_{i_j,i_k}) \mathbf{v}_{i_j} \tag{2}$$

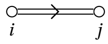
for $k = 1, \dots, m$.

We briefly explain how to obtain Theorem 9. Suppose that $w = s_{i_1} \cdots s_{i_m}$ is a reduced decomposition for $w \in W$ consisting of distinct simple reflections. Then the Schubert variety X_w is isomorphic to the *Bott–Samelson variety* corresponding to the word $(i_1, \dots, i_m)$. Here, a Bott–Samelson variety is a smooth projective variety that can be understood as the total space of an iterated $\mathbb{C}P^1$-bundle whose bundle structure is decided by the decomposition $(i_1, \dots, i_m)$. Moreover, such a Bott–Samelson variety is also toric and its fan structure is described in the paper by Grossberg and Karshon (see [9, Section 3.7]).

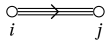
We call the right $m \times m$ submatrix in (1) the *reduced characteristic matrix*. Recall that the Cartan integers $c_{i,j}$ can be read directly from the Dynkin diagram as follows:



$c_{i,j} = -1$, $c_{j,i} = -1$



$c_{i,j} = -2$, $c_{j,i} = -1$



$c_{i,j} = -3$, $c_{j,i} = -1$

Here, we note that an arrow on the Dynkin diagram represents the relative lengths of two simple roots. Indeed, an arrow points to the shorter of two roots. (Hence, one can also consider an arrow as an inequality comparing the length of two roots.) In the above diagrams with multiple edges, we have $\|\alpha_i\| > \|\alpha_j\|$ for two simple roots α_i and α_j associated with the vertices i and j in the Dynkin diagram.

Example 10. Suppose that G is of type A_5. Let $w = s_3 s_1 s_4 s_5 s_2$. Then the reduced characteristic matrix in (1) is given as follows:

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 & -1 \end{bmatrix}.$$

Since $(i_1, i_2, i_3, i_4, i_5) = (3,1,4,5,2)$, the column vectors above are $\mathbf{w}_3, \mathbf{w}_1, \mathbf{w}_4, \mathbf{w}_5, \mathbf{w}_2$ from the left. Therefore we have

$$\mathbf{v}_3 + \mathbf{w}_3 = \mathbf{v}_4 + \mathbf{v}_2, \qquad \mathbf{v}_1 + \mathbf{w}_1 = \mathbf{v}_2, \qquad \mathbf{v}_4 + \mathbf{w}_4 = \mathbf{v}_5, \qquad \mathbf{v}_5 + \mathbf{w}_5 = \mathbf{0}, \qquad \mathbf{v}_2 + \mathbf{w}_2 = \mathbf{0}.$$

Example 11. Suppose that G is of type B_2. As computed in Example 8, there are five toric Schubert varieties. Considering toric Schubert varieties of dimension 2, we obtain the reduced characteristic matrices in (1) as follows:

$$\begin{array}{cc} s_1 s_2 & s_2 s_1 \\ \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} \end{array}$$

For $w = s_1 s_2$, the $(2,1)$-entry of the reduced characteristic matrix is $-c_{i_2,i_1} = -c_{2,1} = -(-1) = 1$. Moreover, for $w = s_2 s_1$, the $(2,1)$-entry of the reduced characteristic matrix is $-c_{i_2,i_1} = -c_{1,2} = -(-2) = 2$. Therefore,

$$\mathbf{v}_1 + \mathbf{w}_1 = \mathbf{v}_2, \qquad \mathbf{v}_2 + \mathbf{w}_2 = \mathbf{0}$$

in the former case and

$$\mathbf{v}_2 + \mathbf{w}_2 = 2\mathbf{v}_1, \qquad \mathbf{v}_1 + \mathbf{w}_1 = \mathbf{0}$$

in the latter case.

1.2. *Fano or weak Fano toric varieties*

There is a combinatorial way to determine whether a smooth projective toric variety is Fano or weak Fano. We say that a smooth projective variety X is Fano (respectively, weak Fano) if the anticanonical divisor $-K_X$ is ample (respectively, nef and big). For a fan Σ, a subset R of the primitive ray vectors is called a *primitive collection* of Σ if

$$\mathrm{Cone}(R) \notin \Sigma \quad \text{but} \quad \mathrm{Cone}(R \setminus \{\mathbf{u}\}) \in \Sigma \quad \text{for every } \mathbf{u} \in R.$$

We denote by $\mathrm{PC}(\Sigma)$ the set of primitive collections of Σ.

Example 12. For the toric Schubert variety X_w in Theorem 9, the primitive ray vectors of the fan Σ of X_w are the column vectors $C := \{\mathbf{v}_{i_1},\dots,\mathbf{v}_{i_m},\mathbf{w}_{i_1},\dots,\mathbf{w}_{i_m}\}$ in (1). Then

$$\mathrm{PC}(\Sigma) = \left\{\{\mathbf{v}_{i_k},\mathbf{w}_{i_k}\} \;\middle|\; k = 1,\dots,m\right\} \tag{3}$$

because a subset of C forms a cone if and only if it does not contain both $\mathbf{v}_{i_k}$ and $\mathbf{w}_{i_k}$ for each k as mentioned in Theorem 9.

For a primitive collection $R = \{\mathbf{u}'_1,\dots,\mathbf{u}'_\ell\}$, we get either $\mathbf{u}'_1+\cdots+\mathbf{u}'_\ell = \mathbf{0}$ or there exists a unique cone σ of positive dimension such that $\mathbf{u}'_1+\cdots+\mathbf{u}'_\ell$ is in the interior of σ. If the sum $\mathbf{u}'_1+\cdots+\mathbf{u}'_\ell$ is in the interior of σ, then

$$\mathbf{u}'_1+\cdots+\mathbf{u}'_\ell = a_1\mathbf{u}_1+\cdots+a_s\mathbf{u}_s, \tag{4}$$

where $\mathbf{u}_1,\dots,\mathbf{u}_s$ are the primitive generators of σ and $a_1,\dots,a_s$ are positive integers. We call (4) a *primitive relation*, and the *degree* $\deg R$ of a primitive collection R is defined to be

$$\deg R := \begin{cases} \ell & \text{if } \mathbf{u}'_1+\cdots+\mathbf{u}'_\ell = \mathbf{0}, \\ \ell-(a_1+\cdots+a_s) & \text{otherwise.} \end{cases} \tag{5}$$

Batyrev [1] provided a criterion for a smooth projective toric variety to be Fano or weak Fano.

Proposition 13 ([1, Proposition 2.3.6]). *A smooth projective toric variety X is Fano (respectively, weak Fano) if and only if* $\deg R > 0$ *(respectively,* $\deg R \geq 0$*) for every primitive collection R of the fan Σ of X.*

A smooth compact toric variety is called a *Bott manifold* if it is isomorphic to the total space of a Bott tower that is an iterated $\mathbb{C}P^1$-bundle starting with a point, where each $\mathbb{C}P^1$-bundle is the projectivization of the Whitney sum of two complex line bundles. It is known that a smooth compact toric variety X is a Bott manifold if and only if the fan Σ of X has primitive collections $\mathrm{PC}(\Sigma)$ of the form in (3). Theorem 9 says that a toric Schubert variety is a Bott manifold. We refer the reader to [9] for Bott towers, and to [14] for details on Bott manifolds.

2. Toric Schubert varieties and directed graphs

In this section, we associate an edge-labeled digraph $\mathcal{G}_w$ to a toric Schubert variety X_w and prove that two toric Schubert varieties are isomorphic as varieties if and only if the associated edge-labeled digraphs are isomorphic (Theorem 18). We also give a simple criterion for determining when X_w is Fano or weak Fano in terms of $\mathcal{G}_w$.

As before, let W be the Weyl group of a simple algebraic group G of rank r and $s_1,\dots,s_r$ the simple reflections in W. Suppose that a Schubert variety X_w ($w \in W$) is toric. Then simple reflections in a reduced decomposition $w = s_{i_1}\cdots s_{i_m}$ for w are mutually distinct by Theorem 6.

Definition 14. *For a reduced expression $w = s_{i_1}\cdots s_{i_m}$ with distinct simple reflections, we define an edge-labeled digraph $\mathcal{G}_w$ as follows:*

- $V(\mathcal{G}_w) = \{i_1,\dots,i_m\}$;
- $(i_j,i_k) \in E(\mathcal{G}_w)$ *if and only if* $c_{i_j,i_k} \neq 0$ *for* $1 \leq k < j \leq m$.

Note that $1 \leq -c_{i_j,i_k} \leq 3$ if $c_{i_j,i_k} \neq 0$. We assign the positive integer $-c_{i_j,i_k}$ to the directed edge $(i_j,i_k) \in E(\mathcal{G}_w)$. When we draw $\mathcal{G}_w$, we omit the label 1 for simplicity.

Example 15.

(1) Let G be of type A. For $w = s_2s_1s_3s_4 = s_2s_3s_1s_4 = s_2s_3s_4s_1$ and $w' = s_3s_4s_2s_1 = s_3s_2s_4s_1 = s_3s_2s_1s_4$, we have

$$\mathcal{G}_w = 1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4 \qquad \mathcal{G}_{w'} = 1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4$$

For $w'' = s_1s_2s_4s_5 = s_1s_4s_2s_5 = s_1s_4s_5s_2 = s_4s_1s_2s_5 = s_4s_1s_5s_2 = s_4s_5s_1s_2$, $\mathscr{G}_{w''}$ is not connected and is given by:

$$\mathscr{G}_{w''} = \begin{tikzcd} 1 & 2 \arrow[l] & & 4 & 5 \arrow[l] \end{tikzcd}$$

(2) Let G be of type C_3. For $w = s_1s_2s_3$ and $w' = s_3s_2s_1$, we have

$$\mathscr{G}_w = \begin{tikzcd} 1 & 2 \arrow[l] & 3 \arrow[l, "2"'] \end{tikzcd} \qquad \mathscr{G}_{w'} = \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \end{tikzcd}$$

(3) Let G be of type D_4. For $w = s_2s_1s_3s_4 = s_2s_3s_1s_4 = s_2s_3s_4s_1 = s_2s_1s_4s_3 = s_2s_4s_1s_3 = s_2s_4s_3s_1$, we have

$$\mathscr{G}_w = \begin{tikzcd} & & 3 \arrow[dl] \\ 1 \arrow[r] & 2 & \\ & & 4 \arrow[ul] \end{tikzcd}$$

We say that two edge-labeled digraphs are isomorphic if there is a bijection between their vertices preserving directed edges and labels on the edges. For example, $\mathscr{G}_w$ and $\mathscr{G}_{w'}$ in Example 15(1) are isomorphic but those in Example 15(2) are not isomorphic as edge-labeled digraphs although they are isomorphic as digraphs.

Remark 16. Suppose that $c_{i_k,i_{k+1}} = 0$ for some $k \in [m-1]$, i.e. s_{i_k} and $s_{i_{k+1}}$ commute. Then replacing the factor $s_{i_k}s_{i_{k+1}}$ in the reduced decomposition $w = s_{i_1}\cdots s_{i_m}$ by $s_{i_{k+1}}s_{i_k}$ is called a 2-*move*. Since w is toric, every reduced decomposition uses each simple reflection at most once, so no braid moves of length ≥ 3 can occur. Therefore, by the Word Property [3, Theorem 3.3.1], any two reduced decompositions are related by a sequence of 2-moves. This implies that the edge-labeled digraph $\mathscr{G}_w$ does not depend on the choice of the reduced decomposition for w.

Let G be a simple algebraic group. We say an edge-labeled digraph $\mathscr{G}$ is an *induced subgraph* of the Dynkin diagram of G if the underlying graph of $\mathscr{G}$ is an induced subgraph of the Dynkin diagram of G, and the edge labels match with the Cartan integers, that is, for each directed edge $(p,q) \in E(\mathscr{G})$, the label $-c_{p,q}$ is assigned. For instance, in type C_3, the first two graphs are induced while the last is not:

$$\begin{tikzcd} 1 & 2 \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} 2 & 3 \arrow[l, "2"'] \end{tikzcd} \qquad \begin{tikzcd} 1 & 2 \arrow[l, "2"'] & 3 \arrow[l, "2"'] \end{tikzcd}$$

Proposition 17. *Let G be a simple algebraic group. An edge-labeled digraph $\mathscr{G}$ is an induced subgraph of the Dynkin diagram of G if and only if $\mathscr{G} = \mathscr{G}_w$ for some $w \in W$ defining a toric Schubert variety. In particular, the underlying graph of $\mathscr{G}$ is the Dynkin diagram of G if and only if $\mathscr{G} = \mathscr{G}_w$ for a Coxeter element w.*

Proof. Suppose that $w \in W$ defines a toric Schubert variety. By definition, $\mathscr{G}_w$ is an induced subgraph of the Dynkin diagram of G.

For the opposite direction, suppose that $\mathscr{G}$ is an induced subgraph of the Dynkin diagram of G with the induced labeling of vertices. We construct a word for w by iteratively removing sinks.

Recall that a vertex i is a *sink* if there is no directed edge of $\mathscr{G}$ emanating from i. Let i_1 be the smallest sink of $\mathscr{G}$. Let $\mathscr{G}_1$ be the induced subgraph obtained from $\mathscr{G}$ by deleting i_1, and let i_2 be the smallest sink of $\mathscr{G}_1$. Continuing this process, we obtain a sequence $i_1,\dots,i_m$, where $m = \left|V(\mathscr{G})\right|$. Set

$$w := s_{i_1}s_{i_2}\cdots s_{i_m}.$$

We claim that $\mathscr{G}_w = \mathscr{G}$.

First, by construction, we have $V(\mathscr{G}_w) = \{i_1,\dots,i_m\} = V(\mathscr{G})$. Next we compare directed edges. Fix $1 \le k < j \le m$. By definition of $\mathscr{G}_w$, there is a directed edge (i_j, i_k) in $\mathscr{G}_w$ if and only if $c_{i_j,i_k} \neq 0$, i.e., the vertices i_j and i_k are adjacent in the Dynkin diagram of G.

On the other hand, since $\mathscr{G}$ is an induced subgraph of the Dynkin diagram, adjacency in the Dynkin diagram is equivalent to adjacency in the underlying graph of $\mathscr{G}$. Moreover, because i_k is a sink in $\mathscr{G}_{k-1}$, there is no directed edge in $\mathscr{G}_{k-1}$ leaving i_k. In particular, for every $j > k$ with $c_{i_j,i_k} \neq 0$, the only possible orientation of the edge between i_j and i_k in $\mathscr{G}_{k-1}$ is toward i_k, i.e., it must be the directed edge (i_j, i_k). Since $\mathscr{G}$ is obtained from $\mathscr{G}_{k-1}$ by adding back the previously removed vertices $i_1,\dots,i_{k-1}$, the orientation of the edge between i_j and i_k remains (i_j, i_k) in $\mathscr{G}$ as well.

Therefore, for every pair $1 \le k < j \le m$ we have:

$$c_{i_j,i_k} \neq 0 \iff i_j \text{ and } i_k \text{ are adjacent in the Dynkin diagram} \iff (i_j, i_k) \in E(\mathscr{G}).$$

This is exactly the edge rule for $\mathscr{G}_w$, and the labels also agree because both $\mathscr{G}$ and $\mathscr{G}_w$ label the directed edge (i_j, i_k) by $-c_{i_j,i_k}$. Hence $\mathscr{G}_w = \mathscr{G}$.

Finally, the underlying graph of $\mathscr{G}$ is the Dynkin diagram of G if and only if $V(\mathscr{G})$ is the full set of vertices of the Dynkin diagram, i.e., $\{i_1,\dots,i_m\} = \{1,\dots,r\}$. In that case $w = s_{i_1}\cdots s_{i_r}$ is a product of all simple reflections, hence a Coxeter element. Conversely, if w is a Coxeter element defining a toric Schubert variety, then $V(\mathscr{G}_w)$ contains all simple reflections, so the underlying graph of $\mathscr{G}_w$ is the Dynkin diagram of G. □

The edge-labeled digraph $\mathscr{G}_w$ has the same information as the primitive relations in (2). Indeed, $\mathscr{G}_w$ can be obtained from the primitive relations in (2) if we take a directed edge (i_j, i_k) whenever c_{i_j,i_k} in (2) is nonzero and put the label $-c_{i_j,i_k}$ on it. Conversely, it is clear that the primitive relations in (2) can be obtained from $\mathscr{G}_w$ through this correspondence.

Theorem 18. *Let W and W' be the Weyl groups of simple algebraic groups G and G', respectively. Then the toric Schubert varieties X_w ($w \in W$) and $X_{w'}$ ($w' \in W'$) are isomorphic as varieties if and only if $\mathscr{G}_w$ and $\mathscr{G}_{w'}$ are isomorphic as edge-labeled digraphs.*

Proof. Note that two smooth complete toric varieties are isomorphic as varieties if and only if they are also isomorphic as toric varieties (see [2]). This is because the automorphism group $\mathrm{Aut}(X)$ of a smooth complete toric variety X is linear algebraic and the algebraic torus acting on X is a maximal torus of X (see [15, Section 3.4] and references therein). Hence, it is enough to prove that X_w and $X_{w'}$ are isomorphic as toric varieties if and only if $\mathscr{G}_w$ and $\mathscr{G}_{w'}$ are isomorphic as edge-labeled digraphs.

First, we prove the "only if" part. Suppose that X_w and $X_{w'}$ are isomorphic as varieties. Then there is an isomorphism φ from the fan Σ of X_w to the fan Σ' of $X_{w'}$. Let

$$\{\mathbf{v}_1,\dots,\mathbf{v}_m,\mathbf{w}_1,\dots,\mathbf{w}_m\} \qquad \text{and} \qquad \{\mathbf{v}'_1,\dots,\mathbf{v}'_m,\mathbf{w}'_1,\dots,\mathbf{w}'_m\} \tag{6}$$

be the ray generators (i.e., column vectors of the matrix in (1)) of the fans Σ and Σ', respectively. (Within this proof, for notational simplicity, we index the vectors by $1,\dots,m$ instead of using $i_1,\dots,i_m$.) Since the isomorphism φ preserves the primitive collections, there exists a bijection $f\colon [m] \to [m]$ such that

$$\big\{\varphi(\mathbf{v}_k), \varphi(\mathbf{w}_k)\big\} = \big\{\mathbf{v}'_{f(k)}, \mathbf{w}'_{f(k)}\big\} \qquad (\forall\, k \in [m]). \tag{7}$$

Therefore,

$$\varphi(\mathbf{v}_k + \mathbf{w}_k) = \varphi(\mathbf{v}_k) + \varphi(\mathbf{w}_k) = \mathbf{v}'_{f(k)} + \mathbf{w}'_{f(k)} = \sum_{p > f(k)} \big(-c_{i'_p, i'_{f(k)}}\big)\mathbf{v}'_p \tag{8}$$

while sending the identity in (2) by φ, we obtain

$$\varphi(\mathbf{v}_k + \mathbf{w}_k) = \sum_{j>k} (-c_{i_j,i_k})\varphi(\mathbf{v}_j). \tag{9}$$

It follows from (8) and (9) that

$$\sum_{p>f(k)} \big(-c_{i'_p,i'_{f(k)}}\big)\mathbf{v}'_p = \sum_{j>k} (-c_{i_j,i_k})\varphi(\mathbf{v}_j). \tag{10}$$

Here, $\varphi(\mathbf{v}_j) = \mathbf{v}'_{f(j)}$ or $\mathbf{w}'_{f(j)}$ by (7).

When $c_{i_j,i_k} \neq 0$, we claim that the latter case does not occur. Assume on the contrary that there exist indices such that $c_{i_j,i_k} \neq 0$ and $\varphi(\mathbf{v}_j) = \mathbf{w}'_{f(j)}$. Take j such that $f(j)$ is minimal among such indices. Comparing the $f(j)$-th entries of both sides in (10), we obtain a contradiction. This is because the vector $\mathbf{w}'_{f(j)}$ has -1 on the $f(j)$-th entry and $-c_{i_j,i_k} \neq 0$, so the vector on the right-hand side of (10) has a negative component while all components of the vector on the left-hand side of (10) are nonnegative. Thus (10) implies

$$\sum_{p>f(k)} \big(-c_{i'_p,i'_{f(k)}}\big)\mathbf{v}'_p = \sum_{j>k} (-c_{i_j,i_k})\mathbf{v}'_{f(j)}.$$

Since $\mathbf{v}'_1,\dots,\mathbf{v}'_m$ are linearly independent, we obtain

$$\big\{f(j) \mid j>k,\ c_{i_j,i_k} \neq 0\big\} = \big\{p \mid p > f(k),\ c_{i'_p,i'_{f(k)}} \neq 0\big\}.$$

Therefore, if we have an edge (i_j, i_k) in $\mathscr{G}_w$ with label $-c_{i_j,i_k}$, then we have $f(j) > f(k)$ with $c_{i'_{f(j)},i'_{f(k)}} = c_{i_j,i_k}$. Hence, we have an edge $(i'_{f(j)}, i'_{f(k)})$ with label $-c_{i_j,i_k}$ in the graph $\mathscr{G}_{w'}$. Therefore, the bijection $f\colon [m] \to [m]$ provides an isomorphism from $\mathscr{G}_w$ to $\mathscr{G}_{w'}$ as edge-labeled digraphs.

Now we prove the "if" part. Suppose that $\mathscr{G}_w$ and $\mathscr{G}_{w'}$ are isomorphic as edge-labeled digraphs. Then there is a bijection $f\colon [m] \to [m]$ which satisfies

$$c_{i_j,i_k} = c_{i'_{f(j)},i'_{f(k)}} \quad \text{for all } 1 \le k < j \le m. \tag{11}$$

We shall observe that the linear automorphism φ of $\mathbb{R}^m$ defined by $\varphi(\mathbf{v}_j) = \mathbf{v}'_{f(j)}$ for $j \in [m]$ provides an isomorphism from the fan Σ to the fan Σ'. First we note

$$\varphi(\mathbf{w}_k) = \mathbf{w}'_{f(k)} \quad (\forall\, k \in [m]). \tag{12}$$

Indeed, (10) holds by (11) and it follows from (2) and (10) that we have

$$\begin{aligned}\varphi(\mathbf{v}_k) + \varphi(\mathbf{w}_k) &= \varphi(\mathbf{v}_k + \mathbf{w}_k)\\ &= \sum_{j>k} (-c_{i_j,i_k})\varphi(\mathbf{v}_j)\\ &= \sum_{f(j)>f(k)} \big(-c_{i'_{f(j)},i'_{f(k)}}\big)\mathbf{v}'_{f(j)}\\ &= \mathbf{v}'_{f(k)} + \mathbf{w}'_{f(k)}.\end{aligned}$$

Since $\varphi(\mathbf{v}_k) = \mathbf{v}'_{f(k)}$ by definition of φ, the identity above implies (12) as we desired. Any maximal cone in Σ is spanned by $\mathbf{u}_1,\dots,\mathbf{u}_m$ where $\mathbf{u}$ denotes either $\mathbf{v}$ or $\mathbf{w}$, and the same is true for Σ'. Therefore, φ sends maximal cones in Σ to those in Σ' bijectively. This means that φ is an isomorphism from Σ to Σ'. Hence X_w and $X_{w'}$ are isomorphic as toric varieties. □

The edge-labeled digraph $\mathscr{G}_w$ encodes all the geometrical information of the toric Schubert variety X_w by Theorem 18. We shall give a simple criterion for when X_w is (weak) Fano in terms of $\mathscr{G}_w$. For each vertex i_k of $\mathscr{G}_w$ (where $k \in [m]$), we define the *indegree* $\deg^-(i_k)$ of i_k to be the sum of labels of the edges going into i_k:

$$\deg^-(i_k) := \sum_{(i_j,i_k)\in E(\mathscr{G}_w)} (-c_{i_j,i_k}) = \sum_{j>k} (-c_{i_j,i_k}). \tag{13}$$

Similarly, the *outdegree* $\deg^+(i_k)$ of i_k is defined to be the sum of labels of the edges going out i_k:

$$\deg^+(i_k) := \sum_{(i_k,i_j)\in E(\mathscr{G}_w)} (-c_{i_j,i_k}) = \sum_{j<k} (-c_{i_j,i_k}).$$

The *degree* of i_k is defined to be the sum of the indegree and outdegree:

$$\deg(i_k) = \deg^-(i_k) + \deg^+(i_k).$$

Theorem 19. *A toric Schubert variety X_w is Fano (respectively, weak Fano) if and only if every vertex of $\mathcal{G}_w$ has indegree at most 1 (respectively, 2).*

Proof. Let $\{\mathbf{v}_{i_1},\dots,\mathbf{v}_{i_m},\mathbf{w}_{i_1},\dots,\mathbf{w}_{i_m}\}$ be the column vectors of the matrix in (1). Then the primitive collections of the fan of X_w are $\{\mathbf{v}_{i_k},\mathbf{w}_{i_k}\}$ for $k=1,\dots,m$ and the primitive relations are

$$\mathbf{v}_{i_k} + \mathbf{w}_{i_k} = \sum_{j>k} (-c_{i_j,i_k})\mathbf{v}_{i_j}.$$

Therefore, it follows from (5) that the degree of the primitive collection $\{\mathbf{v}_{i_k},\mathbf{w}_{i_k}\}$ is given by

$$\deg\{\mathbf{v}_{i_k},\mathbf{w}_{i_k}\} = 2 - \sum_{j>k} (-c_{i_j,i_k}) = 2 - \deg^-(i_k),$$

where the latter equality follows from (13). Thus, the theorem follows from Proposition 13. □

In type A, the edge-labeled digraph $\mathcal{G}_w$ is a union of directed path graphs whose all edges have label 1. Therefore, we have the following.

Corollary 20. *In Type* A*, any toric Schubert variety X_w is weak Fano. Moreover, it is Fano if and only if $\mathcal{G}_w$ has no sink other than leaves.*

Using the graph $\mathcal{G}_w$, one can also determine whether the toric variety X_w decomposes as a product of lower-dimensional toric Schubert varieties.

Proposition 21. *Let $w \in W$. Suppose that a reduced expression of w consists of distinct simple reflections, so that X_w is a toric Schubert variety. Let $\mathcal{G}_1,\dots,\mathcal{G}_k$ be the connected components of the edge-labeled digraph $\mathcal{G}_w$. Then X_w is isomorphic to the product of the toric Schubert varieties corresponding to $\mathcal{G}_1,\dots,\mathcal{G}_k$.*

Proof. Let $w = s_{i_1}\cdots s_{i_m}$ be a reduced expression with mutually distinct simple reflections. Write $V(\mathcal{G}_w) = V(\mathcal{G}_1)\sqcup\cdots\sqcup V(\mathcal{G}_k)$ and set $m_\nu := \left|V(\mathcal{G}_\nu)\right|$ for $1 \le \nu \le k$. To prove the proposition, recall from [7, Proposition 3.1.14] that the fan of a product of toric varieties is the product of the fans of the factors.

As a first step, we use the connected components of $\mathcal{G}_w$ to identify the factors. If $p \in V(\mathcal{G}_\mu)$ and $q \in V(\mathcal{G}_\nu)$ with $\mu \neq \nu$, then there is no edge between p and q in the underlying graph of $\mathcal{G}_w$. Since $\mathcal{G}_w$ is an induced subgraph of the Dynkin diagram, this implies $c_{p,q} = c_{q,p} = 0$, hence s_p and s_q commute. Therefore, by applying a sequence of 2-moves, we may reorder the reduced expression so that it is a concatenation

$$w = w^{(1)}\cdots w^{(k)}, \tag{14}$$

where each $w^{(\nu)}$ is the product of the simple reflections whose indices lie in $V(\mathcal{G}_\nu)$. Notice that a choice of reduced decomposition does not change X_w and it does not change the fan up to a lattice automorphism.

By Theorem 9, the fan of X_w is described by the matrix in (1). The right $m\times m$ block of that matrix has (j,k)-entry $-c_{i_j,i_k}$ for $j>k$. Considering the reduced decomposition in (14), the index set $\{1,\dots,m\}$ is partitioned into consecutive blocks of sizes $m_1,\dots,m_k$, and for $\mu\neq\nu$ we have $c_{p,q}=0$ whenever $p\in V(\mathcal{G}_\mu)$ and $q \in V(\mathcal{G}_\nu)$. Hence, all off-block entries in the right block vanish, so the matrix in (1) becomes block diagonal with k blocks, each block being exactly the matrix associated to $w^{(\nu)}$. Consequently, after applying an appropriate lattice isomorphism, the fan Σ_w of X_w is the product fan

$$\Sigma_w \cong \Sigma_{w^{(1)}} \times \cdots \times \Sigma_{w^{(k)}}$$

under the identification $\mathbb{R}^m \cong \mathbb{R}^{m_1} \oplus \cdots \oplus \mathbb{R}^{m_k}$. Therefore, by [7, Proposition 3.1.14], we obtain a toric isomorphism $X_w \cong X_{w^{(1)}} \times \cdots \times X_{w^{(k)}}$. Since $\mathscr{G}_{w^{(v)}} = \mathscr{G}_v$ for each v by construction, the factors are precisely the toric Schubert varieties corresponding to $\mathscr{G}_1, \ldots, \mathscr{G}_k$. □

3. Enumeration of toric Schubert varieties

As discussed in Section 2, the graph $\mathscr{G}_w$ encodes fruitful geometric information about a toric Schubert variety X_w. In this section, we enumerate the isomorphism classes of Fano or weak Fano toric Schubert varieties using $\mathscr{G}_w$ in Proposition 24.

Let G be a simple algebraic group of rank r with Cartan matrix $C = (c_{i,j})_{1 \le i,j \le r}$. A bijection $\theta\colon [r] \to [r]$ is called a *Dynkin diagram automorphism* if

$$c_{\theta(i),\theta(j)} = c_{i,j}$$

holds for all i, j. For each Lie type, we recall from [10, Section 12.2] the group Γ of Dynkin diagram automorphisms in Table 2. Note that except for D_4, the only nontrivial automorphism group is $\mathbb{Z}_2$. When a generator $\theta \in \Gamma$ is realized by conjugation in W, i.e., when there exists $x \in W$ such that $x s_i x^{-1} = s_{\theta(i)}$ for all $i \in [r]$, we display one such element x in the third column of the table.

Table 2. Dynkin diagram automorphisms.

Type	Γ	x
A_r	$\mathbb{Z}_2$ $(r \ge 2)$	w_0
$\mathsf{B}_r, \mathsf{C}_r$	1	
D_r	$\begin{cases} \mathfrak{S}_3 & (r=4) \\ \mathbb{Z}_2 & (r>4) \end{cases}$	w_0 if r odd
E_6	$\mathbb{Z}_2$	w_0
E_7	1	
E_8	1	
F_4	1	
G_2	1	

An element of W is called a *Coxeter element* if it can be written as a product of all simple reflections $s_1, \ldots, s_r$. Let Cox_W denote the set of all Coxeter elements in W. Using Theorem 18, we obtain the following result.

Proposition 22. *Let W be the Weyl group of a simple algebraic group G. Let $w, w' \in \mathrm{Cox}_W$. The following statements are equivalent:*

(1) *$X_w \cong X_{w'}$ as varieties;*
(2) *$\mathscr{G}_w \cong \mathscr{G}_{w'}$ as edge-labeled digraphs.*

If G is of type A, D_r (r odd), or E_6, then the above statements are equivalent to the following:

(3) *$w' = w$ or $w' = w_0 w w_0$.*

Remark 23. The statement (3) implies the statement (1) in Proposition 22 for any (not necessarily toric) Schubert variety (see [16]). When G is of type A, we consider the composition σ of the two isomorphisms τ, ρ of $\mathrm{GL}_n(\mathbb{C})$ defined by

$$\tau(A) = {}^t A^{-1}, \qquad \rho(A) = w_0 A w_0.$$

Since both τ and ρ send the upper triangular Borel subgroup B to the lower triangular Borel subgroup, their composition σ preserves B and hence induces a variety automorphism of $\mathrm{GL}_n(\mathbb{C})/B$. Since τ fixes permutation matrices, we have $\sigma(BwB) = B\sigma(w)B = Bw_0ww_0B$, which implies $\sigma(X_w) = X_{w_0ww_0}$.

Recall that for a Coxeter element $w \in \mathrm{Cox}_W$, the underlying graph of $\mathscr{G}_w$ is the same as the Dynkin diagram of W. Accordingly, by Theorem 18, the number of isomorphism classes of toric Schubert varieties given by Cox_W is the same as that of orientations and the edge labels of the Dynkin diagram of W.

Proposition 24. *For each simple algebraic group G, the number of isomorphism classes of Fano or weak Fano toric Schubert varieties in G/B given by Coxeter elements is displayed in Table 3.*

Table 3. The cardinalities of the isomorphism classes of toric Schubert varieties given by Coxeter elements.

Type	$\#\{X_w \mid w \in \mathrm{Cox}_W\}/\sim$	weak Fano	Fano
A_r	$\begin{cases} 2^{r-2} & \text{when } r \text{ is even and } r \geq 2 \\ 2^{r-2} + 2^{\frac{r-3}{2}} & \text{when } r \text{ is odd and } r \geq 3 \end{cases}$	$\begin{cases} 2^{r-2} \\ 2^{r-2} + 2^{\frac{r-3}{2}} \end{cases}$	$\begin{cases} \frac{r}{2} \\ \frac{r+1}{2} \end{cases}$
B_r	$\begin{cases} 2^{r-1} & \text{when } r = 2 \text{ or } 3 \\ 7 \times 2^{r-4} & \text{when } r \text{ is even and } r \geq 4 \\ 7 \times 2^{r-4} + 2^{\frac{r-5}{2}} & \text{when } r \text{ is odd and } r \geq 5 \end{cases}$	$\begin{cases} 2^{r-1} \\ 7 \times 2^{r-4} \\ 7 \times 2^{r-4} + 2^{\frac{r-5}{2}} \end{cases}$	1
C_r	$\begin{cases} 4 & \text{when } r = 3 \\ 7 \times 2^{r-4} & \text{when } r \text{ is even and } r \geq 4 \\ 7 \times 2^{r-4} + 2^{\frac{r-5}{2}} & \text{when } r \text{ is odd and } r \geq 5 \end{cases}$	$\begin{cases} 3 \\ 5 \times 2^{r-4} \\ 5 \times 2^{r-4} + 2^{\frac{r-5}{2}} \end{cases}$	$\begin{cases} 2 \\ \frac{r}{2} \\ \frac{r+1}{2} \end{cases}$
D_r	$\begin{cases} 4 & \text{if } r = 4 \\ 3 \cdot 2^{r-3} & \text{otherwise} \end{cases}$	$\begin{cases} 3 \\ 5 \cdot 2^{r-4} \end{cases}$	$\begin{cases} 2 \\ r-1 \end{cases}$
E_6	20	17	4
E_7	2^6	56	7
E_8	2^7	112	8
F_4	2^3	6	2
G_2	2	1	1

Before providing a proof, we recall some terminology on (labeled) digraphs. We say that a vertex k is a *sink* if $\deg^+(k) = 0$, and a *source* if $\deg^-(k) = 0$. For instance, for the digraph $\mathscr{G}_w$ in Example 15(1), the vertices 1 and 4 are sources, 2 is a sink, and the sink set is $\{2\}$. Moreover, if a sink has indegree 1, we call it a *leaf*. For instance, 1 is a leaf in the graph $\mathscr{G}_w$ in Example 15(2).

Proof of Proposition 24. We provide proof by analyzing directed Dynkin diagrams case-by-case. We will use the numbering of the vertices in Table 1. We consider the simply laced cases first and then treat non-simply laced cases.

Before proceeding, we explain the strategy. By Proposition 22, the enumeration reduces to counting orientations up to Dynkin diagram automorphisms.

Type A. For type A, since there are at most two edges at each vertex, any toric Schubert variety is weak Fano by Theorem 19.

We enumerate the isomorphism classes depending on the parity of the rank r. When r is even, consider any orientation on the corresponding Dynkin diagram, which is a path graph having r vertices. The involution on the Dynkin diagram does not preserve the orientation. Therefore, the involution on the Dynkin diagram provides an involution on the set $\{X_w \mid w \in \mathrm{Cox}_W\}$ of toric Schubert varieties, and this involution has no fixed element. Since there are $r-1$ edges, the number of isomorphism classes is $2^{r-1}/2 = 2^{r-2}$.

When r is odd, there exist orientations which are fixed by the involution on the Dynkin diagram. In fact, a directed path graph $\mathscr{G}$ on $[r]$ is fixed by the involution if and only if

$$(i, i+1) \in E(\mathscr{G}) \quad \Longleftrightarrow \quad (r-i+1, r-i) \in E(\mathscr{G}) \quad \text{for any } 1 \le i \le \tfrac{r-1}{2}.$$

Therefore, the number of orientations fixed by the involution is $2^{\frac{r-1}{2}}$. Hence, the number of orientations that are not fixed by the involution is $2^{r-1} - 2^{\frac{r-1}{2}}$. Accordingly, the number of isomorphism classes is

$$(2^{r-1} - 2^{\frac{r-1}{2}})/2 + 2^{\frac{r-1}{2}} = 2^{r-2} + 2^{\frac{r-3}{2}}.$$

Now we enumerate the isomorphism classes of Fano toric Schubert varieties in $\{X_w \mid w \in \mathrm{Cox}_W\}$. By Theorem 19, we consider orientations, where each vertex has at most one incoming edge. Since the underlying graph is a path graph, any sink is a leaf of the graph. Indeed, an internal sink on a path would have two incident edges, both directed toward it, and hence indegree 2, contradicting the Fano condition. Consider the following possibilities.

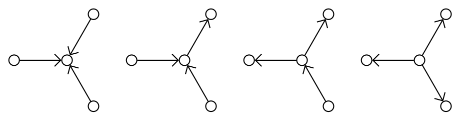
(1) The sink set is $\{1, r\}$: there is a unique source, and it can be any one of the vertices $2, \ldots, r-1$; once the source is chosen, the orientation is forced. Considering the involution reversing the path, there are $\frac{r-1}{2}$ orientations when r is odd; $\frac{r}{2} - 1$ orientations when r is even.

(2) The sink set is $\{1\}$: the vertex r must be a source, and there is a unique orientation having a source at r and sink at 1. Moreover, this is isomorphic to the orientation having a source at 1 and a sink at r.

The above list exhausts all Fano orientations up to the Dynkin diagram involution since the sink set $\{r\}$ is the involution image of sink set $\{1\}$, and no other sink configurations are possible under the indegree ≤ 1 condition on a path. Therefore, we have $\frac{r-1}{2} + 1 = \frac{r+1}{2}$ orientations satisfying the Fano condition (up to isomorphism) when r is odd; $\frac{r}{2} - 1 + 1 = \frac{r}{2}$ orientations satisfying the Fano condition (up to isomorphism) when r is even. This proves the claim.

Type D. For type D_r, when $r = 4$, we have the four directed Dynkin diagrams up to isomorphism:

$\mathscr{G}_w$



maximum of $\deg^-(k)$ 3 2 1 0

When $r > 4$, the Dynkin subgraph consisting of vertices $1, \ldots, r-2$ contributes 2^{r-3} orientations, and the two leaf edges incident to $r-2$ contribute 3 possibilities up to the diagram automorphism swapping $r-1$ and r (both edges oriented into $r-2$, both oriented out of $r-2$, or one in/one out). Accordingly, we have $2^{r-3} \times 3$ directed Dynkin diagrams up to isomorphism.

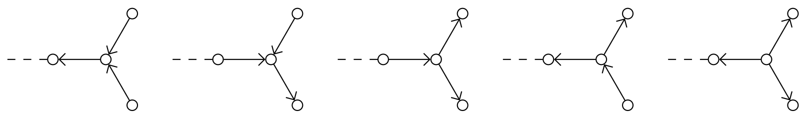
To enumerate Fano or weak Fano toric Schubert varieties, we consider directions on the Dynkin diagram of type D_r. By Theorem 19, to enumerate Fano toric Schubert varieties, it is enough to consider orientations in which each vertex has at most one incoming edge. Accordingly, any sink is a leaf of the graph and there should be at least two sinks. Indeed, an internal sink would force indegree ≥ 2, and if there were only one sink then the trivalent vertex $r-2$ would necessarily have at least two incoming edges, violating the Fano indegree ≤ 1 condition. Consider the following possibilities:

(1) the sink set is $\{1, r\}$: there is only one orientation satisfying the Fano condition; in this case, the vertex $r-1$ is the unique source;
(2) the sink set is $\{1, r-1\}$: there is only one orientation satisfying the Fano condition; in this case, the vertex r is the unique source;
(3) the sink set is $\{r-1, r\}$: there is only one orientation satisfying the Fano condition; in this case, the vertex 1 is the unique source;
(4) the sink set is $\{1, r-1, r\}$: there are $r-3$ orientations satisfying the Fano condition; indeed, each of the vertices $2, \dots, r-2$ can be a source.

Since the orientations in (1) and (2) yield isomorphic digraphs (via the Dynkin diagram automorphism swapping the two leaf vertices r and $r-1$), we have $1+1+r-3 = r-1$ orientations satisfying the Fano condition (up to isomorphism).

Now we enumerate weak Fano toric Schubert varieties. There is only one trivalent vertex in the graph and because of Theorem 19, it is enough to consider the orientations such that the indegree of the trivalent vertex $r-2$ is at most 2. Indeed, since all edge labels are 1 and every other vertex has degree ≤ 2, those vertices automatically have indegree ≤ 2, so the only potential obstruction is the condition $\deg^-(r-2) \le 2$. Considering the orientations on the emanating edges of the vertex $r-2$, there are $2^3 - 1 = 7$ allowed orientations of the three incident edges (excluding the case $\deg^-(r-2) = 3$), and modulo the symmetry swapping $r-1$ and r these fall into five isomorphism types:

$\mathscr{G}_w$



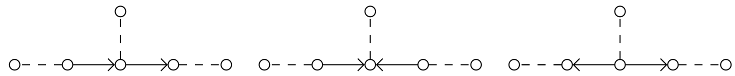
$\deg^-(r-2)$ 2 2 1 1 0

Since there is no condition on the $r-4$ edges in the path subgraph on the vertices $1, \dots, r-3$, there are $5 \cdot 2^{r-4}$ orientations satisfying weak Fano condition.

Type E. For type E_6, the longest element w_0 provides a Dynkin diagram involution. Since the Dynkin diagram of E_6 can be constructed by adding one more edge to the middle vertex in the Dynkin diagram of type A_5, the number of isomorphism classes is the same as twice that of A_5. This is because once an orientation on the A_5-subdiagram is fixed up to the involution, there are two choices for the orientation of the additional edge attached to the branching vertex. Therefore, we obtain $2 \times 10 = 20$. For types E_7 and E_8, there does not exist a nontrivial Dynkin diagram automorphism. Accordingly, there are 2^6 (respectively, 2^7) isomorphism classes in type E_7 (respectively, E_8).

To enumerate (weak) Fano toric Schubert varieties in type E_r, we consider (weak) Fano in type A_{r-1} first and then see how many choices of orientations there are for the extra edge attached to the vertex 4. The extra edge contributes two choices of orientations exactly when the vertex 4 is the unique source in the A_{r-1} orientation; otherwise its direction must point outward the vertex 4. For instance, for type E_6, there are three Fano toric Schubert varieties in type A_5, where they are distinguished by the position of the source. Now we look at the orientation on the extra edge of E_6. If the source is the middle point, that is the vertex 4, then both orientations are fine but otherwise the orientation will be unique. Therefore we obtain $2+1+1$ orientations satisfying Fano condition for Type E_6. A similar observation will work for E_7 and E_8. Indeed, for $r = 7, 8$, an orientation satisfying the Fano condition in type A_{r-1} has only one source. In fact, there is a unique source and it can be any of the vertices $[r] \setminus \{4\}$ on the A_{r-1}-subgraph. Moreover, the Dynkin diagram automorphism on A_{r-1} does not induce that on E_r. Therefore, we have $r-2+2$ orientations satisfying the Fano condition for type E_r. Here, $r-2$ orientations are induced from that on A_{r-1} which do not have the source on the vertex 4; and two orientations coming from that on A_{r-1} which has the source on the vertex 4.

Now we consider weak Fano toric Schubert varieties. For type E_6, we consider orientations in type A_5. There are ten different orientations in type A_5 (up to isomorphism). We have the following three possibilities by looking at the middle vertex (which corresponds to the vertex 4 in E_6):

$\mathscr{G}_w$ 

# of orientations (up to isomorphism)	4	3	3

If the middle vertex is a sink, then we have only one choice for the orientation of the additional edge to be weak Fano but otherwise, both orientations are fine. Indeed, if the vertex 4 already has indegree 2 in the A_5 orientation, then the extra edge must point outward to keep indegree ≤ 2; otherwise either direction works. Accordingly, we have $4 \times 2 + 3 + 3 \times 2 = 17$ orientations satisfying the weak Fano condition on type E_6 (up to isomorphism).

Now we consider E_r for $r = 7, 8$. Considering three edges emanating from the vertex 4, there are $2^3 - 1 = 7$ orientations such that the vertex 4 has at most 2 indegree. Since there are no other conditions on the remaining $r-4$ edges, we have $7 \times 2^{r-4}$ orientations satisfying the weak Fano condition on type E_r for $r = 7, 8$.

For the other Lie types, we use the following observation. Suppose that the Dynkin diagram has a double or triple edge of the following form:

From the definition of the edge labeling on the graph $\mathscr{G}_w$ and the definition of the Cartan integers, s_j appears ahead of s_i in the reduced word of w if and only if the corresponding edge in the graph $\mathscr{G}_w$ has the label $-c_{i,j}$.

Type B. We have the label 2 on the edge $(r-1, r)$ on the graph $\mathscr{G}_w$ if s_r appears before s_{r-1} in the reduced expression of w. In this case, we have $\deg^-(r) = 2$:

$$r-1 \xrightarrow{\;2\;} r$$

Otherwise, we have label 1 on the edge $(r, r-1)$, so $\deg^-(r) = 0$ and $\deg^-(r-1) \geq 1$. When $r = 2$, there are two toric Schubert varieties which are weak Fano and one of which is Fano. When $r = 3$, there are four toric Schubert varieties which are weak Fano and one of which is Fano. Now we assume $r \geq 4$.

We first enumerate the isomorphism classes of toric Schubert varieties.

(1) If we have the edge $(r-1, r)$, then there are 2^{r-2} different digraphs up to isomorphism.
(2) If we have the edge $(r, r-1)$, then the vertex r should be a source. We consider the following two cases separately.
   (a) If the first vertex (with label 1 in the Dynkin diagram) is a sink, then 2^{r-3} digraphs are all different. Indeed, if the vertex 1 is required to be a sink, then the edge $(2, 1)$ is forced. Hence, there is no nontrivial diagram symmetry identifying distinct orientations.
   (b) If the first vertex (with label 1 in the Dynkin diagram) is a source, then the number of isomorphism classes of the directed graphs is the same as that of the directed graphs of Lie type A_{r-2}.

By adding all possible cases above, we obtain that if r is even and $r \geq 4$, then the number of isomorphism classes is $7 \times 2^{r-4}$; if r is odd and $r \geq 5$, then the number of isomorphism classes is $7 \times 2^{r-4} + 2^{\frac{r-3}{2}-1}$.

Moreover, we note that all orientations provide weak Fano toric Schubert varieties. To enumerate the isomorphism classes of Fano toric Schubert varieties, we consider the directed Dynkin diagrams which do not have label 2, that is, we have the edge $(r, r-1)$ in $\mathscr{G}_w$. Since r is a source and (under the indegree ≤ 1 condition) any sink must be a leaf, the only possible sink is vertex 1, which forces the unique orientation $1 \leftarrow 2 \leftarrow \cdots \leftarrow r-1 \leftarrow r$. Indeed, the Coxeter element $s_1 s_2 \cdots s_r$ provides a Fano toric Schubert variety.

Type C. We have the label 2 on the edge $(r, r-1)$ if s_{r-1} appears before s_r in the reduced expression of w. Otherwise, we have the edge $(r-1, r)$ with label 1. Therefore, we obtain the same number of isomorphism classes in a similar way to the case of type B_r. When $r = 3$, there are four toric Schubert varieties, three of which are weak Fano and two of which are Fano. Now we assume $r \geq 4$.

If the graph $\mathscr{G}_w$ has the edge $(r, r-1)$ with label 2 and the edge $(r-2, r-1)$, then $r-1$ is a sink with $\deg^-(r-1) = 1 + 2 = 3$:

$$r-2 \longrightarrow r-1 \xleftarrow{\;2\;} r$$

Therefore, any completion of the remaining edges necessarily violates the weak Fano condition $\deg^-(k) \leq 2$ for any vertex $k \in [r]$. The number of such orientations is 2^{r-3}. Since all the other orientations provide weak Fano, we obtain the number of isomorphism classes of weak Fano toric Schubert varieties as in Table 3.

Recall that X_w is Fano if and only if $\deg^-(k) \leq 1$ for any vertex $k \in [r]$. Therefore, for X_w to be Fano, there is no edge with label 2 in the graph $\mathscr{G}_w$ and no sink on the internal vertices $\{2, \ldots, r-1\}$. Accordingly, the vertex r should be a sink and there exists only one source. Then the problem reduces to the same path-orientation counting as in the type A case. Enumerating such orientations, we obtain $\frac{r+1}{2}$ orientations when r is odd; $\frac{r}{2}$ orientations when r is even.

Type F_4. There are eight directed Dynkin diagrams and we display the value $\max\{\deg^-(k)\}$ in Table 4. Accordingly, there are six isomorphism classes of weak Fano toric Schubert varieties, and two isomorphism classes of Fano toric Schubert varieties.

Table 4. Coxeter elements and corresponding directed Dynkin diagrams in type F_4.

w	$s_1 s_2 s_3 s_4$	$s_2 s_1 s_3 s_4$	$s_1 s_2 s_4 s_3$
$\mathscr{G}_w$	$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4$	$1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4$	$1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4$
maximum of $\deg^-(k)$	1	2	1
w	$s_2 s_1 s_4 s_3$	$s_1 s_3 s_2 s_4$	$s_3 s_2 s_1 s_4$
$\mathscr{G}_w$	$1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4$	$1 \longleftarrow 2 \xrightarrow{2} 3 \longleftarrow 4$	$1 \longrightarrow 2 \xrightarrow{2} 3 \longleftarrow 4$
maximum of $\deg^-(k)$	2	3	3
w	$s_1 s_4 s_3 s_2$	$s_4 s_3 s_2 s_1$	
$\mathscr{G}_w$	$1 \longleftarrow 2 \xrightarrow{2} 3 \longrightarrow 4$	$1 \longrightarrow 2 \xrightarrow{2} 3 \longrightarrow 4$	
maximum of $\deg^-(k)$	2	2	

Type G_2. There are two directed Dynkin diagrams corresponding to two Coxeter elements:

$$\begin{array}{cc} s_1 s_2 & s_2 s_1 \\ 1 \xleftarrow{3} 2 & 1 \longrightarrow 2 \end{array}$$

One provides a Fano toric Schubert variety and the other is not weak Fano. This completes the proof. □

4. The cohomology ring distinguishes toric Schubert varieties for simply-laced types

Throughout this section, we assume that X_w is a toric Schubert variety. We consider those X_w for which all edges in $\mathscr{G}_w$ are labeled by 1. This family includes every toric Schubert variety in G/B for a simple algebraic group G of simply-laced type. We show that for each toric Schubert variety X_w in the family above, the edge-labeled digraph $\mathscr{G}_w$ is recovered (up to isomorphism) from the cohomology ring $H^*(X_w;\mathbb{Z})$.

For the sake of simplicity, we define the following two sets.

$$E_w^-(j) = \{k \mid (k,j) \in E(\mathscr{G}_w)\},$$
$$E_w^+(j) = \{k \mid (j,k) \in E(\mathscr{G}_w)\}.$$

That is, $E_w^-(j)$ corresponds to the set of inward edges to j, and $E_w^+(j)$ corresponds to the set of outward edges from j in the digraph $\mathscr{G}_w$.

Lemma 25. *Let X_w be a toric Schubert variety in G/B. Assume that all edges in $\mathscr{G}_w$ are labeled by* 1. *Then the cohomology ring $H^*(X_w)$ can be described in terms of the graph $\mathscr{G}_w$. If we further assume $w \in \mathrm{Cox}_W$, then*

$$H^*(X_w) = \mathbb{Z}[x_1,\dots,x_r] \Big/ \left(x_j^2 - x_j \sum_{k \in E_w^+(j)} x_k \;\middle|\; j = 1,\dots,r \right),$$

where r is the rank of G.

Proof. Let $w = s_{i_1} \cdots s_{i_m}$. Applying Jurkiewicz's theorem to Theorem 9 (see [11]), we have

$$H^*(X_w) = \mathbb{Z}[y_1,\dots,y_m] \Big/ \left(y_j^2 + y_j \left(\sum_{k<j} c_{i_j,i_k} y_k \right) \;\middle|\; j = 1,\dots,m \right),$$

where c_{i_j,i_k} is a Cartan integer. Since we assume that all the edges in $\mathscr{G}_w$ are labeled by 1, the Cartan integer c_{i_j,i_k} is either 0 or -1. Moreover, $c_{i_j,i_k} = -1$ if and only if the vertices i_j and i_k are adjacent in the digraph $\mathscr{G}_w$. Hence, by changing $y_j \mapsto x_{i_j}$ for each $j = 1,\dots,m$, we get

$$H^*(X_w) = \mathbb{Z}[x_{i_1},\dots,x_{i_m}] \Big/ \left(x_{i_j}^2 - x_{i_j} \sum_{(i_j,i_k) \in E(\mathscr{G}_w)} x_{i_k} \;\middle|\; j = 1,\dots,m \right). \tag{15}$$

If $w \in \mathrm{Cox}_W$, then m is the rank of G, and this proves the lemma. □

Remark 26. From the presentation of $H^*(X_w)$ in (15), we obtain the following decomposition property. Since the defining ideal

$$\left(x_{i_j}^2 - x_{i_j} \sum_{(i_j,i_k) \in E(\mathscr{G}_w)} x_{i_k} \;\middle|\; j = 1,\dots,m \right)$$

depends only on the edge set $E(\mathscr{G}_w)$, if $\mathscr{G}_w$ is disconnected and admits a decomposition

$$\mathscr{G}_w = \mathscr{G}_{w^{(1)}} \sqcup \cdots \sqcup \mathscr{G}_{w^{(p)}},$$

into connected components, then the defining ideal decomposes as the sum of the ideals corresponding to these components. It follows that

$$H^*(X_w) \cong H^*(X_{w^{(1)}}) \otimes \cdots \otimes H^*(X_{w^{(p)}}).$$

We demonstrate Lemma 25 in the following example.

Example 27. Suppose that $w = s_3 s_1 s_4 s_5 s_2 s_7 s_8 \in W_{\mathrm{A}_8}$. Then the digraph $\mathscr{G}_w$ is given as follows:

$$(1) \longleftarrow (2) \longrightarrow (3) \longleftarrow (4) \longleftarrow (5) \qquad\qquad (7) \longleftarrow (8)$$

By Lemma 25, we obtain that the cohomology ring $H^*(X_w)$ is the truncated polynomial ring $\mathbb{Z}[x_1,\dots,x_5,x_7,x_8]/\mathscr{I}$, where the ideal $\mathscr{I}$ is the sum $\mathscr{I}' + \mathscr{I}''$, with $\mathscr{I}'$ generated by

$$x_1^2, \qquad x_2^2 - x_2(x_3+x_1), \qquad x_3^2, \qquad x_4^2 - x_4 x_3, \qquad x_5^2 - x_5 x_4,$$

and $\mathscr{I}''$ generated by

$$x_7^2, \qquad x_8^2 - x_8 x_7.$$

Therefore,

$$\begin{aligned} H^*(X_w) &\cong \mathbb{Z}[x_1,\dots,x_5]/\mathscr{I}' \otimes \mathbb{Z}[x_7,x_8]/\mathscr{I}'' \\ &\cong H^*(X_{w^{(1)}}) \otimes H^*(X_{w^{(2)}}), \end{aligned}$$

where $w^{(1)} = s_3 s_1 s_4 s_5 s_2$, $w^{(2)} = s_7 s_8$, and $w = w^{(1)} w^{(2)}$.

We notice that in Example 27, $x_i^2 = 0$ in the cohomology ring $H^*(X_w)$ if and only if $i = 1, 3$, or 7. On the other hand, the sinks of the digraph $\mathscr{G}_w$ are 1, 3, and 7. We will see that this observation holds in general.

An element $z \in H^2(X_w)$ is *primitive* if z cannot be divisible by an integer greater than 1 and is called *square-zero* if $z^2 = 0$ in $H^*(X_w)$. From the ring presentation in Lemma 25, we see that x_i is a square-zero primitive element if the vertex i is a sink of $\mathscr{G}_w$. There are more square-zero primitive elements in $H^2(X_w)$. For simplicity, we set

$$\alpha_j := \sum_{k \in E_w^+(j)} x_k \quad \text{for } j = 1,\dots,m. \tag{16}$$

Hence if j is a sink, then $\alpha_j = 0$; otherwise it is the sum of at most three x_k's.

Lemma 28. *Let X_w be a toric Schubert variety in G/B. Assume that all the edges in $\mathscr{G}_w$ are labeled by* 1*. A square-zero primitive element in $H^2(X_w)$ is of the form up to sign:*

$$x_j \qquad \text{or} \qquad 2x_k - x_j,$$

where j is a sink of $\mathscr{G}_w$. Moreover, the latter occurs only when $E_w^+(k) = \{j\}$.

Proof. We first notice that $x_j^2 = 0$ in the cohomology ring $H^*(X_w)$ if and only if j is a sink of the digraph $\mathscr{G}_w$. By [5, Corollary 2.1], a square-zero primitive element is of the following form up to sign:

$$x_k - \frac{1}{2}\alpha_k \qquad \text{or} \qquad 2x_k - \alpha_k$$

for some k with $\alpha_k^2 = 0$, where the former case occurs when α_k is divisible by 2 and the latter occurs otherwise. In our case, α_k is divisible by 2 only when $\alpha_k = 0$ (equivalently, k is a sink). If $\alpha_k \neq 0$, then $\alpha_k^2 = 0$ only when α_k equals x_j for a sink j of $\mathscr{G}_w$, that is, $E_w^+(k) = \{j\}$ for a sink j of $\mathscr{G}_w$. This completes the lemma. $\square$

Since the mod 2 reduction of the elements in Lemma 28 is x_j, we can find all sinks of $\mathscr{G}_w$ by looking at square-zero primitive elements in $H^2(X_w)$.

Following [6, Section 6], we call an element $\alpha \in H^2(X_w;\mathbb{Z}_2)$ (possibly $\alpha = 0$) an *eigenelement* if there exists $x \in H^2(X_w;\mathbb{Z}_2)$ such that

$$x^2 = \alpha x, \quad x \neq 0, \quad \text{and} \quad x \neq \alpha.$$

Moreover, we call such x an *eigenvector* associated to α. We define $E(\alpha)$ as the set of elements x with $x^2 = \alpha x$ (x may be 0 or α) for an eigenelement α, which is called the *eigenspace* associated to α. In fact, $E(\alpha)$ forms a vector space because we are working over $\mathbb{Z}_2$. We denote by $\overline{E}(\alpha)$ the quotient of $E(\alpha)$ by the one-dimensional subspace $\langle\alpha\rangle$ spanned by α and call $\overline{E}(\alpha)$ the *reduced eigenspace* associated with α. Note that eigenelements, eigenspaces, and reduced eigenspaces are preserved under a cohomology ring isomorphism. For an eigenelement $\alpha \in H^2(X_w;\mathbb{Z}_2)$, we define its *multiplicity* to be the dimension of $\overline{E}(\alpha)$.

Lemma 29 ([6, Lemma 6.2]). *Let X_w be a toric Schubert variety in G/B. Assume that all the edges in $\mathscr{G}_w$ are labeled by 1. The eigenelements are α_j's (regarded as elements over $\mathbb{Z}_2$) and the reduced eigenspace $\overline{E}(\alpha)$ is spanned by x_j's for each eigenelement $\alpha_j = \alpha$.*

We see an example for nonzero eigenelements and their eigenspaces.

Example 30. Let G be of type D_9 and $w = s_8s_5s_4s_2s_1s_3s_6s_7s_9$. Then we have:

$$\mathscr{G}_w = \begin{tikzcd} & & & & & & & 8 \\ 1 \arrow[r] & 2 & 3 \arrow[l] \arrow[r] & 4 \arrow[r] & 5 & 6 \arrow[l] & 7 \arrow[l] \arrow[ur] & \\ & & & & & & & 9 \arrow[ul] \end{tikzcd}$$

Then $\alpha_2 = \alpha_5 = \alpha_8 = 0$, $\alpha_1 = x_2$, $\alpha_3 = x_2 + x_4$, $\alpha_4 = x_5$, $\alpha_6 = x_5$, $\alpha_7 = x_6 + x_8$, and $\alpha_9 = x_7$. Hence, for nonzero eigenelements of $H^2(X_w;\mathbb{Z}_2)$, their eigenspaces are given as follows:

α ($\neq 0$)	$E(\alpha)$	$\dim \overline{E}(\alpha)$
x_2	$\langle x_1, x_2\rangle$	1
$x_2 + x_4$	$\langle x_3, x_2 + x_4\rangle$	1
x_5	$\langle x_4, x_5, x_6\rangle$	2
$x_6 + x_8$	$\langle x_7, x_6 + x_8\rangle$	1
x_7	$\langle x_7, x_9\rangle$	1

By Lemma 29 and the definition of α_j, any eigenelement α is the element $\alpha_j = \sum_{k\in E_w^+(j)} x_k$ for some j. Hence, α_j is one of the following four forms:

$$0, \qquad x_k, \qquad x_i + x_k, \qquad x_i + x_k + x_\ell,$$

where i, j, k, ℓ are pairwise distinct. In the following lemma, we describe the eigenspace $E(\alpha)$ in each case.

Lemma 31. *Let X_w be a toric Schubert variety in G/B. Assume that all the edges in $\mathscr{G}_w$ are labeled by 1. Then for each eigenelement α, the following hold:*

(1) *if $\alpha = 0$, then $\overline{E}(\alpha) = \langle x_k \mid k \text{ is a sink}\rangle$;*
(2) *if $\alpha = x_k$, then*
$$\overline{E}(\alpha) = \left\langle x_i \;\middle|\; E_w^+(i) = \{k\}\right\rangle;$$
(3) *if $\alpha = x_i + x_k$ with $i \neq k$, then $\overline{E}(\alpha) = \langle x_j\rangle$, where $E_w^+(j) = \{i, k\}$;*
(4) *if $\alpha = x_i + x_k + x_\ell$ with pairwise distinct i, k, ℓ, then $\overline{E}(\alpha) = \langle x_j\rangle$, where $E_w^+(j) = \{i, k, \ell\}$.*

Therefore, if α is nonzero, then we get $1 \le \dim \overline{E}(\alpha) \le 3$.

Proof. Since α is of the form $\alpha_j = \sum_{k\in E_w^+(j)} x_k$, we prove the lemma according to the cardinality of $E_w^+(j)$.

Case 1: Suppose that $\left|E_w^+(j)\right| = 0$. Then j is a sink, and $\alpha_j = 0$. Hence, we get

$$\overline{E}(\alpha_j) = E(\alpha_j) = \langle x_k \mid k \text{ is a sink}\rangle.$$

Hence, $\dim \overline{E}(0)$ is the number of sinks in $\mathscr{G}_w$.

Case 2: Suppose that $\left|E_w^+(j)\right| = 1$. Then $\alpha_j = x_k$, where $j \neq k$. If $E_w^+(i) = \{k\}$, then $\alpha_i = x_k$; otherwise, $\alpha_i \neq x_k$. Therefore, we obtain

$$\overline{E}(x_k) = \left\langle x_i \;\middle|\; E_w^+(i) = \{k\}\right\rangle.$$

Therefore, $1 \le \dim \overline{E}(x_k) \le \deg^-(k) \le 3$.

Case 3: Suppose that $\left|E_w^+(j)\right| = 2$. Then $\alpha_j = x_i + x_k$, where i, j, k are pairwise distinct. If $E_w^-(j) = \emptyset$, then j is a source; otherwise, j is a trivalent vertex of $\mathscr{G}_w$. In any case, $E(\alpha_j) = \langle x_j, x_i + x_k\rangle$ and $\overline{E}(\alpha_j) = \langle x_j\rangle$.

Case 4: Suppose that $\left|E_w^+(j)\right| = 3$. Then $\alpha_j = x_i + x_k + x_\ell$, where i, j, k, ℓ are pairwise distinct. Then j is a trivalent vertex, and it is a source. Furthermore, $E(\alpha_j) = \langle x_j, x_i + x_k + x_\ell\rangle$ and $\overline{E}(\alpha_j) = \langle x_j\rangle$. □

Note that, as we saw in Lemma 31(1) and (2), the same α_j may occur for different indices j; nevertheless, when counted with their indices, the total number of eigenelements is exactly the number of vertices of $\mathscr{G}_w$.

Theorem 32. *Let X_w be a toric Schubert variety in G/B. The edge-labeled digraph $\mathscr{G}_w$ can be recovered (up to isomorphism) from the cohomology ring $H^*(X_w;\mathbb{Z})$ if all the labels in $\mathscr{G}_w$ are* 1.

Recall that for a toric Schubert variety X_w in G/B for a simple Lie group of type A, D, or E, every edge in $\mathscr{G}_w$ is labeled by 1. Hence we get the following result.

Corollary 33. *Let W and W' be the Weyl groups of simple Lie groups G and G' of type* A, D, *or* E. *Let w and w' be elements in W and W', respectively. The following statements are equivalent:*

(1) $H^*(X_w;\mathbb{Z}) \cong H^*(X_{w'};\mathbb{Z})$ *as graded rings;*

(2) $\mathscr{G}_w \cong \mathscr{G}_{w'}$ *as digraphs.*

Remark 34. Richmond and Slofstra [16] studied a relation between isomorphism classes of (not necessarily toric) Schubert varieties and their cohomology rings. Indeed, they proved that two Schubert varieties are isomorphic (as algebraic varieties) if and only if there is a graded cohomology ring isomorphism preserving the Schubert bases. Here, for a toric Schubert variety of dimension m, the cohomology classes $\left\{x_{i_1}\cdots x_{i_k} \;\middle|\; 1 \le i_1 < \cdots < i_k \le m\right\}$ form the Schubert basis in terms of the cohomology ring presentation in Lemma 25.

Remark 35. As mentioned in Remark 5, the assumption in Theorem 32 cannot be weakened. In this remark, we consider the cohomology rings of two toric Schubert varieties $X_{s_3s_2s_1s_4}$ and $X_{s_2s_1s_4s_3}$ of type F_4. The cohomology rings are given as follows:

$$H^*(X_{s_3s_2s_1s_4}) \cong \mathbb{Z}[x_1,x_2,x_3,x_4]/\big(x_1^2 - x_1x_2,\ x_2^2 - 2x_2x_3,\ x_3^2,\ x_4^2 - x_3x_4\big),$$
$$H^*(X_{s_2x_1s_4s_3}) \cong \mathbb{Z}[y_1,y_2,y_3,y_4]/\big(y_1^2 - y_1y_2,\ y_2^2,\ y_3^2 - y_2y_3 - y_3y_4,\ y_4^2\big).$$

The graded ring homomorphism given by $\varphi(x_1) = y_3 - y_2 - y_4$, $\varphi(x_2) = -y_2 - y_4$, $\varphi(x_3) = -y_2$, $\varphi(x_4) = -y_2 + y_1$ provides the graded ring isomorphism between two cohomology rings because

$$\varphi(x_1^2 - x_1x_2) = y_3^2 - y_2y_3 - y_3y_4, \qquad \varphi(x_2^2 - 2x_2x_3) = -y_2^2 + y_4^2,$$
$$\varphi(x_3^2) = y_2^2, \qquad \varphi(x_4^2 - x_3x_4) = y_1^2 - y_1y_2.$$

However, two toric varieties are not isomorphic as proved in Proposition 22 (see Table 4 for the corresponding directed Dynkin diagrams).

Since a directed graph $\mathscr{G}_w$ has at most one trivalent vertex, we divide the proof of Theorem 32 into two cases according to whether $\mathscr{G}_w$ has a trivalent vertex. We first treat the case with no trivalent vertex and then the case with exactly one trivalent vertex. Because the essential idea already appears in type A, we begin with an example of type A to illustrate the argument. We then prove the case without a trivalent vertex. After that, we give an example of type D to indicate the additional issues caused by the presence of a trivalent vertex, and finally complete the proof in the general case.

We begin with an example of type A, in which the directed graph has no trivalent vertex. This simple case already exhibits the essential mechanism of the proof.

Example 36. Let G be of type A and $w = s_9 s_{10} s_5 s_4 s_6 s_7 s_8 s_1 s_2 s_3$. Then we have:

$$\mathscr{G}_w = 1 \leftarrow 2 \leftarrow 3 \rightarrow 4 \rightarrow 5 \leftarrow 6 \leftarrow 7 \leftarrow 8 \rightarrow 9 \leftarrow 10$$

Note that $\alpha_1 = \alpha_5 = \alpha_9 = 0$ and

$$\alpha_2 = x_1, \qquad \alpha_3 = x_2 + x_4, \qquad \alpha_4 = x_5, \qquad \alpha_6 = x_5, \qquad \alpha_7 = x_6, \qquad \alpha_8 = x_7 + x_9, \qquad \alpha_{10} = x_9.$$

Now we recover $\mathscr{G}_w$ from $H^*(X_w)$ by the following three steps.

Step 1. We define the $\mathbb{Z}_2$-vector spaces $V_1, \dots, V_{10}$ using eigenelements $\alpha_1, \dots, \alpha_{10}$:

$$V_j = \begin{cases} \langle x_j \rangle & \text{if } \alpha_j = 0, \\ E(\alpha_j) & \text{otherwise.} \end{cases}$$

Then the spaces V_j's are given as follows:

$$\begin{array}{lllll} V_1 = \langle x_1 \rangle, & V_2 = \langle x_1, x_2 \rangle, & V_3 = \langle x_3, x_2 + x_4 \rangle, & V_4 = \langle x_4, x_5, x_6 \rangle, & V_5 = \langle x_5 \rangle, \\ V_6 = \langle x_4, x_5, x_6 \rangle, & V_7 = \langle x_6, x_7 \rangle, & V_8 = \langle x_8, x_7 + x_9 \rangle, & V_9 = \langle x_9 \rangle, & V_{10} = \langle x_9, x_{10} \rangle. \end{array}$$

We draw a graph whose vertices are labeled by $V_1, \dots, V_{10}$ in the following Steps 2 and 3.

Step 2. We find a connected component starting from V_1, V_5, and V_9, which are one-dimensional. First, for each $j = 1, 5, 9$, we find a space V with $\dim(V \cap V_j) = 1$, and then draw a directed edge $V \to V_j$. Then we get:

$$V_1 \leftarrow V_2, \qquad V_4 \rightarrow V_5 \leftarrow V_6, \qquad V_9 \leftarrow V_{10}.$$

Then V_2, V_4, V_6, and V_{10} are the sources in the current graph. Next, for each source V_j, we find a space V not yet contained in the current graph satisfying $\dim(V \cap V_j) = 1$. Then V_4 and V_6 are the same, and $V_4 \cap V_7 = V_6 \cap V_7 = \langle x_6 \rangle$. We choose one of V_4 and V_6, and the draw a directed edge from V to the chosen source. Note that any choice produces isomorphic graphs, so the construction is well defined up to isomorphism. In the following we choose V_4. Then we get:

$$V_1 \leftarrow V_2, \qquad V_7 \rightarrow V_4 \rightarrow V_5 \leftarrow V_6, \qquad V_9 \leftarrow V_{10}.$$

Let $\mathscr{R}$ be the set of spaces V_j that are not contained in any connected component in the current graph, and let $\mathscr{S}$ be the set of sources in the current graph. Then $\mathscr{R} = \{V_3, V_8\}$ and $\mathscr{S} = \{V_2, V_6, V_7, V_{10}\}$. In this case, there is no more pair of $V \in \mathscr{R}$ and $V' \in \mathscr{S}$ such that $\dim(V \cap V') = 1$.

Now we use a new rule to connect the components constructed in Step 2 by means of the remaining spaces.

Step 3. Let $\mathscr{L}$ be the set of leaves of the connected components in the graph constructed in Step 2, viewed as an undirected graph. We have $\mathscr{L} = \{V_1, V_2, V_6, V_7, V_9, V_{10}\}$. Then for each $V_j = E(\alpha_j) \in \mathscr{R}$, we can choose V_p and V_q in $\mathscr{L}$ minimal with respect to inclusion such that $\alpha_j \in V_p + V_q$. Indeed, since $\alpha_3 = x_2 + x_4 \in V_2 + V_6$ and $\alpha_8 = x_7 + x_9 \in V_7 + V_9 \subset V_7 + V_{10}$, we choose

V_2 and V_6 for V_3, and we choose V_7 and V_9 for V_8. Then we draw the directed edges $V_j \to V_p$ and $V_j \to V_q$. The resulting graph is:

$$V_1 \leftarrow V_2 \leftarrow V_3 \to V_6 \to V_5 \leftarrow V_4 \leftarrow V_7 \leftarrow V_8 \to V_9 \leftarrow V_{10},$$

which is isomorphic to the graph $\mathscr{G}_w$ as a directed graph.

Now we prove Theorem 32 when there is no trivalent vertex.

Proof of Theorem 32 in the absence of a trivalent vertex. Assume $\mathscr{G}_w$ has no trivalent vertex. We recover the graph $\mathscr{G}_w$ from the cohomology ring $H^*(X_w)$ in three steps. In Step 1, we determine the vertex set. In Step 2, we first recover the sinks, and then iteratively recover all vertices except for the sources that are not leaves. In Step 3, we recover the source vertices that are not leaves, which completes the reconstruction of $\mathscr{G}_w$.

For simplicity, we first prove the result for $w \in \mathrm{Cox}_W$, and then treat the general case.

Step 1. For each $j = 1, \dots, r$, we define a $\mathbb{Z}_2$-vector space V_j by

$$V_j = \begin{cases} \langle x_j \rangle & \text{if } \alpha_j = 0, \\ E(\alpha_j) & \text{otherwise.} \end{cases}$$

In Steps 2 and 3, we construct a directed graph with vertices $V_1, \dots, V_r$ using the following statements, which are obtained from Lemma 31.

(1) $V_i \subset V_j$, which occurs only when i is a sink and $E_w^+(j) = \{i\}$;
(2) $\dim(V_i \cap V_j) = 1$, which occurs only when $E_w^+(j) = \{i\}$;
(3) $\alpha_i \in V_a + V_b$, which occurs when $E_w^+(i) = \{a, b\}$.

Step 2. For each index j with $\alpha_j = 0$, we construct a connected component as follows. Fix such a j and initialize the directed graph with the vertex V_j. First, we find a space satisfying the following:

(J_1) a space V satisfies $\dim(V \cap V_j) = 1$.

For such a space V, we draw a directed edge $V \to V_j$. Let $\mathscr{S}$ be the set of sources of the directed graph constructed so far.

Next, we incorporate the remaining spaces one by one. Given a space V not yet contained in the current graph, if there exists a source $V_a \in \mathscr{S}$ such that $\dim(V \cap V_a) = 1$, we choose one such V_a and draw a directed edge $V \to V_a$. We then update the set $\mathscr{S}$ of sources and continue. We repeat this process until no unused space V has a one-dimensional intersection with any source in $\mathscr{S}$.

If the choice of V_a is not unique (which may occur when $V_a = V_{a'}$ for distinct indices), any choice produces isomorphic graphs, so the construction is well defined up to isomorphism.

At this stage, the resulting connected component is a path, and there are three possible forms by

$$\langle x_j \rangle \leftarrow \underbrace{E(\alpha_{j+1})}_{=\langle x_j, x_{j+1}\rangle} \leftarrow \cdots \leftarrow \underbrace{E(\alpha_{j+p})}_{=\langle x_{j+p-1}, x_{j+p}\rangle},$$

$$\underbrace{E(\alpha_{j-q})}_{=\langle x_{j-q}, x_{j-q+1}\rangle} \to \cdots \to \underbrace{E(\alpha_{j-1})}_{=\langle x_{j-1}, x_j\rangle} \to \langle x_j \rangle,$$

$$E(\alpha_{j-q}) \to E(\alpha_{j-q+1}) \to \cdots \to E(\alpha_{j-1}) \to \langle x_j \rangle \leftarrow E(\alpha_{j+1}) \leftarrow \cdots \leftarrow E(\alpha_{j+p-1}) \leftarrow E(\alpha_{j+p}),$$

where

$$\begin{aligned} E(\alpha_{j-1}) &= E(\alpha_{j+1}) = E(x_j) = \langle x_{j-1}, x_j, x_{j+1}\rangle, \\ E(\alpha_{j+t-1}) &= \langle x_{j+t-1}, x_{j+t}\rangle \quad \text{for } 2 \le t \le p, \\ E(\alpha_{j-s+1}) &= \langle x_{j-s}, x_{j-s+1}\rangle \quad \text{for } 2 \le s \le q. \end{aligned}$$

(Note that the multiplicity of the eigenelement x_j is two and the last two identities above make sense when $p \ge 2$, $q \ge 2$.)

If the graph obtained in this step is connected, then it is the desired graph and the construction is complete. Otherwise, some spaces have not yet been incorporated, and we proceed to the next step, where we use a new rule to connect the components constructed in Step 2 by means of the remaining spaces.

Step 3. Let $\mathscr{R}$ be the set of subspaces V_j that are not contained in any connected component constructed in Step 2. Let $\mathscr{L}$ be the set of leaves of those connected components, viewed as an undirected graph. Then every space $V_j \in \mathscr{R}$ satisfies the following statement:

(J_2) for $V_j = E(\alpha_j) \in \mathscr{R}$, there exist $V_p, V_q \in \mathscr{L}$ such that $\alpha_j \in V_p + V_q$.

We choose $V_p, V_q \in \mathscr{L}$ minimal with respect to inclusion, and then we draw directed edges $V_j \to V_p$ and $V_j \to V_q$. In some cases, the choice of V_p and V_q is not uniquely determined. This occurs when $V_p = V_{p'}$ for distinct indices p and p'. In this situation, we have $E^+_w(p) = E^+_w(p') = \{s\}$ for some s, and hence p and p' are symmetric neighbors of s. Consequently, either choice leads to the same resulting graph, and the construction is completed.

It remains to explain why it suffices to treat the case where w is a Coxeter element. For a general element $w = s_{i_1} \cdots s_{i_m}$, the associated graph $\mathscr{G}_w$ may be disconnected, and the indices are given by $i_1, \dots, i_m$ rather than $1, \dots, m$. However, by Remark 26, the cohomology ring $H^*(X_w)$ decomposes as the tensor product of the cohomology rings associated with the connected components of $\mathscr{G}_w$. Therefore, the reconstruction problem reduces to the connected case, namely the case where w is a Coxeter element. The same argument as above then applies on each connected component, with j replaced by i_j. $\square$

We now discuss the modifications needed when $\mathscr{G}_w$ has a trivalent vertex. If the outdegree of the trivalent vertex is at most one, then the reconstruction proceeds by repeating Step 2 exactly as in the case with no trivalent vertex. If the outdegree is two, then after Step 3 has been carried out, one can return to Step 2 and continue the reconstruction. If the trivalent vertex is a source, then the argument is analogous to Step 3, although a new rule is needed to handle the trivalent vertex. The following example of type E illustrates the case where Step 2 must be applied again after Step 3.

Example 37. Let G be of type E_7 and $w = s_7 s_2 s_1 s_3 s_4 s_5 s_6$. Then we have:

$$\begin{tikzcd} & & & 2 & & & \\ \mathscr{G}_w = & 1 & 3 \arrow[l] & 4 \arrow[l] \arrow[u] & 5 \arrow[l] & 6 \arrow[l] \arrow[r] & 7 \end{tikzcd}$$

Then $\alpha_1 = \alpha_2 = \alpha_7 = 0$ and $\alpha_3 = x_1$, $\alpha_4 = x_2 + x_3$, $\alpha_5 = x_4$, $\alpha_6 = x_5 + x_7$.

Step 1. The spaces V_j's are given as follows:

$$\begin{aligned} &V_1 = \langle x_1\rangle, && V_2 = \langle x_2\rangle, && V_3 = \langle x_1, x_3\rangle, && V_4 = \langle x_2 + x_3, x_4\rangle, \\ &V_5 = \langle x_4, x_5\rangle, && V_6 = \langle x_5 + x_7, x_6\rangle, && V_7 = \langle x_7\rangle. \end{aligned}$$

Step 2. We obtain the following connected components:

$$V_1 \leftarrow V_3, \qquad V_2, \qquad V_7.$$

Step 3. We have $\mathscr{R} = \{V_4, V_5, V_6\}$ and $\mathscr{L} = \{V_1, V_2, V_3, V_7\}$. Since $\alpha_4 \in V_2 + V_3$, we combine the connected components obtained from Step 2, we get the following graph:

$$\begin{array}{ccccccc} & & & & V_2 & & & & & & \\ & & & & \uparrow & & & & & & \\ V_1 & \leftarrow & V_3 & \leftarrow & V_4 & & & & & & V_7 \end{array}$$

Unlike in the case with no trivalent vertex, for the spaces V_5 and V_9 in $\mathscr{R}$, there is no pair $V_p, V_q \in \mathscr{L}$ satisfying (J_2). Nevertheless, V_4 satisfies $\dim(V_4 \cap V_5) = 1$, so the criterion from Step 2 still applies.

Step 4. From the condition $V_4 \cap V_5 = \langle x_4 \rangle$, we draw a directed edge from V_5 to V_4. Then, we get the following graph:

$$\begin{array}{ccccccccc} & & & & V_2 & & & & \\ & & & & \uparrow & & & & \\ V_1 & \leftarrow & V_3 & \leftarrow & V_4 & \leftarrow & V_5 & & & & V_7 \end{array}$$

Finally, V_6 satisfies that $\alpha_6 \in V_5 + V_7$, so we can apply the criterion from Step 3 again. We draw the directed edges $V_6 \to V_5$ and $V_6 \to V_7$, which recover the graph $\mathscr{G}_w$:

$$\begin{array}{ccccccccccccc} & & & & V_2 & & & & & & & & \\ & & & & \uparrow & & & & & & & & \\ V_1 & \leftarrow & V_3 & \leftarrow & V_4 & \leftarrow & V_5 & \leftarrow & V_6 & \rightarrow & V_7 \end{array}$$

Let us prove Theorem 32 when there is a trivalent vertex.

Proof of Theorem 32 in the presence of a trivalent vertex. Now we assume that $\mathscr{G}_w$ has a trivalent vertex. As in the case without a trivalent vertex, it is enough to prove when w is a Coxeter element. Then the arguments in Steps 1 and 2 remain unchanged. We will modify Step 3 by adding one more criterion and add Step 4, which repeats Steps 2 and 3.

We remark that if the trivalent vertex is already recovered after Step 2, then its outdegree is necessarily at most 1. Indeed, this happens only when the trivalent vertex itself is a sink, or when there is a sink in the branch determined by the outward edge from the trivalent vertex.

Step 3′. As before, we begin with the sets $\mathscr{R}$ and $\mathscr{L}$, where $\mathscr{R}$ is the set of subspaces V_j that are not contained in any connected component constructed in Step 2, and $\mathscr{L}$ is the set of leaves of those connected components viewed as an undirected graph. Now we add another criterion as follows:

(J_3) for $V_j = E(\alpha_j) \in \mathscr{R}$, there exist $V_p, V_q, V_r \in \mathscr{L}$ such that $\alpha_j \in V_p + V_q + V_r$.

Note that there exists a space $V_j \in \mathscr{R}$ satisfying (J_3) only when the trivalent vertex of $\mathscr{G}_w$ is a source. Then each $V_j \in \mathscr{R}$ satisfies (J_2), (J_3), or neither. Hence, we divide $\mathscr{R}$ into three subsets:

$$\begin{aligned} \mathscr{R}_1 &= \{V_j \in \mathscr{R} \mid V_j \text{ satisfies neither } (J_2) \text{ nor } (J_3)\}, \\ \mathscr{R}_2 &= \{V_j \in \mathscr{R} \mid V_j \text{ satisfies } (J_2)\}, \\ \mathscr{R}_3 &= \{V_j \in \mathscr{R} \mid V_j \text{ satisfies } (J_3)\}. \end{aligned}$$

Case 1: Assume that the trivalent vertex has outdegree at most 1. Then both $\mathscr{R}_1 = \mathscr{R}_3 = \emptyset$, and the graph $\mathscr{G}_w$ can be recovered under the same procedures as in Step 3 for the case with no trivalent vertex.

Case 2: Assume that the trivalent vertex is a source. In this situation, $\mathscr{R}_1 = \emptyset$ and $\mathscr{R}_3$ is a singleton, say $\mathscr{R}_3 = \{V\}$. Then we can choose V_p, V_q, V_r minimal with respect to inclusion. If such a choice is not unique, we choose one of them and draw the directed edges $V \to V_p$, $V \to V_q$, and $V \to V_r$. As in the case with (J_2), either choice leads to the same resulting graph. For each $V \in \mathscr{R}_2$, we draw the two directed edges as in the case with no trivalent vertex. Then we can recover the graph $\mathscr{G}_w$.

Case 3: Assume that the trivalent vertex has outdegree 2. In this case, $\mathscr{R}_1 \neq \emptyset$ and $\mathscr{R}_3 = \emptyset$. If $\mathscr{R}_2 \neq \emptyset$, then we proceed with Step 3 as in the case with no trivalent vertex, and then go to Step 4.

Note that each space in $\mathscr{R}_1$ satisfies (J_1) or (J_2).

Step 4. Let $\mathscr{V}$ be the set of V_j's contained in some connected component of the graph obtained from Step 3′. Since one of the connected components contains the vertex V_b corresponding to the trivalent vertex recovered in Step 3′, Case 3, there exists a unique subspace $V_a = E(\alpha_a) \in \mathscr{R}_1$ such that $\dim(V_a \cap V_b) = 1$ or $\alpha_a \in V_b + V_c$ for some $V_c \in \mathscr{V}$. We distinguish two cases according to the structure of $E_w^+(a)$.

Case 1: Assume $\alpha_a \in V_b + V_c$ for some $V_c \in \mathscr{V}$. We choose such spaces V_b and V_c minimal with respect to inclusion. If such a choice is not unique, we choose one of them and draw $V_a \to V_b$ and $V_a \to V_c$. As before, either choice leads to the same resulting graph. This situation corresponds to $\mathscr{R}_1 = \{V_a\}$, and the resulting graph is the desired one.

Case 2: Assume $\dim(V_a \cap V_b) = 1$. In this case, we draw a directed edge $V_a \to V_b$ as in Step 2. We then continue this procedure as in Step 2, successively attaching spaces whose intersections are one-dimensional. This process terminates either when all remaining spaces have been incorporated, or when a configuration of the form satisfying (J_2) appears. In the latter case, we apply Step 3 once, and the construction is complete.

This completes the proof. □

Remark 38. We cannot extend Theorem 32 to other Lie types by simply following the steps considered in its proof. For a simple Lie group G of type B_2, the cohomology ring $H^*(X_{s_2 s_1})$ is isomorphic to $\mathbb{Z}[x,y]/\langle x^2, y^2\rangle$. Hence, there are two primitive square-zero elements x and y. This produces a graph consisting of two vertices with no edges by following the steps in the proof of Theorem 32. However, the digraph $\mathscr{G}_{s_2 s_1}$ consists of two vertices and a directed edge with label 2. This failure occurs because the procedure relies solely on the cohomology ring data and cannot detect nontrivial edge labels such as the label 2.

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