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On the Fano dimension of an Enriques surface

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Abstract. We construct a family of Fano fourfolds with the derived category of coherent sheaves of a general Enriques surface as semiorthogonal component. This improves a result of Kuznetsov, lowering the Fano dimension of a general Enriques surface from six to four.

Keywords. Enriques surfaces, Fano dimension, degeneracy loci.

2020 Mathematics Subject Classification. 14J45, 14J28, 14F08.

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1. Introduction

The Fano-visitor problem is a natural, yet elusive question in algebraic geometry and derived categories. First posed by Bondal in 2011, it amounts to asking if for any smooth projective variety X, there exists a smooth Fano variety Y together with a full and faithful functor $\mathbf{D}(X) \to \mathbf{D}(Y)$ between the bounded derived categories of coherent sheaves on X and Y respectively. If the answer is positive, X is called Fano-visitor, and Y is called Fano-host of X. This led, in [3], to the definition of the Fano dimension of a variety X, which is the minimal dimension of Fano-hosts of X.

We recall some key results on the topic. Bondal and Orlov in [2] proved that the derived category of a hyperelliptic curve X of genus g can be embedded into the derived category of the intersection of two quadrics in $\mathbb{P}^{2g+1}$. Kuznetsov, in [4], proved that the derived categories of some K3 surfaces are embedded into special cubic fourfolds. Bernardara, Bolognesi, and Faenzi in [1] proved that every smooth plane curve is a Fano-visitor. Segal and Thomas in [8] proved that a general quintic threefold is a Fano-visitor in an eleven-dimensional Fano-host. Finally, Kiem, Kim, H. Lee, and K. Lee, in [3], proved that all smooth projective complete intersections are Fano-visitors.

In [6], Kuznetsov first showed that a certain divisorial family in the moduli space of Enriques surfaces (so-called *nodal Enriques*[1]) can be realized as Fano-visitor for a Fano fourfold, which

[1] Despite the name, these Enriques surfaces are smooth.

can be written as the blow-up of $\mathrm{Gr}(2,4)$ in the same Enriques surface. Then he showed that an Enriques surface general in moduli can be seen as a Fano-visitor for a six-dimensional Fano-host. In summary:

- if S is a *nodal Enriques* surface, then its Fano dimension is ≤ 4;
- if S is a general Enriques surface, then its Fano dimension is ≤ 6.

In our work, we improve the latter bound for general Enriques surfaces.

Theorem 1 (Lemma 2, Corollary 3, Lemma 4). *For a general Enriques surface S there is a smooth Fano complete intersection of hypersurfaces of multidegree $(1,2,0)$ and $(1,0,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ and a semiorthogonal decomposition*

$$\mathbf{D}(Y) = \big\langle \mathbf{D}(S), E_1, \ldots, E_9 \big\rangle,$$

where $E_1, \ldots, E_9$ are exceptional vector bundles. In particular, the Fano dimension of a general Enriques surface is 4.

Moreover, as shown in Corollary 3, the Hodge diamond of Y is diagonal, and $K_0(Y)$ contains a 2-torsion class; in particular, $\mathbf{D}(Y)$ does not have a full exceptional collection.

2. A general Enriques surface and its host

We discuss two Fano hosts for a general Enriques surface: the first is six-dimensional and was described by Kuznetsov in [6]; the second is four-dimensional and is the main result of this paper.

We fix the following notation: let V_3, V_3', W_3 be three-dimensional vector spaces, $X = \mathbb{P}(V_3) \times \mathbb{P}(V_3')$, $\mathscr{F} = \mathscr{O}_X(2,0) \oplus \mathscr{O}_X(0,2)$, $\mathscr{E} = \mathscr{O}_X \otimes W_3$. Consider $\mathbb{P}_X(\mathscr{F}^\vee)$ the projective bundle on X, with $q\colon \mathbb{P}_X(\mathscr{F}^\vee) \to X$ be the natural projection to X, and tautological sequence

$$0 \longrightarrow \mathscr{O}_{\mathscr{F}^\vee}(-1) \longrightarrow q^*\mathscr{F}^\vee \longrightarrow \mathscr{Q}_{\mathscr{F}^\vee} \longrightarrow 0.$$

Let us denote with $\mathrm{D}_k(\varphi)$ the k-th degeneracy locus of a morphism φ, i.e. $\mathrm{D}_k(\varphi) = \big\{x \in X \;\big|\; \mathrm{rk}\,\varphi_x \leq k\big\}$. Recall from [7, Lemma 2.1] that a general Enriques surface can be described as the first degeneracy locus $\mathrm{D}_1(\varphi)$ of a general morphism:

$$\varphi\colon \mathscr{E} \longrightarrow \mathscr{F}.$$

By the projection formula, we obtain

$$H^0(X, \mathscr{E}^\vee \otimes \mathscr{F}) \cong H^0\big(\mathbb{P}_X(\mathscr{F}^\vee), q^*\mathscr{E}^\vee \otimes \mathscr{O}_{\mathscr{F}^\vee}(1)\big). \tag{1}$$

Observe that

$$H^0(X, \mathscr{E}^\vee \otimes \mathscr{F}) \cong \mathrm{Hom}(\mathscr{E}, \mathscr{F}), \qquad H^0\big(\mathbb{P}_X(\mathscr{F}^\vee), q^*\mathscr{E}^\vee \otimes \mathscr{O}_{\mathscr{F}^\vee}(1)\big) \cong \mathrm{Hom}\big(q^*\mathscr{E}, \mathscr{O}_{\mathscr{F}^\vee}(1)\big).$$

Hence, given a morphism $\varphi\colon \mathscr{E} \to \mathscr{F}$, we denote by

$$\varphi_{\mathscr{F}^\vee}\colon q^*\mathscr{E} \longrightarrow \mathscr{O}_{\mathscr{F}^\vee}(1)$$

its image under the isomorphism (1). We then consider its zero locus

$$S = V(\varphi_{\mathscr{F}^\vee}) \subset \mathbb{P}_X(\mathscr{F}^\vee),$$

which is precisely the Enriques surface considered in [6, Lemma 3].

Six-dimensional Fano-host

Let $\mathbb{P}_X(\mathscr{E})$ be the projective bundle on X with $p\colon \mathbb{P}_X(\mathscr{E}) \to X$ be the natural projection to X, and tautological sequence

$$0 \longrightarrow \mathscr{O}_{\mathscr{E}}(-1) \longrightarrow p^*\mathscr{E} \longrightarrow \mathscr{Q}_{\mathscr{E}} \longrightarrow 0.$$

In [6], in order to produce a six-dimensional Fano-host for S, Kuznetsov considered the product

$$\mathbb{P}_X(\mathscr{F}^\vee) \times \mathbb{P}(W_3) \cong \mathbb{P}_X(\mathscr{F}^\vee) \times_X \mathbb{P}_X(\mathscr{E}).$$

The key observation is the isomorphism given by the Künneth formula

$$H^0\big(\mathbb{P}_X(\mathscr{F}^\vee),\, q^*\mathscr{E}^\vee \otimes \mathscr{O}_{\mathscr{F}^\vee}(1)\big) \cong H^0\big(\mathbb{P}_X(\mathscr{F}^\vee) \times_X \mathbb{P}_X(\mathscr{E}), \mathscr{O}_{\mathscr{F}^\vee}(1) \boxtimes \mathscr{O}_{\mathscr{E}}(1)\big). \tag{2}$$

Hence, the same section φ from the previous paragraph is associated to a unique general global section $\widetilde{\varphi}$ of $H^0\big(\mathbb{P}_X(\mathscr{F}^\vee) \times_X \mathbb{P}_X(\mathscr{E}), \mathscr{O}_{\mathscr{F}^\vee}(1) \boxtimes \mathscr{O}_{\mathscr{E}}(1)\big)$. We can then consider its zero locus $T = V(\widetilde{\varphi}) \subset \mathbb{P}_X(\mathscr{F}^\vee) \times_X \mathbb{P}_X(\mathscr{E})$.

In [6, Theorem 4] is proved that T is a Fano sixfold, described as a stratified projective bundle on $\mathbb{P}_X(\mathscr{F}^\vee)$, with general fiber a $\mathbb{P}^1$ which jumps to a $\mathbb{P}^2$-bundle over S. From this, it follows that

$$\mathbf{D}(T) = \big\langle \mathbf{D}(S), \mathbf{D}\big(\mathbb{P}_X(\mathscr{F}^\vee)\big), \mathbf{D}\big(\mathbb{P}_X(\mathscr{F}^\vee)\big)\big\rangle.$$

Four-dimensional Fano-host

The key idea behind this paper is to consider another variety Y which is defined by the very same φ, and to relate it with S. In fact, if $p\colon \mathbb{P}_X(\mathscr{E}) \to X$ is the natural projection, then, as in (2), one gets another canonical isomorphism

$$H^0\big(\mathbb{P}_X(\mathscr{F}^\vee) \times_X \mathbb{P}_X(\mathscr{E}), \mathscr{O}_{\mathscr{F}^\vee}(1) \boxtimes \mathscr{O}_{\mathscr{E}}(1)\big) \cong H^0\big(\mathbb{P}_X(\mathscr{E}), \mathscr{O}_{\mathscr{E}}(1) \otimes p^*\mathscr{F}\big). \tag{3}$$

Denote with $\varphi_{\mathscr{E}}$ the unique global section of $\mathscr{O}_{\mathscr{E}}(1) \otimes p^*\mathscr{F}$ associated to $\widetilde{\varphi}$ via (3). Therefore, we can consider the zero locus $Y = V(\varphi_{\mathscr{E}}) \subset \mathbb{P}_X(\mathscr{E})$.

In the following lemma, we prove that X and Y are birational and show that Y can be viewed as a Fano-host for S.

Lemma 2. *Consider a general section*

$$\varphi_{\mathscr{E}} \in H^0\big(\mathbb{P}_X(\mathscr{E}), \mathscr{O}_{\mathscr{E}}(1) \otimes p^*\mathscr{F}\big).$$

The variety

$$Y = V(\varphi_{\mathscr{E}}) \subset \mathbb{P}_X(\mathscr{E})$$

is a smooth Fano fourfold. Moreover, $Y \cong \mathrm{Bl}_S X$, with S a general Enriques surface.

Proof. By Bertini's theorem for degeneracy loci, $\operatorname{codim}_X \mathrm{D}_1(\varphi) = 2$, and $\operatorname{codim}_X\big(\mathrm{D}_0(\varphi)\big)$ is strictly greater than $\dim X$, and hence $\mathrm{D}_0(\varphi) = \emptyset$. Therefore, the assumptions of [5, Lemma 2.1] are satisfied,[2] and we obtain

$$Y \cong \mathrm{Bl}_{\mathrm{D}_1(\varphi)} X.$$

Moreover, by [7, Lemma 2.1], the surface $S = \mathrm{D}_1(\varphi)$ is a general Enriques surface.

Recall that $\mathbb{P}_X(\mathscr{E})$ is $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, and $\mathscr{O}_{\mathscr{E}}(1) \otimes p^*\mathscr{F}$ is $\mathscr{O}(1,2,0) \oplus \mathscr{O}(1,0,2)$. Thus, Y is the complete intersection of hypersurfaces of multidegree $(1,2,0)$ and $(1,0,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. By the adjunction formula,

$$K_Y = \big(K_{\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2} \otimes \mathscr{O}(1,2,0) \otimes \mathscr{O}(1,0,2)\big)\big|_Y = \mathscr{O}(-1,-1,-1)\big|_Y.$$

Therefore, Y is Fano. □

[2] We remark that [5] adopts the opposite convention for numbering degeneracy loci.

Corollary 3. *The variety Y has the following invariants:*

$$e(Y) = 21, \qquad h^0(-K_Y) = 27, \qquad (-K_Y)^4 = 102.$$

The half-lower Hodge diamond of Y is

$$\begin{array}{ccccccccc} 0 & & 0 & & 13 & & 0 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 3 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

The integral cohomology of Y is:

$$H^k(Y,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0,\ 8, \\ 0 & \text{if } k = 1,\ 7, \\ \mathbb{Z}^{\oplus 3} & \text{if } k = 2,\ 6, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } k = 3,\ 5, \\ \mathbb{Z}^{\oplus 13} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } k = 4. \end{cases}$$

Moreover, the variety Y has the following semiorthogonal decomposition

$$\mathbf{D}(Y) = \langle \mathbf{D}(S), E_1, \ldots, E_9 \rangle$$

where $E_1, \ldots, E_9$ are exceptional bundles. $K_0(Y)$ contains a 2-torsion class; in particular, $\mathbf{D}(Y)$ does not have a full exceptional collection.

Proof. Since Y is a complete intersection, its invariants can be quickly computed using standard exact sequences and the Riemann–Roch theorem.

The semiorthogonal decomposition is given by Orlov's blow-up formula and the fact that $\mathbf{D}(\mathbb{P}^2 \times \mathbb{P}^2)$ is generated by an exceptional collection of length nine.

The Grothendieck group is additive with respect to semiorthogonal decompositions, hence

$$K_0(Y) = K_0(S) \oplus \mathbb{Z}^{\oplus 9}.$$

The 2-torsion class in S gives a 2-torsion class in Y, hence, by [6, Lemma 1], $\mathbf{D}(Y)$ does not admit a full exceptional collection. □

This yields a four-dimensional variety as a solution to the Fano-visitor problem for Enriques surfaces, general in moduli.

Lemma 4. *The Fano dimension of a general Enriques surface is* 4.

Proof. By Corollary 3, the Fano dimension of S is at most 4. Observe that neither $\mathbb{P}^1$ nor a del Pezzo surface can be a Fano host for S. Therefore, if one shows that no Fano threefold can be a Fano host for S, it follows that the Fano dimension of a general Enriques surface is 4.

Assume, by contradiction, that Y is a three-dimensional Fano host for S. Then we obtain the following inequality for the dimensions of Hochschild homology:

$$12 = \dim\big(\mathrm{HH}_0(S)\big) \leq \dim\big(\mathrm{HH}_0(Y)\big).$$

If $\rho(Y)$ denotes the Picard rank of Y, we have

$$\dim \mathrm{HH}_0(Y) = 2 + 2\rho(Y),$$

and hence $\rho(Y) \geq 5$. By the classification of Fano threefolds, there are only eight families of Fano threefolds with Picard rank at least 5. In particular, two are the blow-ups of $\mathbb{P}^3$ and of a quadric threefold, while the other six are products of $\mathbb{P}^1$ with a del Pezzo surface. Thus, all of them admit a full exceptional collection. This implies that $K_0(Y)$ is torsion-free, which contradicts the assumption that $K_0(S) \subset K_0(Y)$. □

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Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

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