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Historical commentary
Algebraic geometry

The posterity of “Residues and duality”

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Abstract. We discuss various developments that occurred in the wake of Robin Hartshorne’s seminar “Residues and duality”. This seminar was based on notes written by Alexander Grothendieck in the summer of 1963, that have recently been published (*Résidus et dualité — Prénotes pour un “Séminaire Hartshorne”* (R. Hartshorne, ed.), Documents Mathématiques, vol. 21, Société Mathématique de France, 2024).

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Introduction

In 1962–1963, Alexander Grothendieck was elaborating a general theory of duality for coherent sheaves. In the spring of 1963, Robin Hartshorne wrote him that he would like to run a seminar at Harvard on this in the fall of the same year and asked him for his permission to do so. Grothendieck agreed and in the summer of 1963 he had completed a long manuscript, entitled *Résidus et dualité ; prénotes pour un Séminaire Hartshorne*, that he sent to Hartshorne to serve as a basis for his seminar. This is a very remarkable document, that has recently been published [22]. The reader will find there historical comments on the genesis of the *prénotes* and a brief discussion of the main constructions. Hartshorne’s seminar was published in 1966 [23].

The present text, written at the suggestion of Hartshorne, was supposed to be included in [22], of which I was to be a co-editor. The editors of the collection had given their agreement. However, in May 2023 they explained to me that Grothendieck’s children would grant their permission to publish only if I withdrew from the project, which I did. Hence, despite the (pleasant) collaboration I had had with Hartshorne on this project from 2019 to 2023, I was not a co-editor, my text was removed, and all references to my name were eradicated, except for two lines of thanks at the end of the preface.

1. Derived categories

Derived and triangulated categories have changed the face of homological algebra. Since their introduction in [23] and Verdier’s notes [48], the theory has undergone considerable developments (among the most notable ones, let me just mention filtered derived categories, t-structures and perverse sheaves, derived ∞ -categories). It is of common use today in almost all parts of algebraic geometry and homotopy theory. I will limit myself here to discussing a few points that are more or less directly related to [22,23].

1.1. Signs

Sign problems have plagued homological algebra since the very beginning. Derived categories and derived functors and the ensuing formalism of *six operations* made it hard to elaborate a coherent system of sign conventions. Deciding whether a given diagram involving canonical maps between various derived functors commutes is often a highly nontrivial matter. Grothendieck didn't pay too much attention to signs, considering that it was a matter of "routine verification". Some discrepancies in [23] were observed by Deligne in [14, Appendice, Signes]. Here he proposes a system of conventions, which he thinks is coherent. They are based on those he developed in [3, XVII, 1.1 and 1.2] (see [13, p. 312] for an erratum). A concise summary of them is given in [6, 0.3]. Sign conventions are also discussed in detail in Conrad's book (see [11, Section 1.3]), the purpose of which is to provide proofs for delicate compatibilities left unproved in [23] and straighten the signs. However, discrepancies were later found in it by Gabber (see Conrad's homepage [12] for corrections and updates).

1.2. Unbounded derived categories

Leaving aside the questions of finiteness (such as coherence or constructibility of cohomology sheaves), it was long tacitly assumed that the definition of the usual derived functors such as Rf_* , Lf^* , $\otimes^L$, etc., required degree restrictions on their sources. For example, for a morphism of schemes $f: X \rightarrow Y$, denoting by $D(X)$ (resp. $D(Y)$) the derived category of $\mathcal{O}_X$ -modules (resp. $\mathcal{O}_Y$ -modules), if no assumption of finiteness or cohomological dimension was made, the functor Rf_* was defined only on the bounded below category $D^+(X)$. In the early 1960s Grothendieck asked whether, if one worked with *good* abelian categories (such as the category of $\mathcal{O}_X$ -modules on a scheme, or a ringed topos, or more generally what is now called a *Grothendieck abelian category* (see [27, 8.3.24])), one could get rid of these restrictions (see e.g. [22, 2.2.3(3)]). It is only in 1988 that a satisfactory solution was proposed, by Spaltenstein [43], using the new notion of *homotopically injective* (resp. *projective*) *resolution*. A more systematic treatment is given in [27, Sections 14 and 18]. However, this theory had no great bearing on duality, as duality theorems (whether global or local) require a combination of boundedness and finiteness assumptions.¹

1.3. Descent in derived categories

As recalled in [22, Introduction to Prénotes], morphisms or objects of derived categories in general do not glue on open covers. A notable exception was discovered long after [23], namely *perverse sheaves* in the étale context (see [5, 3.2.4]). Grothendieck was of course well aware of the difficulty, and that led him to sketch his theory of *pseudo-complexes* in [22, 5.2], which brought a partial solution to the gluing problem. I remember his excitement when in a talk at the IHES in the mid 1960s (see [25]) Deligne explained his theory of *cohomological descent* (later written up by Saint-Donat, see [2, V bis]), which fully solved the problem. However, the conjectural approach via "pseudo-complexes" can be seen as a first attempt to defining a "higher" way of gluing in derived categories, which has nowadays been achieved by the enhancement of derived categories to derived ∞ -categories. Classical derived functors can be enhanced to functors between derived ∞ -categories, and objects of these enhanced categories can be glued on open covers, see [32–34].

¹(Note added in November, 2023.) Bogdan Zavyalov (private communication) points out that nowadays it is possible to drop boundedness assumptions in Poincaré duality type results. For example, see [35, Proposition 2.9.31] for the case of discrete adic spaces (and schemes), and [49, Theorems 1.3.1 and 1.3.2] for étale cohomology of schemes/adic spaces. In fact, the categorical approach used to construct six-functor formalisms in [35] or [49] *requires* functors to be defined on the unbounded categories $\mathcal{D}$.

This formalism, which supersedes that of Deligne, is being increasingly used today. In [41], it is shown how it can be combined with inputs from Huber's adic geometry to yield another approach to the $f^!$ functor of [22,23], and even to a full six-operation formalism of coherent duality. See the end of Section 3 for more references.

2. The avatars of the $f^!$ functor

The central object in [22,23] is the $f^!$ functor. As recalled in [22, Introduction to Prénotes], Grothendieck in [22] discussed two ways of constructing it. In the appendix to [23] Deligne proposed a third construction.

2.1. The three constructions

For a noetherian scheme Z , let $D_{\text{coh}}^+(Z)$ denote the full subcategory of $D^+(Z, \mathcal{O}_Z)$ consisting of complexes with coherent cohomology sheaves. Let $f: X \rightarrow Y$ be a morphism of finite type between noetherian schemes. There are three ways of constructing

$$f^!: D_{\text{coh}}^+(Y) \longrightarrow D_{\text{coh}}^+(X).$$

Each one requires additional hypotheses. Let me briefly recall the constructions.

- (a) *Via embeddings into a smooth scheme.* Assume that we have a factorization $f = gi$, where $i: X \rightarrow Z$ is a closed immersion, and $g: Z \rightarrow Y$ is smooth, of pure relative dimension d . Using the *fundamental local isomorphism* (see [22, 3.1.4] and [23, III 7]), Grothendieck proves that, up to a transitive system of isomorphisms, the composite functor

$$i^!(g^*(-) \otimes \Omega_{Z/Y}^d[d]): D_{\text{coh}}^+(Y) \longrightarrow D_{\text{coh}}^+(X),$$

sending M to $i^!(g^* M \otimes \Omega_{Z/Y}^d[d])$, where $i^!L$ is $R\mathcal{H}om_{\mathcal{O}_Z}(i_*\mathcal{O}_X, L)$ restricted to X , is independent of the factorization. The corresponding limit of this system is denoted by $f^!$. He then shows (see [22, 3.4] and [23, III 11]) that, for f *projective*, $f^!$ is right adjoint to $Rf_*: D_{\text{coh}}^+(X) \rightarrow D_{\text{coh}}^+(Y)$. This uses the construction of a *trace map* $Rf_*f^! \rightarrow \text{Id}$ (and reduction to the projection of a standard projective space onto its base scheme).

- (b) *Using residual complexes and biduality.* Assume that Y admits a *dualizing complex* K_Y , and that X and Y are of finite Krull dimension.² Then K_Y determines a codimension function on Y , hence a canonical filtration of the underlying set of Y by subsets stable under specialization, and a corresponding *Cousin complex* $E(K_Y)$, which is isomorphic to K_Y in $D_{\text{coh}}^b(Y)$. Then $E(K_Y)$ is what Grothendieck calls a *residual complex* on Y , i.e., a dualizing complex, which, as a graded $\mathcal{O}_Y$ -module, is isomorphic to the direct sum over all points y of Y of $i_{y*}I(y)$, where $I(y) = H_{\{y\}}^{d_y}(\mathcal{O}_{Y,y})$ is an injective envelope of the residue field $k(y)$, with $d_y = \dim(\mathcal{O}_{Y,y})$, and $i_y: \{y\} = \text{Spec}(k(y)) \rightarrow Y$. Using his (conjectural) theory of pseudo-complexes and rigidity properties of residual complexes, Grothendieck shows in [22] that the Cousin complexes of the various $E(i^!(g^*(E(K_Y)) \otimes \Omega_{Z/Y}^d[d]))$ for closed embeddings $i: U \hookrightarrow Z$ of open subschemes U of X in schemes Z equipped with a smooth map $g: Z \rightarrow Y$ can be glued to a residual complex $E(K_X)$ on X associated with a dualizing complex K_X . A proof independent of the theory of pseudo-complexes is given in [23, VI]. Issues in [23, IV 3.2 and 3.4] concerning the equivalence between dualizing and residual complexes are fixed in [11, 3.1.3], and a proof of the unchecked compatibilities [23, VI 3.1 VAR 5 and VII 3.4(a)]

²This ensures that dualizing complexes on X and Y are bounded, and that pointwise dualizing complexes are dualizing.

is given in [11, 3.3.1]. Then, taking $D_X := R\mathcal{H}om(-, K_X)$, $D_Y := R\mathcal{H}om(-, K_Y)$, the $f^!$ functor is defined as

$$f^! := D_X Lf^* D_Y : D_{\text{coh}}^+(Y) \longrightarrow D_{\text{coh}}^+(X).$$

For f proper, $f^!$ thus defined is again shown to be a right adjoint to Rf_* , using a trace map defined by means of a *residue homomorphism*. The formulas stated in [23, III 9] for the corresponding *residue symbol* are proved in [11, A2]. Various sign issues concerning it are discussed in [12].

- (c) *Using formal adjunction.* Assume that Y is of finite Krull dimension and f is *compactifiable*, i.e., that there exists a factorization $f = gj$, where $j: X \rightarrow Z$ is an open immersion, and $g: Z \rightarrow Y$ is proper, with Z of finite Krull dimension. Using an analogue of Verdier's construction of the $f^!$ functor in the topological case [45], Deligne shows in [23, Appendix] that the functor $Rg_*: D_{\text{coh}}^+(Z) \rightarrow D_{\text{coh}}^+(Y)$ admits a right adjoint $g^!$. Then $f^!$ is defined as

$$j^* g^!: D_{\text{coh}}^+(Y) \longrightarrow D_{\text{coh}}^+(X).$$

The fact that, up to a transitive system of isomorphisms, $j^* g^!$ is independent of the factorization is non-trivial.

In [23, Appendix] Deligne sketches one method to prove it. For f not necessarily proper, he defines a functor $Rf_i: \text{pro } D_{\text{coh}}^b(X) \rightarrow \text{pro } D_{\text{coh}}^b(Y)$ between categories of pro-objects, by $Rf_i = Rg_* j_i$, where j_i is a pro-coherent version of the extension by zero, obtained by sending a coherent sheaf $\mathcal{F}$ on X to the pro-coherent sheaf “ $\varprojlim I^n \mathcal{F}$ ”, where $\mathcal{F}$ is a coherent extension of $\mathcal{F}$ on Z and I an ideal of $\mathcal{O}_Z$ defining a closed subscheme with support $Z - X$. The independence of $j^* g^!$ of the compactification is then reduced to the independence of Rf_i , which is easy by standard Artin–Rees–Krull arguments. By a different method, a full proof is given by Verdier in [47, Corollary 1].

Verdier's construction of $f^!$ in the topological case as a right adjoint to Rf_i , and its subsequent use by Deligne, first in the coherent context in [23, Appendix], and later in the étale one in [3, XVIII], relies on an elementary lemma [3, XVIII, 3.1.3] to the effect that any contravariant functor from a Grothendieck abelian category to abelian groups transforming arbitrary small inductive limits into projective ones is representable. A precursor of this lemma can be found in Gabriel's thesis [17, II, 6 Théorème 2]. Variants and generalizations for triangulated categories (see [27, 10.5.3 and 14.2.3]), inspired by Brown's representability theorem, were later developed (and applied to coherent duality) by Neeman, Lipman et al. [31, 38]. However, these apply to “classical derived categories”. In the ∞ -categorical setting, the adjoint functor theorem of [32, Corollary 5.5.2.9] for presentable ∞ -categories is in general needed.

2.2. Comparison and base change

For each of the three constructions several natural compatibilities for composition and base change have to be checked. This is done only partially in the original texts. Also, while for f proper the fact that $f^!$ is right adjoint to Rf_* ensures the existence and uniqueness of isomorphisms between the three approaches in their common domain of definition, the extension of these to the non-proper case raises delicate questions.

Comparison between (a) and (b) is carried out in [23] and with many more details in [11], where a main base change theorem for the trace map for proper, flat Cohen–Macaulay maps is proved (see [11, Theorem 3.6.5]).

Comparison between (a) (or (b)) and (c) is more difficult. Some cases (e.g., f compactifiable, flat, and locally of complete intersection) are sketched in [23, Appendix]. The smooth case is (fully) treated by Verdier in [47, Theorem 3]. In addition, Verdier proves a base change theorem for $f^!$ for a flat base change (see [47, Theorem 2]), from which the calculation of $f^! G$, $G \in D_{\text{coh}}^+(Y)$

for f of finite tor-dimension is reduced to the case where $G = \mathcal{O}_Y$. Another approach to this calculation is given in [26]. However, there remained delicate issues concerning explicit forms of the trace and residue maps and compatibility of $f^!$ with non-flat base change. These have been recently addressed by Nayak and Sastry [36,37]. One can also consult the Stacks Project [44].

2.3. Further developments

Let me discuss a few problems mentioned in [22] that were tackled later or are still of interest today.

2.3.1. The non-noetherian case

Grothendieck insisted on eliminating noetherian hypotheses. To him it was the relative structure of a morphism that mattered, not its base. This is already apparent at various places of [22]. Over a non-noetherian scheme the notion of coherent sheaf is no longer adapted. In 3.2.4 Grothendieck introduces the notion of sheaf of modules of ∞ -finite presentation, which he generalizes into that of *pseudo-coherent complex* in 4.1.1(e). He also defines the notion of *pseudo-coherent morphism* in 5.4.5. These notions were to be systematically studied and developed in [7, I–III]. Complexes that are both pseudo-coherent and of finite tor-dimension, considered on p. 80, will be called *perfect* in [7]. They are interesting already in the noetherian case and they have since then played a crucial role in many parts of arithmetic geometry. In 5.8(B) Grothendieck sketches the construction of the *determinant* of a perfect complex, a fundamental invariant, defined and studied in detail by Knudsen and Mumford in [28], and which is still an object of attention today in the framework of derived geometry. Partial generalizations of the duality theorem to the non-noetherian case (using the notion of pseudo-coherence) and to diagrams of schemes were given in [31] (see e.g., Corollary 4.4.2). However, a full six-functor formalism in the non-noetherian case was constructed only recently [35], systematically using derived geometry and condensed mathematics.

2.3.2. Cotangent complex and self-intersection complex

In [22, 5.8], Grothendieck defines the (truncated of length 1) *cotangent complex* of a homomorphism of rings $A \rightarrow B$. This is, historically, the first appearance of this object. As he told me, he got the idea from his work with Dieudonné on *imperfection modules* in [18, 10.6]: he conjectured that these modules should be (in good cases) interpreted as H_1 of a finer invariant, the cotangent complex of $A \rightarrow B$, whose H_0 would be the classical module of Kähler differentials. At about the same time, and independently, Lichtenbaum and Schlessinger defined a (truncated of length 2) cotangent complex of $A \rightarrow B$ [30]. However, the two problems mentioned by Grothendieck in [22, 5.8], namely:

- (i) getting a theory giving a transitivity triangle for a composition $A \rightarrow B \rightarrow C$;
- (ii) globalizing to morphisms of schemes;

were solved only a few years later. As for (ii), that was done by Grothendieck himself for the (truncated of length 1) complex in [19]. Problem (i) was solved by Quillen [39] (a definition and study of the homology and cohomology objects of the cotangent complex was made independently by M. André [1]). The globalization of Quillen's construction was carried out in [24].

Already the cotangent complex played a key role in the formulation of the Riemann–Roch theorem for projective morphisms $f: X \rightarrow Y$ that are lci, i.e., *locally of complete intersection*,³ Y being quasi-compact and admitting an ample line bundle (see [7, VIII, Théorème 3.6]). It is also in [22] that the notion of lci morphism appears for the first time (see [22, 4.8 C]). The

³In this case the truncated cotangent complex agrees with the full one.

basic formula [22, 5.8.2(ii)], to the effect that for $X \rightarrow Y$ flat and lci, of relative dimension d , $f^! \mathcal{O}_Y = \det(L_{X/Y})[d]$ was not reproduced in [23] (perhaps for lack of a suitable reference for the construction of the determinant of a perfect complex), and, curiously, not even in [11]. The flat lci morphisms, later called *syntomic morphisms*, were to play an important role in p -adic Hodge theory.

In [22, p. 60], Grothendieck describes a construction of Cartier of the residue homomorphism for a morphism of rings $A \rightarrow B$, based on what he calls the *self-intersection complex* of B/A , namely $B \otimes_{B \otimes_A B}^L B$, which he denotes by $L_{B/A}$. The analogy of notation with the cotangent complex $L_{B/A}$ of B/A is not fortuitous. Deep relations between this self-intersection complex and $L_{B/A}$, for B flat over A , and in general, with $B \otimes_{B \otimes_A B}^L B$ replaced by $B \otimes_{B \otimes_A^L B}^L B$ were discovered by Quillen (see [39, 8]) in relation with *associative algebra cohomology* and *Hochschild cohomology*. This topic is still actively studied today in relation with p -adic Hodge theory.

2.3.3. Complexes of differential operators

In [22, 3.5], Grothendieck proposes an extension of the duality theorem to complexes of differential operators. As far as I know, the problem has not yet been addressed in general. Over complex manifolds, duality theorems involving $\mathcal{D}$ -modules and complexes of differential operators are established in [40].

2.3.4. Homology, cycle classes, Lefschetz–Verdier formula

Let $f: X \rightarrow S$ be a morphism of finite type between noetherian schemes of finite dimension, where S admits a dualizing complex K_S and $K_X := f^! K_S$ is the corresponding dualizing complex on X . In [22, 8.4], elaborating on ideas he had sketched in his talk at the 1958 ICM, Grothendieck defines, for $L \in D^-(X, \mathcal{O}_X)$, the *homology groups* of X with coefficients in L as

$$H^{-i} R\mathrm{Hom}(L, K_X) = H^{-i}(X, R\mathcal{H}om(L, K_X)) = \mathrm{Ext}^{-i}(L, K_X).$$

This definition was later transposed in the settings of topological spaces (by Verdier, see [15, VI]) and étale cohomology (by Grothendieck in the oral seminar⁴ SGA 5). Since then it has often been abusively called *Borel–Moore homology*, by reference to [8], where no derived categories, a fortiori, no dualizing complexes appear. Grothendieck intended to write in Section 9 properties of variance for these groups, and use them to construct a theory of *cycle classes* in Hodge cohomology, satisfying compatibilities with cup-products and Gysin maps. As explained in [22, Introduction to Prénotes], this part was not written. In [16] El Zein discusses variants of these cycle classes in de Rham and Hodge homology groups.

In his letter to Serre of August 8, 1964 (see [10, p. 165], Grothendieck thus comments on the Woods Hole Lefschetz fixed point formula that Serre had described in the first section of his letter of August 2–3, 1964: “Le n. 1 ne m’excite guère, malgré les jolies applications ; le théorème des points fixes lui-même ne me semble pas plus qu’un exercice sur un air connu !” In [22], at the end of Commentaires, Grothendieck hinted that such fixed point formulas would follow from his duality formalism. It was at that time that Verdier was elaborating his general Lefschetz fixed point formula for cohomological correspondences in étale cohomology, later called the *Lefschetz–Verdier formula* [46] (see also [20, III]), relying on the étale analogue of the duality formalism of [23]. A variant of this formula in the coherent setting was later established in [20, III 6, Appendice], generalizing the Woods Hole formula. Hodge cohomology classes and residues of [23, III 9] play a crucial role in it.

⁴Written up later by Laumon [15, VIII] and Deligne [13, Cycle].

The duality formalism of [23] has inspired similar theories in many other contexts. A full description of them, or even of the state of the art today, would require a whole volume. I will list some of them below, with no references, as the literature is too large:

- It is by far the étale theory that has undergone the most extensive developments (up to today). In many respects, it has served as a model for the others.

For a historical study of Grothendieck’s work, ideas and reflections on the theme of duality in the light of [21], see L. Lafforgue’s essay [29].

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⁵“Formalisme des six opérations” was Grothendieck’s terminology in [21]. Nowadays, “six-functor formalism” is a preferred terminology.

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