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Research article
Algebraic geometry

A note on Azumaya algebras and one-forms

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Abstract. The crystalline differential operators on a smooth variety X give rise to a non-split Azumaya algebra over the cotangent bundle of the Frobenius twist X'. In some cases, this Azumaya algebra splits when restricted to finite covers of X'. In this short note, we show that, whenever X has a non-closed global one-form, there is a degree-1 cover of X' on which the Azumaya algebra does not split, answering a question of Sasha Petrov.

Keywords. Azumaya algebra, positive characteristics.

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1. The question

Throughout, k is an algebraically closed field of characteristic $p > 0$. Let X be a smooth proper variety over k, and let X' be its Frobenius twist. By Bezrukavnikov–Mirkovic–Rumynin [1, Theorem 2.2.3], the sheaf of crystalline differential operators on X defines an Azumaya algebra $\mathscr{A}_X$ on the cotangent bundle T^*X'.

In many useful cases, especially in the characteristic-p geometric Langlands and non-abelian Hodge Theory for curves [2,3,5], and for abelian varieties [6], this Azumaya algebra splits when restricted to spectral varieties; compare also the p-adic analogue [7]. Sasha Petrov asked the following question.

Question 1 ([11, Question 7.3(1)]). *Does there exist a smooth proper variety X together with a smooth subvariety $Z \subset T^*X'$, finite over X', such that $\mathscr{A}_X\big|_Z$ does not split?*

In this note, we answer this question affirmatively in every positive characteristic.

2. The answer

Let $X' := X \times_{k,F_k} k$ be the Frobenius twist. Write $F_X = F_{X/k} \colon X \to X'$ for the relative Frobenius, and $\pi_X \colon X' \to X$ for the projection. Milne's exact sequence, in the form used by Ogus–Vologodsky [10, (4.1.1)], is the following exact sequence of étale sheaves of abelian groups on X':

$$0 \longrightarrow \mathscr{O}_{X'}^* \longrightarrow F_{X,*}\mathscr{O}_X^* \xrightarrow{\mathrm{d}\log} F_{X,*}Z^1_{X/k} \xrightarrow{\pi_X^* - C_X} \Omega^1_{X'/k} \longrightarrow 0. \qquad (1)$$

Here $Z^1_{X/k} \subset \Omega^1_{X/k}$ is the sheaf of closed one-forms and C_X is the Cartier operator. The Yoneda Ext^2 class presented by (1) defines the obstruction homomorphism

$$\mathrm{ob}_{X'}\colon H^0(X',\Omega^1_{X'/k}) \longrightarrow H^2_{\text{ét}}(X',\mathscr{O}^*_{X'}). \tag{2}$$

Let $\mathscr{D}_{X/k}$ be the sheaf of crystalline differential operators on X. The center of $F_{X,*}\mathscr{D}_{X/k}$ is identified with $\mathrm{Sym}_{\mathscr{O}_{X'}} T_{X'/k}$, and $F_{X,*}\mathscr{D}_{X/k}$ has the structure of an Azumaya algebra on T^*X'. This is the algebra denoted $\mathscr{A}_X$ above.

Lemma 2. *Let X/k be smooth, and let $\omega \in H^0(X',\Omega^1_{X'/k})$. Let $i_\omega\colon X' \to T^*X'$ be the section corresponding to ω. Let $\Gamma_\omega := \mathrm{im}(i_\omega) \subset T^*X'$ be the graph. Then,*

$$[i^*_\omega \mathscr{A}_X] = \mathrm{ob}_{X'}(\omega) \quad \textit{in } H^2_{\text{ét}}(X',\mathscr{O}^*_{X'}).$$

In particular, $\mathscr{A}_X\big|_{\Gamma_\omega}$ splits if and only if $\mathrm{ob}_{X'}(\omega) = 0$.

Proof. Ogus–Vologodsky identify $\mathrm{ob}_{X'}(\omega)$ with the class of the $\mathbb{G}_m$-gerbe of line bundles on X with integrable connection and p-curvature ω [10, Proposition 4.2]. The same gerbe is the gerbe of splittings of the restriction of $\mathscr{A}_X$ to the section i_ω by [10, Remark 4.3]. □

The following proposition reduces Question 1 to finding a variety with non-closed global one-forms.

Proposition 3. *Let X/k be a smooth proper connected variety. If not every global one-form on X is closed, i.e. $H^0(X, Z^1_{X/k}) \subsetneq H^0(X,\Omega^1_{X/k})$, then the obstruction map $\mathrm{ob}_{X'}$ as in (2) is nonzero.*

Proof. Let $G := \mathrm{Pic}^{\nabla,\mathrm{rig}}_{X/k}$ be the rigidified Picard scheme of line bundles with integrable connection. Let $V := H^0(X',\Omega^1_{X'/k})$, viewed as a vector group. By [10, Proposition 4.11 and the paragraph following it], taking p-curvature defines a morphism of group schemes

$$\psi\colon G \longrightarrow V.$$

By [10, Proposition 4.2], $\mathrm{ob}_{X'}(\omega) = 0$ if and only if $\omega \in \psi(G(k))$. Therefore, it is enough to show that $\psi(G(k)) \neq V(k)$.

By Cartier descent,

$$\mathrm{Pic}_{X'/k} \longrightarrow G, \quad M \longmapsto (F^*_X M, \nabla_{\mathrm{can}}),$$

identifies $\mathrm{Pic}_{X'/k}$ with $\ker(\psi)$; see [8, Theorem 5.1]. Forgetting the connection fits into the exact sequence of fppf sheaves

$$0 \longrightarrow H^0(X, Z^1_{X/k}) \longrightarrow G \xrightarrow{b} \mathrm{Pic}_{X/k} \longrightarrow H^1(X, Z^1_{X/k}),$$

as in [10, Proposition 4.11]. In particular, if G^0 is the neutral component of G, then

$$\dim G^0 \leq h^0(X, Z^1_{X/k}) + \dim \mathrm{Pic}^0_{X/k}.$$

Let $\mathrm{Im}(\psi|_{(G^0)_{\mathrm{red}}})$ be the algebraic subgroup image. The quotient morphism

$$(G^0)_{\mathrm{red}} \longrightarrow \mathrm{Im}(\psi|_{(G^0)_{\mathrm{red}}})$$

is faithfully flat. Therefore

$$\dim \mathrm{Im}(\psi|_{(G^0)_{\mathrm{red}}}) = \dim (G^0)_{\mathrm{red}} - \dim \ker(\psi|_{(G^0)_{\mathrm{red}}}) \leq h^0(X, Z^1_{X/k}) < \dim V.$$

Finally, passing from G^0 to G only gives finitely many translates: let $G_1 = b^{-1}(\mathrm{Pic}^0_{X/k})$. Since G_1 is of finite type, G_1/G^0 is finite. Moreover, $G(k)\big/G_1(k)$ injects into $\mathrm{NS}(X)$, so its image under the map to $V/\psi(G_1(k))$ induced by ψ is finite: $\mathrm{NS}(X)$ is finitely generated, whereas $V(k)$ is killed by p. □

Theorem 4. *Let X/k be a smooth proper connected variety. If not every global one-form on X is closed, then there is a smooth subvariety $Z \subset T^*X'$, finite over X', such that $\mathscr{A}_X\big|_Z$ does not split. Moreover, the finite morphism $Z \to X'$ is an isomorphism.*

Proof. Combine Lemma 2 and Proposition 3, and take Z to be Γ_ω for some ω such that $\mathrm{ob}_{X'}(\omega) \neq 0$. □

Remarks 5 (Relation with liftability).

(a) If the Hodge-to-de Rham spectral sequence degenerates at E_1, then we must have that $H^0(X, Z^1_{X/k}) = H^0(X, \Omega^1_{X/k})$. Therefore, by Deligne–Illusie [4, Theorem 2.1 and Corollary 2.5], if a smooth proper X has a non-closed global one-form, then it does not lift to the ring of second Witt vectors $W_2(k)$. In particular, all examples in Section 2.1 do not lift to $W_2(k)$.

(b) Conversely, [11, Lemma 7.1] shows that, if a smooth proper X lifts to $W_2(k)$, then $\mathscr{A}_X$ splits on any closed subvariety $Z \subset T^* X'$ that is finite étale over X'. Such a Z is also $W_2(k)$-liftable.

(c) Furthermore, any smooth proper subvariety Z of $T^* X'$ on which $\mathscr{A}_X$ does not split must also have a non-closed global one-form, thus not $W_2(k)$-liftable. Indeed, this follows from the proof of [11, Lemma 7.1].

2.1. *The examples*

2.1.1. *Surface examples*

The existence of smooth projective surfaces with non-closed global one-forms goes back at least to Mumford [9, Corollary on p. 341]. The author would like to thank Sasha Petrov for pointing out this example.

In [12, Corollary 3.4 and the paragraph following it], Takeda constructs some generalized Raynaud surfaces that also have non-closed global one-forms. [12] is written under the assumption that $p > 2$, but one can check that these examples still hold when $p = 2$, see e.g. [13, Section 5] for more details.

2.1.2. *Higher-dimensional examples*

For higher-dimensional examples, we can take X to be any of the surfaces above and take $X \times \mathbf{P}^n_k$. Indeed, if $\eta \in H^0(X, \Omega^1_{X/k})$ is non-closed, then so is the pullback of η to $X \times \mathbf{P}^n_k$.

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Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

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