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# Chebyshev's bias for modular forms

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**Abstract.** We study Chebyshev's bias for the signs of Fourier coefficients of cuspidal newforms on  $\Gamma_0(N)$ . Our main result shows that the bias towards either sign is completely determined by the order of vanishing of the  $L$ -function  $L(s, f)$  at the central point of the critical strip. We then give several examples of modular forms where we explicitly compute the order of vanishing of  $L(s, f)$  at the central point and as a by-product, verify the super-positivity property, in the sense of Yun–Zhang (2017), for these examples.

**Keywords.** Chebyshev's bias, modular forms,  $L$ -functions.

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## 1. Introduction

Chebyshev's bias originally referred to the phenomenon that, even though the primes are equidistributed in the multiplicative residue classes mod 4, there seem to be more primes congruent to 3 mod 4 than 1 mod 4. This was first noted by Chebyshev in a letter to Fuss in 1853, and supported by extensive numerical evidence over the next few decades. However, a rigorous mathematical formulation of the bias remained out of reach for the next century, with many natural formulations running into stumbling blocks. For example, let  $\pi(x; q, a)$  denote the number of primes up to  $x$  congruent to  $a$  modulo  $q$  and let  $S = \{x \in \mathbb{R}_{\geq 2} : \pi(x; 4, 3) - \pi(x; 4, 1) > 0\}$ ; then Knapowski–Turán [15] conjectured that the proportion of positive real numbers lying in the set  $S$  would equal 1 as  $x \rightarrow \infty$ . However, this conjecture was later disproven by Kaczorowski [12] conditionally on the Generalised Riemann Hypothesis, by showing that the limit does not exist.

A breakthrough in this direction of formulating the bias came via the work of Rubinstein and Sarnak [24], who instead considered the logarithmic density

$$\delta(S) := \lim_{X \rightarrow \infty} \frac{1}{\log X} \cdot \int_{t \in S \cap [2, X]} \frac{dt}{t}$$

of  $S$ ; assuming the Generalised Riemann Hypothesis and that the non-negative imaginary parts of zeros of Dirichlet  $L$ -functions are linearly independent over  $\mathbb{Q}$ , they showed that this limit exists and  $\delta(S) = 0.9959\dots$ , hence giving a satisfactory explanation of Chebyshev's observations. The phenomenon of Chebyshev's bias has since been observed in various other situations; see, for instance, the expository article [21] by Mazur discussing Chebyshev's bias in the context of elliptic curves, as well as the annotated bibliography [20] by Martin et al. which contains a detailed overview of related literature on the subject.

In [1], Aoki–Koyama introduced a new framework of studying Chebyshev's bias which crucially relies on the Deep Riemann Hypothesis, a conjecture by Kurokawa et al. about the convergence of Euler products of  $L$ -functions on the critical line. We now briefly explain this conjecture in the setting of automorphic  $L$ -functions. Let  $\pi$  be an irreducible cuspidal automorphic representation of  $\mathrm{GL}_n$  over  $\mathbb{Q}$  with associated  $L$ -function  $L(s, \pi)$ . We can write  $L(s, \pi)$  as

$$L(s, \pi) = \prod_p \prod_{j=1}^n (1 - \alpha_{j,p} p^{-s})^{-1},$$

where, for the unramified primes  $p$ , the  $\alpha_{j,p}$ 's are the Satake parameters for the local representation  $\pi_p$  (see Section 2). We let  $v(\pi) = m(\mathrm{Sym}^2 \pi) - m(\wedge^2 \pi) \in \mathbb{Z}$ , where  $m(\rho)$  denotes the multiplicity of the trivial representation  $\mathbf{1}$  in  $\rho$ . We have  $v(\pi) = -\mathrm{ord}_{s=1} L_2(s, \pi)$ , where  $L_2(s, \pi)$  is the second moment  $L$ -function attached to  $L(s, \pi)$  defined via

$$L_2(s, \pi) = \prod_p \prod_{j=1}^n (1 - \alpha_{j,p}^2 p^{-s})^{-1}. \quad (1)$$

**Conjecture 1 (Kaneko–Koyama–Kurokawa [14]).** *Keep the assumptions and notation as above and assume that  $L(s, \pi)$  is entire. Let  $m = \mathrm{ord}_{s=1/2} L(s, \pi)$ . Then the limit*

$$\lim_{x \rightarrow \infty} \left( (\log x)^m \prod_{p \leq x} \prod_{j=1}^n (1 - \alpha_{j,p} p^{-\frac{1}{2}})^{-1} \right) \quad (2)$$

*satisfies the following conditions:*

- (A) *the limit (2) exists and is non-zero;*
- (B) *the limit (2) satisfies*

$$\lim_{x \rightarrow \infty} \left( (\log x)^m \prod_{p \leq x} \prod_{j=1}^n (1 - \alpha_{j,p} p^{-\frac{1}{2}})^{-1} \right) = \frac{\sqrt{2}^{v(\pi)}}{e^{m\gamma} m!} \cdot L^{(m)}\left(\frac{1}{2}, \pi\right).$$

The constant  $\frac{\sqrt{2}^{v(\pi)}}{e^{m\gamma}}$  in the Euler product asymptotic in Conjecture 1(B) was first discovered by Goldfeld [9] in the setting of  $L$ -functions of elliptic curves, while studying the original version of the Birch and Swinnerton–Dyer conjecture. Euler products on the critical line for general  $L$ -functions were subsequently studied in great depth by Conrad [3], whose results motivated the formulation of Conjecture 1.

Conjecture 1 implies the Generalised Riemann Hypothesis (GRH) for  $L(s, \pi)$ , and is deeper than GRH in a precise quantitative sense; we refer to [14] for more background on the conjecture. There are both theoretical considerations as well as extensive numerical evidence which support Conjecture 1. For instance, the function-field analogue of the conjecture is known [14, Theorem 5.2] and the second author showed in [29] that the GRH implies Conjecture 1 as  $x \rightarrow \infty$  outside a set of finite logarithmic measure.

We now recall the framework of Aoki–Koyama [1] for studying Chebyshev's bias using Conjecture 1.

**Definition 2 (Aoki–Koyama [1]).** *Let  $(c_p)_p \subseteq \mathbb{R}$  be a sequence over primes  $p$  such that*

$$\lim_{x \rightarrow \infty} \frac{\#\{p \mid c_p > 0, p \leq x\}}{\#\{p \mid c_p < 0, p \leq x\}} = 1.$$

We say that  $(c_p)_p$  has a Chebyshev's bias towards being positive (resp. negative) if there exists a positive (resp. negative) constant  $C$  such that

$$\sum_{p \leq x} \frac{c_p}{\sqrt{p}} \sim C \log \log x.$$

On the other hand, we say that  $c_p$  is unbiased if

$$\sum_{p \leq x} \frac{c_p}{\sqrt{p}} = O(1).$$

**Example 3 (Aoki–Koyama [1, Example 3.4]).** Let  $\chi_4$  denote the non-trivial Dirichlet character mod 4, so  $\chi_4(p) = -1$  if  $p \equiv 3 \pmod{4}$  and  $\chi_4(p) = 1$  if  $p \equiv 1 \pmod{4}$ . Assume Conjecture 1 for  $L(s, \chi_4)$ . Then there is a constant  $c$  such that

$$\sum_{p \leq x} \frac{\chi_4(p)}{\sqrt{p}} = -\frac{1}{2} \log \log x + c + o(1).$$

Thus, in the sense of Definition 2, there is a Chebyshev's bias towards primes which are 3 mod 4.

**Example 4 (Koyama–Kurokawa [16, Theorem 2]).** Let  $\tau(n)$  denote Ramanujan's tau function. Assume Conjecture 1 for  $L(s + \frac{11}{2}, \Delta)$ . Then there exists a constant  $c$  such that

$$\sum_{p \leq x} \frac{\tau(p)}{p^6} = \frac{1}{2} \log \log x + c + o(1).$$

Thus, in the sense of Definition 2, the sequence  $(\tau(p)p^{-\frac{11}{2}})_p$  has a Chebyshev's bias towards being positive.

We refer to [1, 13, 23] for further examples of Chebyshev's bias in the framework of Definition 2. The main theorem of this paper is to generalise Example 4 by establishing the desired asymptotic in Definition 2 when  $c_p = a_f(p)$  is the Fourier coefficient of an arbitrary cuspidal newform on  $\Gamma_0(N)$ . As we explain in Section 1.1 below, the signs of the Fourier coefficients  $a_f(p)$  are equidistributed as we range over all primes  $p$  (asymptotically half of them are positive and half of them are negative), so we are indeed in the setting of Definition 2 and it is thus natural to investigate a bias towards either sign.

### 1.1. Statement of the main result

Let  $f(z) = \sum_{n=1}^{\infty} a_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z} \in S_k(\Gamma_0(N))$  be a cusp form (normalised so that  $a_f(1) = 1$ ) with trivial nebentypus. The restriction to trivial nebentypus implies that  $a_f(n)$  is real for all  $n \geq 1$ . We recall that if  $f$  is also an eigenform for all of the Hecke operators and Atkin–Lehner involutions, then  $f$  is called a newform. In this paper, we assume that  $f$  is not a CM form, so there is no imaginary quadratic field  $K$  such that for  $p \nmid N$ ,  $p$  is inert in  $K$  if and only if  $a_f(p) = 0$ . Deligne's proof of the Weil conjectures implies that for each prime  $p$ , there exists an angle  $\theta_p \in [0, \pi]$  such that  $a_f(p) = 2 \cos \theta_p$ . The Sato–Tate conjecture, now proven by Barnet-Lamb, Geraghty, Harris and Taylor [2], asserts that if  $f$  is non-CM, then the sequence  $\{\theta_p\}$  is equidistributed in the interval  $[0, \pi]$  with respect to the measure  $d\mu_{\text{ST}} := (2/\pi) \sin^2 \theta d\theta$ . Equivalently, for an interval  $I \subseteq [0, \pi]$ , one has  $\{p \leq x : \theta_p \in I\} \sim \mu_{\text{ST}}(I) \pi(x)$  as  $x \rightarrow \infty$ . In particular, the Fourier coefficients  $a_f(p)$  satisfy the condition in Definition 2 (the set of primes  $p$  for which  $a_f(p) = 0$  has density zero by [27, Corollaire 2, p. 174]). Let  $\alpha_p = e^{i\theta_p}$  and  $\beta_p = e^{-i\theta_p}$  so that

$$a_f(p) = \alpha_p + \beta_p \quad \text{and} \quad \alpha_p \beta_p = 1. \quad (3)$$

The  $L$ -function of  $f$  is defined to be

$$L(s, f) := \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_p (1 - a_f(p)p^{-s} + p^{-2s})^{-1} = \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1},$$

which satisfies a functional equation relating  $s$  to  $1-s$  and admits an analytic continuation to the entire complex plane.

**Theorem 5.** *Let  $f \in S_k(\Gamma_0(N))$  be a cuspidal newform. Assume Conjecture 1 for  $L(s, f)$  and let  $m(f) = \text{ord}_{s=1/2} L(s, f)$ . Then there exists a constant  $c_f$  such that*

$$\sum_{p \leq x} \frac{a_f(p)}{\sqrt{p}} = \left( \frac{1}{2} - m(f) \right) \log \log x + c_f + o(1).$$

*In particular, in the sense of Definition 2, we conclude that*

- (1) *The sequence  $(a_f(p))_p$  has a Chebyshev's bias towards being positive if  $m(f) = 0$ .*
- (2) *The sequence  $(a_f(p))_p$  has a Chebyshev's bias towards being negative if  $m(f) \geq 1$ .*

**Remark 6.** We remark that Chebyshev's bias for the coefficients of Ramanujan's tau function, as well as for Frobenius traces of elliptic curves, had also been previously studied (from the point of view of the Rubinstein–Sarnak framework [24]) in Sarnak's letter to Mazur [26]; similar topics were also later studied in [5]. It is interesting to note that, by applying Proposition 8 below to [5, Theorem 2.1], GRH implies that the mean value of the limiting logarithmic distribution (in the sense of [5, Definition 1.4]) of the function  $E_f(x) = \frac{\log x}{\sqrt{x}} \sum_{p \leq x} a_f(p)$  is  $1 - 2m(f)$ , i.e., twice the coefficient of the leading term in Theorem 5 above.

To the best of our knowledge, Theorem 5 is the first result in the Aoki–Koyama framework which deals with the question of Chebyshev's bias for the signs of Fourier coefficients of arbitrary cuspidal newforms on  $\Gamma_0(N)$ . Another feature of Theorem 5, which is common with the other results in the Aoki–Koyama framework, is that it does not rely on the Linear Independence Hypothesis; we remark that using [29, Theorem 4.1], the asymptotic in Theorem 5 holds as  $x \rightarrow \infty$  outside a set of finite logarithmic measure assuming only GRH, and so we can in fact obtain a statement which captures the bias assuming only GRH. Since Theorem 5 shows that the bias is completely determined by  $m(f)$ , in Section 3 we explicitly compute  $m(f)$  for various examples of Hecke cusp forms for the group  $\text{SL}_2(\mathbb{Z})$ , thereby obtaining completely explicit versions of Theorem 5 for these examples.

## 2. Proof of Theorem 5

We prove Theorem 5 by applying a general asymptotic about Chebyshev's bias for Satake parameters of cuspidal automorphic representations, proven in [29, Theorem 4.1]. We begin by introducing the relevant definitions and notations.

Let  $\pi = \otimes' \pi_v$  be an irreducible cuspidal automorphic representation of  $\text{GL}_n(\mathbb{A}_{\mathbb{Q}})$ . Outside a finite set of places, for each finite place  $p$ ,  $\pi_p$  is unramified and we can associate to  $\pi_p$  a semisimple conjugacy class  $\{A_\pi(p)\}$  in  $\text{GL}_n(\mathbb{C})$ . Such a conjugacy class is parametrised by its eigenvalues  $\alpha_{1,p}, \dots, \alpha_{n,p}$ . The local Euler factors  $L_p(s, \pi_p)$  are given by

$$L_p(s, \pi_p) = \det(1 - A_\pi(p)p^{-s})^{-1} = \prod_{j=1}^n (1 - \alpha_{j,p}p^{-s})^{-1}.$$

At the ramified finite primes, the local factors are best described by the Langlands parameters of  $\pi_p$  (see for instance the appendix in [25]). They are of the form  $L_p(s, \pi_p) = P_p(p^{-s})^{-1}$ , where  $P_p(x)$  is a polynomial of degree at most  $n$ , and  $P_p(0) = 1$ . We will in this case too write the local factors in the form above, with the convention that we now allow some of the  $\alpha_{j,p}$ 's to be zero. The global  $L$ -function attached to  $\pi$  is given by

$$L(s, \pi) = \prod_p L_p(s, \pi_p) = \prod_p \prod_{j=1}^n (1 - \alpha_{j,p}p^{-s})^{-1}.$$

Up to finitely many local Euler factors, the Rankin–Selberg, symmetric square and exterior square  $L$ -functions are given by

$$\begin{aligned} L(s, \pi \otimes \pi) &\doteq \prod_p \prod_{i=1}^n \prod_{j=1}^n (1 - \alpha_{i,p} \alpha_{j,p} p^{-s})^{-1}, \\ L(s, \text{Sym}^2 \pi) &\doteq \prod_p \prod_{1 \leq i \leq j \leq n} (1 - \alpha_{i,p} \alpha_{j,p} p^{-s})^{-1} \\ \text{and} \quad L(s, \wedge^2 \pi) &\doteq \prod_p \prod_{1 \leq i < j \leq n} (1 - \alpha_{i,p} \alpha_{j,p} p^{-s})^{-1}. \end{aligned}$$

Here,  $\doteq$  means that equality holds up to multiplication by finitely many Euler factors at the ramified places, whose explicit description we omit; the equalities with  $\doteq$  are sufficient for our purposes since we will only be interested in the order of vanishing of the corresponding  $L$ -functions. From the Euler product expansions above, we see that

$$L(s, \pi \otimes \pi) \doteq L(s, \text{Sym}^2 \pi) L(s, \wedge^2 \pi). \quad (4)$$

On the other hand, the second moment  $L$ -function  $L_2(s, \pi)$  defined in (1) can be written as

$$L_2(s, \pi) \doteq \frac{L(s, \text{Sym}^2 \pi)}{L(s, \wedge^2 \pi)}.$$

As explained in [5, Example 1], there exists an open subset  $U \supseteq \{s \in \mathbb{C} : \text{Re}(s) \geq 1\}$  such that  $L_2(s, \pi)$  can be continued to a meromorphic function on  $U$ ; thus  $\text{ord}_{s=1} L_2(s, \pi)$  is well-defined.

**Theorem 7 (Chebyshev's bias for Satake parameters [29, Theorem 4.1]).** *Let  $\pi$  be an irreducible cuspidal automorphic representation of  $\text{GL}_n(\mathbb{A}_{\mathbb{Q}})$  such that  $L(s, \pi)$  is entire, let  $m = \text{ord}_{s=1/2} L(s, \pi)$  and let  $R(\pi) = \text{ord}_{s=1} L_2(s, \pi)$ . Assume Conjecture 1 for  $L(s, \pi)$ . Then there exists a constant  $c_{\pi}$  such that*

$$\text{Re} \left( \sum_{p \leq x} \frac{\alpha_{1,p} + \cdots + \alpha_{n,p}}{\sqrt{p}} \right) = \left( \frac{R(\pi)}{2} - m \right) \log \log x + c_{\pi} + o(1).$$

**Proof.** The asymptotic in the theorem follows by taking logarithms in Conjecture 1, applying a generalised version of Mertens' theorem [29, Lemma 3.6] and then taking real parts; we refer to the proof of [29, Theorem 4.1] for more details.  $\square$

**Proposition 8.** *Let  $f \in S_k(\Gamma_0(N))$  be a cuspidal newform. Then we have that  $R(\pi_f) = 1$ , where  $\pi_f$  denotes the automorphic representation generated by  $f$ .*

**Proof.** By [28, Theorem 1.1] we have that  $L(s, \text{Sym}^2 \pi_f)$  is non-vanishing on the line  $\text{Re}(s) = 1$ . Moreover,  $L(s, \pi_f \otimes \pi_f)$  has a simple pole at  $s = 1$  since  $\pi_f$  is self-dual (see for instance the proof of [5, Theorem 3.4]). On the other hand, we have by (3) that  $L(s, \wedge^2 \pi_f) \doteq \zeta(s)$ . It follows from (4) that  $\text{ord}_{s=1} L(s, \text{Sym}^2 \pi_f) = 0$  and so  $R(\pi_f) = \text{ord}_{s=1} L_2(s, \pi_f) = \text{ord}_{s=1} L(s, \text{Sym}^2 \pi_f) - \text{ord}_{s=1} L(s, \wedge^2 \pi_f) = 0 - (-1) = 1$ .  $\square$

**Remark 9.** If  $\pi$  is an arbitrary self-dual irreducible cuspidal automorphic representation, then  $R(\pi)$  can only equal  $\pm 1$ ; cf. the proof of [5, Theorem 3.4].

**Proof of Theorem 5.** This follows by combining (3), Theorem 7 and Proposition 8.  $\square$

### 3. Explicit examples

Since Theorem 5 shows that the bias is completely determined by  $m(f)$ , in this section we explicitly compute these quantities for various examples of cuspidal Hecke eigenforms for the group  $\text{SL}_2(\mathbb{Z})$ .

### 3.1. Vanishing caused by root number considerations

The completed  $L$ -function of  $f$  is defined by  $\Lambda(s, f) = (2\pi)^{-s - \frac{k-1}{2}} \Gamma(s + \frac{k-1}{2}) L(s, f)$  and satisfies the functional equation

$$\Lambda(s, f) = (-1)^{\frac{k}{2}} \Lambda(1-s, f). \quad (5)$$

#### 3.1.1. The $k \equiv 0 \pmod{4}$ case

When  $k \equiv 0 \pmod{4}$ , it has been conjectured that  $m(f) = 0$ ; see, for instance, the work of Conrey–Farmer [4] which introduces this conjecture and provides numerical evidence. As another piece of evidence, Luo [19] showed using the method of mollifiers that a positive proportion of Hecke eigenforms have non-vanishing central  $L$ -value as  $k \rightarrow \infty$  with  $k \equiv 0 \pmod{4}$ .

#### 3.1.2. The $k \equiv 2 \pmod{4}$ case

If  $k \equiv 2 \pmod{4}$ , then it follows from (5) that  $\Lambda(\frac{1}{2}, f) = 0$ ; thus,  $L(\frac{1}{2}, f) = 0$  as well and  $m(f) \geq 1$ . We thus obtain the following corollary.

**Corollary 10.** *Assume Conjecture 1. If  $f$  is a cuspidal Hecke eigenform for  $\mathrm{SL}_2(\mathbb{Z})$  of weight  $k \equiv 2 \pmod{4}$ , then the sequence  $(a_f(p))_p$  has a Chebyshev’s bias towards being negative.*

It is natural to investigate  $m(f)$  when  $k \equiv 2 \pmod{4}$  as well. To the best of our knowledge, a precise conjecture for this case has not been recorded in the literature, although one expects that  $m(f) = 1$  by the minimalist philosophy that  $L$ -functions do not have extra vanishing at the central point unless there is an underlying deeper geometric reason. Moreover, Liu [18] showed that a positive proportion of Hecke eigenforms satisfy  $L'(\frac{1}{2}, f) \neq 0$  as  $k \rightarrow \infty$  with  $k \equiv 2 \pmod{4}$ ; an explicit proportion was subsequently calculated by Jobrack [11]. In view of the above, we make the following conjecture.

**Conjecture 11.** *Let  $f$  be a cuspidal Hecke eigenform for the group  $\mathrm{SL}_2(\mathbb{Z})$  of weight  $k$ . If  $k \equiv 2 \pmod{4}$ , then  $m(f) = 1$ .*

**Remark 12.** The above discussion yields the following interpretation of Theorem 5: the Fourier coefficients  $a_f(p)$  have a bias towards being positive (resp. negative) when the root number of  $L(s, f)$  is positive (resp. negative).

### 3.2. Super-positivity

In this section, we recall the notion of super-positivity of  $L$ -functions; we shall see in Section 3.3 below how this notion is closely related to the problem of computing  $m(f)$  in certain examples. Super-positivity for self-dual automorphic  $L$ -functions was introduced by Yun–Zhang (see [30, Appendix B]) in their celebrated work on higher Gross–Zagier formulae over function fields, and asserts that the value of any derivative of the corresponding completed  $L$ -function at the central point must be non-negative. In the case of modular forms, this property is formulated as follows.

**Definition 13 (Super-positivity).** *Let  $f$  be a cuspidal Hecke eigenform for the group  $\mathrm{SL}_2(\mathbb{Z})$  with associated completed  $L$ -function  $\Lambda(s, f)$ . We say that  $\Lambda(s, f)$  satisfies the super-positivity property if  $\Lambda^{(m)}(\frac{1}{2}, f) \geq 0$  for all  $m \geq 0$ .*

Yun–Zhang show that GRH implies the super-positivity property [30, Theorem B.2]. In the next subsection, we verify super-positivity *unconditionally* for some explicit examples of modular forms (including Ramanujan’s Delta function). Our argument also computes  $m(f)$  in these examples, and we thus obtain a completely explicit version of Theorem 5 for these examples.

### 3.3. Mizumoto's positivity

To study Definition 13, we introduce another positivity property which was studied by Mizumoto in [22].

**Definition 14 (Mizumoto's positivity).** We say that a Hecke eigenform  $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$  satisfies Mizumoto's positivity if  $f(it) > 0$  for all  $t \in \mathbb{R}$  with  $t > 1$ .

**Example 15.** It follows from the product expansion  $\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$ , where  $q = e^{2\pi iz}$  and  $z \in \mathbb{H}$ , that Ramanujan's Delta function satisfies Mizumoto's positivity.

For  $k \geq 2$ , we let

$$E_{2k}(z) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

be the Eisenstein series of weight  $2k$  for the group  $\mathrm{SL}_2(\mathbb{Z})$ , where  $\sigma_t(n) := \sum_{d|n} d^t$  for  $t \in \mathbb{N}$  and the  $k$ -th Bernoulli number is defined via the formula  $\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$ . We recall that  $E_{2k}$  satisfies the automorphy relation  $E_{2k}\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} E_{2k}(z)$  for all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  and  $z \in \mathbb{H}$ . We refer to [17, Section 9.2], for instance, for more details on Ramanujan's Delta function and Eisenstein series.

**Proposition 16.** For all  $k \geq 2$ ,  $E_{2k}(z)$  satisfies Mizumoto's positivity.

**Proof.** Applying the automorphy relation using the matrix  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , we obtain  $E_{2k}\left(-\frac{1}{t}\right) = (-1)^k E_{2k}(i)$ . Thus if  $k = 2m + 1$  for some  $m \geq 1$ , then  $E_{2k}(i) = E_{4m+2}(i) = 0$ . Since  $B_{4m+2} > 0$ , it follows that  $E_{2k}(it) > E_{2k}(i) = 0$  for all  $t > 1$ . If  $k = 2m$  for some  $m \geq 1$ , then since  $B_{4m} < 0$  we have  $E_{2k}(it) = E_{4m}(it) = 1 - \frac{8m}{B_{4m}} \sum_{n=1}^{\infty} \sigma_{4m-1}(n) e^{-2\pi n t} > 0$  for all  $t > 1$ .  $\square$

For  $k \in \{12, 16, 20, 18, 22, 26\}$ , we define

$$\Delta_k(z) = \Delta(z) \cdot E_{k-12}(z) = \sum_{n=1}^{\infty} \tau_k(n) q^n,$$

where we use the convention that  $E_0 = 1$ . Since  $\dim_{\mathbb{C}} S_k(\mathrm{SL}_2(\mathbb{Z})) = 1$  for  $k \in \{12, 16, 20, 18, 22, 26\}$ , it follows that  $\Delta_k$  is a Hecke eigenform. By the main result of [6–8],  $\Delta_k$  for  $k \in \{12, 16, 20, 18, 22, 26\}$  are the only eigenforms for  $\mathrm{SL}_2(\mathbb{Z})$  that are the product of two eigenforms.

**Corollary 17.** For  $k \in \{12, 16, 20, 18, 22, 26\}$ ,  $\Delta_k$  satisfies Mizumoto's positivity.

**Proof.** This follows by combining Example 15 and Proposition 16.  $\square$

**Theorem 18.** The following hold true:

- (1) for  $k \in \{12, 16, 20, 18, 22, 26\}$ ,  $\Lambda(s, \Delta_k)$  satisfies the super-positivity property;
- (2) if  $k = 12, 16$  or  $20$ , then  $m(\Delta_k) = 0$ ;
- (3) if  $k = 18, 22$  or  $26$ , then  $m(\Delta_k) = 1$ .

**Proof.** Expanding the definition of  $\Lambda(s, \Delta_k)$ , making the substitution  $y \mapsto \frac{1}{y}$  and applying the automorphy relation for  $\Delta_k$  yields for  $\mathrm{Re}(s) > 1$  (cf. [17, Theorem 9.7])

$$\begin{aligned} \Lambda(s, \Delta_k) &= \int_0^{\infty} \left( \sum_{n=1}^{\infty} \tau_k(n) e^{-2\pi n y} \right) y^{s + \frac{k-1}{2} - 1} dy \\ &= \int_0^{\infty} \Delta_k(iy) y^{s + \frac{k-1}{2} - 1} dy \\ &= \int_0^1 \Delta_k(iy) y^{s + \frac{k-1}{2} - 1} dy + \int_1^{\infty} \Delta_k(iy) y^{s + \frac{k-1}{2} - 1} dy \\ &= \int_1^{\infty} \Delta_k\left(\frac{i}{y}\right) y^{-s - \frac{k-1}{2} - 1} dy + \int_1^{\infty} \Delta_k(iy) y^{s + \frac{k-1}{2} - 1} dy \end{aligned}$$

$$\begin{aligned}
&= (-1)^{\frac{k}{2}} \int_1^\infty \Delta_k(iy) y^{\frac{k-1}{2}-s} dy + \int_1^\infty \Delta_k(iy) y^{s+\frac{k-1}{2}-1} dy \\
&= \int_1^\infty \Delta_k(iy) \left( (-1)^{\frac{k}{2}} y^{\frac{k-1}{2}-s} + y^{s+\frac{k-3}{2}} \right) dy.
\end{aligned}$$

The last integral above converges for all  $s \in \mathbb{C}$  (since  $\Delta_k(iy)$  decays exponentially as  $y \rightarrow \infty$ ). We thus get

$$\Lambda^{(m)}\left(\frac{1}{2}, \Delta_k\right) = \left(1 + (-1)^{\frac{k}{2}+m}\right) \int_1^\infty \Delta_k(iy) (\log y)^m y^{\frac{k-2}{2}} dy. \quad (6)$$

The assertions (1), (2) and (3) follow by combining (6) with Corollary 17.  $\square$

**Corollary 19.** *Assume Conjecture 1 for  $L(s, \Delta_k)$ . We have that*

$$\sum_{p \leq x} \frac{\tau_k(p)}{p^{\frac{k}{2}}} = \begin{cases} \frac{1}{2} \log \log x + c_k + o(1) & \text{if } k = 12, 16, 20, \\ -\frac{1}{2} \log \log x + c_k + o(1) & \text{if } k = 18, 22, 26, \end{cases}$$

for some constant  $c_k$ . In particular, in the sense of Definition 2, the sequence  $(\tau_k(p)p^{-\frac{k-1}{2}})_p$  has a Chebyshev's bias towards being positive if  $k \in \{12, 16, 20\}$  and a Chebyshev's bias towards being negative if  $k \in \{18, 22, 26\}$ .

**Proof.** This follows by combining Theorems 5 and 18.  $\square$

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## Appendix A. Modular forms with real Fourier coefficients

**Theorem 20.** *Let  $f \in S_k(\Gamma_0(N), \chi)$  be a newform with  $a_f(n) \in \mathbb{R}$  for all  $n \geq 1$ . Then either  $\chi$  is trivial or  $f$  is a CM form.*

**Proof.** By [10, Equation (6.57)], we have that  $\overline{a_f(n)} = \chi(n)a_f(n)$  for all  $n$  with  $(n, N) = 1$ . Thus, if  $a_f(n) \in \mathbb{R}$  for all  $n \geq 1$ , then either  $\chi$  is trivial or  $a_f(n) = 0$  whenever  $\chi(n) \neq 1$ . In the latter case, we must have  $a_f(p) = 0$  for a set of primes having non-zero density. This implies, by a result of Serre [27, Corollaire 2, p. 174], that  $f$  is a CM form.  $\square$

Chebyshev's bias in the framework of Definition 2 requires all the Fourier coefficients of the newform to be real; Theorem 20 implies that any such form must have trivial nebentypus or must be a CM form. On the other hand, the properties of Fourier coefficients of CM forms differ from those of non-CM forms since, as explained in Section 1.1, they satisfy  $a_f(p) = 0$  for a set of primes having density  $1/2$  unlike the case of non-CM forms for which such a set of primes has density zero. Moreover,  $L$ -functions associated to CM forms are  $L$ -functions associated to

Hecke characters of imaginary quadratic fields for which Proposition 8 need not hold (since the corresponding exterior square  $L$ -function may be entire, cf. the proof of [5, Theorem 3.7]). These observations justify why we restrict to non-CM forms with trivial nebentypus in the paper.

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