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On the stochastic proof of the Blaschke–Santaló inequality

Joseph Lehec[a]

[a] Université de Poitiers, CNRS, LMA, France
E-mail: joseph.lehec@univ-poitiers.fr

Abstract. In 2024, Courtade, Fathi and Mikulincer gave a proof of the symmetrized Talagrand inequality based on stochastic calculus, in the spirit of Borell's proof of the Prékopa–Leindler inequality. The symmetrized Talagrand inequality can be seen as a dual form of the functional Santaló inequality. The modest purpose of this note is to give a simplified version of the Courtade, Fathi and Mikulincer argument. Namely we first recall briefly Borell's original argument, and we then explain a simple twist in his proof that allows to recover the functional Santaló inequality directly, rather than in its dual form.

Keywords. Functional inequalities, Blaschke–Santaló inequality, Brownian motion.

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Introduction

Given a convex body K in $\mathbb{R}^n$, namely a compact convex set with non empty interior, we let $K^\circ = \left\{y \in \mathbb{R}^n\colon x\cdot y \le 1,\ \forall\ x \in K\right\}$ be its polar body. The volume product of K is the quantity $|K|\cdot|K^\circ|$ where $|\cdot|$ denotes the Lebesgue measure. The Blaschke–Santaló inequality is a central inequality in the asymptotic theory of convex bodies. It asserts that among convex bodies having their barycenter at 0, the volume product is maximal when K is the Euclidean ball (which is the unique fixed point of the map $K \mapsto K^\circ$). This was proved by Blaschke in dimension 3 and Santaló [22] in the general case, using variational methods. More recent proofs [16,21] rely on symmetrization arguments. In this paper we shall discuss the following functional version of the Blaschke–Santaló inequality.

Theorem 1 (Functional Santaló inequality). *Let f, g be non-negative measurable functions on $\mathbb{R}^n$ satisfying*

$$f(x)g(y) \le \mathrm{e}^{-x\cdot y}, \quad \forall\ x,y \in \mathbb{R}^n. \tag{1}$$

If f (or g) has its barycenter at 0, namely if $\int_{\mathbb{R}^n} x f(x)\,\mathrm{d}x = 0$, then

$$\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x \int_{\mathbb{R}^n} g(y)\,\mathrm{d}y \le (2\pi)^n.$$

This implies in particular that given a convex function h such that e^{-h} has its barycenter at 0, and letting h^* be the Legendre transform of h, the *functional volume product* $\int \mathrm{e}^{-h} \cdot \int \mathrm{e}^{-h^*}$ is maximal when $h(x) = |x|^2/2$ (and again this is the unique fixed point of the Legendre transform). The main point of the functional version is that it allows to recover the set version easily, by applying it to $f = \exp\big(-\|\cdot\|_K^2/2\big)$ where $\|\cdot\|_K$ is the gauge function of K. The functional inequality was first proved by Ball [2] under the additional assumption that f is even. Theorem 1 is essentially due to Artstein, Klartag and Milman [1], with the caveat that in their paper the function having its barycenter at 0 should be log-concave. These works relied on the geometric version of the inequality, by applying to level sets of the function f. The first direct proof, based on some induction on the dimension, is due to the author in [11]. See also [12] where a more general inequality, originally due to Fradelizi and Meyer [9], is established. Lastly, the functional Santaló inequality admits a dual formulation, which is an improved form of Talagrand's transport inequality for measures having their barycenter at 0, see [8] for a precise formulation.

Recently, Nakamura and Tsuji [18] gave a new proof of the functional Santaló inequality, based on a semigroup argument. Namely, they proved that when f is even the expression $\int_{\mathbb{R}^n} f^\circ(y)\,\mathrm{d}y$ is monotone increasing along the Fokker–Planck semigroup. Here f° denotes the polar function of f, namely the largest function g such that (1) is satisfied. This recovers Santaló (for even functions) by sending time to $+\infty$. See also [6], in which the argument is somewhat simplified.

In 2024, Courtade, Fathi and Mikulincer [7] found a proof based on stochastic calculus of the dual version of the functional Santaló inequality (the improved Talagrand inequality). Stochastic proofs of functional inequalities seem to have started with Borell's seminal paper [5] on the Prékopa–Leindler inequality, which is a functional counterpart of the Brunn–Minkowski inequality. The modest purpose of this note is to give a slightly simplified version of the Courtade, Fathi, and Mikulincer argument. In particular we prove the genuine functional Santaló inequality, rather than its dual form, which makes the connection with Borell's work more apparent.

1. Borell's formula

Throughout this section we are given a standard Brownian motion on $\mathbb{R}^n$, denoted $(B_t)_{t\in[0,1]}$ (all our processes will be indexed by the finite time interval $[0,1]$). In this context we call *drift* a progressively measurable process (u_t) such that

$$\mathbb{E}\int_0^1 |u_t|^2\,\mathrm{d}t < +\infty.$$

The standard Gaussian measure is denoted γ_n. Borell's proof of the Prékopa–Leindler inequality relies on the following representation formula.

Lemma 2 (Borell's formula). *Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ be measurable and bounded from above. Then*

$$\log\left(\int_{\mathbb{R}^n} \mathrm{e}^{\varphi}\,\mathrm{d}\gamma_n\right) = \sup\left\{\mathbb{E}\left[\varphi\left(B_1 + \int_0^1 u_t\,\mathrm{d}t\right) - \frac{1}{2}\int_0^1 |u_t|^2\,\mathrm{d}t\right]\right\}$$

where the supremum is taken over all drifts (u_t).

In [5] Borell proves this formula and gives a nice application to the Prékopa–Leindler inequality, to which we shall come back below. A dual version is given in our previous work [13] together with further applications to functional inequalities. A byproduct of the later version is the following additional information on the optimal drift, which we will need later on.

Lemma 3. *The supremum in Borell's formula is attained. Moreover the optimal drift (u_t) has constant expectation, equal to the barycenter of* e^{φ}*:*

$$\mathbb{E}u_t = \frac{\int_{\mathbb{R}^n} x\mathrm{e}^{\varphi(x)}\,\gamma_n(\mathrm{d}x)}{\int_{\mathbb{R}^n} \mathrm{e}^{\varphi(x)}\,\gamma_n(\mathrm{d}x)}, \quad \forall\, t\in[0,1].$$

2. Property (τ)

Property (τ) is a concentration inequality introduced by Maurey in [15]. It was mainly motivated by some delicate isoperimetric inequality for the product of symmetric exponential measures in dimension n, but here we only need the Gaussian version of the inequality, which states as follows.

Lemma 4 (Property (τ) for the Gaussian measure). *Suppose $\varphi, \psi \colon \mathbb{R}^n \to \mathbb{R}$ satisfy*

$$\varphi(x) + \psi(y) \leq \frac{1}{4}|x-y|^2, \quad \forall\, x, y \in \mathbb{R}^n. \tag{2}$$

Then

$$\left(\int_{\mathbb{R}^n} \mathrm{e}^{\varphi}\,\mathrm{d}\gamma_n\right)\left(\int_{\mathbb{R}^n} \mathrm{e}^{\psi}\,\mathrm{d}\gamma_n\right) \leq 1. \tag{3}$$

Proof. Let us specialize Borell's proof of the Prékopa–Leindler inequality, of which the Gaussian property (τ) is a particular case. By monotone convergence it is certainly enough to prove the result for functions φ and ψ which are bounded from above. Let (B_t) be a standard Brownian motion on $\mathbb{R}^n$ and let (u_t) and (v_t) be two drifts. Using the hypothesis, the Cauchy–Schwarz inequality, and the convexity of the Euclidean norm squared we get

$$\begin{aligned}\varphi\left(B_1 + \int_0^1 u_t\,\mathrm{d}t\right) + \psi\left(B_1 + \int_0^1 v_t\,\mathrm{d}t\right) &\leq \frac{1}{4}\left|\int_0^1 u_t - v_t\,\mathrm{d}t\right|^2 \\ &\leq \frac{1}{4}\int_0^1 |u_t - v_t|^2\,\mathrm{d}t \\ &\leq \frac{1}{2}\int_0^1 |u_t|^2\,\mathrm{d}t + \frac{1}{2}\int_0^1 |v_t|^2\,\mathrm{d}t.\end{aligned}$$

Taking expectation and then the supremum over (u_t) and (v_t) yields the desired inequality by Borell's formula. □

The constant $1/4$ in the hypothesis (2) is largest possible. This can be seen by taking a linear function for φ and an appropriate quadratic function for ψ. However, as pointed out in [1], the functional Santaló inequality amounts to saying that under an additional centering condition, this constant can be replaced by $1/2$. More precisely setting $f(x) = \mathrm{e}^{\varphi(x)+|x|^2/2}$ in the functional Santaló inequality (and similarly for g) leads to the following statement.

Theorem 5 (Reformulation of the functional Santaló inequality). *Suppose $\varphi, \psi \colon \mathbb{R}^n \to \mathbb{R}$ satisfy*

$$\varphi(x) + \psi(y) \leq \frac{1}{2}|x-y|^2, \quad \forall\, x, y \in \mathbb{R}^n \tag{4}$$

and assume additionally that

$$\int_{\mathbb{R}^n} x\mathrm{e}^{\varphi(x)}\,\gamma_n(\mathrm{d}x) = 0.$$

Then the inequality (3) *holds true.*

Again the constant $1/2$ in (4) cannot be improved. In [10] a direct proof of this inequality is given under the stronger assumption that φ is even, using the Poincaré inequality for even functions in Gauss space and some symmetrization argument.

As we mentioned earlier these inequalities admit dual formulations in terms of transport/entropy inequality. In particular, Theorem 5 can be reformulated as an improved Talagrand inequality under a certain centering condition. Again we refer to [8] for the details. This dual version is the approach taken by Courtade, Fathi and Mikulincer. Here we stick to the direct version, and we show that the stochastic method of Borell allows to recover the full statement quite easily.

3. Stochastic proof of Santaló

We prove the second formulation, in terms of property (τ). We need to leverage the barycenter assumption so as to gain a factor $1/2$ in Borell's argument. Let (u_t) be the optimal drift in Borell's formula applied to the first function φ, and with (B_t) as driving Brownian motion. By Lemma 3 and the barycenter assumption we have $\mathbb{E}u_t = 0$ for all $t \in [0,1]$. The main trick, which we borrow from the Courtade, Fathi, Mikulincer paper mentioned above, is to use a different Brownian motion for the second function, namely the process $(\widehat{B}_t)$ given by $\widehat{B}_t = B_1 - B_{1-t}$ for all $t \in [0,1]$. Observe that $(\widehat{B}_t)$ is also a standard Brownian motion, and that $\widehat{B}_1 = B_1$. Now let $(\widehat{v}_t)$ be any drift with respect to the reversed process $(\widehat{B}_t)$. Since $\widehat{B}_1 = B_1$ we get from the hypothesis (4)

$$\varphi\left(B_1 + \int_0^1 u_t \,\mathrm{d}t\right) + \psi\left(\widehat{B}_1 + \int_0^1 \widehat{v}_t \,\mathrm{d}t\right) \leq \frac{1}{2}\left|\int_0^1 u_t \,\mathrm{d}t - \int_0^1 \widehat{v}_t \,\mathrm{d}t\right|^2. \tag{5}$$

Reversing time in the last integral and applying Cauchy–Schwarz we obtain

$$\left|\int_0^1 u_t \,\mathrm{d}t - \int_0^1 \widehat{v}_t \,\mathrm{d}t\right|^2 = \left|\int_0^1 u_t \,\mathrm{d}t - \int_0^1 \widehat{v}_{1-t} \,\mathrm{d}t\right|^2 \leq \int_0^1 |u_t - \widehat{v}_{1-t}|^2 \,\mathrm{d}t. \tag{6}$$

Let $(\mathscr{F}_t)$ and $(\widehat{\mathscr{F}}_t)$ be the natural filtrations of (B_t) and $(\widehat{B}_t)$ respectively. From the independence of the Brownian increments it is easily seen that for any fixed $t \in [0,1]$ the σ-fields $\mathscr{F}_t$ and $\widehat{\mathscr{F}}_{1-t}$ are independent. As a result u_t and $\widehat{v}_{1-t}$ are independent. Since the drift (u_t) has expectation 0 for all time we obtain

$$\mathbb{E}\langle u_t, \widehat{v}_{1-t}\rangle = \langle \mathbb{E}u_t, \mathbb{E}\widehat{v}_{1-t}\rangle = 0, \quad \forall\, t \in [0,1]. \tag{7}$$

Combining together (5), (6) and (7) yields

$$\mathbb{E}\left[\varphi\left(B_1 + \int_0^1 u_t \,\mathrm{d}t\right) + \psi\left(\widehat{B}_1 + \int_0^1 \widehat{v}_t \,\mathrm{d}t\right)\right] \leq \mathbb{E}\left[\frac{1}{2}\int_0^1 |u_t|^2 \,\mathrm{d}t + \frac{1}{2}\int_0^1 |\widehat{v}_t|^2 \,\mathrm{d}t\right].$$

Recalling that (u_t) is the optimal drift for φ and taking the supremum in $(\widehat{v}_t)$ yields the result.

4. Concluding remarks

4.1. *Equality cases*

It is known (see e.g. [1]) that there is equality in Theorem 1 if and only if

$$f(x) = c \cdot \mathrm{e}^{-\frac{1}{2}\langle Ax,x\rangle} \quad \text{and} \quad g(x) = \frac{1}{c} \cdot \mathrm{e}^{-\frac{1}{2}\langle A^{-1}x,x\rangle} \tag{8}$$

for almost every x, and for some positive definite matrix A, and some positive constant c. It is possible to derive this equality case from our proof. We sketch the argument below, omitting the technical details. If there is equality in Theorem 5, and if (u_t) and $(\widehat{v}_t)$ are the optimal drifts for φ and ψ, with driving Brownian motions (B_t) and $(\widehat{B}_t)$ respectively, then we must have equality in (6). This implies that the map $t \mapsto u_t - \widehat{v}_{1-t}$ must be constant, almost surely. In particular, for every $s \leq t$, the random vector $u_t - u_s = \widehat{v}_{1-t} - \widehat{v}_{1-s}$ is measurable with respect to $\widehat{\mathscr{F}}_{1-s}$, hence independent of $\mathscr{F}_s$. So the drift (u_t) must have constant expectation, equal to 0, and independent increments. This implies that (u_t) is a square integrable martingale with deterministic quadratic variation (in particular (u_t) must be a Gaussian process), and similarly for $(\widehat{v}_t)$. By plugging this information into the explicit form of the optimal drift in Borell's formula (from [5] or [13]) we then seen that this can only happen if the functions φ and ψ are quadratic forms (plus maybe a constant). The result then follows from an explicit computation.

4.2. *Improved Brascamp–Lieb inequalities for centered functions*

In the aforementioned article [18], Nakamura and Tsuji obtain the functional Santaló inequality as a limit case of a certain family of functional inequalities. More precisely they establish an improved form of the reversed hypercontractivity inequality of Borell [4] for even functions. Moreover, they pushed this line of work further in [19], in which they show that functions satisfying suitable centering conditions satisfy stronger forms of the reversed Brascamp–Lieb inequalities (of which reversed hypercontractivity is a particular case) than generic functions. See [3] and the references therein for the background on reversed Brascamp–Lieb inequalities. Besides, it was shown by E. Milman [17] that these improved reversed Brascamp–Lieb inequalities contain the celebrated Gaussian correlation inequality of Royen [20] as a special case.

The Brascamp–Lieb inequalities and their reversed forms are known to be amenable to the stochastic approach initiated by Borell, see [14]. It is natural to ask whether the centered versions discovered by Nakamura and Tsuji could also be proven this way. Unfortunately, we were unable to complete this task, and we leave it as an open problem.

Declaration of interests

The author does not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and has declared no affiliations other than their research organizations.

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