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A van der Waerden-free proof of Rado's theorem

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Abstract. We present a proof of the sufficiency of Rado's condition for the partition regularity of linear Diophantine equations that avoids any use of van der Waerden's theorem. The proof is based on fundamental properties that are common knowledge in combinatorics of numbers and is entirely “elementary”.

Keywords. Rado's condition, van der Waerden, arithmetic Ramsey theory.

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Introduction

A fundamental result in arithmetic Ramsey theory is *Rado's theorem* that characterizes the linear Diophantine equations $\sum_{i=1}^n c_i x_i = 0$ that are partition regular as those where a partial sum of the coefficients equals zero. This latter property is known as “Rado's condition”. The original proof of Rado's theorem is grounded on the observation that if Rado's condition holds, then configurations made of long enough arithmetic progressions and appropriate multiples of the common difference contain solutions to $\sum_{i=1}^n a_i x_i = 0$; to complete the proof, one then applies a strengthening of van der Waerden's theorem about the existence of arbitrarily long monochromatic arithmetic solutions. To the best of our knowledge, no elementary proof of the sufficiency of Rado's condition avoiding any use of strengthenings of van der Waerden's theorem has ever been given, where by “elementary proof” we mean a proof that does not appeal to any advanced machinery coming from other mathematical areas and that are often used in arithmetic number theory, such as ergodic theory, discrete Fourier analysis, algebra on the space of ultrafilters, or nonstandard analysis. This paper is aimed at filling this void.

We aim to keep this paper self-contained, so in the first two sections we recall the few notions and properties that we need to give our proof of Rado's theorem, presented in Section 3. In the following Section 4 we develop the “elementary approach” and give an alternative proof of a result about the possibility of combining partition regularity of different equations, that have been previously proved by using algebra in the space of ultrafilters; as an example of application

we give a second elementary proof of Rado's theorem. In the final Section 5 we outline possible generalizations of our work, and pose two open problems.

1. Terminology

Let us now recall the basic notions and fix terminology.

Definition 1. A finite coloring of a set X is a finite partition $X = C_1 \cup \dots \cup C_r$. The pieces C_i of the partition are called colors.

Given a finite coloring as above, a set F is called monochromatic if all elements of F have the same color, i.e. $F \subseteq C_i$ for some i .

Definition 2. A family $\mathcal{F} \subseteq \mathcal{P}(X)$ is called partition regular on X if for every finite coloring of X there exists an element $F \in \mathcal{F}$ which is monochromatic.

A family $\mathcal{F}$ is called strongly partition regular if for every $A \in \mathcal{F}$ and for every coloring $A = C_1 \cup \dots \cup C_r$, one of the colors $C_i \in \mathcal{F}$.

In what follows, we will be interested in the case where $X = \mathbb{N}$ is the set of natural numbers; we agree that $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ denotes the set of non-negative integers.

Historically, the first example of a partition regular property involving equations was provided by a result of Schur [10] that appeared in 1916.

Theorem 3 (Schur). For every finite coloring of $\mathbb{N} = C_1 \cup \dots \cup C_r$ there exists a monochromatic “Schur triple” $a, b, a + b \in C_i$.

A fundamental result in this area was then proved by van der Waerden [12] in 1927. With an ingenious and rather intricate double induction procedure, he proved the following.¹

Theorem 4 (van der Waerden). For every $\ell \in \mathbb{N}$ and for every finite coloring of $\mathbb{N} = C_1 \cup \dots \cup C_r$ there exists a monochromatic ℓ -term arithmetic progression $a, a + d, \dots, a + (\ell - 1)d \in C_i$. Consequently, one color contains arbitrarily long arithmetic progressions.

A strengthening of van der Waerden's result, that includes Schur's theorem as a particular case, allows also a fixed multiple of the common distance of the arithmetic progression to be taken of the same color.²

Theorem 5. For every $\ell, s \in \mathbb{N}$ and for every coloring of $\mathbb{N} = C_1 \cup \dots \cup C_r$ there exists a monochromatic ℓ -term arithmetic progression $a, a + d, \dots, a + (\ell - 1)d \in C_i$ so that also $sd \in C_i$.

We remark that also the multiplicative versions of the above results hold, where additive Schur's triples are replaced by multiplicative triples “ a, b, ab ”, where arithmetic progressions are replaced by geometric progressions $a, ad, ad^2, \dots, ad^{\ell-1}$, and where multiples of the common difference are replaced by powers of the common ratio, respectively.

Since its publication, van der Waerden's result has become one of the cornerstones of arithmetic Ramsey theory, and a fundamental tool that is found in many proofs. A typical example is given by Rado's theorem (see below) about partition regular equations.

Definition 6. Let $f: \mathbb{N}^k \rightarrow \mathbb{N}_0$. We say that the equation “ $f(x_1, \dots, x_k) = 0$ ” is partition regular on $\mathbb{N}$ to mean that the family of solutions $\mathcal{F}_f = \{\{n_1, \dots, n_k\} \in [\mathbb{N}]^k \mid f(n_1, \dots, n_k) = 0\}$ is partition regular on $\mathbb{N}$. (This means that for every coloring $\mathbb{N} = C_1 \cup \dots \cup C_r$ there exist monochromatic $n_1, \dots, n_k \in C_i$ which are a solution $f(n_1, \dots, n_k) = 0$.)

¹A detailed proof can be found e.g. in [5, Section 2.1].

²A detailed proof can be found e.g. in [5, Section 2.2].

In this case, by abusing terminology, sometimes we will directly say the function f is partition regular on $\mathbb{N}$.

Partition regularity can be seen as a property that indicates the existence of a large and “well-spread” set of solutions, in the sense that in every given finite partition of the natural numbers one always finds elements of the same color that are a solution.

Note that Schur’s theorem states the partition regularity of the equation $x_1 + x_2 - x_3 = 0$. A few years later, in 1933 R. Rado [9] found a simple full characterization of all linear Diophantine equations that are partition regular.³

Theorem 7 (Rado). *Let $a_1, \dots, a_n \in \mathbb{Z} \setminus \{0\}$. Then the following are equivalent:*

- (1) *the equation $\sum_{i=1}^n a_i x_i = 0$ is partition regular on $\mathbb{N}$;*
- (2) *the following “Rado’s condition” holds:*

There exists a nonempty $I \subseteq \{1, \dots, n\}$ such that $\sum_{i \in I} a_i = 0$.

The necessity of Rado’s condition can be proven directly via elementary arithmetic arguments; see for instance [5, Section 3.2] for details. As for the converse implication, the original proof is based on the observation that a given linear equation $\sum_{i=1}^n a_i x_i = 0$ that satisfies Rado’s condition can be solved, for appropriate s, ℓ , inside sets of the form $\{a, sd, a + d, \dots, a + \ell d\}$; the partition regularity property then follows by an application of Theorem 5. Our goal is to provide an elementary proof of the sufficiency of Rado’s condition that avoids the use of van der Waerden’s theorem or its strengthenings.

2. Preliminaries

In this section we will recall all the needed basic notions and properties involved in our proof of the sufficiency of Rado’s condition. In order to make this paper self-contained, and to show that all the arguments we need are indeed elementary, we will also include most of the proofs of these properties.

For $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}_0$, the *leftward additive shift* of A by n and the *rightward additive shift* by n are the following sets, respectively:

$$A - n = \{m \in \mathbb{N} \mid n + m \in A\} \quad \text{and} \quad n + A = \{n + m \in \mathbb{N} \mid m \in A\}.$$

The *leftward* and *rightward multiplicative shift* are defined similarly:

$$A/n = \{m \in \mathbb{N} \mid n \cdot m \in A\} \quad \text{and} \quad n \cdot A = \{n \cdot m \mid m \in A\}.$$

Shifts can be used to introduce several elemental notions in combinatorics of numbers, including that of Banach density, sometimes called “uniform density”.

Definition 8. *Let $A \subseteq \mathbb{N}$. The upper Banach density (or simply the Banach density) of A is defined by setting:*

$$\text{BD}(A) = \limsup_{n \rightarrow \infty} \left(\max_{x \in \mathbb{N}} \frac{|(A - x) \cap [1, n]|}{n} \right) = \limsup_{n \rightarrow \infty} \left(\max_{x \in \mathbb{N}} \frac{|A \cap [x + 1, x + n]|}{n} \right).$$

The above limit superior can be proved to be an actual limit. Indeed, observe that the sequence $a_n := \max_{x \in \mathbb{N}} |(A - x) \cap [1, n]|$ is subadditive, i.e., $a_{n+m} \leq a_n + a_m$ for all $n < m$; then by Fekete’s lemma one has that $(a_n \mid n \in \mathbb{N})$ converges and $\lim_{n \rightarrow \infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n}$. So we have:

$$\text{BD}(A) = \lim_{n \rightarrow \infty} \left(\max_{x \in \mathbb{N}} \frac{|(A - x) \cap [1, n]|}{n} \right) = \inf_{n \in \mathbb{N}} \left(\max_{x \in \mathbb{N}} \frac{|(A - x) \cap [1, n]|}{n} \right).$$

³A detailed historical account of the early years of Ramsey theory, and in particular the discovery of the theorems of Schur, van der Waerden and Rado, can be found in [11].

Remark 9. A more commonly used notion of density is that of *upper asymptotic density*, which is defined by setting:

$$\overline{d}(A) = \limsup_{n \rightarrow \infty} \frac{|A \cap [1, n]|}{n}.$$

Note that $\text{BD}(A) \geq \overline{d}(A)$ since in Banach density one considers arbitrary intervals instead of just initial segments $[1, n]$. It is not difficult to find examples of sets A such that $\overline{d}(A) = 0$ but $\text{BD}(A) = 1$.

Clearly $0 \leq \text{BD}(A) \leq 1$. Note that Banach density is invariant under additive shifts; besides, multiplicative shifts preserve non-zero density.

Proposition 10. *Let $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$. Then:*

- (1) $\text{BD}(A - n) = \text{BD}(n + A) = \text{BD}(A)$;
- (2) $\text{BD}(n \cdot A) = \frac{\text{BD}(A)}{n}$.

It is readily seen that $\text{BD}(A \cup B) \leq \text{BD}(A) + \text{BD}(B)$ for all $A, B \subseteq \mathbb{N}$.

Banach density is not additive; indeed one easily finds examples of disjoint sets $A \cap B = \emptyset$ such that $\text{BD}(A \cup B) < \text{BD}(A) + \text{BD}(B)$. However, it behaves like a finitely additive measure in the case of families of pairwise disjoint shifts of a given set.

Proposition 11. *Let $A \subseteq \mathbb{N}$ and let $x_1 < \dots < x_k$. If $(A - x_i) \cap (A - x_j) = \emptyset$ for all $i \neq j$ then $\text{BD}(\bigcup_{i=1}^k (A - x_i)) = k \cdot \text{BD}(A)$.*

Proof. Let us prove the property when $k = 2$; the general case is similar. For every n , pick an interval I_n of length n such that $a_n := |A \cap I_n| = \max_{x \in \mathbb{N}} |(A - x) \cap [1, n]|$, so that $\text{BD}(A) = \lim_{n \rightarrow \infty} \frac{a_n}{n}$. Let ε_n, η_n be such that $a_n = |(A - x_1) \cap I_n| + \varepsilon_n = |(A - x_2) \cap I_n| + \eta_n$. Clearly $|\varepsilon_n| \leq x_1$ and $|\eta_n| \leq x_2$, so $\lim_{n \rightarrow \infty} \frac{\varepsilon_n + \eta_n}{n} = 0$. We have the following chain of inequalities:

$$\begin{aligned} 2 \cdot \text{BD}(A) &= \lim_{n \rightarrow \infty} \frac{a_n + a_n}{n} \\ &= \lim_{n \rightarrow \infty} \left(\frac{|(A - x_1) \cap I_n|}{n} + \frac{|(A - x_2) \cap I_n|}{n} + \frac{\varepsilon_n + \eta_n}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{|((A - x_1) \cup (A - x_2)) \cap I_n|}{n} \\ &\leq \text{BD}((A - x_1) \cup (A - x_2)) \\ &\leq \text{BD}(A - x_1) + \text{BD}(A - x_2) \\ &= 2 \cdot \text{BD}(A). \end{aligned}$$

□

A notion of largeness that is widely used in combinatorics of numbers is obtained by considering sets of differences.

Definition 12. *The Delta-set of a set $X \subseteq \mathbb{N}$ is defined as:*

$$\Delta(X) =: \{x' - x \mid x < x' \text{ in } X\}.$$

A set $A \subseteq \mathbb{N}$ is Delta-large if $A \supseteq \Delta(X)$ for an infinite X .

Proposition 13. *The family of Delta-large sets is strongly partition regular.*

Proof. This is a straight application of Ramsey's theorem. Indeed let $A = C_1 \cup \dots \cup C_r$ be a finite coloring of a Delta-large set A . Pick an infinite $X = \{x_1 < x_2 < \dots\}$ such that $\Delta(X) \subseteq A$, and define the coloring of the pairs $[\mathbb{N}]^2 = D_1 \cup \dots \cup D_r$ by letting $\{n < m\} \in D_i \Leftrightarrow x_m - x_n \in C_i$. By Ramsey's theorem there exists an infinite $H \subseteq \mathbb{N}$ such that $[H]^2 \subseteq D_i$ is monochromatic. This means that if we let $X' := \{x_n \mid n \in H\}$ then $\Delta(X') \subseteq C_i$. □

Definition 14. *A set $A \subseteq \mathbb{N}$ is called thick if it contains arbitrarily long intervals.*

We conclude this subsection by recalling two well-known facts that connect sets of positive Banach density, thick sets, and Delta-large sets.

Proposition 15. *Let $A \subseteq \mathbb{N}$.*

- (1) $\text{BD}(A) = 1$ *if and only if* A *is thick.*
- (2) *If A is thick then A is Delta-large.*

Proof. (1) The fact that thick sets have Banach density 1 is straightforward from the definitions. Conversely, if a set A is not thick, pick $n \in \mathbb{N}$ such that A does not contain n consecutive natural numbers. It is readily seen that this forces $\text{BD}(A) \leq 1 - \frac{1}{n}$.

(2) We recursively define an increasing sequence $(x_n \mid n \in \mathbb{N})$ such that $x_m - x_n \in A$ for all $n < m$. Pick any $x_1 \in A$. At the inductive step, pick an interval $I = [y, y'] \subseteq T$ of length $y' - y > x_n$, and let $x_{n+1} = y'$. Then one has $x_{n+1} - x_i \in A$ for every $i \leq n$ because $y < y' - x_n \leq x_{n+1} - x_i < y'$. $\square$

Proposition 16. *Let $A \subseteq \mathbb{N}$ have positive Banach density $\text{BD}(A) = \alpha > 0$. Then $\Delta(A) \cap \Delta(X) \neq \emptyset$ for all infinite $X \subseteq \mathbb{N}$.*

Proof. Consider elements $x_1 < x_2 < \dots < x_n$ in X where $n > 1/\alpha$. If we show that there exist $i < j \leq n$ with $(A - x_i) \cap (A - x_j) \neq \emptyset$ then we are done. Indeed in this case we can pick $y \in (A - x_i) \cap (A - x_j)$, and if $a, a' \in A$ are such that $y = a - x_i = a' - x_j$ then $x_j - x_i = a' - a \in \Delta(A) \cap \Delta(X)$. So, assume towards a contradiction that $(A - x_i) \cap (A - x_j) = \emptyset$ for all $i < j \leq n$. Then, by Proposition 11 we would have $\text{BD}(\bigcup_{i=1}^n (A - x_i)) = n \cdot \alpha > 1$. $\square$

3. Proof of Rado's theorem

We are to give an elementary proof of Rado's theorem based on the facts that we recalled above.

For equations in two variables, Rado's condition reduces to the fact that the polynomial $P(x_1, x_2)$ is of the form $c(x_1 - x_2)$ for some $c \in \mathbb{Z}$.⁴ In this case, the partition regularity is trivial by considering any constant solution $P(a, a) = 0$.

Similarly, also polynomials $P(x_1, \dots, x_n)$ where the sum of all coefficients $\sum_{i=1}^n c_i = 0$ admit constant solutions $P(a, \dots, a) = 0$, and hence they are trivially partition regular.

The following proposition deals with the first non-trivial case in which there are three variables and where the sum of all coefficients may be different from 0.

Proposition 17. *For every $c \in \mathbb{N}$ and for every $d \in \mathbb{Z}$, the Diophantine equation $cx - cy - dz = 0$ is partition regular on $\mathbb{N}$.*

Proof. As already observed above, if $d = 0$ the result is trivial. If $d \neq 0$ then, by reordering the terms of the equation and by exchanging the variables x and y if necessary, we can assume without loss of generality that $d \in \mathbb{N}$. Let $\mathbb{N} = C_1 \cup \dots \cup C_s \cup C_{s+1} \cup \dots \cup C_r$ be a finite coloring of $\mathbb{N}$, where we arranged colors so that the Banach density $\text{BD}(C_i) > 0$ if and only if $i \leq s$. The set $D =: C_1 \cup \dots \cup C_s$ has Banach density 1, and so it is Delta-large by Proposition 15. Since Delta-large sets are strongly partition regular, there exists $i \leq s$ and an infinite W such that $\Delta(W) \subseteq C_i$. Now, $\text{BD}(C_i) > 0$ implies that $\text{BD}(c \cdot C_i) > 0$, and so $\Delta(c \cdot C_i) \cap \Delta(d \cdot W) \neq \emptyset$, by Proposition 16. This means that there exist $a_1 > a_2$ in C_i and $a'_1 > a'_2$ in W such that $ca_1 - ca_2 = da'_1 - da'_2 = da_3$ where $a_3 =: a'_1 - a'_2 \in \Delta(W)$. Finally, observe that $a_1, a_2, a_3 \in C_i$ are monochromatic. $\square$

Remark 18. The idea of using density arguments combined with the strong partition regularity of Delta-sets is certainly not new. To the authors' knowledge, the first use of such arguments goes back to Bergelson's paper [2] of 1986, where he proved a density version of Schur's theorem.

⁴This characterization can be extended to the partition regularity of arbitrary nonlinear equations $P(x_1, x_2) = 0$: in fact such equations are partition regular if and only if they are multiples of $x_1 - x_2$, as was proved in [1].

We can now fulfill our original goal.

Theorem 19. *If a linear homogeneous polynomial $P \in \mathbb{Z}[x_1, \dots, x_k]$ satisfies Rado's condition, then it is partition regular on $\mathbb{N}$.*

Proof. By renaming the variables if necessary, without loss of generality we can assume that $P(x_1, \dots, x_k) = \sum_{i=1}^k c_i x_i$ where $\sum_{i=1}^m c_i = 0$ for some $1 \leq m \leq k$. If $m = k$ the equation admits constant solutions, so we restrict to the case $m < k$.

Let $I = \{1, \dots, m\}$, let $I_1 = \{i \in \{1, \dots, m\} \mid c_i > 0\}$, and let $I_2 = I \setminus I_1$. Observe that $\sum_{i \in I_1} c_i = -\sum_{i \in I_2} c_i$ by Rado's condition. Let $c = \sum_{i \in I_1} c_i$, $d = -\sum_{i=m+1}^k c_i$. Given any finite coloring of $\mathbb{N}$, by Proposition 17 we can find monochromatic y_1, y_2, y_3 so that $c y_1 - c y_2 - d y_3 = 0$. By letting $x_i = y_1$ for $i \in I_1$, $x_i = y_2$ for $i \in I_2$ and $x_i = y_3$ for $i \in \{m+1, \dots, k\}$, we obtain a monochromatic solution of $P(x_1, \dots, x_k) = 0$. $\square$

Remark 20. In the previous proof, the monochromatic solutions we obtain are not constant, but neither are they pairwise different. In fact, they have at least two different values, since we can choose $y_1 > y_2$, but it could happen that $y_3 = y_1$ or $y_3 = y_2$. We observe that this is already enough to obtain a proof of the 3-term van der Waerden's theorem, because we have monochromatic solutions to the equation $x + y - 2z = 0$ where $x > y$, and hence $y < z < x$ form a non-trivial arithmetic progression.

4. Piecewise syndetic sets and partition regularity

In this section we will show an example that gives further evidence to the fact that the largeness notion of piecewise syndeticity for sets of integers is instrumental in partition regularity problems. Precisely, we will consider a strong property for combining partition regularity of different equations. While its known proof [7] heavily relies on the ultrafilter space $\beta\mathbb{N}$ and its algebra, here we will show that an alternative proof can be given that is only based on the fundamental notions introduced so far, and is entirely “elementary”. At the end of the section, we will also show an alternative proof of Rado's theorem based on the above mentioned property.

4.1. Thickness and syndeticity

We already introduced the notion of (additively) thick set, as a set that contains arbitrarily long intervals. The multiplicative counterpart is the following.

Definition 21. *A set $A \subseteq \mathbb{N}$ is multiplicatively thick if it contains arbitrarily large sets of multiples.*

Recall the following elemental notions of combinatorics of numbers.

Definition 22.

- *A is additively syndetic if it has “bounded gaps”, i.e. if there exists $k \in \mathbb{N}$ such that for every interval I of length $|I| \geq k$ one has that $A \cap I \neq \emptyset$.*
- *A is multiplicatively syndetic if it has “bounded gaps on multiples”, i.e. if there exists $k \in \mathbb{N}$ such that for every $x \in \mathbb{N}$ one has that $A \cap \{x, x \cdot 2, \dots, x \cdot k\} \neq \emptyset$.*
- *A is additively piecewise syndetic if it is the intersection of an additively thick set with an additive syndetic set.*
- *A is multiplicatively piecewise syndetic if it is the intersection of a multiplicatively thick set with a multiplicatively syndetic set.*

We itemize below the first properties, that are basic knowledge in this area. Proofs are directly obtained from the definitions and are omitted.

Proposition 23. *Let $A \subseteq \mathbb{N}$. Then:*

- (1) *A is additively syndetic if and only if the complement A^c is not additively thick if and only if $\mathbb{N} = \bigcup_{i=1}^{\ell} (A - n_i)$ is a finite union of leftward additive shifts of A ;*
- (2) *A is multiplicatively syndetic if and only if the complement A^c is not multiplicatively thick if and only if $\mathbb{N} = \bigcup_{i=1}^{\ell} A/n_i$ is a finite union of leftward multiplicative shifts of A .*

A well-known relevant property of piecewise syndetic sets that is neither satisfied by thick nor by syndetic sets, is partition regularity.

Proposition 24. *The family of additively (or multiplicatively) piecewise syndetic sets is strongly partition regular.*

Proof. Below, we do not specify whether we are considering the additive or the multiplicative notion, since the arguments used in the proof are exactly the same for both cases.

Let $A \subseteq \mathbb{N}$ be piecewise syndetic. By induction on r we will show that in every partition $A = C_1 \cup \dots \cup C_r$ one of the pieces C_i is piecewise syndetic. The base case $r = 1$ is trivial. At the inductive step $r > 1$, let $C' := C_1 \cup \dots \cup C_{r-1}$. We note that, using the inductive hypothesis, it suffices to show that one of C' and C_r is piecewise syndetic.

Pick a syndetic set S and a thick set T such that $A = S \cap T = C' \cup C_r$. We distinguish two cases. If $S \setminus C'$ is syndetic, then $C_r = (S \setminus C') \cap T$ is piecewise syndetic as the intersection of a syndetic and a thick set; if instead $S \setminus C'$ is not syndetic then the complement $(S \setminus C')^c = S^c \cup C'$ is thick, and in this case $C' = S \cap (S^c \cup C')$ is piecewise syndetic. $\square$

Remark 25. The previous result is a particular case of a more general property. Precisely, along the same lines, one can prove the following:

- *Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a nonempty family that is upward closed with respect to inclusion. Then the following family is partition regular on X :⁵*

$$\mathcal{F}' := \{A \cap B \mid A \in \mathcal{F}, B^c \notin \mathcal{F}\}.$$

4.2. The principle of compactness

A well-known tool in combinatorics is given by the compactness principle, a general property that connects infinitary and finitary properties. There are several different formulations of that fundamental principle; the one we will use in the sequel is the following.

Theorem 26 (Compactness principle). *Suppose that $\mathcal{F} \subseteq \mathcal{P}(X)$ only contains finite sets. Then for every $r \in \mathbb{N}$, the family $\mathcal{F}$ is r -partition regular on X if and only if there exists a finite $Y \subseteq X$ such that $\mathcal{F} \cap \mathcal{P}(Y)$ is r -partition regular on Y .*

By “ r -partition regularity” we mean that the partition regularity property holds for r -colorings, i.e., for colorings into r -many colors.

Proof. One implication is immediate. To prove that the other implication is “elementary”, we include its proof here. The argument is the same as, e.g., in [5, Section 1.5].

Consider the topological space of r -colorings $\{1, \dots, r\}^X$ obtained as the topological product of the discrete space $\{1, \dots, r\}$. By Tychonoff’s theorem, such a space is compact.⁶ Now assume for the sake of contradiction that for every finite $Y \subseteq X$, the family $\mathcal{F} \cap \mathcal{P}(Y)$ is not r -partition regular on Y , i.e., the set C_Y of the r -colorings such that no monochromatic $F \in \mathcal{F}$ is contained in Y is nonempty. Now observe that $C_{Y_1 \cup \dots \cup Y_n} \subseteq C_{Y_1} \cap \dots \cap C_{Y_n}$, and so the family $\{C_Y \mid Y \subseteq X \text{ finite}\}$ has the finite intersection property. Since all sets C_Y are closed, by compactness the intersection $C_X = \bigcap_{Y \subseteq X, Y \text{ finite}} C_Y$ is nonempty, contradicting the r -partition regularity of $\mathcal{F}$ on X . $\square$

⁵The family $\mathcal{F}^* = \{B^c \mid B \in \mathcal{F}\}$ is usually called the *dual family* of $\mathcal{F}$.

⁶Note that here we are using the *axiom of choice*.

For example, by compactness one has that Schur's theorem 3 implies the following finitary statement (the converse implication is trivial):⁷

- For every $r \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for every r -coloring $\{1, \dots, n\} = C_1 \cup \dots \cup C_r$ there exists a monochromatic triple $a, b, a + b \in C_i$.

4.3. Combining partition regularity of different equations

Let us start with a disclaimer: all the results we will prove in this subsection (although sometimes formulated slightly differently) have already been proved in [4, 7].⁸ However, all the proofs there were carried within the ultrafilter space $\beta\mathbb{N}$ and used in an essential way its algebraic properties; instead, here we will provide alternative proofs that use only the elementary tools introduced so far.

The crucial tool we will need is a sort of universal property of the piecewise syndetic sets with respect to partition regularity. To the authors' knowledge, it was first proven in [8, Theorem 1.2] by using compactness of the topological spaces $\{1, \dots, r\}^{\mathbb{N}}$, but has since been considered several times in the literature; for example it can be obtained as a direct consequence of [7, Theorem 4.13], which was proven using the ultrafilter method. We give here a short proof as a straight application of combinatorial compactness.

Proposition 27. *Let $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ be a partition regular family of finite sets. Then every additive (or multiplicative) piecewise syndetic set contains an additive (or multiplicative, resp.) shift of an element of $\mathcal{F}$.*

Proof. We consider here only the multiplicative case, because the additive one is proved by using exactly the same arguments.

Given a multiplicatively piecewise syndetic set A , let $r \in \mathbb{N}$ be such that the union $T := \bigcup_{i=1}^r A/i$ is multiplicatively thick. Since $\mathcal{F}$ only contains finite sets, by compactness there exists n such that for every r -coloring of $[1, n]$ there exists a monochromatic set $F \in \mathcal{F} \cap \mathcal{P}([1, n])$. By multiplicative thickness, there exists m such that the multiplicative shift $m \cdot [1, n] \subseteq T$. Now consider the r -coloring $[1, n] = C_1 \cup \dots \cup C_r$ where for every $x \in [1, n]$ one lets $x \in C_j$ if and only if $j = \min\{i \leq k \mid mx \in A/i\}$. Let $F \in \mathcal{F} \cap \mathcal{P}([1, n])$ be monochromatic, say $F \subseteq C_i$. Then, by definition, for every $x \in F$ we have $mx \in A/i$, and hence the multiplicative shift $(im) \cdot F \subseteq A$, as desired. $\square$

Remark 28. With essentially the same proof, the result above can be extended to any semigroup $(S, \star)$ in place of $(\mathbb{N}, +)$ or $(\mathbb{N}, \cdot)$. Some caution is needed in the non-commutative case, where one needs to consider the one-sided notions of right (or left) thick set, and of right (or left, resp.) piecewise syndetic set.⁹

The central role of multiplicatively piecewise syndetic sets for our purposes is shown by the following property. We prove it as a direct consequence of the universal property of piecewise syndetic sets that we have seen above.

Proposition 29. *Let $P \in \mathbb{Z}[x_1, \dots, x_k]$ be a homogeneous polynomial. Then the following are equivalent:*

- (1) *the equation $P(x_1, \dots, x_k) = 0$ is partition regular on $\mathbb{N}$;*

⁷Actually, both the original Schur's theorem [10] and van der Waerden's theorem were stated in their finitary versions.

⁸Precisely, Proposition 27 is a straight consequence of [7, Theorem 4.13], Proposition 29 was proven in [7, Corollary 6.14], Proposition 30 was proven in [4, Lemma 2.1], Lemma 31 (in a much-strengthened way) can be found in [4, Theorem 2.7], and the argument in the proof of Theorem 32 is similar to the one used to prove [4, Corollary 2.5 and Theorem 2.11].

⁹See [3, Section 2] or [6, Section 4.4].

- (2) *for every multiplicatively piecewise syndetic set $A \subseteq \mathbb{N}$ there exist elements $a_1, \dots, a_k \in A$ that are a solution, i.e., $P(a_1, \dots, a_k) = 0$.*

Proof. (2) $\Rightarrow$ (1) directly follows from the partition regularity of multiplicatively piecewise syndetic sets (see Proposition 24).

(1) $\Rightarrow$ (2). Consider the family

$$\mathcal{F} =: \{ \{a_1, \dots, a_k\} \in [\mathbb{N}]^k \mid P(a_1, \dots, a_k) = 0 \}.$$

By the hypothesis, $\mathcal{F}$ is partition regular on $\mathbb{N}$ and so, by the previous Proposition 27, there exists a set $\{b_1, \dots, b_k\} \in \mathcal{F}$ and $y \in \mathbb{N}$ such that the multiplicative shift $y \cdot \{b_1, \dots, b_k\} \subseteq A$. If we let $a_i := y \cdot b_i \in A$ then we have $P(a_1, \dots, a_k) = y^\ell \cdot P(b_1, \dots, b_k) = 0$, where ℓ is the degree of homogeneity of P . $\square$

Combining the two previous properties, one obtains a valuable tool that allows combining different equations in such a way as to preserve partition regularity.

Theorem 30. *Suppose the homogeneous polynomials $P_i(x_{i,1}, \dots, x_{i,k_i})$ are partition regular on $\mathbb{N}$ for $i = 1, \dots, N$. Then in every multiplicatively piecewise syndetic set A one finds elements b and $a_{i,2}, \dots, a_{i,k_i}$ such that $P_i(b, a_{i,2}, \dots, a_{i,k_i}) = 0$ for $i = 1, \dots, N$.*

Proof. For $i = 1, \dots, N$, consider the set:

$$\Gamma_i := \{ y \in A \mid \forall y_2, \dots, y_{n_i} \in A, P_i(y, y_2, \dots, y_{n_i}) \neq 0 \}.$$

We observe that Γ_i is not multiplicatively piecewise syndetic because it does not contain any solution of P_i , and therefore neither is the finite union $\Gamma := \bigcup_{i=1}^N \Gamma_i \subseteq A$. By strong partition regularity, it follows that $A \setminus \Gamma$ is multiplicatively piecewise syndetic, and hence nonempty. Finally, by definition, if b is any element of $A \setminus \Gamma$ then for every $i = 1, \dots, N$ there exist elements $a_{i,2}, \dots, a_{i,n_i} \in A$ such that $P_i(b, a_{i,2}, \dots, a_{i,n_i}) = 0$, as desired. $\square$

As a consequence of the previous theorem we obtain the following general property about partition regularity.

Proposition 31. *Assume the homogeneous polynomial $P \in \mathbb{Z}[x_1, \dots, x_k]$ is partition regular on $\mathbb{N}$. Then for every $q \in \mathbb{Q} \setminus \{0\}$ also the polynomial*

$$P_q(x_1, \dots, x_{k-1}, x_k, x_{k+1}) = P(x_1, \dots, x_{k-1}, x_k + qx_{k+1})$$

is partition regular on $\mathbb{N}$.

Proof. Let A be a multiplicatively piecewise syndetic, and let $q = \frac{d}{c}$ where $c \in \mathbb{N}$ and $d \in \mathbb{Z} \setminus \{0\}$. By Proposition 17, the homogeneous polynomial $Q(y_1, y_2, y_3) := cy_1 - cy_2 - dy_3$ is partition regular. Now apply Proposition 30 to the polynomials P and Q , and obtain the existence of elements b and $a_1, \dots, a_{k-1}, b_2, b_3$ in A such that $P(a_1, \dots, a_{k-1}, b) = 0$ and $Q(b, b_2, b_3) = cb - cb_2 - db_3 = 0$. Since $b = b_2 + qb_3$, the elements $a_1, \dots, a_{k-1}, b_2, b_2 + qb_3 \in A$ are a solution of P_q , as desired. $\square$

Several applications of the above theorem and its consequences can be found in [4]. Here, to illustrate its use, we give a second, alternative elementary proof of the sufficiency of Rado's condition.

Theorem 32. *If the linear homogeneous polynomial $P \in \mathbb{Z}[x_1, \dots, x_k]$ satisfies Rado's condition, then it is partition regular on $\mathbb{N}$.*

Proof. By renaming the variables if necessary, without loss of generality we can assume that $P(x_1, \dots, x_k) = \sum_{i=1}^k c_i x_i$ where $\sum_{i=1}^m c_i = 0$ for some $1 \leq m \leq k$. The polynomial $Q(x_1, \dots, x_m) := c_1 x_1 + \dots + c_m x_m$ is trivially partition regular, as it is solved by every constant m -tuple.

Then, by the previous proposition, also the following polynomial is partition regular:

$$Q\left(x_1, \dots, x_{m-1}, x_m + \frac{c_{m+1}}{c_m} x_{m+1}\right) = c_1 x_1 + \dots + c_m x_m + c_{m+1} x_{m+1}.$$

By iterating the procedure for $k - m$ steps, we finally obtain that P is partition regular. □

5. Conclusions

The version of Rado's theorem considered here is a special case of the general formulation that characterizes the partition regularity on $\mathbb{N}$ of arbitrary (finite) systems of Diophantine equations. We have not been able to adapt our methods to the general formulation of Rado's theorem, hence our first question is the following.

Question 1. Is it possible to give an elementary proof of the general Rado's theorem for systems of linear equations, without resorting to van der Waerden's theorem?

In our proof of Theorem 19, the monochromatic solutions we produce are “almost constant”, in the sense that they attain only three different values (at most). Often, in Ramsey theory we prefer to avoid such restrictions. Hence, our final question is the following.

Question 2. Is it possible to give an alternative elementary proof of Rado's theorem that proves also the existence of injective monochromatic solutions (i.e., solutions whose coordinates are pairwise distinct)?

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