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Quantitative weak-BV stability of “wild” solutions to compressible Euler equations, with a view towards higher systems

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Abstract. For hyperbolic systems of conservation laws, including important physical models from continuum mechanics, the question of stability for large data solutions remains a challenging open problem. In recent work [arXiv:2507.23645] the authors introduce a framework for showing Hölder stability of potentially “wild” large data solutions, relative to a class of BV solutions, for systems with two conserved quantities. This is referred to as “weak-BV” stability. In this paper, we give a short introduction to the methods while applying them to the “full” Euler system with three conserved quantities. We discuss applications to future work for higher systems (i.e. systems with more than two conserved quantities).

Keywords. Stability, systems of conservation law, full Euler system.

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1. Introduction

In this paper, we consider the Hölder stability of hyperbolic conservation laws in the space of L^2 . The general system of hyperbolic conservation laws is in the form of

$$u_t + (f(u))_x = 0, \quad u(x, 0) = u^0, \quad (1)$$

where $u = (u_1, \dots, u_n) \in \mathcal{V} \subset \mathbb{R}^n$ are the unknowns and $u^0: \mathbb{R} \rightarrow \mathcal{V}$ is the initial data. By assumption, the state space $\mathcal{V}$ is an open and bounded convex set. We assume $f = (f_1, \dots, f_n)$ is C^4 on $\mathcal{V}$ up to the boundary. Moreover, we assume there exists a strictly convex entropy $\eta \in C^3(\overline{\mathcal{V}})$ and entropy-flux $q \in C^3(\overline{\mathcal{V}})$ verifying

$$q' = \eta' f' \quad (2)$$

in $\mathcal{V}$. We restrict ourselves to entropy-weak solutions, i.e. solutions verifying in a distributional sense

$$(\eta(u))_t + (q(u))_x \leq 0. \quad (3)$$

One fundamental example in the form of (1) is the compressible Euler equation, which is widely used to describe the compressible inviscid flow, such as gas dynamics. Compressible Euler equations are among the oldest partial differential equations (PDEs) written down. It first appeared in published form in Euler's article in 1757 [18]. In Euler's original work, the system of equations consisted of the momentum and continuity equations. An additional equation, which was called the adiabatic condition, was supplied by Pierre-Simon Laplace in 1816 [24] (for an extended history see Dafermos [16, pp. XVII–XXXII]).

The 1-d compressible Euler equation, in Lagrangian coordinates, is

$$\begin{aligned} \tau_t - w_x &= 0, \\ w_t + p_x &= 0, \\ \left(\frac{1}{2} w^2 + \mathcal{E} \right)_t + (w p)_x &= 0, \end{aligned} \quad (4)$$

which can be written in the form of hyperbolic conservation laws (1) with $(u_1, u_2, u_3) = (\tau, w, \frac{1}{2} w^2 + \mathcal{E})$. In the literature this system is referred to as “full Euler” or “non-isentropic Euler”. Here τ is the specific volume (equivalent to $1/\rho$, for a density ρ), p is pressure, w is fluid velocity, and $\mathcal{E}$ is the specific internal energy.

The system is closed by an additional equation of state. For convenience, we consider the polytropic ideal γ -law gas, with equation of state

$$\mathcal{E} = c_v \theta = \frac{p \tau}{\gamma - 1} \quad \text{and} \quad p \tau = \bar{R} \theta, \quad \text{so} \quad p = \bar{K} e^{S/c_v} \tau^{-\gamma}. \quad (5)$$

Here S is the entropy, θ is the temperature, $\bar{R}$, $\bar{K}$, c_v are positive constants, and $\gamma > 1$ is the adiabatic gas constant. These state variables satisfy the Gibbs relation

$$\theta dS = d\mathcal{E} + p d\tau. \quad (6)$$

We use the entropy and entropy-flux pair

$$\eta = -S = c_v \left((1 - \gamma) \ln \tau - \ln \mathcal{E} + \ln \frac{\bar{K}}{\gamma - 1} \right), \quad q = 0.$$

Remark (3) is satisfied since the entropy function S increases after passing any moving 1-shock or 3-shock, by the Lax entropy condition (see [29, pp. 341–342] or [8]). The solution of (3)–(4) is equivalent to the one coming from the Eulerian coordinates, even in the class of weak solutions (see [32]).

Due to the absence of dissipative effects, classical (i.e. C^1) solutions cannot be sustained, and generically, shock waves form in finite time, where the conserved variable becomes discontinuous. Breakdown of classical solutions has been long known. For scalar equations, it goes back to Stokes [30] and Riemann [27]. Shocks are physical phenomenon. For example, the “sonic boom” created by supersonic aircraft is characterized by rapid discontinuities in pressure. Once a shock wave forms, one must study weak solutions, and the analysis becomes very difficult, especially when there are many or even infinite shock waves. A natural space to study the solution is the space of *bounded variation* or BV solutions, where the amount of oscillation is controlled. Following the timeline, there are currently three major approaches to studying the global well-posedness of BV solutions in one space dimension: Glimm Scheme, L^1 theory, and L^2 theory.

The first global existence result was given by [21], followed by other methods such as the front tracking scheme in the 1990s [3, 16, 22], and the vanishing viscosity method (artificial viscosity [2] and from isentropic Navier–Stokes to Euler [10]). The stability theory is more involved, after

the seminal L^1 theory in the 1990s [3,5,6,26], a new approach showing the stability in L^2 has been established relatively recently [31]. Until the work [9], these results were limited to giving qualitative stability in L^2 (see e.g. [11–13]).

The current paper is dedicated to giving a short introduction to a new framework developed for showing *quantitative* Hölder-1/2 stability in L^2 [9]. Our work is based on the method of relative entropy first introduced by Dafermos [15] and DiPerna [17], and later extended to a -contraction theory by Kang and Vasseur [23]. Due to the presence of shock waves, Hölder-1/2 is the optimal rate of stability result one can expect in L^2 (see [9, Section 1.2]). This work considers systems (1) with two conserved quantities, i.e. $n = 2$, also called 2×2 systems. More precisely, we show the following result.

Theorem 1 (L^2 -Hölder stability for 2×2 conservation laws, a summary [9]). *Consider a large class of 2×2 conservation laws, including isentropic Euler and the system of shallow water waves. Fix $T, R > 0$. Consider a weak solution v arising as a limit of a sequence of front tracking approximations. Consider also a “wild” solution u , in a class of solutions analogous to $\mathcal{S}_{\text{weak}}$ (see (9), below). We then have*

$$\|u(\cdot, \tau) - v(\cdot, \tau)\|_{L^2((-R, R))} \leq K \sqrt{\|u(\cdot, 0) - v(\cdot, 0)\|_{L^2((-R-sT, R+sT))}} \quad (7)$$

for all $\tau \in [0, T]$, for a universal constant K , and where $s > 0$ is the speed of information (see e.g. (36), below).

For a general 2×2 system, the solution v can be a limit of small-BV front tracking approximations. For isothermal Euler, the solution v may be a limit of front tracking schemes with large but finite BV-norm.

This is related to the question of uniqueness of solutions in L^∞ (see Bressan’s Open Problem #6 [4, p. 558]).

The estimate (7) has absolutely no dependence on the BV-norm of the weak solution u , which may be infinite. This result links the L^2 theory with the L^1 -Lipschitz quantitative estimates for BV solutions in the L^1 theory.

As a consequence of Theorem 1, for the isothermal system we have uniqueness of certain large L^∞ solutions with initial data which is *everywhere* locally infinite BV, i.e. the BV norm of the initial data, when restricted to any open set, is infinite (see [9, Corollary 1]). Compare with the other very recent work [7], which also gives a uniqueness result for certain initial data with BV norm equal to infinity, for certain small L^∞ solutions for general 2×2 genuinely nonlinear systems.

In this paper, we prove the Hölder stability of the full 3×3 (non-isentropic) Euler system with three equations (see our Theorem 3, below). This new result has a direct application to the uniqueness of solutions in a wide space, and also provides a basis for many other future approaches such as the inviscid limit from non-isentropic Navier–Stokes to the Euler equation.

In the proof of Theorem 1 in the paper [9], we use the Bressan–Colombo theory [5] as it is instrumental in the analysis of large-BV front tracking schemes (see Colombo–Risebro [14]). However, the Bressan–Colombo theory is fundamentally limited to the 2×2 systems and furthermore is quite delicate (with e.g. the use of sophisticated homotopy arguments to measure distance in L^1). In this paper, we have a view towards higher systems with many conserved quantities. We use the Bressan–Liu–Yang theory [6] which is simpler and, moreover, applies to general $n \times n$ systems with n conserved quantities (see Section 3, below). Within this simplified framework, we aim to convey the main ideas and give a concise proof of our Theorem 3, quoting some more technical details from other works when necessary.

A key part of proving Hölder stability involves carefully controlling the entropy dissipation at a shock (see Proposition 5, below). In related work, the second author of this present paper has

started a new program to study the dissipation at shocks in all wave families and for general $n \times n$ systems [19].

A final point is that the L^2 theory generally and [9] in particular only require that the entropy inequality (3) hold for one single entropy η . This is a natural assumption as for many physical systems only one entropy exists (see [16, pp. 13–14] and [16, pp. 54–55]). This is true for the Euler system (4).

Now we come to state our main results.

We restrict our study to the solutions verifying what the literature often refers to as the “Strong Trace Property”.

Definition 2 (Strong Trace Property). *Let $u \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$. We say that u verifies the Strong Trace Property if for any Lipschitzian curve $t \rightarrow X(t)$, there exist two bounded functions $u_-, u_+ \in L^\infty(\mathbb{R}^+)$ such that for any $T > 0$,*

$$\lim_{n \rightarrow \infty} \int_0^T \sup_{y \in (0, 1/n)} |u(X(t) + y, t) - u_+(t)| dt = \lim_{n \rightarrow \infty} \int_0^T \sup_{y \in (-1/n, 0)} |u(X(t) + y, t) - u_-(t)| dt = 0.$$

We will consider the following classes of solutions. Fix an open, convex subset $\mathcal{W}$ of $\mathcal{V}$ such that $\overline{\mathcal{W}} \subset \mathcal{V}$, fix any $d \in \mathcal{W}$, and fix a small $\epsilon > 0$. Then define

$$\mathcal{S}_{\text{BV}, \epsilon}^0 := \left\{ \text{functions } u^0: \mathbb{R} \rightarrow \mathcal{W} \mid \|u^0 - d\|_{L^\infty(\mathbb{R})} \leq \epsilon \text{ and } \|u^0\|_{\text{BV}(\mathbb{R})} \leq \epsilon \right\}. \quad (8)$$

We also define the following class of potentially very “wild” solutions without smallness.

$$\mathcal{S}_{\text{weak}} := \left\{ u \in L^\infty(\mathbb{R} \times [0, T]; \mathcal{V}) \mid u \text{ is a weak solution to (3)–(4) satisfying Definition 2} \right\}, \quad (9)$$

for some $T > 0$. This is a large class, beyond BV or BV_{loc} . See Figure 1. Then, we have the following result.

Theorem 3 (L^2 -Hölder stability for full Euler). *Fix $R, T > 0$. Then consider $u \in \mathcal{S}_{\text{weak}}$ solving the system (4). Given any initial data $v^0 \in \mathcal{S}_{\text{BV}, \epsilon}^0$, where $\epsilon > 0$ is sufficiently small, there exists a classical small-BV solution v (via e.g. front tracking). Then for all $\tau \in [0, T]$ we have the bound*

$$\|u(\cdot, \tau) - v(\cdot, \tau)\|_{L^2((-R, R))} \leq K \sqrt{\|u(\cdot, 0) - v(\cdot, 0)\|_{L^2((-R-s\tau, R+s\tau))}}, \quad (10)$$

for a universal constant $K > 0$ and where $s > 0$ is the speed of information (see equation (36)).

In a nutshell, the idea of the proof is the following: by artificially changing shock speeds, we can keep an approximate front tracking solution (denoted ψ , see Section 3) L^2 -stable with respect to a “wild” solution from $\mathcal{S}_{\text{weak}}$. The artificial “shifting” of shocks comes at a price, which we keep track of using a dissipation estimate (Proposition 5). Taking the limit of the approximate front tracking solutions gives our result.

Notation. K will denote a universal constant, which may change from line to line during computations.

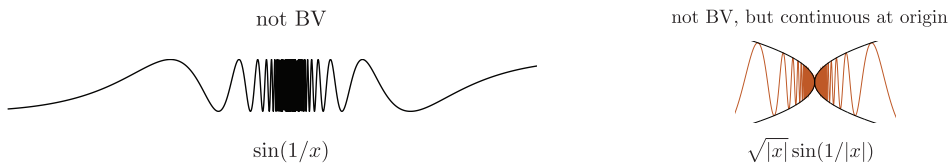


Figure 1. Solutions which behave like $\sin(1/x)$ are outside the BV theory. However, solutions which behave like $\sqrt{|x|} \sin(1/|x|)$ are also outside the BV class, but due to having more controlled oscillations, verify the Strong Trace Property (Definition 2), live in $\mathcal{S}_{\text{weak}}$, and fit in our L^2 theory.

2. L^2 stability for shock, contact and rarefaction

Given our entropy, entropy-flux pair η, q , and states $(u, v) \in \mathcal{V} \times \mathcal{V}$ we define the relative quantities

$$\eta(u|v) := \eta(u) - \eta(v) - \nabla\eta(v) \cdot (u - v), \quad q(u;v) := q(u) - q(v) - \nabla\eta(v) \cdot (f(u) - f(v)). \quad (11)$$

For fixed v these quantities then constitute an entropy flux pair for the system (1). We recall the following lemma.

Lemma 4 ([25, Lemma 1]). *Let W be a compact subset of $\mathcal{V}$. There exists a constant $C^* > 0$ such that for all $(u, v) \in \mathcal{V} \times W$ we have*

$$\frac{1}{C^*} |u - v|^2 \leq \eta(u|v) \leq C^* |u - v|^2. \quad (12)$$

This lemma states that controlling integrals of our relative entropy is equivalent to controlling the L^2 difference of u and v . From this, given an i -shock $(u_L, u_R, \lambda_i(u_L, u_R))$, where $\lambda_i(u_L, u_R)$ is the shock velocity dictated by the Rankine–Hugoniot condition, and a solution $u \in \mathcal{S}_{\text{weak}}$ we consider the weighted relative entropy

$$E_t(u; a_1, a_2, h) := \int_{-\infty}^{h(t)} a_1 \eta(u|u_L) dx + \int_{h(t)}^{\infty} a_2 \eta(u|u_R) dx \quad (13)$$

with the constants $a_1, a_2 > 0$ and h a Lipschitz curve. Differentiating and using both Definition 2 and inequality (3) we find the dissipation functional D_t ,

$$\begin{aligned} \frac{d}{dt} E_t(u; a_1, a_2, h) &\leq a_2 \overbrace{\left[q(u(h(t)+, t); u_R) - \dot{h}(t) \eta(u(h(t)+, t) | u_R) \right]}^{\text{dissipation from right side of } h} \\ &\quad - a_1 \overbrace{\left[q(u(h(t)-, t); u_L) - \dot{h}(t) \eta(u(h(t)-, t) | u_L) \right]}^{\text{dissipation from left side of } h} =: D_t(u; a_1, a_2, h). \end{aligned} \quad (14)$$

Thus, showing the existence of a triple (a_1, a_2, h) which gives bounds on D_t then gives us bounds on the L^2 distance of u and (u_L, u_R, h) , which is no longer a shock but rather a shifted front. The forthcoming propositions give us such a triple for the shocks and contact discontinuities of (4).

Proposition 5 ([9, Proposition 3.1]). *Consider system (3)–(4). Let $d \in \mathcal{V}$. Then there exist constants $\hat{\lambda}, K, C_1, \varepsilon$, and $\alpha > 0$ such that the following holds.*

Consider any shock (u_L, u_R) with $|u_L - d| + |u_R - d| \leq \varepsilon$. Consider also any $u \in \mathcal{S}_{\text{weak}}$, any $T > 0$, $T_{\text{start}} \in [0, T]$, and any $x_0 \in \mathbb{R}$. Let $s_0 = |u_L - u_R|$ be the size of the shock. Then for any $a_1, a_2 > 0$ satisfying

$$1 - 2C_1 s_0 \leq \frac{a_2}{a_1} \leq 1 - \frac{C_1 s_0}{2} \quad \text{if } (u_L, u_R) \text{ is a 1-shock}, \quad (15)$$

$$1 + \frac{C_1 s_0}{2} \leq \frac{a_2}{a_1} \leq 1 + 2C_1 s_0 \quad \text{if } (u_L, u_R) \text{ is a 3-shock}, \quad (16)$$

there exists a Lipschitz shift function $h: [T_{\text{start}}, T] \rightarrow \mathbb{R}$, with $h(T_{\text{start}}) = x_0$, such that our dissipation functional D_t satisfies the bound

$$\frac{d}{dt} E_t(u; a_1, a_2, h) \leq D_t \leq -a_2 K s_0 (\dot{h}(t) - \lambda_i(u_L, u_R))^2 \quad (17)$$

for almost all $t \in [T_{\text{start}}, T]$, where $\lambda_i(u_L, u_R)$ is the Rankine–Hugoniot velocity of the i -shock (u_L, u_R) , $i = 1$ or 3 .

Moreover, if (u_L, u_R) is a 1-shock, then for almost all $t \in [T_{\text{start}}, T]$:

$$-\frac{\hat{\lambda}}{2} \leq \dot{h}(t) \leq -\alpha < 0. \quad (18)$$

Similarly, if (u_L, u_R) is a 3-shock, then for almost all $t \in [T_{\text{start}}, T]$:

$$0 < \alpha \leq \dot{h}(t) \leq \frac{\hat{\lambda}}{2}. \quad (19)$$

At contact discontinuities we experience weighted entropy decay without shift due to [28, Theorem 4.1].

Lemma 6 ([28, Theorem 4.1]). *Consider a contact discontinuity (u_L, u_R) of the system (4) located at $x = \beta$ and a solution $u \in \mathcal{S}_{\text{weak}}$. Then the associated relative entropy decays without shift when we select our weights to be the absolute temperature at u_L and u_R :*

$$\frac{d}{dt} E_t(u; \theta_L, \theta_R, \beta) \leq D_t \leq 0. \quad (20)$$

We note that this lemma differs from Proposition 5 in that it requires the ratio

$$\frac{a_2}{a_1} = \frac{\theta_R}{\theta_L} \quad \text{for a contact,} \quad (21)$$

where a_1, a_2 are from the context of Proposition 5. This impacts the construction of our space-time weight in Section 3, where we must be careful to preserve this exact equality.

Finally, for dissipation at the discretized rarefaction fans which arise in front tracking, we have the following.

Proposition 7 ([11, Proposition 4.4]). *There exists a constant $C > 0$ such that the following is true. For any rarefaction wave $\bar{u}(y)$, $v_L \leq y \leq v_R$, of (4), let*

$$\begin{aligned} \delta &= |v_L - v_R| + \sup_{y \in [v_L, v_R]} |u_L - \bar{u}(y)|, \\ \bar{u}(v_L) &= u_L, \\ \bar{u}(v_R) &= u_R. \end{aligned}$$

Then for any $u \in \mathcal{S}_{\text{weak}}$, any $v_L \leq v \leq v_R$, and any $t > 0$ we have the bound

$$\int_0^t \left\{ q(u(tv+, t); u_R) - q(u(tv-, t); u_L) - v(\eta(u(tv+, t) | u_R) - \eta(u(tv-, t) | u_L)) \right\} dt \leq C\delta |u_L - u_R| t.$$

3. The modified front tracking scheme and weight function $a(x, t)$

One key step in the proof of the main theorem is to estimate the difference between a front tracking v -approximate solution $u(x, t)$ with true Rankine–Hugoniot speeds (up to the $\mathcal{O}(v)$ error) and a *shifted* v -approximate solution $\psi(x, t)$ where, given a fixed $u \in \mathcal{S}_{\text{weak}}$, shocks (u_L, u_R) move according to the function h from Proposition 5 except for small perturbations to avoid interactions of more than two waves.

Here we use the same definition of front tracking scheme in [12] to define the v -approximate solutions: both the un-shifted and shifted front tracking schemes.

We note the un-shifted front tracking scheme in [12] is a special case of the v -approximate front tracking scheme used in [6] (named as “ ϵ -approximate” front tracking scheme in [6]).

In fact, in the front tracking scheme of [12], there exists a small constant v , controlling three types of errors, as in [6]:

- errors in the speeds of shock and rarefaction fronts, to avoid interaction of more than two waves;
- the maximum strength of any rarefaction front;
- the total strength of all non-physical waves.

In the front tracking scheme, after discretizing the initial data to a piecewise-constant approximation, we let initial waves to propagate until the next interaction. And we define two Riemann solvers (accurate solver and simplified solver) to treat each wave interaction. In these two solvers, we use four types of wave fronts: shock, rarefaction jump, contact discontinuity and non-physical wave. Thanks to the decay of the Glimm functional in Proposition 8 below, the total number of wave interactions is finite and the BV norm of approximate solutions (shifted or un-shifted) has a uniform bound. So one can prove the convergence of the scheme to a BV solution using Helly's theorem. We refer the reader to [3,12] for details.

For any v -approximate (shifted or un-shifted) solution u , we use standard notions L and Q for total variation and Glimm potential at any time t :

$$L(u) := \text{TV}(u)(t), \quad Q(u) := \sum_{i,j: \text{approaching waves}} |\sigma_i| |\sigma_j| (t), \quad (22)$$

where the summation runs over all pairs of approaching waves, with strengths $|\sigma_i|$ and $|\sigma_j|$, at time t (see [6, p. 5] for details).

Proposition 8 ([12]). *There exists $\kappa > 0$ such that for any ϵ small enough, where the initial total variation of the (shifted, or un-shifted) v -approximate solution u is less than ϵ , then the functional $Y(u) := L(u) + \kappa Q(u)$ is decreasing in time.*

We need to construct a space-time weight so that when we calculate the dissipation of entropy for ψ at shocks and contact discontinuities, we have the proper weighting of the left and right states at each discontinuity (e.g. a_1, a_2 in Proposition 5 or θ_L, θ_R in Lemma 6). See Section 5.1, below, for details.

Space-time weight. Now we revisit the weight function $a(x, t)$ in [12] and will use it for the current paper.

First, we give our definition of $\bar{\sigma}$. In particular, for a 2-wave, we define

$$\bar{\sigma} = \frac{1}{C_1 J} \Delta \theta, \quad (23)$$

where $\Delta g = g_R - g_L$ measures the difference of g on two sides of waves, the constant C_1 is defined in Proposition 5 and $J = \frac{1}{2} \inf_{\overline{\mathcal{W}}} \theta > 0$ where the temperature θ defined in (5) always has a positive lower bound. This infimum includes all possible states our approximation u_v can achieve when ϵ in (8) is chosen sufficiently small.

For any 1-wave and 3-wave, we choose $\bar{\sigma} = \pm |\Delta u|$, where $\bar{\sigma}$ takes negative sign on a shock and positive sign on a rarefaction front. For non-physical wave, we choose $\bar{\sigma} = |\Delta u|$. Finally, at each discontinuity $x_i(t)$, we label the corresponding $\bar{\sigma}$ as $\bar{\sigma}^{(i)}$.

We introduce the following measure $\mu(\cdot, t)$ as a sum of Dirac measures in x :

$$\mu(x, t) = - \sum_{i: 1\text{-shock}} |\bar{\sigma}^{(i)}| \delta_{\{x_i(t)\}} + \sum_{i: 3\text{-shock}} |\bar{\sigma}^{(i)}| \delta_{\{x_i(t)\}} + \sum_{i: 2\text{-wave}} \frac{J \hat{a}}{\hat{\theta}} \bar{\sigma}^{(i)} \delta_{\{x_i(t)\}}$$

where $\hat{a}$ and $\hat{\theta}$ denote the a and θ values to the left of the contact. Then

$$a(x, t) = 1 + C_1 \left(L(t) + \kappa Q(t) + \int_{-\infty}^x \mu(x, t) dx \right), \quad (24)$$

where the constant C_1 is defined in Proposition 5. The weight a is constant across the two sides of a rarefaction or a non-physical wave.¹

By [12], we know the weight $a(x, t)$ defined in (24) satisfies (15), (16), (21) at shocks and contact discontinuities, respectively, and the time decay

$$a(x, t+) \leq a(x, t-), \quad (25)$$

except at any point of interaction.

4. L^1 distance between solutions with real and shifted shock speeds and error estimate

In this section, we will give some key estimate for the weighted L^1 distance, proposed by [6], between a front tracking v -approximate solution $u(x, t)$ with true Rankine–Hugoniot speeds (up to the $\mathcal{O}(v)$ error) and a *shifted* v -approximate solution $\psi(x, t)$ whose shocks are moved by prescribed shifts (following Proposition 5). Both u and ψ take initial values in $\mathcal{S}_{\text{BV}, \varepsilon}^0$ defined in (8).

We adopt the notation in [3] to define the weighted distance for hyperbolic conservation laws (1) in the general $n \times n$ case, which includes the Euler equation as a special example ($n = 3$).

First, by $\sigma \mapsto S_i(\sigma)(u_0) \equiv S_i(u_0)$, we denote the parametrized i -shock curve through the point u_0 , with $S_i(0)(u_0) = u_0$. If the i -th characteristic field is linearly degenerate (e.g. second family of Euler equation), the curve S_i will be parametrized by arc-length. If the i -th field is genuinely non-linear (first and third families for Euler equations), the parametrization is chosen as follows. We choose the right eigenvector $r_i(u)$ corresponding to the eigenvalue $\lambda_i(u)$ of $Df(u)$, so that $r_i \cdot \nabla \lambda_i \equiv 1$. Moreover, we choose the parametrization of the i -shock curve $\sigma \mapsto S_i(\sigma)(u_0)$ and i -rarefaction curve $\sigma \mapsto R_i(\sigma)(u_0)$ so that

$$\frac{d}{d\sigma} \lambda_i(S_i(\sigma)(u_0)) \equiv 1, \quad \frac{d}{d\sigma} \lambda_i(R_i(\sigma)(u_0)) \equiv 1, \quad \lambda_i(S_i(\sigma)(u_0)) = \lambda_i(R_i(\sigma)(u_0)) = \lambda_i(u_0) + \sigma.$$

Same as before, $S_i(\sigma)(u_0)$ represents the entropy shock curve when $\sigma < 0$.

For any two small BV functions v and u , we define the scalar functions q_i implicitly by:

$$v(x) = S_n(q_n(x)) \circ \cdots \circ S_1(q_1(x))(u(x)).$$

Then

$$\frac{1}{K} |v(x) - u(x)| \leq \sum_{i=1}^n |q_i(x)| \leq K |v(x) - u(x)| \quad (26)$$

for some $K > 0$.

Then we define the weighted distance as:

$$\Phi(u, v) := \sum_{i=1}^n \int_{-\infty}^{\infty} |q_i(x)| W_i(x) dx$$

where the weights W_i are defined as:

$$W_i(x) := 1 + \kappa_1 A_i(x) + \kappa_2 [Q(u) + Q(v)].$$

¹ More precisely, we need to define a inductively from the extreme left. Assume $a(\cdot, t)$ consists of n constant states $a^{(1)}$ to $a^{(n)}$ from left to right. Define $a^{(1)}(t) = 1 + C_1(L(t) + \kappa Q(t))$, then

$$a^{(i+1)}(t) = a^{(i)}(t) + C_1 \left(-(\bar{\sigma}_1^{(i)})_- + (\bar{\sigma}_3^{(i)})_- + \frac{J\hat{a}}{\hat{\theta}} \bar{\sigma}_2^{(i)} \right),$$

where there are at most three waves $\bar{\sigma}_1^{(i)}, \bar{\sigma}_2^{(i)}, \bar{\sigma}_3^{(i)}$ at the point when a changes from $a^{(i)}$ to $a^{(i+1)}$. Here $g_- = -\min(g, 0)$ for any function g , and $\hat{a}$ and $\hat{\theta}$ denote the a and θ values to the left of the contact, in particular, between outgoing 1- and 2-waves right after an interaction.

Here Q is the interaction potential defined in (22), and we denote $\mathcal{J}(u), \mathcal{J}(v)$ as the sets of all jumps in u and in v respectively and $\mathcal{J} = \mathcal{J}(u) \cup \mathcal{J}(v)$. For any wave with strength $|\sigma_\alpha|$ located at x_α , we use k_α to denote its family. Then

$$A_i(x) := \left[\sum_{\alpha \in \mathcal{J}, x_\alpha < x, i < k_\alpha \leq n} + \sum_{\alpha \in \mathcal{J}, x_\alpha > x, 1 \leq k_\alpha < i} \right] |\sigma_\alpha|$$

if the i -th field is linearly degenerate. For the general genuinely nonlinear field, $A_i(x)$ includes an additional term

$$\begin{aligned} & \left[\sum_{k_\alpha = i, \alpha \in \mathcal{J}(u), x_\alpha < x} + \sum_{k_\alpha = i, \alpha \in \mathcal{J}(v), x_\alpha > x} \right] |\sigma_\alpha| \quad \text{if } q_i(x) < 0, \\ & \left[\sum_{k_\alpha = i, \alpha \in \mathcal{J}(v), x_\alpha < x} + \sum_{k_\alpha = i, \alpha \in \mathcal{J}(u), x_\alpha > x} \right] |\sigma_\alpha| \quad \text{if } q_i(x) \geq 0. \end{aligned}$$

When ϵ is small enough, $1 \leq W_i \leq 2$, so by (26), we know

$$\frac{1}{K} \cdot \|v - u\|_{L^1} \leq \Phi(u, v) \leq 2K \cdot \|v - u\|_{L^1}. \quad (27)$$

We now give an important estimate for our theory.

Lemma 9 (L^1 estimate for shifted front tracking). *Let u be a v -approximate front tracking solution without shifts and let ψ be a modified v -approximate front tracking solution whose shocks follow shift functions $h_\alpha(t)$ according to Proposition 5 (see Section 3, above). Then, in the sense of distributions,*

$$\frac{d}{dt} \Phi(u(\cdot, t), \psi(\cdot, t)) \leq K \sum_{\alpha} |\psi(h_\alpha(t)+, t) - \psi(h_\alpha(t)-, t)| |\dot{h}_\alpha(t) - \dot{h}_{\text{true}, \alpha}(t)| + \mathcal{O}(v). \quad (28)$$

Here the sum is over the shocks of ψ at time t , K is a universal constant, $\dot{h}_\alpha$ is the imposed (shifted) shock speed in ψ , and $\dot{h}_{\text{true}, \alpha}$ is the Rankine–Hugoniot speed the same discontinuity would have in the un-shifted scheme.

Proof. Now we consider the distance $\Phi(u, \psi)$, with v -approximate solution u and shifted v -approximate solution ψ .

Since there are in total only finitely many waves in u and ψ , there are in total only finitely many interaction times when two waves in u interact or two waves in ψ interact. If one wave in u attaches to a wave in ψ for a while, we can perturb the speed of the wave in u a bit to avoid it, since we allow v error in the speeds of waves in u .

Case 1. Now we first consider one of finitely many open time intervals (t_j, t_{j+1}) , in which there are no interactions between any two waves in u or in ψ .

For any $t \in (t_j, t_{j+1})$, all jumps in u and ψ are denoted as $\{x_\alpha = x_\alpha(t), \alpha \in \mathcal{J} = \mathcal{J}(u) \cup \mathcal{J}(\psi)\}$ where we use $J(u)$ and $J(\psi)$ to denote all jumps in u and ψ , respectively.

Then it is easy to get

$$\frac{d}{dt} \Phi(u(\cdot, t), \psi(\cdot, t)) = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^n \{ |q_i(x_\alpha-) W_i(x_\alpha-) - |q_i(x_\alpha+) W_i(x_\alpha+) \} \dot{x}_\alpha \quad (29)$$

(see [6] or [3, equation (8.16)]).

On the other hand, starting from the profile of $\psi(\cdot, t^*)$ for any fixed $t^* \in (t_j, t_{j+1})$, we consider the corresponding v -approximate solution $\tilde{\psi}$ with initial data $\psi(\cdot, t^*)$, in a short open time interval including t^* . The function $\tilde{\psi}$ will have wave speeds $\tilde{x}_\alpha$, including almost correct

Rankine–Hugoniot for shocks, allowing up to ν error. We remark that $\dot{\bar{x}}_\alpha$ may only be different from $\dot{x}_\alpha$ for shock waves. We still have

$$\begin{aligned} \frac{d}{dt}\Phi(u(\cdot, t), \bar{\psi}(\cdot, t)) &= \sum_{\alpha \in \mathcal{J}(u)} \sum_{i=1}^n \{ |q_i(x_\alpha-) | W_i(x_\alpha-) - |q_i(x_\alpha+) | W_i(x_\alpha+) \} \dot{x}_\alpha \\ &\quad + \sum_{\alpha \in \mathcal{J}(\bar{\psi})} \sum_{i=1}^n \{ |q_i(\bar{x}_\alpha-) | W_i(\bar{x}_\alpha-) - |q_i(\bar{x}_\alpha+) | W_i(\bar{x}_\alpha+) \} \dot{\bar{x}}_\alpha, \end{aligned} \quad (30)$$

where we use $J(\bar{\psi})$ to denote all jumps in $\bar{\psi}$.

By the result in [6, equation (3.7)] or [3, equation (8.22)],

$$\frac{d}{dt}\Phi(u(\cdot, t), \bar{\psi}(\cdot, t)) = \mathcal{O}(\nu).$$

Note, at $t = t^*$, $\bar{x}_\alpha = x_\alpha$, and $\bar{\psi}$ and ψ share the same profile, so by (29) and (30),

$$\begin{aligned} \left. \frac{d}{dt}\Phi(u(\cdot, t), \psi(\cdot, t)) \right|_{t=t^*} - \left. \frac{d}{dt}\Phi(u(\cdot, t), \bar{\psi}(\cdot, t)) \right|_{t=t^*} \\ = \left(\sum_{\alpha \in \mathcal{J}(\psi)} \sum_{i=1}^n \{ |q_i(x_\alpha-) | W_i(x_\alpha-) - |q_i(x_\alpha+) | W_i(x_\alpha+) \} (\dot{x}_\alpha - \dot{\bar{x}}_\alpha) \right) \Big|_{t=t^*}. \end{aligned}$$

Now we prove the right-hand side of this equation is less than

$$K \sum_{\alpha} |\psi(x_\alpha+) - \psi(x_\alpha-)| |\dot{x}_\alpha - \dot{\bar{x}}_\alpha| \Big|_{t=t^*} + \mathcal{O}(\nu), \quad (31)$$

where the sum is over shocks of ψ at time $t = t^*$.

To prove (31), we use the fact that, for the genuinely nonlinear or linearly degenerate family

$$\begin{aligned} W_i(x_\alpha+) - W_i(x_\alpha-) &= \mathcal{O}(1)\sigma_\alpha \quad \text{if } q_i(x_\alpha+)q_i(x_\alpha-) > 0; \\ q_i(x_\alpha-) &= \mathcal{O}(1)|\sigma_\alpha| \quad \text{if } q_i(x_\alpha+)q_i(x_\alpha-) \leq 0; \end{aligned} \quad (32)$$

and

$$q_i(x_\alpha+) - q_i(x_\alpha-) = \mathcal{O}(1)|\sigma_\alpha| \quad (33)$$

(see page 162 of [3]). Then by (32) and (33), we know

$$q_i(x_\alpha+) = \mathcal{O}(1)|\sigma_\alpha| \quad \text{if } q_i(x_\alpha+)q_i(x_\alpha-) \leq 0.$$

Then

$$\begin{aligned} &|q_i(x_\alpha-) | W_i(x_\alpha-) - |q_i(x_\alpha+) | W_i(x_\alpha+) \\ &= -W_i(x_\alpha+) (|q_i(x_\alpha+) | - |q_i(x_\alpha-) |) - (W_i(x_\alpha+) - W_i(x_\alpha-)) |q_i(x_\alpha-) | \\ &= \mathcal{O}(1)|\sigma_\alpha|. \end{aligned}$$

Hence (31) holds, using the equivalence between $|\sigma_\alpha|$ and $|\psi(x_\alpha+) - \psi(x_\alpha-)|$ and $\dot{x}_\alpha = \dot{\bar{x}}_\alpha$ for corresponding rarefactions and non-physical shocks in ψ and $\bar{\psi}$.

On the other hand, we already know from [3,6] that

$$\left. \frac{d}{dt}\Phi(u(\cdot, t), \bar{\psi}(\cdot, t)) \right|_{t=t^*} = \mathcal{O}(\nu).$$

Since t^* is arbitrary, we have the estimate

$$\frac{d}{dt}\Phi(u(\cdot, t), \psi(\cdot, t)) \leq K \sum_{\alpha} |\psi(x_\alpha+, t) - \psi(x_\alpha-, t)| |\dot{x}_\alpha - \dot{\bar{x}}_\alpha| + \mathcal{O}(\nu), \quad (34)$$

with the sum over shocks of ψ at time $t \in (t_j, t_{j+1})$.

Case 2. When $t = t_j$ with an interaction, it is easy to see

$$\Phi(u(\cdot, t+), \psi(\cdot, t+)) \leq \Phi(u(\cdot, t-), \psi(\cdot, t-)).$$

In fact, we still find the v -approximate solution $\bar{\psi}$ in a short open time interval including t_j , starting from the profile of $\psi(\cdot, t_j)$. Then it is clear that

$$\Phi(u(\cdot, t+), \psi(\cdot, t+)) = \Phi(u(\cdot, t+), \bar{\psi}(\cdot, t+)) \leq \Phi(u(\cdot, t-), \bar{\psi}(\cdot, t-)) = \Phi(u(\cdot, t-), \psi(\cdot, t-)).$$

Here using the Lax entropy condition, when we trace back the solution in time at any interaction between two waves in ψ , we know incoming waves of ψ and $\bar{\psi}$ share the same configuration (see [9]). $\square$

5. Proof of Theorem 3

Here we present the proof of our main theorem. For brevity and clarity, we leave out some very technical details and refer the reader to [12], [9, Section 6], and [20, Section 6].

5.1. Step 1: Dissipation computation

Given a fixed $v^0 \in \mathcal{S}_{BV, \epsilon}^0$ for $\epsilon > 0$ sufficiently small there exists a sequence of front tracking approximations v_v from [3, Chapter 7] such that $v_v(\cdot, t) \rightarrow v(\cdot, t)$ in L^2 for all t , including $v_v(\cdot, 0) \rightarrow v^0$ in L^2 as $v \rightarrow 0$. Fixing our parameter v , let ψ be a shifted front tracking solution as described in [12] such that $\psi(\cdot, 0) = v_v(\cdot, 0)$. Associated to this shifted approximation ψ , we also fix a weight $a(x, t)$ as defined in Section 3.

Given consecutive interaction times of waves $t_j < t_{j+1}$ for our shifted solution ψ we define $h_1, \dots, h_N: [t_j, t_{j+1}] \rightarrow \mathbb{R}$ to be the positions of the discontinuities of ψ , where $h_i < h_{i+1}$ for all i . Furthermore, since we are restricting ourselves to the information cone we are able to define

$$h_0(t) := -R + s(t - \tau), \quad h_{N+1}(t) := R - s(t - \tau) \quad (35)$$

where $s > 0$ is our speed of information, taken sufficiently large to verify

$$|q(a; b)| \leq s\eta(a|b) \quad (36)$$

for all states $a \in \mathcal{V}$ and states b attained by our shifted approximation ψ . We further suppose that s is larger than $\hat{\lambda}$, which results in h_0, h_{N+1} moving faster than any artificial shift constructed by Proposition 5, rarefaction front, or non-physical front.

For $t \in [t_j, t_{j+1}]$, in any fixed quadrilateral

$$Q = \{(x, r) \mid t_j < r < t, h_i(r) < x < h_{i+1}(r)\} \quad (37)$$

the functions $\psi|_Q$ and $a|_Q$ are both constant. Integrating (3) over Q with entropy $\eta(\cdot | \psi|_Q)$, entropy-flux $q(\cdot; \psi|_Q)$, and using that u has the Strong Trace Property (Definition 2) we find²

$$\begin{aligned} \int_{h_i(t)}^{h_{i+1}(t)} a(x, t-) \eta(u(x, t) | \psi(x, t)) dx \\ \leq \int_{h_i(t_j)}^{h_{i+1}(t_j)} a(x, t_j+) \eta(u(x, t_j) | \psi(x, t_j)) dx + \int_{t_j}^t F_i^+(r) - F_{i+1}^-(r) dr, \end{aligned} \quad (38)$$

where the various F_i^+, F_i^- are the right/left parts of the dissipation functional (14) for the fronts bounding the left/right edges of Q . Summing all quadrilaterals left-to-right between t_j, t we can

²We remark that to simplify presentation of this proof we are ignoring a technical issue of stopping and restarting the clock, which requires use of approximate limits in this computation. See [11, Section 7] for discussion of this technicality.

now pair the traces at each front (matching the left- and right-sided dissipation, as in (14)), and by virtue of the bounds in Section 2 we find

$$\begin{aligned}
& \int_{-R-s(t-\tau)}^{R+s(t-\tau)} a(x, t-\tau) \eta(u(x, t) | \psi(x, t)) dx \\
& \leq \int_{h_0(t_j)}^{h_{N+1}(t_j)} a(x, t_j) \eta(u(x, t_j) | \psi(x, t_j)) dx \\
& \quad - \frac{1}{K} \int_{t_j}^t \sum_{i \in \mathcal{S}(r)} |\psi(h_i(r)+, r) - \psi(h_i(r)-, r)| (\lambda_i^s - \dot{h}_i(r))^2 dr \quad (39) \\
& \quad + K(t - t_j) \left[v + \sum_{i \in \mathcal{NP}(t)} |\sigma_i| \right].
\end{aligned}$$

where $\mathcal{S}(t)$, $\mathcal{NP}(t) \subset \{1, \dots, N\}$ correspond to the fronts h_i of shocks and non-physical fronts of ψ at time t , respectively, and $\lambda_i^s = \lambda_{\alpha_i}(\psi(h_i(r)+, r), \psi(h_i(r)-, r))$ is the Rankine–Hugoniot velocity of the i -th shock, which is of family $\alpha_i \in \{1, 3\}$. We remark that $\mathcal{S}(t)$, $\mathcal{NP}(t)$ are constant between times of interaction and that the errors caused by perturbing wave velocities (to prevent more than two waves from interacting at once) and discretized rarefactions are bounded by $Kv(t - t_j)$ in the above expression.

Fixing $\tau \in [0, T]$ we now sum (39) over each time interval $[t_j, t_{j+1}]$ where $t_1 < \dots < t_{J-1}$ are consecutive times of front interaction prior to τ and $t_{J-1} \leq t_J := \tau$. Using the time decay of our weight a (25) and the convexity of η , we receive the bound

$$\begin{aligned}
& \int_{-R}^R a(x, \tau) \eta(u(x, \tau) | \psi(x, \tau)) dx \\
& \leq \int_{-R-s\tau}^{R+s\tau} a(x, 0) \eta(u(x, 0) | \psi(x, 0)) dx \\
& \quad - \frac{1}{K} \int_0^\tau \sum_{i \in \mathcal{S}(r)} |\psi(h_i(r)+, r) - \psi(h_i(r)-, r)| (\lambda_i^s - \dot{h}_i(r))^2 dr \quad (40) \\
& \quad + K \left[v + \sup_{t \in [0, \tau]} \sum_{i \in \mathcal{NP}(t)} |\sigma_i| \right],
\end{aligned}$$

where the v term contains all errors due to the discretization of rarefaction fans and perturbation of wave speeds to avoid multiple simultaneous interactions in ψ .

5.2. Step 2: Bound on shifting

Recalling that $v_v(\cdot, 0) = \psi(\cdot, 0)$ and using Lemma 9, we have

$$\Phi(v_v(\cdot, \tau), \psi(\cdot, \tau)) \leq K \int_0^\tau \sum_{i \in \mathcal{S}(r)} |\psi(h_i(r)-, r) - \psi(h_i(r)+, r)| |\lambda_i^s - \dot{h}_i(r)| + \mathcal{O}(v),$$

where $\dot{h}_i$ is the shifted velocity of the i -th shock front in ψ . The Cauchy–Schwartz inequality gives us

$$\begin{aligned}
& \Phi(v_v(\cdot, \tau), \psi(\cdot, \tau)) \\
& \leq K \left[\int_0^\tau \sum_{i \in \mathcal{S}(r)} |\psi(h_i(r)-, r) - \psi(h_i(r)+, r)| dr \right]^{1/2} \\
& \quad \times \left[\int_0^\tau \sum_{i \in \mathcal{S}(r)} |\psi(h_i(r)-, r) - \psi(h_i(r)+, r)| |\lambda_i^s - \dot{h}_i(r)|^2 dr \right]^{1/2} + \mathcal{O}(v) \quad (41)
\end{aligned}$$

$$\leq K\sqrt{\epsilon} \left[\int_{-R-s\tau}^{R+s\tau} a(x,0)\eta(u(x,0) \mid \psi(x,0)) dx + K \left[v + \sup_{t \in [0,\tau]} \sum_{i \in \mathcal{N}} |\sigma_i| \right] \right]^{1/2} + \mathcal{O}(v)$$

where the last bound follows by (40) and the uniform bound on $\|\psi(\cdot, t)\|_{BV(\mathbb{R})}$ from Proposition 8.

5.3. Step 3: Putting it together

The triangle inequality gives us

$$\|u(\cdot, \tau) - v_v(\cdot, \tau)\|_{L^1((-R, R))} \leq \|u(\cdot, \tau) - \psi(\cdot, \tau)\|_{L^1((-R, R))} + \|\psi(\cdot, \tau) - v_v(\cdot, \tau)\|_{L^1((-R, R))}. \quad (42)$$

Using the inclusion $L^2 \hookrightarrow L^1$ for finite measure spaces we find

$$\|u(\cdot, \tau) - v_v(\cdot, \tau)\|_{L^1((-R, R))} \leq \sqrt{2R} \|u(\cdot, \tau) - \psi(\cdot, \tau)\|_{L^2((-R, R))} + \|\psi(\cdot, \tau) - v_v(\cdot, \tau)\|_{L^1((-R, R))}.$$

We further note that by $v_v(\cdot, 0) = \psi(\cdot, 0)$, the equivalence (27), dissipation estimate (40), Lemma 4, and the bound (41) that

$$\|u(\cdot, \tau) - v_v(\cdot, \tau)\|_{L^1((-R, R))} \leq K \left[\|u(\cdot, 0) - v_v(\cdot, 0)\|_{L^2((-R-s\tau, R+s\tau))}^2 + K \left[v + \sup_{t \in [0,\tau]} \sum_{i \in \mathcal{N}} |\sigma_i| \right] \right]^{1/2} + \mathcal{O}(v).$$

Finally, recalling that $\|f\|_{L^2} \leq \sqrt{\|f\|_{L^1} \|f\|_{L^\infty}}$ and taking the limit $v \rightarrow 0$ we arrive at

$$\|u(\cdot, \tau) - v(\cdot, \tau)\|_{L^2((-R, R))} \leq K \sqrt{\|u(\cdot, 0) - v(\cdot, 0)\|_{L^2((-R-s\tau, R+s\tau))}}, \quad (43)$$

where we note the term summing the strength of non-physical fronts converges to zero (see [1, Lemma 3.1]).

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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