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Volume 364 (2026), p. 665-672

Online since: 6 October 2026

<https://doi.org/10.5802/crmath.857>

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www.centre-mersenne.org — e-ISSN : 1778-3569



Research article
Combinatorics

Sumset problem for distinct sets of integers

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Abstract. Let A and B be two non-empty finite subsets of integers and $n, k \in \mathbb{Z}^+$. The sum of dilates of sets A and B can be defined: $n \cdot A + k \cdot B = \{na + kb : a \in A, b \in B\}$. Under certain assumptions, in 2017, Mistri [Notes Number Theory Discrete Math. **23** (2017), no. 4] proved that $|A + k \cdot B| \geq 4|A| - 4$, where A and B are non-empty finite subsets of integers and $|k| \geq 3$. In this article, we consider the sumset $p \cdot A + k \cdot B$, with $k(> p) \in \mathbb{Z}^+$, where p is an odd prime and $(p, k) = 1$, and prove that $|p \cdot A + k \cdot B| \geq 4|A| - 4$, where A and B are non-empty finite subsets of integers satisfying some properties. We also rectify the proof of Kaur and Singh [Comptes Rendus. Mathématique **362** (2024)] and generalize this extended inverse result for arbitrary values of $k \in \mathbb{Z}^+ (\geq 2)$.

Keywords. Sum of dilates, additive combinatorics, sumsets.

2020 Mathematics Subject Classification. 11B13, 11B75.

Funding. The research is supported by NBHM, Department of Atomic Energy, Government of India, under the research project (Sanction No. 02011/49/2023/R&D-II/13983).

Manuscript received 9 October 2025, revised 11 August 2026, accepted 27 August 2026, online since 6 October 2026.

1. Introduction and preliminaries

Let A and B be two non-empty finite subsets of integers. For some $n \in \mathbb{Z}^+$, n dilations of the set A is denoted by $n \cdot A$ and can be defined as $n \cdot A = \{na : a \in A\}$. For any positive integers n and k , the sum of dilates of A and B can be defined as $n \cdot A + k \cdot B = \{na + kb : a \in A, b \in B\}$. For $n = k = 1$, the sumset $A + B = \{a + b : a \in A, b \in B\}$ is called the Minkowski sumset. In additive number theory, problems involving sumsets are generally classified into three categories: a direct problem, an inverse problem, and an extended inverse problem. A direct problem for the sum of dilates $n \cdot A + k \cdot B$ is to find the lower bound for the cardinality of $n \cdot A + k \cdot B$ and an inverse problem is to find the structure of the sets A and B when the lower bound is given. An extended inverse problem for this sumset is to determine the structure of the sets A and B when the arbitrary value of $|n \cdot A + k \cdot B|$ is given. Several researchers have investigated the above types of sumset problems and proved many important results in this field, thereby opening up new directions for future research. In 1935, Davenport [9] proved a result for a prime number p and two non-empty finite subsets $A, B \subseteq \mathbb{Z}_p$,

$$|A + B| \geq \min\{p, |A| + |B| - 1\},$$

which is also known as the Cauchy–Davenport theorem [4,9]. In the same year, Chowla [6] generalized this result and proved that, for an integer $n \geq 2$ and two non-empty finite subsets

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$A, B \subseteq \mathbb{Z}_n$, $|A+B| \geq \min\{m, |A|+|B|-1\}$. In 1996, Nathanson [18] proved that $|A+B| \geq |A|+|B|-1$, where A and B are non-empty finite subsets of integers and equality holds if and only if A and B are in arithmetic progressions with the same common difference. In 2008, Bukh [3] proved that for any finite subset A of integers and $(\alpha_1, \alpha_2, \dots, \alpha_h) \in \mathbb{Z}^k$, such that $(\alpha_1, \alpha_2, \dots, \alpha_h) = 1$, then

$$|\alpha_1 \cdot A + \alpha_2 \cdot A + \dots + \alpha_h \cdot A| \geq (|\alpha_1| + |\alpha_2| + \dots + |\alpha_h|)|A| - o(|A|).$$

In 2009, Cilleruelo et al. [7] proved that $|A+k \cdot A| \geq (k+1)|A| - \lceil (k^2+2k)/4 \rceil$, where $k = p$ (p is any prime) and $|A| \geq 3(k-1)^2(k-1)!$. In 2010, Cilleruelo et al. [8] proved that $|A+3 \cdot A| \geq 4|A|-4$ and the equality holds for $A = \{0, 1, 3\}$ or $\{0, 1, 4\}$ or $3 \cdot \{0, \dots, n\} \cup (3 \cdot \{0, \dots, n\} + 1)$ and for all affine transforms of these sets. They also conjectured that if A is an arbitrarily large subset of integers and $k \in \mathbb{Z}^+$, then $|A+k \cdot A| \geq (k+1)|A| - \lceil (k^2+2k)/4 \rceil$. In 2014, Du et al. [10] confirmed the conjecture for a non-empty finite subset A of integers with $|A| \geq (k-1)^2 k!$ and $k = p^\eta$ (p is any prime), $\eta \in \mathbb{Z}^+$. They also settled the conjecture for $k = pq$ (where p and q are distinct primes). In 2024, Kaur et al. [13] proved the conjecture for all positive integers k except of the form $2^{n_1} p^{n_2}$ or $2^{n_1} 3^{n_2} 5^{n_3}$, where $n_i \geq 1$ for $i = 1, 2, 3$ and p is any odd prime. Several other authors are interested in finding the cardinality of $n \cdot A + k \cdot A$ for specific integer values of n and k . In 2011, Hamidoune and Rue [12] considered the case $n = 2$ and proved that $|2 \cdot A + k \cdot A| \geq (k+2)|A| - k^2 - k + 2$ with $|A| > 8k^k$ (k is an odd prime), where A is a finite subset of $\mathbb{Z}$. In 2013, Ljubic [16] proved the same bound for k to be power of an odd prime and product of two distinct odd primes. In 2014, Balog and Shakan [1] obtained that $|n \cdot A + k \cdot A| \geq (n+k)|A| - (nk)^{(n+k-3)(n+k)+1}$, where $A \subseteq \mathbb{Z}$ and $n < k$ are relatively primes. In 2014, Freiman et al. [11] proved that $|A+k \cdot A| \geq 4|A|-4$ for any subset A of $\mathbb{Z}$. In 2019, Bhanja et al. [2] gave the alternative proof of the inequality $|A+k \cdot A| \geq 4|A|-4$, $k \geq 3$ and also proved extended inverse theorem for $|A+2 \cdot B|$. In 2020, Chahal and Pandey [5] considered the case when $n = 3$ and $k \geq 5$ be a prime number, and also proved the result for the cardinality of $n \cdot A + k \cdot A$, where $k > n$ is any odd prime. In 2017, Mistri [17] proved that for non-empty finite subsets A and B of integers with $|k| \geq 3$ under some conditions, $|A+k \cdot B| \geq 4|A|-4$.

Throughout the paper, we use the following notations and assumptions.

Let $A = \{x_0 < x_1 < \dots < x_{q-1}\}$ and $B = \{y_0 < y_1 < \dots < y_{s-1}\}$ be finite subsets of integers. Let $\ell(A) = \max(A) - \min(A)$ and $d(A) = \gcd(x_1 - x_0, x_2 - x_0, \dots, x_{q-1} - x_0)$, where $q \geq 2$. By, h_A we denote the number of holes in the set A , $h_A = \ell(A) - |A| + 1$.

Let $\hat{A} = \{\bar{x} = x + k\mathbb{Z} : x \in A\}$ and $|\hat{A}| = r$ such that $A_1, A_2, \dots, A_r$ be the distinct congruence classes of A modulo k .

In 2024, Kaur and Singh [14] proved that $|A+k \cdot B| \geq 3|A|+|B|-4$ for $k \in \mathbb{Z}^+ (\geq 3)$, A and B are non-empty finite subsets of integers with some certain conditions. They also proved the extended inverse result, given below.

Theorem 1 ([14, Theorem 2]). *Let $A, B \subseteq \mathbb{Z}$ be non-empty finite subsets such that $|\hat{A}| = r$ and $0 \in A, B$ with properties:*

- (1) $d(A) = d(B) = 1$;
- (2) $\ell(A) \leq \ell(B)$;
- (3) $h_A \leq h_B$.

If $|A+3 \cdot B| = |A| + r(|B|-1) + t \leq 2|A| + r(|B|-2)$ for some integer t , then both the sets A and B are subsets of arithmetic progressions of length at most $|B|+t = |A+3 \cdot B| - |A| - (r-1)|B| + r \leq |A|+|B|-3$.

Observe that the above result is not true for any arbitrary set. Consider the following example.

Example 2. Take $A = \{0, 1\}$ and $B = \{0, 1, 3\}$. These sets satisfy all the conditions of [14, Theorem 2]. Now $|A+3 \cdot B| = 6$, so $6 = 2 + 2(3-1) + t \leq 4 + 2(3-2) = 6$, the condition holds. Then by [14, Theorem 2], both the sets A and B are subsets of arithmetic progressions of length at most $3+0 \leq 2+3-3$, which is not true. Hence [14, Theorem 2] is not valid in general.

In this paper, we prove a direct result for the sumset $p \cdot A + k \cdot B$ and an extended inverse result for the sumset $A + k \cdot B$. As a consequence, we rectify the error in the extended inverse result [14, Theorem 2] and improve this result. We will frequently use the following well-known results of Liv [15] and Stanchescu [19] to prove our main results.

Theorem 3 ([15,19]). *Let A and B be finite subsets of $\mathbb{N}$ such that $0 \in A \cap B$. Define*

$$\delta_{A,B} = \begin{cases} 1 & \text{if } \ell(A) = \ell(B), \\ 0 & \text{if } \ell(A) \neq \ell(B). \end{cases}$$

Then the followings holds:

- (1) *if $\ell(A) = \max(\ell(A), \ell(B)) \geq |A| + |B| - 1 - \delta_{A,B}$ and $d(A) = 1$, then*

$$|A + B| \geq |A| + 2|B| - 2 - \delta_{A,B};$$

- (2) *if $\max(\ell(A), \ell(B)) \leq |A| + |B| - 2 - \delta_{A,B}$, then*

$$|A + B| \geq (|A| + |B| - 1) + \max(h_A, h_B) = \max(\ell(A) + |B|, \ell(B) + |A|).$$

2. Direct problem

In this section, we prove the result related to a direct problem for the sumset $p \cdot A + k \cdot B$.

Theorem 4. *Let $k \in \mathbb{Z}^+$ such that $k > p$ (p is odd prime), $(p, k) = 1$ and $A, B \subseteq \mathbb{Z}$ are non-empty finite subsets satisfying the following conditions:*

- (1) $|A| \leq |B|$ and $l(A) \leq l(B)$;
- (2) $d(A) = d(B) = 1$ if $|A|, |B| \geq 2$;
- (3) $h_A \leq h_B$.

Then $|p \cdot A + k \cdot B| \geq 4|A| - 4$.

Proof. Let $B = \{y_0 < y_1 < \dots < y_{s-1}\}$ be an arbitrary subset of integers. We prove the result by considering different cases depending upon the cardinality of A .

If $|A| = 1$, the result holds trivially. Assume that $A = \{x_0, x_1\}$ such that $x_0 < x_1$. Then

$$p \cdot A + k \cdot B = \{px_0 + ky_j, px_1 + ky_j : 1 \leq j \leq (s-1)\}.$$

Clearly $px_0 + ky_0, px_1 + ky_0, px_0 + ky_{s-1}, px_1 + ky_{s-1}$ are four different elements of $p \cdot A + k \cdot B$. Hence $|p \cdot A + k \cdot B| \geq 4 = 4|A| - 4$ and the result holds true for $|A| = 2$.

Assume that $|A| \geq 3$. Since the affine transformation of sets A and B does not affect the cardinality of $p \cdot A + k \cdot B$. Without loss of generality, we assume that A and B are subsets of non-negative integers with $0 \in A \cap B$. Let $A_1, A_2, \dots, A_r$ be distinct congruence classes of A modulo k such that $A = A_1 \cup A_2 \cup \dots \cup A_r$. Now

$$|p \cdot A + k \cdot B| = \sum_{i=1}^r |p \cdot A_i + k \cdot B| \geq \sum_{i=1}^r |p \cdot A_i| + |k \cdot B| - 1 = |A| + r(|B| - 1).$$

If $r > 2$, then $|p \cdot A + k \cdot B| \geq |A| + 3(|B| - 1) \geq 4|A| - 4$, and the result holds true.

Assume that $r = 2$.

Let $|B| \geq \frac{3}{2}|A| - 1$, then $|p \cdot A + k \cdot B| \geq |A| + 2(|B| - 1) \geq |A| + 2(\frac{3}{2}|A| - 1 - 1) = 4|A| - 4$. Next, suppose that $|B| < \frac{3}{2}|A| - 1$.

Let $A = A_1 \cup A_2$, where A_1 and A_2 are non-empty and $A_1 \cap A_2 = \emptyset$. Suppose $|A_1| = m_1 \geq 1$, $|A_2| = m_2 \geq 1$ and $|A| = m_1 + m_2 = m$ (say). Without loss of generality, we assume that $m_1 \geq m_2$. Define $A_1^* = \frac{p}{k}(A_1 - \min(A_1))$ and $A_2^* = \frac{p}{k}(A_2 - \min(A_2))$. Clearly $|A_i| = |A_i^*|$ for $i = 1, 2$. Thus,

$$\begin{aligned} |p \cdot A + k \cdot B| &= |p \cdot (A_1 \cup A_2) + k \cdot B| \\ &= |(p \cdot A_1 + k \cdot B) \cup (p \cdot A_2 + k \cdot B)| \\ &= |p \cdot A_1 + k \cdot B| + |p \cdot A_2 + k \cdot B| - |(p \cdot A_1 + k \cdot B) \cap (p \cdot A_2 + k \cdot B)| \\ &= |p \cdot A_1 + k \cdot B| + |p \cdot A_2 + k \cdot B| \\ &= |A_1^* + B| + |A_2^* + B|. \end{aligned}$$

Therefore, $\ell(A_i) \geq k(m_i - 1) > p(m_i - 1) \geq 3(m_i - 1)$ and $\ell(A_i^*) = \frac{p}{k}\ell(A_i) < \ell(A_i) \leq \ell(A) \leq \ell(B)$. To prove the result, we will consider the following cases.

Case 1. Suppose $m_2 = 1$.

Let $\ell(B) \geq |B| + |A_1^*| - 1$. By Theorem 3, we have $|B + A_1^*| \geq |B| + |A_1^*| - 2$. Clearly $|B + A_2^*| \geq |B|$. Thus, $|m \cdot A + k \cdot B| = |A_1^* + B| + |A_2^* + B|$ and $|p \cdot A + k \cdot B| \geq 4m - 4$.

Now, let $\ell(B) \leq |B| + |A_1^*| - 1$. By Theorem 3, $|B + A_1^*| \geq \ell(B) + |A_1^*|$. Thus $|B + A_1^*| \geq \ell(B) + |A_1^*| \geq 3m_1 - 3 + m_1 = 4m_1 - 3$. Hence

$$|p \cdot A + k \cdot B| \geq 4m_1 + |B| - 3 \geq 4m_1 + 4m_2 + |B| - 3 - 4 = 4m - 4.$$

Case 2. Suppose $m_1 \geq m_2 \geq 2$.

We have three subcases: either $\ell(B) \geq |B| + |A_1^*| - 1$ or $|B| + |A_2^*| - 1 \leq \ell(B) \leq |B| + |A_1^*| - 2$ or $\ell(B) \leq |B| + |A_2^*| - 2$.

Subcase 2(i). Let $\ell(B) \geq |B| + |A_1^*| - 1 \geq |B| + |A_2^*| - 1$. By Theorem 3, $|B + A_i^*| \geq |B| + 2|A_i^*| - 2 = m + 2m_i - 2$ for $i = 1, 2$. Therefore,

$$|p \cdot A + k \cdot B| = |B + A_1^*| + |B + A_2^*| \geq (m + 2m_1 - 2) + (m + 2m_2 - 2) = 4m - 4.$$

Subcase 2(ii). Assume that $|B| + |A_2^*| - 1 \leq \ell(B) \leq |B| + |A_1^*| - 2$.

Then $|A_1^*| - 2 \geq |A_2^*| - 1$ and $m_1 \geq m_2 + 1$. By Theorem 3,

$$|B + A_1^*| \geq \ell(B) + |A_1^*| \geq \ell(A) + |A_1^*| \geq 3m_1 - 3 + m_1 = 4m_1 - 3,$$

and

$$|B + A_2^*| \geq |B| + 2|A_2^*| - 2 \geq m + 2m_2 - 2 \geq m_1 + 3m_2 - 2 \geq 4m_2 - 1.$$

Therefore,

$$|p \cdot A + k \cdot B| = |B + A_1^*| + |B + A_2^*| \geq 4m_1 - 3 + 4m_2 - 1 = 4m - 4.$$

Subcase 2(iii). Let $\ell(B) \leq |B| + |A_2^*| - 2$. We know that $\ell(B) \geq \ell(A) \geq \ell(A_i) \geq \ell(A_i^*)$ for $i = 1, 2$. Thus

$$|p \cdot A + k \cdot B| = |B + A_1^*| + |B + A_2^*| \geq \ell(B) + |A_1^*| + \ell(B) + |A_2^*| = 2\ell(B) + m.$$

If $m_1 = m_2$, then $\ell(B) \leq |B| + |A_2^*| - 2 < \frac{3}{2}|A| - 1 + |A_2^*| - 2 = 2m - 3$. Hence

$$\frac{3}{2}m - 3 = 3m_i - 3 \leq \ell(A_i) \leq \ell(A) \leq \ell(B) \leq 2m - 4 \text{ for } i = 1, 2.$$

We claim that $\ell(B) \geq \frac{3}{2}n - 2$. Indeed, if $\ell(B) \leq \frac{3}{2}n - 3$, then

$$\ell(A_i) = \ell(A) = \max(A) \quad \text{for } i = 1, 2.$$

Also $\max(A) \notin A_i$ for some i , it follows that $\ell(A_i) < \ell(A)$, which is a contradiction. Hence our claim is true. Thus

$$|p \cdot A + k \cdot B| \geq 2\ell(B) + m \geq 2\left(\frac{3}{2}m - 2\right) + m = 4m - 4.$$

Let $m_1 \neq m_2$. Then $m_1 \geq \frac{m+1}{2}$ and

$$|p \cdot A + k \cdot B| \geq 2\ell(B) + m \geq 2(3m_1 - 3) + m \geq 6\left(\frac{m+1}{2}\right) + m - 6 \geq 4m - 4.$$

This completes the proof of the theorem. $\square$

3. Extended inverse problem

In this section, we prove the following extended inverse problem of the sumset $A + k \cdot B$.

Theorem 5. Let $A, B \subseteq \mathbb{Z}$ be non-empty finite subsets and $k \in \mathbb{Z}^+$ ($k \geq 2$). Let $|\widehat{A}| = r \geq 2$, $|A| \geq r + 1$ and $0 \in A \cap B$ such that:

- (1) $\ell(A) \leq \ell(B)$;
- (2) $d(A) = d(B) = 1$;
- (3) $h_A \leq h_B$.

If $|A + k \cdot B| = |A| + r(|B| - 1) + t < 2|A| + r(|B| - 2)$ for some integer t , then both the sets A and B are subsets of arithmetic progressions of length at most $|B| + t = |A + k \cdot B| - |A| - (r - 1)|B| + r \leq |A| + |B| - (r + 1)$.

Proof. Let $A = \{x_0 < x_1 < \dots < x_{q-1}\}$ and $B = \{y_0 < y_1 < \dots < y_{s-1}\}$ be arbitrary subsets of integers. Without loss of generality, assume that $x_0 = y_0 = 0$. Let $A_1, A_2, \dots, A_r$ be the distinct congruence classes of A modulo k such that $A = A_1 \cup A_2 \cup \dots \cup A_r$, where $A_i \subseteq k\mathbb{Z} + (i - 1)$, for all $1 \leq i \leq r$. Since $0 \in A_1 \subseteq A$ and $|A_i| = m_i$ for all $1 \leq i \leq r$.

For $1 \leq i \leq r$, define

$$A_r = \{kx_{r,1} + (r - 1) < kx_{r,2} + (r - 1) < \dots < kx_{r,m_r} + (r - 1)\}.$$

Let

$$A'_r = \frac{1}{k}(A_r - (r - 1) - x_{r,1}) = \{0 < x_{r,2} - x_{r,1} < x_{r,3} - x_{r,1} < \dots < x_{r,m_r} - x_{r,1}\}.$$

Clearly $\ell(A'_i) < \ell(A)$ for all $1 \leq i \leq r$. Therefore,

$$\begin{aligned} |A + k \cdot B| &= |(A_1 \cup A_2 \cup \dots \cup A_r) + k \cdot B| \\ &= |A_1 + k \cdot B| + |A_2 + k \cdot B| + \dots + |A_r + k \cdot B| \\ &= |k \cdot A'_1 + k \cdot B| + |k \cdot (A'_2 + x_{2,1}) + 1 + k \cdot B| + \dots + |k \cdot (A'_r + x_{r,1}) + r + k \cdot B| \\ &= |A'_1 + B| + |A'_2 + B| + \dots + |A'_r + B|. \end{aligned}$$

Thus,

$$|A + k \cdot B| = |A'_1 + B| + |A'_2 + B| + \dots + |A'_r + B|. \quad (1)$$

Now, we have following cases depending upon the value of $\max\{m_1, m_2, \dots, m_r\}$.

Case 1. Let $\max\{m_1, m_2, \dots, m_r\} = m_1$. We prove the following claims.

Claim 1(i). $\ell(B) \leq s + m_1 - 2 \leq s + q - 3$.

Proof. On contrary, assume that $\ell(B) \geq s + m_1 - 1 = |B| + |A'_1| - 1$. Since $\ell(B) \geq \ell(A) > \ell(A'_i)$ for $1 \leq i \leq r$, therefore $\delta_{B, A'_i} = 0$ for $1 \leq i \leq r$. Thus

$$\ell(B) \geq s + m_1 - 1 \geq s + m_i - 1 = |B| + |A'_i| - 1.$$

By Theorem 3, $|A'_1 + B| \geq 2m_1 + s - 2$. Also $\ell(B) \geq s + m_i - 1 = |B| + |A'_i| - 1$ for $2 \leq i \leq r$, thus

$$|A'_i + B| \geq 2|A_i| + |B| - 2. \quad (2)$$

By (1) and (2), $|A + k \cdot B| \geq r|B| + 2|A| - 2r = r|B| + 2(|A| - r)$, which contradicts our hypothesis. This proves the claim.

Claim 1(ii). $|A + k \cdot B| \geq |A| + r(|B| - 1) + h_B$.

Proof. By Claim 1(i), $\ell(B) \leq s + m_1 - 2$. By applying Theorem 3, it follows that $|A'_1 + B| \geq (|A'_1| + s - 1) + h_B$. Therefore,

$$\begin{aligned} |A + k \cdot B| &= |A'_1 + B| + |A'_2 + B| + \cdots + |A'_r + B| \\ &\geq (|A'_1| + |B| - 1) + h_B + (|A'_2| + |B| - 1) + \cdots + (|A'_r| + |B| - 1) \\ &\geq (m_1 + s - 1) + h_B + (m_2 + s - 1) + \cdots + (m_r + s - 1) \\ &\geq |A| + r(|B| - 1) + h_B. \end{aligned}$$

Hence $0 \leq h_B \leq |A + k \cdot B| - |A| - r|B| + r = t \leq |A| - (r + 1) = q - (r + 1)$. Also, $B \subseteq \{y_0, y_1, \dots, y_{s-1}\} \subseteq \{0, 1, \dots, y_{s-1}\}$ implies that B is subset of an arithmetic progression of length at most $y_{s-1} + 1 = s + h_B \leq s + t \leq s + q - (r + 1) = |A| + |B| - (\widehat{A} + 1)$.

Also, $\ell(A) \leq \ell(B)$. Therefore the set A is also contained in an arithmetic progression of length at most $s + q - (r + 1) = |A| + |B| - (\widehat{A} + 1)$.

Hence, the result holds in this case.

Case 2. Suppose that $\max\{m_1, m_2, \dots, m_r\} = m_2$.

Claim 2(i). We claim that $\ell(B) \leq s + m_2 - 2 \leq s + q - 3$.

Proof. Assume the claim is false. Then $\ell(B) \geq s + m_2 - 1 = |B| + |A'_2| - 1$. Since $\ell(B) \geq \ell(A) > \ell(A'_i)$ for $1 \leq i \leq r$. Therefore $\delta_{B, A'_i} = 0$ for $1 \leq i \leq r$. By Theorem 3, $|A'_2 + B| \geq 2|m_2| + |B| - 2$ and for $1 \leq i \leq r$ ($i \neq 2$),

$$|A'_i + B| \geq 2|A_i| + |B| - 2. \quad (3)$$

Since m_2 is maximum, $m_2 \geq m_i$ for all i . By (1) and (3),

$$|A + k \cdot B| \geq r|B| + 2|A| - 2r = r|B| + 2(|A| - r).$$

This contradicts our hypothesis and our claim is true.

Claim 2(ii). Here, we claim that $|A + k \cdot B| \geq |A| + r(|B| - 1) + h_B$.

Proof. By Claim 2(i), $\ell(B) \leq s + m_2 - 2$. By Theorem 3, $|A'_2 + B| \geq (|A'_2| + s - 1) + h_B$. Therefore,

$$\begin{aligned} |A + k \cdot B| &= |A'_1 + B| + |A'_2 + B| + \cdots + |A'_r + B| \\ &\geq (|A'_2| + |B| - 1) + h_B + (|A'_1| + |B| - 1) + \cdots + (|A'_r| + |B| - 1) \\ &\geq (m_2 + s - 1) + h_B + (m_1 + s - 1) + \cdots + (m_r + s - 1) \\ &\geq |A| + r(|B| - 1) + h_B. \end{aligned}$$

Thus, $0 \leq h_B \leq |A + k \cdot B| - |A| - r|B| + r = t \leq |A| - (r + 1)$. Hence $B \subseteq \{y_0, y_1, \dots, y_{s-1}\} \subseteq \{0, 1, \dots, y_{s-1}\}$. It follows that B is a subset of an arithmetic progression of length at most $y_{s-1} + 1 = s + h_B \leq s + t \leq s + q - (r + 1) = |B| + |A| - (\widehat{A} + 1)$. Since $\ell(A) \leq \ell(B)$, it implies that the set A is also contained in an arithmetic progression of length at most $s + q - (r + 1) = |B| + |A| - (\widehat{A} + 1)$. Hence, the result holds true in this case.

Using similar arguments, we prove the result for $\max\{m_1, m_2, \dots, m_r\} = m_i$ for any i . $\square$

Recently, in 2024, Kaur and Singh [14, Theorem 2] proved the extended inverse result for the cardinality of the sumset $A + 3 \cdot B$. In Corollary 6, we refine the statement of this result and correct the proof of the extended inverse result [14, Theorem 2].

Corollary 6. Let $A, B \subseteq \mathbb{Z}$ be non-empty finite subsets such that $|\widehat{A}| = r \geq 2$, $|A| \geq r + 1$ and $0 \in A, B$ with properties:

- (1) $\ell(A) \leq \ell(B)$;
- (2) $d(A) = d(B) = 1$;

$$(3) \quad h_A \leq h_B.$$

If $|A + 3 \cdot B| = |A| + r(|B| - 1) + t < 2|A| + r(|B| - 2)$ for some integer t , then both the sets A and B are subsets of arithmetic progressions of length at most $|B| + t = |A + 3 \cdot B| - |A| - (r - 1)|B| + r \leq |A| + |B| - (r + 1)$.

Proof. By Theorem 5, $k = 3$ implies that the sets A and B are subsets of arithmetic progression of length at most $|B| + t = |A + 3 \cdot B| - |A| - (r - 1)|B| + r \leq |A| + |B| - (r + 1)$. $\square$

Acknowledgments

The first author gratefully acknowledges the financial support received from Akal University, Talwandi Sabo, for carrying out this research work.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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