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A note on Gersten’s conjecture for étale cohomology over two-dimensional henselian regular local rings

Une note sur la conjecture de Gersten pour la cohomologie étale sur des anneaux locaux réguliers henséliens à deux dimensions

Makoto Sakagaito

Abstract. We prove Gersten’s conjecture for étale cohomology over two dimensional henselian regular local rings without assuming equi-characteristic. As an application, we obtain the local-global principle for Galois cohomology over mixed characteristic two-dimensional henselian local rings.

Résumé. Nous montrons la conjecture de Gersten pour la cohomologie étale sur des anneaux locaux réguliers henséliens sans supposer de caractère équicaractéristique. En application, nous obtenons le principe local-global pour la cohomologie de Galois sur des anneaux locaux henséliens à deux dimensions de caractéristique mixte.

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1. Introduction

Let $R$ be an equi-characteristic regular local ring, $k(R)$ the field of fractions of $R$, $l$ a positive integer which is invertible in $R$ and $\mu$, the étale sheaf of $l$-th roots of unity. Then the sequence of étale cohomology groups

$$0 \longrightarrow H_{\text{ét}}^{n+1} \left( R, \mu_l^\otimes n \right) \longrightarrow H_{\text{ét}}^{n+1} \left( k(R), \mu_l^\otimes n \right)$$

$$\longrightarrow \bigoplus_{p \in \text{Spec } R \atop \text{ht}(p)=1} H_{\text{ét}}^{n} \left( k(p), \mu_l^\otimes (n-1) \right)$$

$$\longrightarrow \bigoplus_{p \in \text{Spec } R \atop \text{ht}(p)=2} H_{\text{ét}}^{n-1} \left( k(p), \mu_l^\otimes (n-2) \right) \longrightarrow \cdots$$

(1)

is exact by Bloch–Ogus ([2]) and Panin ([10]). Here $k(p)$ is the residue field of $p \in \text{Spec } R$.

By using the exactness of the complex (1) at the first two terms, Harbater–Hartmann–Krashen ([7]) and Hu ([8]) proved the local-global principle as follows.

Let $K$ be a field of one of the following types:

(a) (semi-global case) The function field of a connected regular projective curve over the field of fractions of a henselian excellent discrete valuation ring $A$.

(b) (local case) The function field of a two-dimensional henselian excellent normal local domain $A$.

Then the following question was raised by Colliot-Thélène ([3]):

Let $n \geq 1$ be an integer and $l$ a positive integer which is invertible in $R$. Is the natural map

$$H_{\text{ét}}^{n+1} \left( K, \mu_l^\otimes n \right) \longrightarrow \prod_{v \in \Omega_K} H_{\text{ét}}^{n+1} \left( K_v, \mu_l^\otimes n \right)$$

(2)

injective?

Here $\Omega_K$ is the set of normalized discrete valuations on $K$ and $K_v$ is the corresponding henselization of $K$ for each $v \in \Omega_K$.

Suppose that $A$ is equi-characteristic. Harbater–Hartmann–Krashen ([7, Theorem 3.3.6]) proved that the local-global map (2) is injective in the semi-global case. Later, Hu ([8, Theorem 2.5]) proved that the local-global map (2) is injective in both the semi-global case and the local case by an alternative method.

If the sequence (1) is exact (at the first two terms) in the case where $R$ is a mixed characteristic two-dimensional excellent henselian local ring, then the local-global map (2) is injective even without assuming equi-characteristic (cf. [7, Remark 3.3.7] and [8, Remark 2.6(2)]).

In the case where $R$ is a local ring of a smooth algebra over a (mixed characteristic) discrete valuation ring, the sequence (1) is exact (cf. [6, Theorem 1.2 and Theorem 3.2 b])

In this paper, we show the following result:

**Theorem 1 (Theorem 9).** Let $R$ be a mixed characteristic two-dimensional excellent henselian local ring and $l$ a positive integer which is invertible in $R$. Then Gersten's conjecture for étale cohomology with $\mu_l^\otimes n$ coefficients holds over $\text{Spec } R$. That is, the sequence (1) is exact.

See Remark 8(iii) for the reason why we assume $\dim(R) = 2$ in Theorem 1. We obtain the following result as an application of Theorem 1:

**Theorem 2.** With notations as above, assume that $A$ is mixed characteristic and $l$ is a positive integer which is invertible in $A$.

In both the semi-global case and the local case, the local-global principle for the Galois cohomology group $H^{n+1} \left( K, \mu_l^\otimes n \right)$ holds for $n \geq 1$. That is, the local-global map (2) is injective for $n \geq 1$.

V. Suresh also proved Theorem 2 by an alternative method (cf. [8, Remark in Theorem 1.2]).
1.1. Notations

For a scheme \( X \), \( X^{(i)} \) is the set of points of codimension \( i \), \( k(X) \) is the ring of rational functions on \( X \) and \( \kappa(p) \) is the residue field of \( p \in X \). If \( X = \text{Spec} \ R \), \( k(\text{Spec} \ R) \) is abbreviated as \( k(R) \). The symbol \( \mu_l \) denotes the étale sheaf of \( l \)-th roots of unity.

2. Proof of the main result (Theorem 1)

In this section, we use the following results (Theorem 3 and Theorem 4) repeatedly:

**Theorem 3 (cf. [4, Theorem B.2.1 and Examples B.1.1.(2)].** Let \( A \) be a discrete valuation ring, \( K \) the function field of \( A \) and \( l \) a positive integer which is invertible in \( A \). Then the homomorphism
\[
H^i_{\text{ét}}(A, \mu_l^{\otimes n}) \rightarrow H^i_{\text{ét}}(K, \mu_l^{\otimes n})
\]
is injective for any \( i \geq 0 \).

**Theorem 4 (The absolute purity theorem [5, p. 159, Theorem 2.1.1].** Let \( Y \rightarrow X \) be a closed immersion of noetherian regular schemes of pure codimension \( c \). Let \( n \) be an integer which is invertible on \( X \), and let \( \Lambda = \mathbb{Z}/n \). Then the cycle class (cf. [5, 1.1]) give an isomorphism
\[
\Lambda Y \sim \rightarrow R^i\Lambda(c)[2c]
\]
in \( D^+(Y_{\text{ét}}, \Lambda) \). Here \( D^+(Y_{\text{ét}}, \Lambda) \) is the derived category of complexes bounded below of étale sheaves of \( \Lambda \)-modules on \( Y \).

In this section, we use Theorem 4 in the case where \( \dim X \leq 2 \). In this case, Theorem 4 was proved much earlier by Gabber in 1976. See also [11, §5, Remark 5.6] for a published proof.

**Proposition 5.** Let \( R \) be a henselian regular local ring, \( m \) the maximal ideal of \( R \) and \( K \) the function field of \( R \). Let \( l \) be a positive integer such that \( l \notin m \). Then the homomorphism
\[
H^i_{\text{ét}}(\text{Spec} \ R, \mu_l^{\otimes n}) \rightarrow H^i_{\text{ét}}(\text{Spec} \ K, \mu_l^{\otimes n})
\]
is injective for any \( i \geq 0 \).

**Proof.** We prove the statement by induction on \( \dim(R) \). Let \( R \) be a discrete valuation ring (which does not need to be henselian). Then the homomorphism (3) is injective by Theorem 3.

Assume that the statement is true for a henselian regular local ring of dimension \( d \).

Let \( R \) be a henselian regular local ring of dimension \( d + 1 \), \( a \in m \setminus m^2 \) and \( p = (a) \). Then \( R/p \) is a henselian regular local ring of dimension \( d \) and
\[
k(R/p) = R_p/pR_p
\]
where \( k(R/p) \) is the function field of \( R/p \).

Therefore the diagram
\[
\begin{array}{ccc}
H^i_{\text{ét}}(\text{Spec} R, \mu_l^{\otimes n}) & \rightarrow & H^i_{\text{ét}}(\text{Spec} R_p, \mu_l^{\otimes n}) \\
\downarrow & & \downarrow \\
H^i_{\text{ét}}(\text{Spec} R/p, \mu_l^{\otimes n}) & \rightarrow & H^i_{\text{ét}}(\text{Spec} k(R/p), \mu_l^{\otimes n})
\end{array}
\]
is commutative. Then the left vertical map in the diagram (4) is an isomorphism by [1, p. 93, Theorem (4.9)] and the bottom horizontal map in the diagram (4) is injective by the induction hypothesis. Hence the homomorphism
\[
H^i_{\text{ét}}(\text{Spec} R, \mu_l^{\otimes n}) \rightarrow H^i_{\text{ét}}(\text{Spec} R_p, \mu_l^{\otimes n})
\]
is injective. Moreover the homomorphism

\[ H^i_{\text{ét}}(\text{Spec } R_p, \mu^{\otimes n}_l) \longrightarrow H^i_{\text{ét}}(\text{Spec } K, \mu^{\otimes n}_l) \]

is injective by Theorem 3. Therefore the statement follows.

\[ \square \]

**Proposition 6** (cf. [12, Proposition 4.7]). Let \( R \) be a regular local ring and \( l \) a positive integer which is invertible in \( R \). Suppose that \( \dim(R) = 2 \). Then the sequence

\[ H^i_{\text{ét}}(R, \mu^{\otimes n}_l) \longrightarrow H^i_{\text{ét}}(k(R), \mu^{\otimes n}_l) \oplus \bigoplus_{p \in (\text{Spec } R)^{(1)}} H^{i-1}_{\text{ét}}(k(p), \mu^{(n-1)}_l) \]

is exact for any \( i \geq 0 \).

**Proof.** Let \( A \) be a Dedekind ring, \( q \) a maximal ideal of \( A \). Then

\[ H^{i+1}_q((\text{Spec } A)_{\text{ét}}, \mu^{\otimes n}_l) = H^{i-1}_{\text{ét}}(k(q), \mu^{(n-1)}_l) \]

by Theorem 4. Hence the sequence

\[ H^i_{\text{ét}}(A, \mu^{\otimes n}_l) \longrightarrow H^i_{\text{ét}}(U, \mu^{\otimes n}_l) \longrightarrow \bigoplus_{q \in Z^{(1)}} H^{i-1}_{\text{ét}}(k(q), \mu^{(n-1)}_l) \]

is exact where \( Z \) is a closed subscheme of Spec \( A \) and \( U = \text{Spec } R \setminus Z \). Since

\[ \lim_{\text{Spec } R \setminus Z} H^i_{\text{ét}}(U, \mu^{\otimes n}_l) = H^i_{\text{ét}}(k(A), \mu^{\otimes n}_l) \]

by [9, pp. 88–89, III, Lemma 1.16], the sequence

\[ H^i_{\text{ét}}(A, \mu^{\otimes n}_l) \longrightarrow H^i_{\text{ét}}(k(A), \mu^{\otimes n}_l) \longrightarrow \bigoplus_{q \in (\text{Spec } A)^{(1)}} H^{i-1}_{\text{ét}}(k(q), \mu^{(n-1)}_l) \]

(5)

is exact.

Let \( \mathfrak{m} \) be the maximal ideal of \( R \). Let \( g \in \mathfrak{m} \setminus \mathfrak{m}^2 \), \( p = (g) \) and \( Z = \text{Spec } R / p \). Then \( R / p \) is a regular local ring and we have

\[ H^{i+1}_Z((\text{Spec } R)_{\text{ét}}, \mu^{\otimes n}_l) = H^{i-1}_{\text{ét}}(R / p, \mu^{(n-1)}_l) \]

by Theorem 4.

We consider the commutative diagram

\[ \begin{array}{cccccc}
H^i_{\text{ét}}(R, \mu^{\otimes n}_l) & \longrightarrow & H^i_{\text{ét}}(R_g, \mu^{\otimes n}_l) & \longrightarrow & H^{i-1}_{\text{ét}}(R / p, \mu^{(n-1)}_l) & \longrightarrow 0 \\
0 & \longrightarrow & \text{Ker}(\ast) & \longrightarrow & H^i_{\text{ét}}(R_g, \mu^{\otimes n}_l) & \longrightarrow H^{i-1}_{\text{ét}}(k(R / p), \mu^{(n-1)}_l) \\
\end{array} \]

(6)

where

\[ H^{i-1}_{\text{ét}}(R / p, \mu^{(n-1)}_l) = \text{Im} \left( H^i_{\text{ét}}(R_g, \mu^{\otimes n}_l) \longrightarrow H^{i-1}_{\text{ét}}(R / p, \mu^{(n-1)}_l) \right) \]

and

\[ H^i_{\text{ét}}(R_g, \mu^{\otimes n}_l) = \text{Ker} \left( H^i_{\text{ét}}(k(R_g), \mu^{\otimes n}_l) \longrightarrow \bigoplus_{q \in (\text{Spec } R_g)^{(1)}} H^{i-1}_{\text{ét}}(k(q), \mu^{(n-1)}_l) \right) . \]

Then the rows in the diagram (6) are exact by Theorem 4. Since \( R_g \) is a Dedekind domain, the middle map in the diagram (6) is surjective by (5). Moreover, since

\[ H^{i-1}_{\text{ét}}(R / p, \mu^{(n-1)}_l) \subset H^{i-1}_{\text{ét}}(R / p, \mu^{(n-1)}_l) \]

and \( R / p \) is a discrete valuation ring, the right map in the diagram (6) is injective by Theorem 3. Therefore the statement follows from the snake lemma.

\[ \square \]
Corollary 7. Let $R$ be the henselization of a regular local ring which is essentially of finite type over a mixed characteristic discrete valuation ring. Suppose that $\dim(R) = 2$. Then
\[ H^{n+1}_{\text{Zar}}(R, \mathbb{Z}/l(n)) = 0 \]
for a positive integer $l$ which is invertible in $R$. Here $\mathbb{Z}(n)$ is Bloch’s cycle complex and $\mathbb{Z}/l(n) = \mathbb{Z}(n) \otimes \mathbb{Z}/l$ (cf. [6, p. 779]).

Proof. Let $m$ be the maximal ideal of $R$. Let $g \in m \setminus m^2$ and $p = (g)$. Then the homomorphism
\[ H^{n+1}_{\text{et}}(R, \mu^m_l) \longrightarrow H^{n+1}_{\text{et}}(R, \mu^m_l) \]
is injective by Proposition 5. Hence the homomorphism
\[ H^{n}_{\text{et}}(R, \mu^m_l) \longrightarrow H^{n-1}_{\text{et}}(R/p, \mu^m_l) \]
is surjective by Proposition 4. Therefore the homomorphism
\[ H^{n}_{\text{Zar}}(R, \mathbb{Z}/l(n)) \longrightarrow H^{n-1}_{\text{Zar}}(R/p, \mathbb{Z}/l(n-1)) \]
is surjective by [6, p. 774, Theorem 1.2] and [14]. Moreover the homomorphism
\[ H^{n+1}_{\text{Zar}}(R, \mathbb{Z}/l(n)) \longrightarrow H^{n+1}_{\text{Zar}}(R, \mathbb{Z}/l(n)) \]
is injective by the localization theorem [6, p. 779, Theorem 3.2]. We consider the commutative diagram
\[
\begin{array}{cccc}
H^{n+1}_{\text{Zar}}(R, \mathbb{Z}/l(n)) & \longrightarrow & H^{n+1}_{\text{et}}(R, \mathbb{Z}/l(n)) \\
\downarrow & & \downarrow \\
H^{n+1}_{\text{Zar}}(k(R_g), \mathbb{Z}/l(n)) & \longrightarrow & H^{n+1}_{\text{et}}(k(R_g), \mathbb{Z}/l(n)).
\end{array}
\]
Then the upper map in the commutative diagram (7) is injective by the Beilinson–Lichtenbaum conjecture ([6, p. 774, Theorem 1.2], [14]) and the right map in the commutative diagram (7) is injective by the commutative diagram (6) in the proof of Proposition 6. Hence the homomorphism
\[ H^{n+1}_{\text{Zar}}(R, \mathbb{Z}/l(n)) \longrightarrow H^{n+1}_{\text{Zar}}(k(R_g), \mathbb{Z}/l(n)) \]
is injective and the homomorphism
\[ H^{n+1}_{\text{Zar}}(R, \mathbb{Z}/l(n)) \longrightarrow H^{n+1}_{\text{Zar}}(k(R_g), \mathbb{Z}/l(n)) \]
is also injective. Since
\[ H^{n+1}_{\text{Zar}}(k(R_g), \mathbb{Z}/l(n)) = 0, \]
we have
\[ H^{n+1}_{\text{Zar}}(R, \mathbb{Z}/l(n)) = 0. \]
This completes the proof. \qed

Remark 8.

(i) If $R$ is a local ring of a smooth algebra over a discrete valuation ring, then
\[ H^{n}_{\text{Zar}}(R, \mathbb{Z}/m(n)) = 0 \]
for $i > n$ and any positive integer $m$ (cf. [6, p. 786, Corollary 4.4]).

(ii) If we have
\[ H^{n+1}_{\text{Zar}}(R, \mathbb{Z}/l(n)) = 0 \]
for any regular local ring $R$ which is finite type over a discrete valuation ring and a positive integer $l$ which is invertible in $R$, we can show that the homomorphism
\[ H^{n+1}_{\text{et}}(R, \mu^m_l) \longrightarrow H^{n+1}_{\text{et}}(k(R), \mu^m_l) \]
is injective by a similar argument as in the proof of [13, Theorem 4.2].
(iii) The reason why we assume \( \dim(R) = 2 \) in Proposition 6 and Theorem 9 is that we have to show that the middle map in the diagram (6), i.e., the homomorphism
\[
H_{\text{et}}^{n+1}(R, \mu_l^{\otimes n}) \longrightarrow H_{\text{et}}^{n+1}(R, \mu_l^{\otimes n})'
\]
is surjective for an element \( g \) of \( m \setminus m^2 \). Here \( m \) is the maximal ideal of \( R \) and
\[
H_{\text{et}}^{n+1}(R, \mu_l^{\otimes n})' = \text{Ker}
\[
\bigl(\bigoplus_{q \in (\text{Spec } R_{\text{et}})^{(1)}} H_{\text{et}}^n(k(q), \mu_l^{\otimes (n-1)})\bigr).
\]
If we have
\[
H_{\text{Zar}}^{n+1}(R, \mathbb{Z}/l(n)) = H_{\text{Zar}}^{n+2}(R, \mathbb{Z}/l(n)) = 0
\]
for any regular local ring \( R \) which is finite type over a discrete valuation ring and a positive integer \( l \) which is invertible in \( R \), then
\[
H_{\text{Zar}}^{n+1}(R, \mathbb{Z}/l(n)) = H_{\text{Zar}}^{n+2}(R, \mathbb{Z}/l(n)) = 0
\]
by the localization theorem ([6, p. 779, Theorem 3.2]) and we can show that
\[
H_{\text{et}}^{n+1}(R, \mu_l^{\otimes n}) = \Gamma(\text{Spec } R, R^{n+1} \epsilon_*(\mu_l^{\otimes n})) = H_{\text{et}}^{n+1}(R, \mu_l^{\otimes n})'
\]
and Proposition 6 holds. Here \( \epsilon : (\text{Spec } R)_{\text{et}} \rightarrow (\text{Spec } R)_{\text{Zar}} \) is the change of site maps.

**Theorem 9.** Let \( R \) be a henselian regular local ring with \( \dim(R) = 2 \) and \( l \) a positive integer which is invertible in \( R \). Then the sequence
\[
0 \longrightarrow H_{\text{et}}^i(R, \mu_l^{\otimes n}) \longrightarrow H_{\text{et}}^i(k(R), \mu_l^{\otimes n}) \longrightarrow \bigoplus_{p \in (\text{Spec } R)^{(1)}} H_{\text{et}}^{i-1}(k(p), \mu_l^{\otimes (n-1)}) \longrightarrow \bigoplus_{p \in (\text{Spec } R)^{(2)}} H_{\text{et}}^{i-2}(k(p), \mu_l^{\otimes (n-2)}) \longrightarrow 0 \quad (8)
\]
is exact for any \( i \geq 0 \).

**Proof.** The exactness of the complex (8) at the first two terms follows from Proposition 5 and Proposition 6.

We consider the coniveau spectral sequence
\[
H_{1}^{p,q} = \prod_{x \in (\text{Spec } R)^{(p)}} H_{x}^{p+q}(R, \mu_l^{\otimes n}) \Rightarrow H_{\text{et}}^{p+q}(R, \mu_l^{\otimes n}) = H_{\text{et}}^{p+q}
\]
(cf. [4, §1]). Then we have a filtration
\[
0 \subset H_{p+q}^{p+q} \subset \cdots \subset H_{1}^{p+q} \subset H_{0}^{p+q} = H_{p+q},
\]
such that
\[
H_{p}^{p+q} / H_{p+1}^{p+q} \simeq H^{p,q}_{\infty}.
\]
By Theorem 4, it suffices to show that
\[
H_{2}^{1,i-1} = H_{2}^{2,i-2} = 0.
\]
By Proposition 5, the morphism
\[
H^i \longrightarrow H_{\infty}^0
\]
is injective and
\[
H_{\infty}^i = H_2^i = 0.
\]
Hence we have
\[
H_{\infty}^{1,i-1} = H_{\infty}^{2,i-2} = 0.
\]
Since
\[
H_{r}^{0,i-p+1} = 0
\]
for $p \geq 3$ and
\[ H^{1-r, i+r-2}_r = 0 \]
for $r \geq 2$, we have
\[ H^{i-1}_2 = H^{i-1}_\infty = 0. \]
By the exactness of the complex (8) at the second term, we have
\[ H^{0, i-1}_2 = H^{0, i-1}_\infty = H^{i-1}_2 \]
and
\[ \text{Im} \left( H^{0, i-1}_2 \xrightarrow{d_2} H^{2, i-2}_2 \right) = 0. \]
Hence we have
\[ H^{2, i-2}_2 = H^{2, i-2}_3. \]
Moreover, since
\[ H^{2-r, i+r-3}_r = 0 \]
for $r \geq 3$, we have
\[ H^{2, i-2}_{r+1} = \frac{\text{Ker}(d^{2, i-2}_r)}{\text{Im}(d^{2-r, i+r-3}_r)} = H^{2, i-2}_r \]
for $r \geq 3$. Therefore
\[ H^{2, i-2}_2 = H^{2, i-2}_3 = H^{2, i-2}_\infty = 0. \]
This completes the proof. \qed

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