# Comptes Rendus 

## Mathématique

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Volume 358, issue 6 (2020), p. 721-725
Published online: 2 October 2020
https://doi.org/10.5802/crmath. 90
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Les Comptes Rendus. Mathématique sont membres du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

# Combinatorial property of all positive measures in dimensions 2 and 3 

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#### Abstract

We prove multi-parameter dyadic embedding theorem for Hardy operator on the multi-tree. We also show that for a large class of Dirichlet spaces of holomorphic functions in bi-disc and tri-disc this proves the embedding theorem of those spaces on bi- and tri-disc. We completely describe the Carleson measures for such embeddings. Funding. We acknowledge the support of the following grants: NSF grant DMS-1900286, Theorem 2 was obtained in the frameworks of the Russian Science Foundation grant 17-11-01064-P, the third author was supported also by Alexander von Humboldt foundation.


Manuscript received 10th February 2020, revised 25th June 2020, accepted 26th June 2020.

## Version française abrégée

Un $n$-arbre $T^{n}, n \geq 1$, est un produit cartésien de $n$ arbres dyadiques identiques avec un ordre partiel induit par la structure du produit. Etant donné un point $\beta \in T^{n}$, nous définissons son successeur en posant $\mathscr{S}(\beta)=\left\{\alpha \in T^{n}: \alpha \leq \beta\right\}$. Soient $w, \mu$ deux fonctions positives sur $T^{n}$, nous définissons la constante de boîte comme le plus petit nombre $[\omega, \mu]_{B o x}$ tel que

$$
\begin{equation*}
\mathscr{E}_{\mathscr{S}(\beta)}[\mu]:=\sum_{\alpha \leq \beta} w(\alpha)\left(\mathbf{I}^{*} \mu(\alpha)\right)^{2} \leq[w, \mu]_{B o x} \mu(\mathscr{S}(\beta)), \quad \forall \beta \in T^{n} . \tag{1}
\end{equation*}
$$

La constante de plongement de Carleson est la plus petite constante $[\omega, \mu]_{C E}$ telle que l'inégalité suivante ait lieu:

$$
\begin{equation*}
\mathscr{E}[\psi \mu] \leq[w, \mu]_{C E} \sum_{\omega \in T^{n}}|\psi(\omega)|^{2} \mu(\omega) \tag{4}
\end{equation*}
$$

Le résultat principal de cet article est le théorème suivant:

[^0]Theorem 1. Soit $\mu: T^{n} \rightarrow \mathbb{R}_{+}, n=1,2,3$ et soit $w: T^{n} \rightarrow[0, \infty)$ un poids d'une forme tensorielle. Alors l'inégalité suivante a lieu

$$
[w, \mu]_{C E} \lesssim[w, \mu]_{B o x}
$$

## 1. Hardy inequality on the $n$-tree and energy of measures

A (finite) tree $T$ is a finite partially ordered set such that for every $\omega \in T$ the set $\{\alpha \in T: \alpha \geq \omega\}$ is totally ordered (here we identify the tree with its vertex set). In what follows we consider rooted dyadic trees, i.e. there is a unique maximal element in $T$, and every element (except for the minimal ones) has exactly two children. An $n$-tree $T^{n}, n \geq 1$ is a Cartesian product of $n$ identical dyadic trees with order induced by the product structure. In what follows no estimate will depend on the depth of the tree. A subset $\mathscr{U}$ (resp. $\mathscr{D}$ ) of a partially ordered set $T^{n}$ is called an $u p$-set (resp. down-set) if, for every $\alpha \in \mathscr{U}$ and $\beta \in T$ with $\alpha \leq \beta$ (resp. $\beta \leq \alpha$ ), we also have $\beta \in \mathscr{U}$ (resp. $\beta \in \mathscr{D}$ ). Given a point $\beta \in T^{n}$ we define its successor set $\mathscr{S}(\beta)=\left\{\alpha \in T^{n}: \alpha \leq \beta\right\}$, clearly it is a down-set.

From now on we assume that the weight $w: T^{n} \rightarrow \mathbb{R}_{+}$is fixed. The Hardy operator associated with $w$ is defined by

$$
\mathbf{I}_{w} \phi(\gamma):=\sum_{\gamma^{\prime} \geq \gamma} w\left(\gamma^{\prime}\right) \phi\left(\gamma^{\prime}\right) \quad \text { and } \quad \mathbf{I}^{*} \psi(\gamma)=\sum_{\gamma^{\prime} \leq \gamma} \psi\left(\gamma^{\prime}\right)
$$

For a measure (non-negative function) $\mu$ on $T^{n}$ we define the ( $w$-) potential to be

$$
\mathbf{V}_{w}^{\mu}(\alpha):=\left(\mathbf{I}_{w} \mathbf{I}^{*} \mu\right)(\alpha), \quad \alpha \in T^{n}
$$

again we usually drop the index $w$. Let $E \subset T^{n}$ and $\mu$ be a measure on $T^{n}$. The $E$-truncated energy of $\mu$ is

$$
\mathscr{E}_{E}[\mu]:=\sum_{\alpha \in E}\left(\mathbf{I}^{*} \mu\right)^{2}(\alpha) w(\alpha) .
$$

If $E=T^{n}$, we write $\mathscr{E}[\mu]$ instead, and so

$$
\mathscr{E}[\mu]=\int_{T^{n}} \mathbf{V}^{\mu} \mathrm{d} \mu:=\sum_{\alpha \in T^{n}} \mathbf{V}^{\mu}(\alpha) \mu(\alpha) .
$$

If $E$ is a $\delta$-level set of $\mathbf{V}^{\mu}$ for some $\delta>0$, i.e. $E=\left\{\alpha: \mathbf{V}^{\mu} \leq \delta\right\}$, then we write $\mathscr{E}_{\delta}[\mu]:=\mathscr{E}_{E}[\mu]$.
We define the box constant to be the smallest number $[w, \mu]_{B o x}$ such that

$$
\begin{equation*}
\mathscr{E}_{\mathscr{S}(\beta)}[\mu]:=\sum_{\alpha \leq \beta} w(\alpha)\left(\mathbf{I}^{*} \mu(\alpha)\right)^{2} \leq[w, \mu]_{B o x} \mu(\mathscr{S}(\beta)), \quad \forall \beta \in T^{n} . \tag{1}
\end{equation*}
$$

The Carleson constant is the smallest number $[w, \mu]_{C}$ such that

$$
\begin{equation*}
\mathscr{E}_{\mathscr{D}}[\mu] \leq[w, \mu]_{C} \mu(\mathscr{D}), \quad \forall \mathscr{D} \subset T^{n} \text { down-set. } \tag{2}
\end{equation*}
$$

The hereditary Carleson constant (or restricted energy condition constant, or REC constant) is the smallest constant $[w, \mu]_{H C}$ such that

$$
\begin{equation*}
\mathscr{E}\left[\mu \rrbracket_{E}\right] \leq[w, \mu]_{H C} \mu(E), \quad \forall E \subset T^{n} . \tag{3}
\end{equation*}
$$

Finally the Carleson embedding constant is the smallest constant $[w, \mu]_{C E}$ such that the adjoint embedding

$$
\begin{equation*}
\mathscr{E}[\psi \mu] \leq[w, \mu]_{C E} \sum_{\omega \in T^{n}}|\psi(\omega)|^{2} \mu(\omega) \tag{4}
\end{equation*}
$$

holds for all functions $\psi$ on $T^{n}$. If $[w, \mu]_{C E}<+\infty$, we call $(w, \mu)$ the trace pair for the weighted Hardy inequality on $T^{n}$.

For positive numbers $A, B$, we write $A \lesssim B$ if $A \leq C B$ with an absolute constant $C$, that in particular does not depend on the pair $(w, \mu)$. The inequalities $[w, \mu]_{B o x} \leq[w, \mu]_{C} \leq[w, \mu]_{H C} \leq$ $[w, \mu]_{C E}$ are obvious. The converse inequalities for 1 -trees were proved in [6]. Our main result is the extension to the 2 - and 3 -trees. The reader can see details in preprints [3], [4] and partially in [1] for 2-tree.

Theorem 1. Let $\mu: T^{n} \rightarrow \mathbb{R}_{+}, n=1,2,3$. Let $w: T^{n} \rightarrow[0, \infty)$ be of tensor product form. Then the reverses of the above inequalities also hold:

$$
[w, \mu]_{C E} \lesssim[w, \mu]_{H C} \lesssim[w, \mu]_{C} \lesssim[w, \mu]_{B o x} .
$$

The key element in the proof of all these inequalities is the so-called surrogate Maximum Principle. To elaborate let us consider the one-parameter case. Then $V_{\delta}^{\mu}(\alpha) \leq \delta$ for any $\alpha \in T$, so, in particular, for any measure $\rho$ on $T$ one has trivially

$$
\int_{T} V_{\delta}^{\mu} \mathrm{d} \rho \leq \delta|\rho|
$$

where $|\rho|=\rho(T)$ is the total mass of $\rho$. In higher dimensions the situation is different, i.e. $\mathbf{V}_{\delta}^{\mu}$ can blow up at some points, however one can still measure the size of the blow-up set in potentialtheoretic terms. We conjecture that the following holds for all $n$.
Theorem 2. For $n=1,2,3$ and a measure $\rho$ on $T^{n}$ one has

$$
\int_{T^{n}} \mathbf{v}_{\delta}^{\mu} \mathrm{d} \rho \lesssim(\delta|\rho|)^{\frac{2}{n+1}}\left(\mathscr{E}_{\delta}[\mu] \mathscr{E}_{[ }[\rho]\right)^{\frac{n-1}{n+1}}
$$

## 2. Applications to holomorphic Sobolev spaces on the polydisc

The Hardy inequality can be interpreted with regards to its connections to certain problems in the theory of Hilbert spaces of analytic functions on the poly-disc. These connections actually motivated the study of the Hardy operator in [2], [5] and [7].

We start with some additional notation. Given an integer $n \geq 1$ and $s=\left(s_{1}, \ldots, s_{n}\right) \in \mathbb{R}^{n}$ we consider a Hilbert space $\mathscr{H}_{s}\left(\mathbb{D}^{n}\right)$ of analytic functions on the poly-disc $\mathbb{D}^{n}$ with the norm

$$
\|f\|_{\mathscr{H}_{s}\left(\mathbb{D} \mathbb{D}^{n}\right)}^{2}:=\sum_{k_{1}, \ldots, k_{n} \geq 0}\left|\widehat{f}\left(k_{1}, \ldots, k_{n}\right)\right|^{2}\left(k_{1}+1\right)^{s_{1}} \cdots \cdots\left(k_{n}+1\right)^{s_{n}},
$$

where $f(z)=\sum_{k_{1}, \ldots, k_{n} \geq 0} \widehat{f}\left(k_{1}, \ldots, k_{n}\right) z_{1}^{k_{1}} \cdots . z_{n}^{k_{n}}, \quad z=\left(z_{1}, \ldots, z_{n}\right) \in \mathbb{D}^{n}$. Observe, that, clearly $\mathscr{H}_{s}\left(\mathbb{D}^{n}\right)=\otimes_{j=1}^{n} \mathscr{H}_{s_{j}}(\mathbb{D})$. In particular, the choice $s=(0, \ldots, 0)$ gives a classical Hardy space on the poly-disc, on the other hand $s=(1, \ldots, 1)$ corresponds to the Dirichlet space.
Definition 3. A measure $v$ on $\mathbb{D}^{n}$ is called a Carleson measure for $\mathscr{H}_{s}$, if there exists a constant $C$ such that

$$
\begin{equation*}
\int_{\mathbb{D}^{n}}|f(z)|^{2} \mathrm{~d} v(z) \leq C\|f\|_{\mathscr{P} e_{s}\left(\mathbb{D}^{n}\right)}^{2}, \tag{5}
\end{equation*}
$$

or, in other words, the embedding Id : $\mathscr{H}_{s}\left(\mathbb{D}^{n}\right) \rightarrow L^{2}\left(\mathbb{D}^{n}, \mathrm{~d} v\right)$ is bounded. The smallest constant $C$ such that (5) holds is denoted by $[v]_{s}$.

Trace pairs for the weighted Hardy inequality on $n$-tree and Carleson measures for $\mathscr{H}_{s}$ are closely related. Below we give a brief overview of this relationship. We start by assuming that $s \in(0,1]^{n}$ (so that $\mathscr{H}_{s}\left(\mathbb{D}^{n}\right)$ is a weighted Dirichlet space on the poly-disc), and that $\operatorname{supp} v \subset r \mathbb{D}^{d}$ for some $r<1$ (the latter is just a convenience assumption). It is well known that $\mathscr{H}_{s_{j}}(\mathbb{D}), 1 \leq j \leq n$, is a reproducing kernel Hilbert space (RKHS) with kernel $K_{s_{j}}$ satisfying (possibly after a suitable change of norm) $\left|K_{s_{j}}\right|\left(z_{j}, \zeta_{j}\right) \approx|1-z \bar{\zeta}|^{1-s_{j}}, 0<s_{j}<1,\left|K_{s_{j}}\right|\left(z_{j}, \zeta_{j}\right) \approx \log |1-z \bar{\zeta}|^{-1}, s_{j}=1$. It follows immediately that $\mathscr{H}_{s}\left(\mathbb{D}^{n}\right)$ is RKHS as well, and

$$
K_{s}(z, \zeta)=\prod_{j=1}^{n} K_{s_{j}}\left(z_{j}, \zeta_{j}\right), \quad z, \zeta \in \mathbb{D}^{n}
$$

Going back to the Carleson embedding we see that Id : $\mathscr{H}_{s}\left(\mathbb{D}^{n}\right) \rightarrow L^{2}\left(\mathbb{D}^{n}, \mathrm{~d} v\right)$ is bounded if and only if its adjoint $\Theta$ is bounded as well. Let us compute its action on a function $g \in L^{2}\left(\mathbb{D}^{n}, \mathrm{~d} v\right)$

$$
(\Theta g)(z)=\left\langle\Theta g, K_{s}(z, \cdot)\right\rangle_{\mathscr{H}_{s}\left(\mathbb{D}^{n}\right)}=\left\langle g, K_{s}(z, \cdot)\right\rangle_{L^{2}\left(\mathbb{D}^{n}, \mathrm{~d} v\right)}=\int_{\mathbb{D}^{n}} g(\zeta) \overline{K_{s}(z, \zeta)} \mathrm{d} v(\zeta) .
$$

Hence, for $\Theta$ to be bounded it must satisfy

$$
\begin{equation*}
\|g\|_{L^{2}\left(\mathbb{D}^{n}, \mathrm{~d} v\right)}^{2} \gtrsim\|\Theta g\|_{\mathscr{P}_{s}\left(\mathbb{D}^{n}\right)}=\langle g, \Theta g\rangle_{L^{2}\left(\mathbb{D}^{n}, \mathrm{~d} v\right)}=\int_{\mathbb{D}^{2 n}} g(z) \overline{g(\zeta)} K_{s}(z, \zeta) \mathrm{d} v(z) \mathrm{d} v(\zeta) . \tag{6}
\end{equation*}
$$

Observe now that if $1-s_{j}, j=1, \ldots, n$ is small enough, we can replace the reproducing kernel with its real part.

Lemma 4. For any $n \geq 1$ there exists a number $\varepsilon_{n}>0$ such that if $\sup _{1 \leq j \leq n}\left(1-s_{j}\right) \leq \varepsilon_{n}$, then

$$
\begin{equation*}
\left|K_{s}(z, \zeta)\right| \leq C(n) \Re K_{s}(z, \zeta) . \tag{7}
\end{equation*}
$$

In this case that (6) is equivalent to

$$
\begin{equation*}
\|g\|_{L^{2}\left(\mathbb{D}^{n}, \mathrm{~d} v\right)}^{2} \gtrsim \int_{\mathbb{D}^{2 n}} g(z) g(\zeta) \Re K_{s}(z, \zeta) \mathrm{d} v(z) \mathrm{d} v(\zeta), \quad \forall g: \mathbb{D}^{n} \rightarrow \mathbb{R}_{+} . \tag{8}
\end{equation*}
$$

Notice that, in particular, for $s_{j}=1, j=1, \ldots, n$ the inequality (7) holds for any $n$, meaning that we work with unweighted Dirichlet space on any poly-disc. We assume that the parameter $s$ satisfies the assumptions of Lemma 4.

Consider now the classical Whitney decomposition of $\mathbb{D}$ into dyadic Carleson half-boxes. Clearly there is a one-to-one correspondence between these boxes and the vertices of a dyadic tree $T$. Consequently the Whitney decomposition of $\mathbb{D}^{n}$ generated by Cartesian products of the respective coordinate decompositions can be encoded by vertices of $T^{n}$, i.e. each (multi-)box $q$ corresponds to a point $\alpha_{q} \in T^{n}$, and vice-versa, each $\alpha \in T^{n}$ has a unique counterpart $q_{\alpha}$. As a result we can define a canonical map $\Lambda: \operatorname{Meas}^{+}\left(\mathbb{D}^{n}\right) \rightarrow$ Meas $^{+}\left(T^{n}\right)$ given by $\Lambda v(\alpha)=v\left(q_{\alpha}\right)$. Similarly, given a function $g \in L^{2}\left(\mathbb{D}^{n}, \mathrm{~d} v\right)$ we write $\Lambda g(\alpha):=\frac{1}{v\left(q_{\alpha}\right)} \int_{q_{\alpha}} g(z) \mathrm{d} v(z)$. Observe that any $z \in q_{\alpha}, \zeta \in q_{\beta}$ one clearly has

$$
\left|K_{s}(z, \zeta)\right| \approx \sup _{z \in q_{\alpha}, \zeta \in q_{\beta}}\left|K_{s}(z, \zeta)\right|=: K_{s}(\alpha, \beta) .
$$

It follows that (8) is satisfied, if and only if

$$
\begin{equation*}
\sum_{\alpha \in T^{n}} \sum_{\beta \in T^{n}} \Lambda g(\alpha) \Lambda g(\beta) K_{s}(\alpha, \beta) \Lambda v(\alpha) \Lambda v(\beta) \lesssim\|\Lambda g\|_{L^{2}\left(T^{n}, d \Lambda v\right)}^{2} . \tag{9}
\end{equation*}
$$

Let $w_{j}(\alpha)=\left|\alpha_{j}\right|^{s_{j}-1}, \alpha_{j} \in T^{n}$, where $\left|\alpha_{j}\right|$ is just the length of corresponding dyadic interval, and define a weight $w$ on $T^{n}$ by $w_{s}(\alpha):=\prod_{j=1}^{n} w_{j}\left(\alpha_{j}\right), \alpha=\left(\alpha_{1}, \ldots, \alpha_{j}\right) \in T^{n}$. An elementary computation gives

$$
\left(\mathbf{I}_{w_{s}} \mathbf{l}\right)(\alpha \vee \beta) \lesssim K_{s}(\alpha, \beta),
$$

where $\alpha \vee \beta$ is the least common ancestor of $\alpha$ and $\beta$ in geometry of $T^{d}$, and $\mathbf{l} \equiv 1$. The inverse inequality is generally not true pointwise (due to the difference between hyperbolic geometry on the unit disc and that of a dyadic tree), however it holds "on average", and we use this

$$
\sum_{\alpha \in T^{d}} \sum_{\beta \in T^{n}} \Lambda g(\alpha) \Lambda g(\beta) \mathbf{I}_{w_{s}} \mathbf{l}(\alpha \vee \beta) \Lambda v(\alpha) \Lambda v(\beta)=\int_{T^{n}}\left(\mathbf{I}^{*}(\Lambda g \Lambda v)\right)^{2} \mathrm{~d} w_{s},
$$

to show that (9) follows from (4) with $\psi=\Lambda g, \mu=\Lambda v$ and $w=w_{s}$.
Theorem 5. Let $v$ be a measure on the polydisc $\mathbb{D}^{n}$. Assume that $s \in(0,1]^{n}$ is such that $\sup _{1 \leq j \leq n}\left(1-s_{j}\right) \leq \varepsilon_{n}$, where $\varepsilon_{n}$ is from Lemma 4 . Then $v$ is a Carleson mesure for the space $\mathscr{H}_{s}$ if and only if $\left(w_{s}, \Lambda v\right)$ is a trace pair for the Hardy inequality on $T^{n}$. In particular, if $n=1,2,3$, then $v$ is Carleson, if and only if the pair $\left(w_{s}, \Lambda v\right)$ satisfies the box condition (1), and $\left[w_{s}, \Lambda v\right]_{\text {Box }} \approx[v]_{s}$.

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