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Elliptical Partial Differential Equations / Équations aux dérivées partielles elliptiques

A Liouville theorem for the fractional Ginzburg–Landau equation

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Abstract. In this paper, we are concerned with a Liouville-type result of the nonlinear integral equation


$$u(x) = \int_{\mathbb{R}^n} \frac{u(1-|u|^2)}{|x-y|^{n-\alpha}} \, dy,$$

where $u : \mathbb{R}^n \to \mathbb{R}^k$ with $k \geq 1$ and $1 < \alpha < n/2$. We prove that $u \in L^2(\mathbb{R}^n) \Rightarrow u \equiv 0$ on $\mathbb{R}^n$, as long as $u$ is a bounded and differentiable solution.

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If a harmonic function $u$ is bounded on $\mathbb{R}^n$, then $u \equiv$ Const. (this is the Liouville theorem). Moreover, if $u$ is integrable (i.e. $u \in L^s(\mathbb{R}^n)$ for some $s \geq 1$), then $u \equiv 0$ on $\mathbb{R}^n$.

In 1994, Brezis, Merle and Rivière [2] studied the quantization effect of the following equation

\begin{equation}
-\Delta u = u(1-|u|^2) \quad \text{on} \quad \mathbb{R}^2.
\end{equation}

Here $u : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector valued function. It is the Euler–Lagrange equation of the Ginzburg–Landau energy

$$E_{GL}(u) = \frac{1}{2} \|\nabla u\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{4} \|1-|u|^2\|_{L^2(\mathbb{R}^2)}^2.$$

In particular, they proved the finite energy solution (i.e., $u$ satisfies $\nabla u \in L^2(\mathbb{R}^2)$) is bounded (see also [4] and [6])

$$|u| \leq 1 \quad \text{on} \quad \mathbb{R}^n.$$

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(Here $n = 2$.) Based on this result, they obtained a Liouville type theorem for finite energy solutions (cf. [2, Theorem 2]):

Let $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a classical solution of (1). If $\nabla u \in L^2(\mathbb{R}^2)$, then either $u \in L^2(\mathbb{R}^2)$ which implies $u \equiv 0$, or $1 - |u|^2 \in L^1(\mathbb{R}^2)$ which implies $u \equiv C$ with $|C| = 1$.

The boundedness and the integrability of solutions are the important conditions which ensure that the Liouville theorem holds. The Pohozaev identity plays a key role in the proof.

In this paper, we are concerned with the integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{u(1-|u|^2)}{|x-y|^{n-\alpha}} dy.$$  \hspace{1cm} (3)

Here $u : \mathbb{R}^n \rightarrow \mathbb{R}^k$, $k \geq 1$, $n \geq 3$, and $1 < \alpha < n/2$. We also apply the integral form of the Pohozaev identity (which was used for the Lane–Emden equations in [3], [5] and [12]) to establish a Liouville theorem.

**Theorem 1.** Assume that $u : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is bounded and differentiable, and solves (3) with $\alpha \in (1, n/2)$. If $u \in L^2(\mathbb{R}^n)$, then $u(x) \equiv 0$.

**Proof.** For convenience, we denote $B_R(0)$ by $B_R$ here.

**Step 1.** We claim that the improper integral

$$\int_{\mathbb{R}^n} \frac{z \cdot \nabla [u(z)(1-|u(z)|^2)]}{|x-z|^{n-\alpha}} dz$$ \hspace{1cm} (4)

is convergent at each $x \in \mathbb{R}^n$.

In fact, since $u \in L^2(\mathbb{R}^n)$, we can find $R = R_j \rightarrow \infty$ such that

$$R \int_{\partial B_R} |u(z)|^2 ds \rightarrow 0.$$ \hspace{1cm} (5)

Since $u$ is bounded, by the Hölder inequality, we obtain that for sufficiently large $R$, there holds

$$R \left| \int_{\partial B_R} \frac{u(z)(1-|u(z)|^2)}{|x-z|^{n-\alpha}} ds \right| \leq CR^{1-n+\alpha} \int_{\partial B_R} |u(z)| ds \leq CR^{1-n+\alpha} \left( R \int_{\partial B_R} |u(z)|^2 ds \right)^{\frac{1}{2}} R^{\frac{1}{2} + \frac{n-1}{2}}.$$ \hspace{1cm} (6)

Let $R = R_j \rightarrow \infty$. Noting $\alpha < n/2$, and using (5) we get

$$R \int_{\partial B_R} \frac{u(z)(1-|u(z)|^2)}{|x-z|^{n-\alpha}} ds \rightarrow 0$$ \hspace{1cm} (7)

when $R = R_j \rightarrow \infty$.

Next, we claim that the improper integral

$$I(\mathbb{R}^n) := \int_{\mathbb{R}^n} \frac{u(z)(1-|u(z)|^2)(x-z) \cdot z}{|x-z|^{n+\alpha+2}} dz$$ \hspace{1cm} (7)

absolutely converges for each $x \in \mathbb{R}^n$.

In fact, we observe that the defect points of $I(\mathbb{R}^n)$ are $x$ and $\infty$. When $z$ is near $\infty$, we have

$$|I(\mathbb{R}^n \setminus B_r)| \leq C \int_{\mathbb{R}^n \setminus B_r} \frac{|u(z)| dz}{|x-z|^{n-\alpha}} \leq C \left( \int_{\mathbb{R}^n} |u|^2 dz \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} \rho^{n-2(n-\alpha)} \frac{d\rho}{\rho} \right)^{\frac{1}{2}}.$$ \hspace{1cm} (8)

In view of $u \in L^2(\mathbb{R}^n)$ and $\alpha < n/2$, we get

$$|I(\mathbb{R}^n \setminus B_r)| < \infty.$$ \hspace{1cm} (9)

When $z$ is near $x$, we first take

$$s \in \left( \frac{n}{\alpha-1}, \infty \right).$$
Clearly, $1 < \alpha < n/2$ implies $s > 2$. In addition,

$$u \in L^s(\mathbb{R}^n)$$

because $u$ is bounded and $u \in L^2(\mathbb{R}^n)$. Note that

$$|I(B_\rho(x))| \leq C \int_{B_\rho(x)} \frac{|u(z)|dz}{|x-z|^{n+\alpha}} \leq C \left( \int_{\mathbb{R}^n} |u|^s dz \right)^{1/s} \left( \int_0^\rho \rho^{n-\frac{s}{2}(n-\alpha+1)} \frac{d\rho}{\rho} \right)^{1-\frac{1}{s}}.$$

By (9) and (10), we get

$$|I(B_\rho(x))| < \infty.$$

Combining this with (8), we prove that (7) is absolutely convergent.

Finally, we prove that (4) is convergent. Integrating by parts yields

$$\int_{B_R} z \cdot \nabla [u(z)(1-|u(z)|^2)] \frac{dz}{|x-z|^{n-\alpha}} = R \int_{\partial B_R} \frac{u(z)(1-|u(z)|^2)}{|x-z|^{n-\alpha}} dz - n \int_{B_R} \frac{u(z)(1-|u(z)|^2)}{|x-z|^{n-\alpha}} dz - (n-\alpha) \int_{B_R} \frac{u(z)(1-|u(z)|^2)(x-z) \cdot z}{|x-z|^{n-\alpha+2}} dz. \quad (11)$$

Letting $R = R_j \to \infty$ in (11) and using (3) and (6), we can see that

$$\int_{\mathbb{R}^n} z \cdot \nabla [u(z)(1-|u(z)|^2)] \frac{dz}{|x-z|^{n-\alpha}} = -nu(x) + (\alpha-n)I(\mathbb{R}^n),$$

and hence it is convergent at each $x \in \mathbb{R}^n$.

**Step 2. Proof of Theorem 1.** For any $\lambda > 0$, from (3) it follows

$$u(\lambda x) = \lambda^a \int_{\mathbb{R}^n} u(\lambda z)(1-|u(\lambda z)|^2) \frac{dz}{|x-z|^{n-\alpha}}.$$

Differentiating both sides with respect to $\lambda$ yields

$$x \cdot \nabla u(\lambda x) = \alpha \lambda^{a-1} \int_{\mathbb{R}^n} \frac{u(\lambda z)(1-|u(\lambda z)|^2)}{|x-z|^{n-\alpha}} dz + \lambda^a \int_{\mathbb{R}^n} \frac{(z \cdot \nabla u(\lambda z))(1-|u(\lambda z)|^2) + u(\lambda z)(-2u(\lambda z)z \cdot \nabla u(\lambda z))}{|x-z|^{n-\alpha}} dz.$$

Letting $\lambda = 1$ yields

$$x \cdot \nabla u(x) = ax(u(x) + \int_{\mathbb{R}^n} z \cdot \nabla [u(1-|u|^2)] \frac{dz}{|x-z|^{n-\alpha}}. \quad (12)$$

Since $u$ is bounded and $u \in L^2(\mathbb{R}^n)$, it follows that $u \in L^4(\mathbb{R}^n)$, and hence

$$R \int_{\partial B_R} |u|^4 dz \to 0 \quad (13)$$

for some $R = R_j \to \infty$. Thus, integrating by parts and using (5) and (13), we respectively obtain

$$\int_{\mathbb{R}^n} u(x)(x \cdot \nabla u(x)) dx = \frac{-n}{2} \int_{\mathbb{R}^n} |u(x)|^2 dx, \quad (14)$$

and

$$\int_{\mathbb{R}^n} u(x)|u(x)|^2(x \cdot \nabla u(x)) dx = \frac{-n}{4} \int_{\mathbb{R}^n} |u(x)|^4 dx. \quad (15)$$

These results show that

$$\int_{\mathbb{R}^n} u(x)(1-|u(x)|^2)(x \cdot \nabla u(x)) dx < \infty. \quad (16)$$

Multiply (12) by $u(x)(1-|u(x)|^2)$ and integrate over $B_R$. Letting $R = R_j \to \infty$, from $u \in L^2(\mathbb{R}^n) \cap L^4(\mathbb{R}^n)$ and (16), we get

$$\int_{\mathbb{R}^n} u(x)(1-|u(x)|^2) \int_{\mathbb{R}^n} z \cdot \nabla [u(1-|u|^2)] \frac{dz}{|x-z|^{n-\alpha}} dx < \infty,$$
and
\[
\int_{\mathbb{R}^n} u(x)(1 - |u(x)|^2)(x \cdot \nabla u(x))dx - \alpha \int_{\mathbb{R}^n} |u(x)|^2(1 - |u(x)|^2)dx
= \int_{\mathbb{R}^n} u(x)(1 - |u(x)|^2) \int_{\mathbb{R}^n} \frac{z \cdot \nabla [u(1 - |u|^2)]}{|x - z|^{n-\alpha}}dzdx. \tag{17}
\]

We use the Fubini theorem and (3) to handle the right hand side term. Thus,
\[
\int_{\mathbb{R}^n} u(x)(1 - |u(x)|^2) \int_{\mathbb{R}^n} \frac{z \cdot \nabla [u(1 - |u|^2)]}{|x - z|^{n-\alpha}}dzdx
= \int_{\mathbb{R}^n} z \cdot \nabla [u(1 - |u|^2)] \int_{\mathbb{R}^n} \frac{u(x)(1 - |u(x)|^2)}{|x - z|^{n-\alpha}}dzdx
= \int_{\mathbb{R}^n} (x \cdot \nabla [u(1 - |u|^2)])u(x)dx
= \int_{\mathbb{R}^n} u(x)(1 - |u|^2)(x \cdot \nabla u(x))dx - \int_{\mathbb{R}^n} |u|^2(x \cdot \nabla |u|^2)dx.
\tag{18}
\]

Inserting this result into (17), and using (15) we have
\[
\int_{\mathbb{R}^n} |\alpha|u(x)|^2 + (\frac{n}{2} - \alpha)|u(x)|^4]dx = 0. \tag{19}
\]

In view of \(\alpha \in (1, n/2)\), (19) leads to \(|u| \equiv 0\). Theorem 1 is proved. \(\square\)

**Remark 2.** Clearly, (19) implies
\[
\int_{\mathbb{R}^n} |\alpha|(|u(x)|^2 - |u(x)|^4) + \frac{n}{2}|u(x)|^4]dx = 0. \tag{20}
\]

When \(u\) satisfies (2), (20) also implies \(|u| \equiv 0\). For (1), the bound \(|u| \leq 1\) for solutions \(u: \mathbb{R}^n \to \mathbb{R}^k\) was first proved by Brezis (cf. [1]). Ma also pointed out that (2) holds true (cf. [6]).

**Remark 3.** In 2016, Ma [7] proved (2) for the Ginzburg–Landau-type equation with fractional Laplacian
\[
(-\Delta)^{\alpha/2} u = (1 - |u|^2)u \quad \text{on } \mathbb{R}^n \tag{21}
\]
under the assumption
\[
1 - |u|^2 \in L^2(\mathbb{R}^n), \tag{22}
\]
where \(n \geq 2\) and \(0 < \alpha < 2\). The physical background of (21) can be found in [9] and [11]. Such an equation with \(\alpha = 1\) was well studied in [8]. Recall the definition of fractional Laplacian on \(\mathbb{R}^n\). Let \(n \geq 2\) and \(0 < \alpha < 2\). Write
\[
E = L_\alpha \cap C^{1,1}_{\text{loc}}(\mathbb{R}^n),
\]
where \(L_\alpha = \{u \in L^1_{\text{loc}}(\mathbb{R}^n), \int_{\mathbb{R}^n} \frac{|u(x)|dx}{1 + |x|^\alpha} < \infty\}\). For a vector value function \(u \in E\) from \(\mathbb{R}^n\) to \(\mathbb{R}^k\), define
\[
(-\Delta)^{\alpha/2} u := C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}}dy = C_{n,\alpha} \lim_{\varepsilon \to 0^+} \int_{|x - y| \geq \varepsilon} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}}dy. \tag{23}
\]

Here \(C_{n,\alpha}\) is a positive constant.

Clearly, (22) and \(u \in L^2(\mathbb{R}^n)\) are incompatible.

**Remark 4.** Another definition of the fractional order Laplacian involves the Riesz potential (cf. [10, Chapter 5]). Assume \(\alpha \in (0, n)\), and \(u, u(1 - |u|^2) \in \mathcal{S}'(\mathbb{R}^n)\), then (21) can be explained as (3). In fact, (3) is equivalent to
\[
\tilde{u}(\xi) = (|x|^{a-n} * |u(1 - |u|^2)|^\wedge)(\xi) = C|\xi|^{-\alpha} |u(1 - |u|^2)|^\wedge(\xi), \tag{24}
\]
where \(C\) is a positive constant. By the property of the Riesz potential, we have
\[
[(-\Delta)^{\alpha/2} u]^\wedge(\xi) = C|\xi|^a \tilde{u}(\xi), \tag{25}
\]

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where \( C \) is another positive constant. Therefore, the above equality (24) amounts to (21). In addition, let \( u \in E \) be a solution of (21) with \( 0 < \alpha < 2 \). From (23), it follows that (25) is still true. If the Fourier inversion formula of (24) holds, then \( u \) also solves (3) (if we omit the constants).

**Remark 5.** If \( u \) is a finite energy solution of (1), then [2] shows that

\[
\int_{\mathbb{R}^n} |u|^2 (1 - |u|^2) \, dx < \infty.
\]

(26)

Therefore, we sometimes call \( u \) a finite energy solution of (3) if \( u \) satisfies (26). Moreover, if \( u \) is uniformly continuous, we can see that either \( u \in L^2(\mathbb{R}^n) \) or \( 1 - |u|^2 \in L^1(\mathbb{R}^n) \) by the same argument of (3.9) and (3.10) in [2]. Therefore, if a bounded, uniformly continuous, differentiable function \( u \) is a finite energy solution of (3), then either \( u \equiv 0 \), or \( |u(x)| \to 1 \) when \( |x| \to \infty \).

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**References**