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Jinsong Xu

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Birational Geometry / *Géométrie birationnelle*

A remark on the rank of finite p -groups of birational automorphisms

Jinsong Xu^a

^a Department of Mathematical Sciences, Xi'an Jiaotong-Liverpool University, No.111,
Ren'ai Road, SIP, Suzhou, Jiangsu Province, China
E-mail: jinsong.xu@xjtlu.edu.cn

Abstract. In this note, we improve a result of Prokhorov and Shramov on the rank of finite p -subgroups of the group of birational transformations of a rationally connected variety. Known examples show that the bounds obtained are optimal in many cases.

Résumé. Dans cette note, nous améliorons un résultat de Prokhorov et Shramov sur le rang des p -sous-groupes finis du groupe des transformations birationnelles d'une variété rationnellement connexe. Des exemples connus montrent que les bornes obtenues sont optimales dans de nombreux cas.

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The goal of this note is to prove the following theorem on finite groups of birational transformations.

Theorem (Main Theorem). *Let X be a rationally connected variety of dimension n over an algebraically closed field \mathbf{k} of characteristic 0. Let p be a prime number and let G be a finite p -subgroup of the group of birational transformations $\text{Bir}(X)$. If $p > n + 1$, then G is abelian and the rank of G is at most n .*

We shall see below in Example 1 that the inequality $p > n + 1$ is optimal when $n + 1$ is prime. The same result, but with a non-optimal inequality $p > L(n)$, was obtained in a series of papers by Prokhorov and Shramov using – and in fact motivating – Birkar's proof of the boundedness of weak Fano varieties (see [2, 10]). Thus, our theorem provides a positive and optimal answer to a question of Serre (see [13, Section 6] and [12, Question 1.1]). We refer to [12, Proposition 1.7] and [11, 12] for surfaces and threefolds; for instance, in [12] it is shown that a finite p -subgroup of $\text{Bir}(\mathbb{P}_{\mathbf{k}}^3)$ is abelian and of rank ≤ 3 as soon as $p \geq 17$; here we improve the inequality to $p \geq 5$.

Our main tool is a remarkable fixed point theorem of Hauton (see [3, Theorem 1.2.1]).

Theorem (Hauton's Theorem). *Let \mathbf{k} be an algebraically closed field, of arbitrary characteristic. Let X be a projective variety over \mathbf{k} , and G be a finite p -group acting by automorphisms on X . Assume that one of the following conditions holds*

- (i) G is cyclic;
- (ii) $\text{char}(\mathbf{k}) = p$;
- (iii) or $\dim X < p - 1$.

Then $X(\mathbf{k})^G = \emptyset$ if and only if the Euler characteristic $\chi(X, \mathcal{F})$ of every G -equivariant coherent \mathcal{O}_X -module \mathcal{F} is divisible by p .

Proof of the Main Theorem. Given a variety X , recall that it is *rationally connected* if for any general points $x, y \in X$, we may find a rational curve $C \subset X$ passing through x and y . This property is a birational invariant. Therefore passing to a smooth regularization of G [9, Lemma-Definition 3.1], we may assume that X is nonsingular and projective, and G acts regularly on X .

Observe that the structure sheaf \mathcal{O}_X is G -equivariant. Moreover, for a nonsingular projective rationally connected variety X , one has $\chi(\mathcal{O}_X) = 1$, because $\dim H^i(X, \mathcal{O}_X) = \dim H^0(X, \Omega_X^i) = 0$ for all $i \geq 1$ (see [4, Chapter IV, 3.8]).

Therefore by case (iii) in Houton's Theorem, G has a fixed point $x \in X(\mathbf{k})$. The action of G on the Zariski tangent space $T_{x,X}$ is faithful because G is finite (see [6, Lemma 4]); so, G embeds into the general linear group $\mathrm{GL}(n, \mathbf{k})$. From the assumption that $p > n + 1$, we deduce that $T_{x,X}$ contains no irreducible G -submodule of dimension > 1 because any such G -submodule has dimension divisible by p . Hence the action of G on $T_{x,X}$ is diagonalizable, and we conclude that G is abelian of rank $\leq n$. \square

Example 1. The lower bound $p > n + 1$ in the Main Theorem is sharp when $n + 1$ is prime: the Fermat hypersurface $X \subset \mathbb{P}_{\mathbf{k}}^{n+1}$ of degree $n + 1$, defined by the equation

$$x_0^{n+1} + x_1^{n+1} + \cdots + x_{n+1}^{n+1} = 0,$$

is a smooth Fano variety, hence rationally connected, and it admits a faithful action of $(\mathbb{Z}/(n+1)\mathbb{Z})^{n+1}$, an abelian group of rank $n + 1$.

Example 2. The rank r of an elementary p -subgroup of $\mathrm{Bir}(X)$ can be greater than n if $p \leq n + 1$.

- (1) For rational surfaces, it is known that $r \leq 4$ if $p = 2$, and $r \leq 3$ if $p = 3$; both of these upper bounds are sharp, see [1].
- (2) Now, suppose that X is a rationally connected threefold. If $p = 2$, then $r \leq 6$ [8, Theorem 1.2], and this is sharp. If $p = 3$, the product of the Fermat cubic surface with $\mathbb{P}_{\mathbf{k}}^1$ gives an example showing that $(\mathbb{Z}/3\mathbb{Z})^4$ embeds into $\mathrm{Bir}(\mathbb{P}_{\mathbf{k}}^3)$. Therefore, our main theorem is sharp in dimension 3. Note that we also have $r \leq 5$ when $p = 3$ [7, Theorem 1.2], but it is unclear whether $r = 5$ can be reached. In a recent manuscript, Kuznetsova obtained some new results on the rank of 3-groups. We refer the interested readers to [5].

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