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**Determinants of Laplacians on discretizations of flat surfaces and analytic torsion**


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Determinants of Laplacians on discretizations of flat surfaces and analytic torsion

Déterminants de laplaciens sur les discrétisations de surfaces plates et torsion analytique.

Siarhei Finski

Abstract. We study the asymptotic expansion of the determinants of the graph Laplacians associated to discretizations of a half-translation surface endowed with a unitary flat vector bundle. By doing so, over the discretizations, we relate the asymptotic expansion of the number of spanning trees and the partition function of cycle-rooted spanning forests, weighted by the monodromy of the unitary connection of the vector bundle, to the corresponding zeta-regularized determinants.


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1. Introduction

In this note, over the discretizations of a given surface, we study the asymptotic behaviour of the number of spanning trees and the partition function of cycle-rooted spanning forests, weighted by the monodromy of the unitary connection on a vector bundle, as the mesh of the discretization of the surface goes to zero.

More precisely, by a spanning tree in a graph we mean a subtree covering all the vertices. By a cycle-rooted spanning forest (CRSF in what follows) on a graph we mean a subset of edges, spanning all vertices and with the property that each connected component of the subset has as many vertices as edges (in particular, it has a unique cycle), cf. Kenyon [16].
The number of spanning trees on a finite graph $G$ is often called the complexity of the graph, denoted here by $t(G)$. In what is now called the matrix-tree theorem, Kirchhoff showed that for a finite graph $G$, the product of nonzero eigenvalues of the combinatorial Laplacian of $G$ is equal to $t(G)$, multiplied by the number of vertices. Forman in [11, Theorem 1] extended the theorem of Kirchoff to the setting of a line bundle with a unitary connection on a graph. Kenyon in [16, Theorems 8 and 9] generalized Forman’s theorem to vector bundles of rank 2 endowed with $\text{SL}_2(\mathbb{C})$ connections. More precisely, for a vector bundle $V$ of rank 2 on a finite graph $G$, and a unitary connection $\nabla$ on $F$, Kenyon in [16, Theorems 8 and 9], cf. Kassel-Kenyon [13, Theorem 15], proved that

$$\sqrt{\det\Delta_{\nabla}^G} = \sum_{T \in \text{CRSF}(G)} \prod_{\gamma \in \text{cycles}(T)} \left(2 - \text{Tr}(w_{(T)})\right),$$

where $\Delta_{\nabla}^G$ is the graph Laplacian, twisted by $\nabla$ (see (3)), $\text{CRSF}(G)$ is the set of all CRSF’s on $G$ and $\text{Tr}(w_{(T)})$ is the trace of the monodromy of $\nabla$ evaluated along the cycle $\gamma$ of a CRSF.

In this paper, instead of considering a single graph $G$, we consider a family $\Sigma_n$, $n \in \mathbb{N}^*$ of graphs, constructed as approximations of a given flat surface $\Sigma$, possibly with conical singularities and corners on the boundary. The vector bundles with connections on $\Sigma_n$ are constructed by restriction of the unitary flat vector bundle from $\Sigma$. Our goal is to understand the asymptotics of $t(\Sigma_n)$ and the mentioned weighted sum over CRSF’s, as $n \to \infty$, and to see how the geometry of $\Sigma$ is reflected in this asymptotics. From the results above, it is enough to study the asymptotics of the related determinants.

Our main result shows that up to some universal contribution, depending only on the angles of conical points and interior angles of the corners on the boundary of $\Sigma$, the normalized logarithm of the determinant of the discrete Laplacian converges to the logarithm of the analytic torsion of the surface, which is an invariant introduced by Ray–Singer in [21]. In particular, this gives a complete answer to Open problem 2 and a partial answer to Open problem 4 in Kenyon [14, §8].

This note is organized as follows. In Section 2, we introduce the main objects and state the main results. In Section 3, we give a short outline of the proof of the main theorem. Details of the results announced here are developed in [9] and [10].

\textbf{Figure 1.} A spanning tree and a CRSF on a square-grid graph approximating an annulus. The edges of the graph are not drawn, but they connect the nearest neighbors. The CRSF has two components. The dotted component is non-contractible in the annulus and the non-dotted one is contractible. (Un arbre couvrant et un CRSF sur un graphe approximant un anneau. Les arêtes du graphe relient les voisins les plus proches. Le CRSF a deux composantes. La composante pointillée est non contractile dans l’anneau et la composante non pointillée est contractile.)
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2. Asymptotics of the determinant of discrete Laplacians

In this section we introduce the main objects of this paper and state our main results.

By a vector bundle $F$ on a graph $G = (V(G), E(G))$, we mean the choice of a vector space $F_v$ for any vertex $v \in V(G)$, so that for any $v, v' \in V(G)$, the vector spaces $F_v$ and $F_{v'}$ are isomorphic. The set of sections $\text{Map}(V(G), F)$ of $F$ is defined by

$$\text{Map}(V(G), F) = \bigoplus_{v \in V(G)} F_v.$$  \hfill (2)

A connection $\nabla^F$ on a vector bundle $F$ is the choice for each edge $e = (v, v') \in E(G)$ of an isomorphism $\phi_{vv'}$ between the corresponding vector spaces $F_v \to F_{v'}$, with the property that $\phi_{vv'} = \phi_{v'v}^{-1}$. This isomorphism is called the parallel transport of vectors in $F_v$ to vectors in $F_{v'}$.

A Hermitian metric $h^F$ on the vector bundle $F$ is the choice of a positive-definite Hermitian metric $h_v$ on $F_v$ for each $v \in V(G)$. We say that a connection $\nabla^F$ is unitary with respect to $h^F$ if the associated parallel transport preserves $h^F$.

The Laplacian $\Delta^F_G$ associated to a graph $G$ and a vector bundle with a connection $(F, \nabla^F)$ is the linear operator $\Delta^F_G : \text{Map}(V(G), F) \to \text{Map}(V(G), F)$, defined for $f \in \text{Map}(V(G), F)$ by

$$\Delta^F_G f(v) = \sum_{(v, v') \in E(G)} (f(v) - \phi_{vv'} f(v')) , \quad \forall \ v \in V(G).$$  \hfill (3)

In the case where $(F, \nabla^F)$ is trivial, we recover the combinatorial Laplacian $\Delta_G$, given by the difference of the degree operator and the adjacency matrix. If one assumes that the connection $\nabla^F$ is unitary with respect to $h^F$, then $\Delta^F_G$ becomes self-adjoint, cf. Kenyon [16, §3.3].

We fix a half-translation surface $(\Sigma, g^{T\Sigma})$ with piecewise geodesic boundary. By a half-translation surface $(\Sigma, g^{T\Sigma})$, we mean a surface endowed with a flat metric $g^{T\Sigma}$ which has conical singularities of angles $k\pi$, $k \in \mathbb{N}^* \setminus \{2\}$, cf. Zorich [23]. By a conical singularity we mean a neighborhood isometric to

$$C_\theta := \{(r, t) : r > 0; t \in \mathbb{R}/\theta \mathbb{Z}\} ,$$  \hfill (4)

for $\theta \neq 2\pi$, endowed with the metric

$$ds^2 = dr^2 + r^2 dt^2.$$  \hfill (5)

We denote by $\text{Con}(\Sigma)$ the set of conical points of the surface $\Sigma$, and by $\text{Ang}(\Sigma)$ the set of points where two different smooth components of the boundary meet (corners). We denote by $\angle : \text{Con}(\Sigma) \to \mathbb{R}$ the function which associates to a conical point its angle and by $\angle : \text{Ang}(\Sigma) \to \mathbb{R}$ the function which associates the interior angle of the corner. Denote also by $\text{Ang}^{\pi/2}(\Sigma)$ (resp. $\text{Ang}^{-\pi/2}(\Sigma)$) the subset of $\text{Ang}(\Sigma)$ corresponding to points with angles $\neq \frac{\pi}{2}$ (resp. $= \frac{\pi}{2}$).

For example, if $\Sigma$ is a rectangular planar domain, then there are no conical angles and the angles of the corners are either equal to $\frac{\pi}{2}$, $\frac{3\pi}{2}$ or $2\pi$.

We fix a unitary flat vector bundle $(F, h^F, \nabla^F)$ on the compactification $\overline{\Sigma} := \Sigma \cup \text{Con}(\Sigma)$. By this we mean that the monodromies of $\nabla^F$ over the contractible loops in $\overline{\Sigma}$ vanish and that the connection $\nabla^F$ preserves the metric $h^F$, cf. Demailly [4, p. 263].

We suppose that $\Sigma$ can be tiled completely and without overlaps over subsets of positive Lebesgue measure by flat squares of the same size and area 1. In particular, the boundary $\partial \Sigma$ gets tiled by the boundaries of the tiles, and the corners on the boundary have angles of the form $\frac{k\pi}{2}$,
It is well-known that because of the conical singularities and non-smoothness of the boundary, the Laplacian $\Delta^F_\Sigma$ is not necessarily \textit{essentially self-adjoint} even after specifying the boundary condition (8), cf. Cheeger [3] and Mooers [17]. Thus, to define the spectrum of $\Delta^F_\Sigma$, we are obliged

\[ \Delta^F_\Sigma := (\nabla^F)^* \nabla^F. \]  

If $(F, h^F, \nabla^F)$ is trivial, and $\Sigma$ is a rectangular domain, it is easy to see that $\Delta^F_\Sigma$ coincides with the usual Laplacian, given by the formula $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$.

In this paper we always consider $\Delta^F_\Sigma$ with \textit{Neumann boundary conditions} on $\partial\Sigma$. In other words, the sections $f$ from the domain of our Laplacian satisfy

\[ \nabla^F_\Sigma f = 0 \quad \text{over } \partial\Sigma. \]  

For example, in the case where $\Sigma$ is a rectangular domain in $\mathbb{C}$ with corners at integer points, the family of graphs $\Sigma_n$ coincides with subgraphs of $\frac{1+\sqrt{-1}}{2n} + \frac{1}{n^2} z^2$, which stays inside $\Sigma$.

**Figure 2.** An $L$–shape and its discretizations. It is a rectangular domain in $\mathbb{C}$ with a single corner with angle $\frac{3\pi}{2}$ and 5 corners with angles $\frac{\pi}{2}$. (Une forme en L et sa discrétisation. C’est un domaine rectangulaire de $\mathbb{C}$ avec un coin d’angle $\frac{3\pi}{2}$ et 5 coins d’angle $\frac{\pi}{2}$.)
to specify the self-adjoint extension of $\Delta^F_{\Sigma}$ we are working with. We choose the Friedrichs extension of $\Delta^F_{\Sigma}$, cf. Reed–Simon [22, Theorem X.23], and by an abuse of notation, we denote it by the same symbol $\Delta^F_{\Sigma}$.

As in the case of smooth domains, the spectrum of $\Delta^F_{\Sigma}$ is discrete (cf. [9, Proposition 2.3]), in other words, we may write (by convention, the sets in this article take into account the multiplicity)

$$\text{Spec}(\Delta^F_{\Sigma}) = \{\lambda_1, \lambda_2, \ldots\},$$

where $\lambda_i, i \in \mathbb{N}^*$ form a non-decreasing sequence.

The zeta-regularized determinant of $\Delta^F_{\Sigma}$ (also called the analytic torsion) is defined non-rigorously by the following non-convergent infinite product

$$\det' \Delta^F_{\Sigma} := \prod_{\lambda \in \text{Spec}(\Delta^F_{\Sigma}) \setminus \{0\}} \lambda.$$ (10)

More formally, one can consider the associated zeta-function $\zeta^F_{\Sigma}(s)$, defined for $s \in \mathbb{C}, \text{Re}(s) > 1$, by the formula (the sum below converges by the Weyl’s law, cf. [10, Corollary 2.8])

$$\zeta^F_{\Sigma}(s) = \sum_{\lambda \in \text{Spec}(\Delta^F_{\Sigma}) \setminus \{0\}} \frac{1}{\lambda^s}.$$ (11)

Similarly to the case of smooth manifolds, $\zeta^F_{\Sigma}$ extends meromorphically to $\mathbb{C}$ and 0 is a holomorphic point of this extension (see Cheeger [3], cf. [10, Proposition 2.7]). Following Ray–Singer [20, 21], we define the analytic torsion $\det' \Delta^F_{\Sigma}$ by

$$\det' \Delta^F_{\Sigma} := \exp \left(- (\zeta^F_{\Sigma})'(0) \right).$$ (12)

The value $\zeta^F_{\Sigma}(0)$ is also interesting, and it can be evaluated as follows (cf. [10, (1.12)])

$$\zeta^F_{\Sigma}(0) = - \dim H^0(\Sigma, F) + \frac{\text{rk}(F)}{12} \left( \sum_{P \in \text{Con}(\Sigma)} \frac{4 \pi^2 - \angle(P)^2}{2 \pi \angle(P)} + \sum_{Q \in \text{Ang}(\Sigma)} \frac{\pi^2 - \angle(Q)^2}{2 \pi \angle(Q)} \right),$$ (13)

where $H^0(\Sigma, F)$ is the vector space of flat sections of $(F, \nabla^F)$. We will use the Catalan constant defined by

$$G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots.$$ (14)

We define the normalized logarithm $\log(\det' \Delta^F_{\Sigma_n})$, $n \in \mathbb{N}^*$ by

$$\widetilde{\log}(\det' \Delta^F_{\Sigma_n}) := \log(\det' \Delta^F_{\Sigma_n}) - \frac{4G}{\pi} \cdot \text{rk}(F) \cdot A(\Sigma) \cdot n^2 - \frac{\log(\sqrt{2} - 1)}{2} \cdot \text{rk}(F) \cdot |\partial \Sigma| \cdot n + 2 \zeta^F_{\Sigma}(0) \cdot \log(n),$$ (15)

where $\det' \Delta^F_{\Sigma_n}$ is the product of the non-zero eigenvalues of $\Delta^F_{\Sigma_n}$, and $A(\Sigma), |\partial \Sigma|$ are area and perimeter of $(\Sigma, g^{T\Sigma})$ respectively.

**Main Theorem.** For any $k, K \in \mathbb{N}$, the following asymptotic bound holds

$$\widetilde{\log}(\det' \Delta^F_{\Sigma_n}) = o(\log(n)), \quad \text{for } n = k^l \cdot K, \ l \in \mathbb{N}.$$ (16)

Also, there is a sequence $A_n$, $n \in \mathbb{N}^*$, which depends only on the set of conical angles $\angle(\text{Con}(\Sigma))$ and the set of angles $\angle(\text{Ang}^{\pi/2}(\Sigma))$, such that, as $n \to \infty$, we have

$$\log(\det' \Delta^F_{\Sigma_n}) - \text{rk}(F) \cdot A_n \to \log(\det' \Delta^F_{\Sigma}) - \frac{\log(2)}{16} \cdot \text{rk}(F) \cdot \# \text{Ang}^{\pi/2}(\Sigma).$$ (17)

Moreover, the sequence $A_n$ depends additively on the sets $\angle(\text{Con}(\Sigma)), \angle(\text{Ang}^{\pi/2}(\Sigma))$. 

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Remark 1.

(a) Duplantier–David in [8, (4.7) and (4.23)] have obtained (17) for squares by using an explicit evaluation of the spectrum. This is used in the final step of our proof.

(b) Kenyon in [14, Theorem 1 and Corollary 2] obtained a similar asymptotic expansion for simply-connected rectangular domains \( \Sigma \) in \( \mathbb{C} \) (and hence trivial \( (F, h^F, \nabla^F) \)). He related the constant term to the so-called regularized Dirichlet energy of the average Thurston height function on \( \Omega \), [14, §2.3 and §2.4]. Our proofs are very different. By combining our theorems, we see that this Dirichlet energy coincides with the analytic torsion, which gives a partial answer to Open problem 4 in Kenyon [14, §8]. The asymptotic expansion (17) for multiply connected domains is an answer to Open problem 2 from [14, §8].

**Corollary 2 (Relative asymptotic complexity formula).** Let \( (\Sigma, g^{T \Sigma}) \) and \( (\Sigma', g^{T \Sigma'}) \) be two half-translation surfaces satisfying the same assumptions as in the Main theorem. Construct the graphs \( \Sigma_n \) and \( \Sigma'_n \), \( n \in \mathbb{N}^* \) by the same procedure as in the Main theorem. Endow \( \Sigma \) (resp. \( \Sigma' \)) with a unitary flat vector bundle \((F, h^F, \nabla^F) \) (resp. \((G, h^G, \nabla^G) \)) and induce unitary vector bundles \((F_n, h^{F_n}, \nabla^{F_n}) \) (resp. \((G_n, h^{G_n}, \nabla^{G_n}) \)) on the graphs \( \Sigma_n \) (resp. \( \Sigma'_n \)).

Suppose that \( A(\Sigma) = A(\Sigma') \), \( |\partial \Sigma| = |\partial \Sigma'| \), \( \angle(C(\Sigma)) = \angle(C(\Sigma')) \), \( \angle(Ang(\Sigma)) = \angle(Ang(\Sigma')) \), \( \operatorname{rk}(F) = \operatorname{rk}(G) \) and \( \dim H^0(\Sigma, F) = \dim H^0(\Sigma', G) \). Then

\[
\lim_{n \to \infty} \frac{\det' \Delta_{\Sigma}^{F_n}}{\det' \Delta_{\Sigma}^{G_n}} = \frac{\det' \Delta_{\Sigma}^{F}}{\det' \Delta_{\Sigma}^{G}}.
\]

(18)

If we apply (18) for \( (F, h^F, \nabla^F), (G, h^G, \nabla^G) \) trivial, we obtain

\[
\lim_{n \to \infty} \frac{t(\Sigma_n)}{t(\Sigma'_n)} = \frac{\det' \Delta_{\Sigma}^{F}}{\det' \Delta_{\Sigma'}^{F}}.
\]

(19)

**Remark 3.**

(a) For simply-connected rectangular domains in \( \mathbb{C} \) (and, consequently, trivial \( (F, h^F, \nabla^F) \)), the existence of the limit (18) was proved by Kenyon in [14, Corollary 2, Remark 4] (cf. also Remark 1). However, no relation with the analytic torsion was given in [14].

(b) For \( \Sigma = \Sigma' \), and \( F, G \) of rank 2, the fact that the limit (18) exists was proved by Kassel–Kenyon in [13, Theorem 17, §4.1]. Their proof is different from ours. For \( \Sigma = \Sigma' \) tori, Dubédat–Gheissari [6, Proposition 4] established (18) for non-trivial unitary flat line bundles \((F, h^F, \nabla^F), (G, h^G, \nabla^G) \), see also Dubédat [5, (5.39)] for a related result.

As one application of Corollary 2, we see that (19) makes a connection between the maximization of the asymptotic complexity and the maximization of the analytic torsion. In the realm of polygonal domains in \( \mathbb{C} \), the last problem has been considered by Aldana–Rowlett [2, Conjecture 1 and Theorem 5]. See also Osgood–Philips–Sarnak [19] for a similar maximization problem. See also [10] for some applications of the Main theorem and Corollary 2 to random geometry.

### 3. An outline of the proof of the Main theorem

The main goal of this section is to give an outline of the proof of the Main theorem. We conserve the notations from the previous section.

The general idea is to establish first the asymptotic relation between the rescaled spectrum

\[
\text{Spec}(n^2 \cdot \Delta_{\Sigma_n}^{F_n}) = \{\lambda_1^n, \lambda_2^n, \ldots\}
\]

(20)

of the discretization \( \Sigma_n \), ordered non-decreasingly for each \( n \in \mathbb{N}^* \), and the spectrum (9). Then we use this relation along with some properties of the meromorphic extension of \( \zeta_{\Sigma}(s) \) to deduce the convergence of the respective determinants. The first step is
Theorem 4 ([9, Theorem 1.1]). For any \( i \in \mathbb{N}^* \), as \( n \to \infty \), the following limit holds
\[
\lambda_i^n \to \lambda_i. \tag{21}
\]

The proof of Theorem 4 relies on the two technical ingredients. First, we prove in [9, Theorem 3.5] that up to some linear combination of a finite number of explicit functions, the eigenvectors of \( \Delta_{\Sigma}^F \) have bounded second derivative. To prove this, we use Grisvard’s weak elliptic regularity estimates from [12], a version of Sobolev’s embedding theorem for spaces satisfying cone property from Adams [1] and a description of the domain of \( \Delta_{\Sigma}^F \) due to Cheeger [3] and Mooers [17].

Another ingredient is the discrete Harnack-type inequality, [9, Theorem 3.11], which essentially says that a sequence of discrete functions, which is “asymptotically harmonic”, is “asymptotically continuous”. We obtain it by using some results from potential theory on lattices, due to Adams [1] and a description of the domain of \( \Delta_{\Sigma} \) from Grisvard [2].

Now, recall that we are trying to get the asymptotic expansion (17), which involves a product of terms, the number of which, \( \text{rk}(F) \dim(\Sigma_n) - \dim H^0(\Sigma, F), n \in \mathbb{N}^* \), tends to infinity quite quickly, as \( n \to \infty \). Thus, there is almost no chance to get (17) by studying simply the convergence of individual eigenvalues, as it would require much stronger convergence result compared to what we obtained in Theorem 4. Moreover, the analytic torsion, which appears on the right-hand side of (17), is defined not through the normalized product of the first eigenvalues, but through the zeta-regularization procedure, see (12). For this reason, for \( s \in \mathbb{C} \), we denote
\[
\zeta_{\Sigma_n}^{F_n}(s) := \frac{1}{\sum_{\lambda \in \text{Spec}(\Delta_{\Sigma_n}^F)} \frac{1}{(\lambda^2)^s}} = \frac{1}{\sum_{\lambda_i^n \neq 0} \frac{1}{(\lambda_i^n)^s}}. \tag{22}
\]

We would like to compare the zeta functions (11) and (22). The first result in this direction is

Theorem 5 ([10, Corollary 2.16]). For any \( s \in \mathbb{C}, \text{Re}(s) > 1 \), as \( n \to \infty \), we have
\[
\zeta_{\Sigma_n}^{F_n}(s) \to \zeta_{\Sigma}^{F}(s). \tag{23}
\]

A central statement in the proof of Theorem 5 is the uniform weak Weyl’s law for the discrete Laplacians, which we prove in [10, Theorem 2.15]. It essentially says that the eigenvalues \( \lambda_i^n \) increase at least asymptotically linearly in \( i \) uniformly in \( n \).

We obviously would like to extend the convergence in Theorem 5 to the whole complex plane \( s \in \mathbb{C} \). This is impossible to do directly, as we know that the functions \( \zeta_{\Sigma_n}^{F_n}(s), n \in \mathbb{N}^* \), are not holomorphic over \( \mathbb{C} \) and the function \( \zeta_{\Sigma}^{F}(s) \) is only meromorphic. To cross the barrier \( \text{Re}(s) > 1 \), we were inspired a lot by the approach used by Müller in [18] in his proof of the Ray–Singer conjecture (now Cheeger–Müller theorem).

More precisely, we consider a covering of \( \Sigma \) by a union of open sets \( U_\alpha, \alpha \in I \), which are themselves half-translation surfaces which can be tiled by squares of area 1. Endow \( U_\alpha \) with the restriction of the vector bundle \((F, h^F, V^F)\), which we denote by the same symbol by an abuse of notation. We take a subordinate partition of unity \( \phi_\alpha, \alpha \in I \) of \( \Sigma \) and consider the normalized zeta function, defined for \( s \in \mathbb{C}, \text{Re}(s) > 1 \), by
\[
\zeta_{\Sigma}^{F_n, \text{nor}}(s) := \text{Tr} \left[ (\Delta_{\Sigma}^{F_n})^{-s} - \sum_{\alpha \in I} \phi_\alpha \cdot (\Delta_{U_\alpha}^{F_n})^{-s} \right], \tag{24}
\]
where \((\Delta^n)^{-s}(\Sigma, F)\) is a power of \(\Delta^n\), restricted to the vector space spanned by the eigenvectors corresponding to non-zero eigenvalues, and \((\Delta_{U_a,n}^n)^{-s}\) are viewed as operators acting on \(L^2(\Sigma, F)\) by the trivial extension. Using some standard techniques, we prove in [10, Proposition 2.17] that \(\zeta_{\Sigma}^{\text{Eor}}(s)\) extends \textit{holomorphically} to the whole complex plane \(\mathbb{C}\).

Now, similarly, for \(s \in \mathbb{C}\), we construct the normalized discrete zeta function
\[
\zeta_{\Sigma}^{\text{Eor}}(s) := \text{Tr} \left[ (n^2 \cdot \Delta_{n}^n)^{-s} - \sum_{a \in I} \phi_a \cdot (n^2 \cdot \Delta_{U_a,n}^n)^{-s} \right],
\]
where the powers \((\Delta^n)^{-s}, (\Delta_{U_a,n}^n)^{-s}\) have to be understood as powers of the respective Laplacians, restricted to the vector spaces spanned by the eigenvectors corresponding to non-zero eigenvalues, the operators \((\Delta_{U_a,n}^n)^{-s}\) are viewed as operators on \(\text{Map} (V(\Sigma_n), F_n)\) by the obvious inclusion \(V(U_{a,n}) \hookrightarrow V(\Sigma_n)\), and \(\phi_a\) are given by pointwise multiplications on the elements of \(V(U_{a,n}) \hookrightarrow U_a\). By using methods of Müller in [18], we establish

**Theorem 6 ([10, Theorem 2.18]).** For any compact \(K \subset \mathbb{C}\), there is \(C > 0\) such that for any \(s \in K\), \(n \in \mathbb{N}^+\), the following bound holds
\[
|\zeta_{\Sigma_n}^{\text{Eor}}(s)| \leq C.
\]

Now, recall a classical result from complex analysis, stating that a sequence of uniformly locally bounded holomorphic functions converges on a connected domain if and only if it converges on some subdomain. By this, Theorems 5 and 6, and considerations similar to those used in the proof of Theorem 5, we deduce that for any \(s \in \mathbb{C}\), as \(n \to \infty\), the following limit holds
\[
\zeta_{\Sigma_n}^{\text{Eor}}(s) \to \zeta_{\Sigma}^{\text{Eor}}(s).
\]

Now, similarly to (12), we see that the following quantity is well-defined
\[
\text{tr} \left[ \phi \cdot \log(\Delta_{\Sigma}^n) \right] := -\frac{\partial}{\partial s} \text{tr} \left[ \phi \cdot (\Delta_{\Sigma}^n)^{-s} \right] \bigg|_{s=0}. \tag{28}
\]
By Cauchy formula, (12) and (27), we see that, as \(n \to \infty\), we have
\[
\log \left( \det(n^2 \cdot \Delta_{\Sigma_n}^n) \right) - \sum_{a \in I} \text{tr} \left[ \phi_a \cdot \log(n^2 \cdot \Delta_{U_{a,n}}^n) \right] \to \log(\det(\Delta^\Sigma)) - \sum_{a \in I} \text{tr} \left[ \phi_a \cdot \log(\Delta_{U_{a,n}}^n) \right]. \tag{29}
\]
Remark that in (29), for the first time in our analysis we see the analytic torsion.

Now, in our final step we choose \(U_a\) and \(\phi_a\) in a special way so that the terms appearing in the sum in the left-hand side of (29) become relatively easy to handle. As we have assumed that \(\Sigma\) has flat geometry, we can cover it by euclidean squares and a finite number of model spaces (which cover corners and conical singularities). Moreover, since we assumed that \((F, \nabla^F, h^F)\) is unitary flat, its restriction over \(U_a\) can be trivialized. By using this and the calculations of Duplantier–David [8], cf. Remark 1, in [10, Theorem 2.21], we completely determined the asymptotics of the terms corresponding to the squares in the left-hand side of (29). This with (29) essentially finishes the proof of the Main theorem.

**References**


