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Complex algebraic geometry, in memory of Jean-Pierre Demailly /

Géométrie algébrique complexe, en mémoire de Jean-Pierre Demailly

Guest editor / *Rédacteur en chef invité*

Claire Voisin

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Avant-propos / *Foreword*
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Géométrie algébrique complexe, en mémoire de Jean-Pierre Demailly / *Complex algebraic geometry, in memory of Jean-Pierre Demailly*

Géométrie algébrique complexe, en mémoire de Jean-Pierre Demailly : Avant-propos

Complex algebraic geometry, in memory of Jean-Pierre Demailly: Foreword

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The English version is available after the French version

Demailly est un spécialiste d'analyse et de géométrie algébrique complexes. Son œuvre s'inscrit dans une grande tradition mathématique, remontant à Riemann, qui étudie les variétés algébriques sur le corps des nombres complexes sous l'angle de la géométrie différentielle complexe et leur applique des méthodes qui peuvent être très analytiques. Par exemple, Hodge développa la théorie des formes harmoniques et l'appliqua aux variétés kählériennes compactes, produisant le fameux théorème de décomposition de Hodge, qui reste de nos jours l'énoncé le plus qualitatif dont on dispose concernant la topologie des variétés algébriques projectives sur les nombres complexes. Il fut suivi de peu par Kodaira et son magnifique théorème de plongement, donnant la généralisation optimale du théorème de plongement de Riemann pour les surfaces de Riemann compactes. Demailly s'inscrit dans cette tradition et plusieurs de ses contributions majeures sont liées aux travaux de Kodaira qu'elles généralisent d'une façon spectaculaire et extrêmement importante pour la géométrie algébrique moderne.

Demailly appartient aussi à l'école de Lelong, qui utilise l'analyse pour étudier des objets beaucoup moins, voire pas du tout, réguliers, à savoir des courants au lieu de formes différentielles. On sait que les fonctions holomorphes sur une variété complexe compacte connexe sont constantes. On leur substitue donc des sections holomorphes de fibrés en droites holomorphes, le quotient de deux telles sections fournissant une fonction méromorphe. C'est la « positivité » de ce fibré en droites qui garantit l'existence de telles sections non identiquement nulles. Mais de quelle notion de positivité s'agit-il ? Dans le théorème de Kodaira, la positivité est donnée par le choix d'une métrique de classe C^∞ sur le fibré en droites, telle que la forme de Chern, ou courbure de la connexion de Chern associée, soit positive dans le sens le plus fort possible, c'est-à-dire soit une forme de Kähler. La conclusion est alors que le fibré en droites est ample, ce qui est aussi la plus forte notion de positivité pour un fibré en droites dont on dispose en géométrie algébrique.

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Demailly a utilisé la théorie et l'analyse des courants de courbure associés à des métriques moins régulières et cela lui a permis d'introduire et caractériser des notions moins restrictives de positivité, telles que la pseudo-effectivité, pour les fibrés en droites. C'est via les estimées L^2 à la Hörmander qu'il caractérise la pseudo-effectivité. En combinant ce type de techniques avec la résolution d'équation de Monge–Ampère à second membre singulier, il a également été un pionnier sur le problème de la grande amplitude effective, où l'on demande quelles puissances tensorielles d'un fibré en droites ample possèdent suffisamment de sections pour fournir un plongement dans l'espace projectif.

Une variété complexe ou algébrique lisse possède toujours au moins un fibré en droites holomorphe, à savoir son fibré canonique (qui peut être trivial). La géométrie birationnelle dans sa forme moderne étudie les propriétés du fibré canonique. C'est un fait remarquable que les formes pluricanoniques des variétés projectives lisses sont contravariantes sous les applications rationnelles dominantes entre variétés de même dimension. L'une des grandes conjectures du domaine est qu'une variété projective lisse est uniréglée, c'est-à-dire couverte par une famille de courbes rationnelles (ou surfaces de Riemann de genre 0), si et seulement si elle ne possède aucune forme pluricanonique non nulle (le « seulement si » étant facile). Dans le magnifique article *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension* [1], Boucksom, Demailly, Păun et Peternell montrent qu'une variété projective lisse est uniréglée si et seulement si son fibré canonique n'est pas pseudo-effectif, ce qui est une condition plus forte que l'annulation des plurigenres, mais l'énoncé constitue néanmoins un pas important vers cette conjecture. Cet article fournit aussi une caractérisation duale (dans l'esprit de Moishezon–Nakai) extrêmement intéressante du cône des diviseurs pseudo-effectifs.

Une autre contribution majeure de Demailly est l'article *Numerical characterization of the Kähler cone of a compact Kähler manifold* [2] écrit avec Păun, où ils démontrent un superbe résultat généralisant le critère de Moishezon–Nakai pour l'amplitude des fibrés en droites. Le critère de Moishezon–Nakai dit qu'un fibré en droites est ample s'il est de degré strictement positif sur toutes les courbes contenues dans la variété et plus généralement, les puissances de sa forme de courbure (ou première classe de Chern) sont d'intégrale strictement positive sur tout fermé algébrique (ou analytique) de la variété. Le théorème de Demailly–Păun étend ce résultat à la positivité des classes de formes fermées de type $(1,1)$ sur une variété kählérienne compacte. Dans ce cas, la variété peut ne contenir aucune sous-variété complexe propre, mais leur résultat est que le cône des classes $(1,1)$ positives est une composante connexe du cône déterminé par la positivité de toutes ces intégrales.

Demailly est également un leader en analyse complexe et il fait partie des rares mathématiciens dont l'œuvre a une grande influence scientifique dans plusieurs domaines. Son école, l'ensemble de ses étudiants et leurs orientations mathématiques, témoignent largement de cette ouverture. La raison pour laquelle je n'ai mentionné ci-dessus que certains de ses résultats liés à la géométrie complexe (algébrique ou kählérienne) est non seulement le fait que je ne suis pas moi-même compétente dans la partie « analyse complexe », mais aussi que le présent volume rassemble des articles relevant pour la plupart de la géométrie algébrique complexe. Un autre volume en hommage à Demailly, d'inspiration plus analytique, sera publié au PAMQ.

Jean-Pierre Demailly était un grand scientifique inspiré à la fois par l'analyse et la géométrie. Il laisse une œuvre magnifique d'un impact considérable. Ce volume qui lui est consacré célèbre la partie de son œuvre touchant la géométrie algébrique et rend hommage à une personnalité exceptionnelle à tous points de vue.

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English version

Demailly's work is mainly devoted to complex analysis and algebraic geometry. It follows a great mathematical tradition, going back to Riemann, where algebraic varieties over the field of complex numbers are studied via complex differential geometry, using methods from analysis. For example, Hodge developped the theory of harmonic forms and applied it to compact Kähler manifolds, proving the famous Hodge decomposition theorem, which is still today the most qualitative theorem concerning the topology of projective algebraic varieties over the field of complex numbers. Soon after, Kodaira proved his magnificent embedding theorem, establishing the optimal generalization in higher dimension of the Riemann embedding theorem for compact Riemann surfaces. Demailly continues this tradition and several of his major contributions, which are related to the work of Kodaira, generalize it in a spectacular and extremely important way for modern algebraic geometry.

Demailly also belongs to the Lelong school, where analysis is used to study objects with very low regularity, namely currents instead of differential forms. It is known that holomorphic functions on a compact connected complex manifold are constant. We thus use as a substitute holomorphic sections of holomorphic line bundles, the quotient of two such sections being a meromorphic function. The "positivity" of this line bundle guarantees the existence of such nonzero sections. The question is "which notion of positivity do we use ?". In the Kodaira theorem, positivity is given by the choice of a C^∞ metric on the line bundle, such that the Chern form, or curvature of the associated Chern connection, is positive in the strongest possible sense, namely is a Kähler form. The conclusion then is that the line bundle is ample, which is also the strongest positivity notion for a line bundle that appears in algebraic geometry. Demailly used the theory and the analysis of curvature currents associated to less regular metrics and this led him to introduce and characterize less restrictive notions of positivity, such as pseudo-effectivity, for line bundles. He succeeded characterizing pseudo-effectivity via L^2 estimates à la Hörmander. Combining this type of technics with the resolution of singular Monge–Ampère equations, he obtained pioneering results on the problem of effective very ampleness, where one asks which powers of an ample line bundle have enough global sections to provide an embedding in projective space.

A complex manifold or smooth algebraic variety always carries at least one holomorphic (algebraic) line bundle, namely its canonical bundle (which can be trivial). Modern birational geometry studies the properties of the canonical bundle. It is a remarkable fact that pluricanonical forms on smooth projective varieties are contravariant under the dominant rational maps between varieties of the same dimension. A big conjecture in the field is that a smooth projective variety is uniruled, that is swept-out by a family of rational curves (or genus 0 Riemann surfaces), if and only if it has no nonzero pluricanonical form (the "only if" being easy). In the superb article *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension* [1], Boucksom, Demailly, Păun and Peternell prove that a smooth projective variety is uniruled if and only if its canonical bundle is not pseudo-effective, a condition which is stronger than the vanishing of the plurigenera, but the statement is nevertheless an important step towards the conjecture. This paper also provides a very interesting dual characterization (in the spirit of Moishezon–Nakai) of the cone of pseudo-effective divisors.

Another major contribution of Demailly is the paper *Numerical characterization of the Kähler cone of a compact Kähler manifold* [2], written with Păun, where they prove a superb result generalizing the Moishezon–Nakai criterion for the ampleness of the line bundle. The Moishezon–Nakai criterion says that a line bundle an a smooth projective variety is ample if has strictly positive degree on all curves contained in the variety and more generally, the powers of its curvature form (or first Chern class) have strictly positive integral on any closed algebraic (or analytic) subset of the variety. The Demailly–Păun theorem extends this result to the positivity of classes of closed forms of type $(1,1)$ on a compact Kähler manifold. In this case, the manifold may not contain any proper closed analytic subset, but their result is that the cone of positive $(1,1)$-classes is a connected component of the cone determined by the positivity of all these integrals.

Demailly is also a leader in complex analysis and he is one of the few mathematicians whose work is greatly influential in several areas. His school, his students and their mathematical orientations, illustrate this breadth. The reason why I mentioned above only his results related to complex geometry (algebraic or Kähler) is not only the fact that I am not myself competent in the analytic aspects of his work, but also that mots papers presented in this volume are related to complex algebraic geometry. Another volume dedicated to Demailly, with more emphasis on analysis, will be published in PAMQ.

Jean-Pierre Demailly was a great scientist inspired both by analysis and geometry. His mathematical work is splendid and highly influential. This volume dedicated to him emphasizes the algebrogeometric aspects of his work and honors an exceptional personality.

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Complex algebraic geometry, in memory of Jean-Pierre Demailly /
Géométrie algébrique complexe, en mémoire de Jean-Pierre Demailly

Non-Archimedean Green's functions and Zariski decompositions

Fonctions de Green non-archimédiennes et décompositions de Zariski

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To the memory of Jean-Pierre Demailly, with admiration

Abstract. We study the non-Archimedean Monge–Ampère equation on a smooth projective variety over a discretely or trivially valued field. First, we give an example of a Green's function, associated to a divisorial valuation, which is not $\mathbb{Q}$-PL (i.e. not a model function in the discretely valued case). Second, we produce an example of a function whose Monge–Ampère measure is a finite atomic measure supported in a dual complex, but which is not invariant under the retraction associated to any snc model. This answers a question by Burgos Gil et al. in the negative. Our examples are based on geometric constructions by Cutkosky and Lesieutre, and arise via base change from Green's functions over a trivially valued field; this theory allows us to efficiently encode the Zariski decomposition of a pseudoeffective numerical class.

Résumé. Nous étudions l'équation de Monge–Ampère non-archimédienne sur une variété projective lisse sur un corps de valuation discrète ou triviale. Tout d'abord, nous donnons un exemple de fonction de Green, associée à une valuation divisorielle, qui n'est pas $\mathbb{Q}$-PL (i.e. pas une fonction modèle, dans le cas de valuation discrète). Ensuite, nous produisons un exemple de fonction dont la mesure de Monge–Ampère est à support dans un complexe dual, mais qui n'est invariante par la rétraction associée à aucun modele snc. Ceci répond négativement à une question de Burgos Gil et al. Nos exemples sont basés sur des constructions géométriques de Cutkosky et Lesieutre, et sont produits par changement de base à partir de fonctions de Green sur un corps trivialement valué ; cette théorie nous permet d'encoder de façon efficace la décomposition de Zariski de toute classe pseudo-effective.

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Introduction

In the seminal paper [43], Yau studied the Monge–Ampère equation

$$(\omega + dd^c\varphi)^n = \mu \tag{MA}$$

on a compact n-dimensional Kähler manifold (X,ω), where μ is a smooth, strictly positive measure on X of mass $\int \omega^n$, and φ a smooth function on X such that the $(1,1)$-form $\omega + dd^c\varphi$ is positive. Yau proved that there exists a smooth solution φ, unique up to a constant. If ω is a rational class, say $\omega = c_1(L)$ for an ample line bundle L, then φ can be viewed as a positive metric on L, and $(\omega + dd^c\varphi)^n$ its the curvature measure.

As observed by Kontsevich, Soibelman, and Tschinkel [31, 32], when studying degenerating 1-parameter families of Kähler manifolds, it can be fruitful to use non-Archimedean geometry in the sense of Berkovich over the field $\mathbb{C}((\varpi))$ of complex Laurent series. In this context, a Monge–Ampère operator was introduced by Chambert–Loir [19], and a version of (MA) was solved by the authors and Favre [11]; see below. Uniqueness of solutions was proved earlier by Yuan and Zhang [44].

Now, the method in [11] is variational in nature, inspired by [4] in the complex case. It has the advantage of being able to deal with more general measures μ, but the drawback of providing less regularity information on the solution. In fact, [11] only gives a continuous solution, and is thus closer in spirit to the work of Kołodziej [30] than to [43].

It is therefore interesting to ask whether we can say more about the regularity of φ in (MA), at least for special measures μ. In the non-Archimedean setting, there are many possible regularity notions; to describe the one we are focusing on, we first need to make the non-Archimedean version of (MA) more precise, following [10, 11].

Let X be a smooth projective variety over $K = \mathbb{C}((\varpi))$ of dimension n. Consider a simple normal crossing (snc) model $\mathcal{X}$ of X, over the valuation ring $K^\circ = \mathbb{C}[[\varpi]]$. The dual complex $\Delta_{\mathcal{X}}$ embeds in the Berkovich analytification X^{an}, and there is a continuous retraction $p_{\mathcal{X}} \colon X^{\mathrm{an}} \to \Delta_{\mathcal{X}}$.

A semipositive closed (1,1)-form on X^{an} in the sense of loc. cit. is represented by a nef relative numerical class $\omega \in \mathrm{N}^1(\mathcal{X}/\operatorname{Spec} K^\circ)$ for some snc model $\mathcal{X}$. We assume that the image $[\omega]$ of ω in $\mathrm{N}^1(X)$ is ample. In this case, there is a natural space $\mathrm{CPSH}(\omega) = \mathrm{CPSH}(X,\omega)$ of continuous ω-plurisubharmonic (psh) functions, and a Monge–Ampère operator taking a function $\varphi \in \mathrm{CPSH}(\omega)$ to a positive Radon measure $\varphi \to (\omega + dd^c\varphi)^n$ on X^{an} of mass $[\omega]^n$; see also [20] for a local theory. When $[\omega]$ is rational, so that $[\omega] = c_1(L)$ for an ample ($\mathbb{Q}$-)line bundle L on X, we can view any $\varphi \in \mathrm{CPSH}(\omega)$ as a semipositive continuous metric on L^{an}, with curvature measure $(\omega + dd^c\varphi)^n$.

As in [11], let us normalize the Monge–Ampère operator and write

$$\mathrm{MA}_\omega(\varphi) := \tfrac{1}{[\omega]^n}(\omega + dd^c\varphi)^n.$$

The main result in [11] is that if μ is a Radon probability measure on X^{an} supported in some dual complex, then there exists $\varphi \in \mathrm{CPSH}(\omega)$, unique up to an additive real constant, such that $\mathrm{MA}_\omega(\varphi) = \mu$. More precisely, this was proved assuming that X is defined over an algebraic curve, an assumption that was later removed in [18]. Here we want to study whether for special measures μ, the solution is regular in some sense.

We first consider the class of *piecewise linear* (PL) functions. A function $\varphi \in \mathrm{C}^0(X^{\mathrm{an}})$ is ($\mathbb{Q}$-)PL if it is associated to a vertical $\mathbb{Q}$-divisor on some snc model, and PL functions are also known as *model functions*. The set $\mathrm{PL}(X)$ of PL functions is a dense $\mathbb{Q}$-linear subspace of $\mathrm{C}^0(X^{\mathrm{an}})$, and it is closed under taking finite maxima and minima.

If $\varphi \in \mathrm{PL}(X) \cap \mathrm{CPSH}(\omega)$, then the measure $\mu = \mathrm{MA}_\omega(\varphi)$ is a rational divisorial measure, i.e. a rational convex combination of Dirac masses at divisorial valuations. For example, when $[\omega] = c_1(L)$ is rational, the space $\mathrm{PL}(X) \cap \mathrm{CPSH}(\omega)$ can be identified with the space of semipositive *model metrics* on L^{an}, represented by a nef model $\mathscr{L}$ of L, and $\mathrm{MA}_\omega(\varphi)$ can be computed in terms of intersection numbers of $\mathscr{L}$.

Assuming ω rational, one may ask whether, conversely, the solution to $\mathrm{MA}_\omega(\varphi) = \mu$, with μ a rational divisorial measure, is necessarily PL. Here we focus on the case when $\mu = \delta_x$ is a Dirac measure, where $x \in X^{\mathrm{div}}$ is a divisorial valuation. In this case, it was proved in [11] that the solution $\varphi_x \in \mathrm{CPSH}(\omega)$ to the Monge–Ampère equation

$$\mathrm{MA}_\omega(\varphi_x) = \delta_x, \quad \varphi_x(x) = 0 \tag{$\star$}$$

is the *Green's function* of x, given by $\varphi_x = \sup\{\psi \in \mathrm{CPSH}(\omega) \mid \psi(x) \le 0\}$.

Theorem A. *Assume that ω is a rational semipositive closed (1,1)-form with* $[\omega]$ *ample, and that* $x \in X^{\mathrm{div}}$ *is a divisorial valuation. Let* $\varphi_x \in \mathrm{CPSH}(\omega)$ *be the Green's function satisfying* $(\star)$ *above. Then:*

- (i) *in dimension 1,* $\varphi_x \in \mathrm{PL}(X)$*;*
- (ii) *in dimension* ≥ 2*, it may happen that* $\varphi_x \notin \mathrm{PL}(X)$.

Writing $[\omega] = c_1(L)$, Theorem A says that the metric on L^{an} corresponding to φ_x is a model metric in dimension 1, but not necessarily in dimension 2 and higher. This answers a question in [11], see Remark 8.8 in loc. cit.

Here (i) is well known, for example from the work of Thuillier [42]; see Section 8.5. As for (ii), we present one example where X is an abelian surface, and another one where $X = \mathbb{P}^3$; see Examples 99 and 100.

We will discuss the structure of these examples shortly, but mention here that they are both $\mathbb{R}$-*PL*, i.e. they belong to the smallest $\mathbb{R}$-linear subspace $\mathbb{R}\mathrm{PL}(X)$ of $\mathrm{C}^0(X^{\mathrm{an}})$ containing $\mathrm{PL}(X)$ and stable under max and min. The question then arises whether also in higher dimension, the solution φ_x to $(\star)$ is $\mathbb{R}$-PL for any divisorial valuation x. While we don't have a counterexample to this exact question (with ω rational, but see Example 67), we prove that the situation can be quite complicated in dimensions three and higher.

Namely, let us say that a function $\varphi \in \mathrm{C}^0(X^{\mathrm{an}})$ is *invariant under retraction* if $\varphi = \varphi \circ p_{\mathscr{X}}$ for some (and hence any sufficiently high) snc model $\mathscr{X}$. For example, a function on X^{an} is $\mathbb{R}$-PL iff it is invariant under retraction and its restriction to any dual complex $\Delta_{\mathscr{X}}$ is $\mathbb{R}$-PL in the sense that it is affine on the cells of some subdivision of $\Delta_{\mathscr{X}}$ into real simplices.

If $\varphi \in \mathrm{CPSH}(\omega)$ is invariant under retraction, say $\varphi = \varphi \circ p_{\mathscr{X}}$, then the Monge–Ampère measure $\mathrm{MA}_\omega(\varphi)$ is supported in $\Delta_{\mathscr{X}}$. However, if μ is supported in $\Delta_{\mathscr{X}}$, then the solution φ to $\mathrm{MA}_\omega(\varphi) = \mu$ may not satisfy $\varphi = \varphi \circ p_{\mathscr{X}}$, see [25, Appendix A]. Still, one may ask whether φ is invariant under retraction, that is, $\varphi = \varphi \circ p_{\mathscr{X}'}$ for any sufficiently high snc model $\mathscr{X}'$, see Question 2 in loc. cit.. A version of this question (see Remark 77) in the context of Calabi–Yau varieties plays a key role in the recent work of Yang Li [36], see also [1, 28, 37]. Our next result provides a negative answer in general.

Theorem B. *Let* $X = \mathbb{P}^3_K$*, with* $K = \mathbb{C}((\varpi))$*, and let ω be the closed (1,1)-form associated to the numerical class of* $\mathcal{O}(1)$ *on* $\mathbb{P}^3_{K^\circ}$*. Then there exists* $\varphi \in \mathrm{CPSH}(\omega)$ *such that* $\mathrm{MA}_\omega(\varphi)$ *has finite support in some dual complex, but φ is not invariant under retraction. In particular,* $\varphi \notin \mathbb{R}\mathrm{PL}(X)$.

Let us now say more about the examples underlying Theorem B and Theorem A (ii). They all arise in the *isotrivial case*, when the variety X over K is the base change of a smooth projective variety Y over $\mathbb{C}$, and the $(1,1)$-form ω is defined by the pullback of an ample numerical class $\theta \in \mathrm{N}^1(Y)$ to the trivial (snc) model Y_{K° of $X = Y_K$. In this case, we can draw on the global pluripotential theory over a trivially valued field developed in [13], a theory which interacts well with algebro-geometric notions such as diminished base loci and Zariski decompositions of pseudoeffective classes.

Specifically, given a smooth projective complex variety Y, and an ample numerical class $\theta \in \mathrm{N}^1(Y)$, we have a convex set $\mathrm{CPSH}(\theta) = \mathrm{CPSH}(Y,\theta) \subset \mathrm{C}^0(Y^{\mathrm{an}})$ of continuous θ-psh functions, where Y^{an} now denotes the Berkovich analytification of Y with respect to the *trivial* absolute value on $\mathbb{C}$. A *divisorial valuation* on Y is of the form $v = t\,\mathrm{ord}_E$, where $t \in \mathbb{Q}_{\geq 0}$ and $E \subset Y'$ is a prime divisor on a smooth projective variety Y' with a proper birational morphism $Y' \to Y$. When instead $t \in \mathbb{R}_{\geq 0}$, we say that v is a *real divisorial valuation.* If $\Sigma \subset Y^{\mathrm{an}}$ is a finite set of real divisorial valuations, then we consider the Green's function of Σ, defined as

$$\varphi_\Sigma := \sup\{\varphi \in \mathrm{CPSH}(Y,\theta) \mid \varphi|_\Sigma \leq 0\}.$$

By [13], $\varphi_\Sigma \in \mathrm{CPSH}(Y,\theta)$, and the Monge–Ampère measure of φ_Σ is supported in Σ.

The base change $X = Y_{\mathbb{C}((\varpi))} \to Y$ induces a surjective map $\pi\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$, and this map admits a canonical section $\sigma\colon Y^{\mathrm{an}} \to X^{\mathrm{an}}$, called *Gauss extension*, and whose image consists of all $\mathbb{C}^\times$-invariant points in X^{an}. For any $\varphi \in \mathrm{CPSH}(Y,\theta)$ we have $\pi^\star\varphi \in \mathrm{CPSH}(X,\omega)$, and

$$\mathrm{MA}_\omega(\pi^\star\varphi) = \sigma_\star \mathrm{MA}_\theta(\varphi).$$

In particular, if $v \in Y^{\mathrm{div}}$, then $\pi^\star\varphi_{\{v\}}$ is the Green's function for $x := \sigma(v) \in X^{\mathrm{div}}$. As both $\pi^\star$ and $\sigma^\star$ preserve the classes of $\mathbb{Q}$-PL and $\mathbb{R}$-PL functions, we see that in order to prove Theorem A (ii), it suffices to find a surface Y and $v \in Y^{\mathrm{div}}$, such that $\varphi_v := \varphi_{\{v\}}$ is not $\mathbb{Q}$-PL.

Further, to prove Theorem B, it suffices to find a finite set Σ of real divisorial valuations on $Y = \mathbb{P}^3_{\mathbb{C}}$ such that $\pi^\star\varphi_\Sigma$ fails to be invariant under retraction. Indeed, the Gauss extension map σ takes real divisorial valuations to Abhyankar valuations, and these are exactly the ones that lie in a dual complex. We then use the following criterion. Define the *center* of any function $\varphi \in \mathrm{PSH}(Y,\theta)$ by

$$Z_Y(\varphi) := c_Y\{\varphi < \sup\varphi\},$$

where $c_Y\colon Y^{\mathrm{an}} \to Y$ is the center map, see Section 3. We show that if $\pi^\star\varphi$ is invariant under retraction, then $Z_Y(\varphi) \subset Y$ is a strict Zariski closed subset, see Corollary 97. It therefore suffices to find a Green's function φ_Σ whose center is Zariski dense.

Our analysis of the Green's functions φ_Σ is based on a relation between θ-psh functions and families of b-divisors. Namely, we can pick a proper birational morphism $\rho\colon Y' \to Y$, with Y' smooth, prime divisors $E_i \subset Y'$, and $c_i \in \mathbb{R}_{>0}$, such that $\Sigma = \{c_i^{-1}\,\mathrm{ord}_{E_i}\}$. If we set $D := \sum_i c_i^{-1} E_i$, then we can express φ_Σ in terms of the *b-divisorial Zariski decomposition* of the numerical class $\rho^\star\theta - \lambda[D]$, for $\lambda \in (-\infty, \lambda_{\mathrm{psef}}]$, where $\lambda_{\mathrm{psef}} \in \mathbb{R}$ is the largest λ such that this class is pseudoeffective (psef), see Theorem 57. The analysis of the Zariski decomposition of a psef class θ in terms of θ-psh functions is of independent interest.

Let us first consider the case of dimension two. The Zariski decomposition of $\rho^\star\theta - \lambda D$ is then an $\mathbb{R}$-PL function of λ, and this implies that the Green's function φ_Σ is $\mathbb{R}$-PL. On the other hand, φ_Σ need not be $\mathbb{Q}$-PL. In fact, we prove in Theorem 60 that φ_Σ is $\mathbb{Q}$-PL iff the pseudoeffective threshold λ_{psef} is a rational number. To prove Theorem A (ii), it therefore suffices to find a divisorial valuation v on a surface Y such that λ_{psef} is irrational, and such examples can be found with Y an abelian surface, and $v = \mathrm{ord}_E$ for a prime divisor E on Y.

Using a geometric construction by Cutkosky [21], we also give an example of a divisorial valuation v on $Y = \mathbb{P}^3$ such that φ_v is $\mathbb{R}$-PL but not $\mathbb{Q}$-PL for $\theta = c_1(\mathcal{O}(1))$, see Example 65. Being $\mathbb{R}$-PL, this example is invariant under retraction. As explained above, in order to prove Theorem B, it suffices to find Σ such that the center $c_Y(\varphi_\Sigma)$ is a Zariski dense subset of Y. Using the notation above, we show that the center contains the image on Y of the diminished base locus of the pseudoeffective class $\rho^\star\theta - \lambda_{\mathrm{psef}}[D]$ on Y'. We can then use a construction of Lesieutre [35], who showed that if $Y = \mathbb{P}^3$, $\theta = c_1(\mathcal{O}(1))$, and $\rho\colon Y' \to Y$ is the blowup at nine very general points, then there exists an effective $\mathbb{R}$-divisor D on Y' supported on the exceptional locus on ρ, such that the diminished base locus of $\rho^\star\theta - D$ is Zariski dense. If we write $D = \sum_{i=1}^{9} c_i E_i$, then we can take $\Sigma = \{c_i^{-1}\,\mathrm{ord}_{E_i}\}$.

Structure of the paper

The article is organized as follows. In Section 1 we recall some facts from birational geometry and pluripotential theory over a trivially valued field. This is used in Section 2 to relate θ-psh functions and suitable families of b-divisors, after which we study the center of a θ-psh function in Section 3. In Section 4 we define the extremal function $V_\theta \in \mathrm{PSH}(\theta)$ associated to a psef class: by evaluating this function at divisorial valuations we recover the minimal vanishing order of θ along a valuation. The extremal function is also closely related to various notions of Zariski decomposition of a psef class, as explored in Section 5. After all this, we are finally ready to study Green's functions in Section 6 and Section 7. Finally, in Section 8 and Section 9 we turn to the discretely valued case and prove Theorems A and B.

Notation and conventions

A *variety* over a field F is a geometrically integral F-scheme of finite type. We use the abbreviations *usc* for "upper semicontinuous", *lsc* for "lower semicontinuous", and *iff* for "if and only if".

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1. Preliminaries

Throughout the paper (except in Section 8) X denotes a smooth projective variety over an algebraically closed field k of characteristic 0.

1.1. *Positivity of numerical classes and base loci*

We denote by $\mathrm{N}^1(X)$ the (finite dimensional) vector space of numerical equivalence classes $\theta = [D]$ of $\mathbb{R}$-divisors D on X. It contains the following convex cones, corresponding to various positivity notions for numerical classes:

- the *pseudoeffective cone* $\mathrm{Psef}(X)$, defined as the closed cone generated by all classes of effective divisors;
- the *big cone* $\mathrm{Big}(X)$, the interior of $\mathrm{Psef}(X)$;
- the *nef cone* $\mathrm{Nef}(X)$, equal to the closed convex cone generated by all classes of basepoint free line bundles;
- the *ample cone* $\mathrm{Amp}(X)$, the interior of $\mathrm{Nef}(X)$;
- the *movable cone* $\mathrm{Mov}(X)$, the closed convex cone generated by all classes of line bundles with base locus of codimension at least 2.

These cones satisfy

$$\mathrm{Nef}(X) \subset \mathrm{Mov}(X) \subset \mathrm{Psef}(X),$$

where the first (resp. second) inclusion is an equality when $\dim X \le 2$ (resp. $\dim X \le 1$), but is in general strict for $\dim X > 2$ (resp. $\dim X > 1$). We will make use of the following simple property:

Lemma 1. *If $\theta \in \mathrm{N}^1(X)$ is movable, then $\theta|_E \in \mathrm{N}^1(E)$ is pseudoeffective for any prime divisor $E \subset X$.*

The *asymptotic base locus* $\mathbb{B}(D) \subset X$ of a $\mathbb{Q}$-divisor D is defined as the base locus of $\mathcal{O}_X(mD)$ for any $m \in \mathbb{Z}_{>0}$ sufficiently divisible. The *diminished* (or *restricted*) *base locus* and the *augmented base locus* of an $\mathbb{R}$-divisor D are respectively defined as

$$\mathbb{B}_-(D) := \bigcup_A \mathbb{B}(D+A) \quad \text{and} \quad \mathbb{B}_+(D) := \bigcap_A \mathbb{B}(D-A),$$

where A ranges over all ample $\mathbb{R}$-divisors such that $D-A$ (resp. $D+A$) is a $\mathbb{Q}$-divisor. Since ampleness is a numerical property, these loci only depend on the numerical class $\theta = [D] \in \mathrm{N}^1(X)$, and will be denoted by $\mathbb{B}_-(\theta) \subset \mathbb{B}_+(\theta)$.

The augmented base locus $\mathbb{B}_+(\theta)$ is Zariski closed, and satisfies

$$\theta \in \mathrm{Big}(X) \Longleftrightarrow \mathbb{B}_+(\theta) \neq X \quad \text{and} \quad \theta \in \mathrm{Amp}(X) \Longleftrightarrow \mathbb{B}_+(\theta) = \emptyset.$$

The diminished base locus satisfies

$$\mathbb{B}_-(\theta) = \bigcup_{\varepsilon \in \mathbb{Q}_{>0}} \mathbb{B}_+(\theta + \varepsilon\omega) \tag{1}$$

for any $\omega \in \mathrm{Amp}(X)$. It is thus an at most countable union of subvarieties, which is not Zariski closed in general, and can even be Zariski dense (see [35]). We further have

$$\theta \in \mathrm{Psef}(X) \Longleftrightarrow \mathbb{B}_-(\theta) \neq X;$$
$$\theta \in \mathrm{Nef}(X) \Longleftrightarrow \mathbb{B}_-(\theta) = \emptyset;$$
$$\theta \in \mathrm{Mov}(X) \Longleftrightarrow \operatorname{codim} \mathbb{B}_-(\theta) \geq 2.$$

1.2. *The Berkovich space*

We use [13, §1] as a reference. The *Berkovich space* X^{an} is defined as the Berkovich analytification of X with respect to the trivial absolute value on k [3]. We view it as a compact (Hausdorff) topological space, whose points are *semivaluations*, i.e. valuations $v\colon k(Y)^\times \to \mathbb{R}$ for some subvariety $Y \subset X$. We denote by $v_{Y,\mathrm{triv}} \in X^{\mathrm{an}}$ the trivial valuation on $k(Y)$, and set $v_{\mathrm{triv}} := v_{X,\mathrm{triv}}$. These trivial semivaluations are precisely the fixed points of the scaling action $\mathbb{R}_{>0} \times X^{\mathrm{an}} \to X^{\mathrm{an}}$ given by $(t, v) \mapsto tv$.

We denote $X^{\mathrm{div}} \subset X^{\mathrm{an}}$ the (dense) subset of *divisorial valuations*, of the form $v = t \operatorname{ord}_E$ with $t \in \mathbb{Q}_{\geq 0}$ and E a prime divisor on a birational model $\pi\colon Y \to X$ (the case $t = 0$ corresponding to $v = v_{\mathrm{triv}}$, by convention). In the present work, where $\mathbb{R}$-divisors arise naturally, it will be convenient to allow t to be real, in which case we will say that $v = t \operatorname{ord}_E$ is a *real divisorial valuation.* We denote by

$$X^{\mathrm{div}}_{\mathbb{R}} = \mathbb{R}_{>0} X^{\mathrm{div}}$$

the set of real divisorial valuations. It is contained in the space $X^{\mathrm{lin}} \subset X^{\mathrm{an}}$ of *valuations of linear growth* (see [17] and [13, §1.5]).

1.3. *Rational and real piecewise linear functions*

In [13], various classes of $\mathbb{Q}$-PL functions on X^{an} were introduced, and the purpose of what follows is to discuss their $\mathbb{R}$-PL counterparts.

First, any ideal $\mathfrak{b} \subset \mathcal{O}_X$ defines a homogeneous function

$$\log|\mathfrak{b}|\colon X^{\mathrm{an}} \longrightarrow [-\infty, 0]$$

such that $\log|\mathfrak{b}|(v) := -v(\mathfrak{b})$ for $v \in X^{\mathrm{an}}$.

Second, any *flag ideal* $\mathfrak{a}$, i.e. a coherent fractional ideal sheaf on $X \times \mathbb{A}^1$ invariant under the $\mathbb{G}_m$-action on $\mathbb{A}^1$ and trivial on $X \times \mathbb{G}_m$, defines a continuous function

$$\varphi_{\mathfrak{a}} : X^{an} \longrightarrow \mathbb{R}$$

given by $\varphi_{\mathfrak{a}}(v) = -\sigma(v)(\mathfrak{a})$, where $\sigma : X^{an} \to (X \times \mathbb{A}^1)^{an}$ is the *Gauss extension*, defined as follows. If v is a valuation on $k(Y)$ for some subvariety $Y \subset X$, then $\sigma(v)$ is the unique valuation on $k(Y \times \mathbb{A}^1) = k(Y)(\varpi)$ with the following property: if $f = \sum_j f_j \varpi^j \in k(Y)[\varpi]$, then $\sigma(v)(f) = \min_j\{v(f_j) + j\}$.

Concretely, any flag ideal can be written $\mathfrak{a} = \sum_{\lambda \in \mathbb{Z}} \mathfrak{a}_\lambda \varpi^{-\lambda}$ for a decreasing sequence of ideals $\mathfrak{a}_\lambda \subset \mathcal{O}_X$ such that $\mathfrak{a}_\lambda = \mathcal{O}_X$ for $\lambda \ll 0$ and $\mathfrak{a}_\lambda = 0$ for $\lambda \gg 0$, and then $\varphi_{\mathfrak{a}} = \max_\lambda(\log|\mathfrak{a}_\lambda| + \lambda)$.

We denote by:

- $\mathrm{PL}^+_{\mathrm{hom}}(X)$ the set of $\mathbb{Q}_+$-linear combinations of functions of the form $\log|\mathfrak{b}|$ with $\mathfrak{b} \subset \mathcal{O}_X$ a nonzero ideal;
- $\mathrm{PL}^+(X)$ the set of functions $\varphi \in \mathrm{C}^0(X^{an})$ of the form $\varphi = \max_i\{\psi_i + \lambda_i\}$ for a finite family $\psi_i \in \mathrm{PL}^+_{\mathrm{hom}}(X)$ and $\lambda_i \in \mathbb{Q}$; equivalently, functions of the form $\varphi = \frac{1}{m}\varphi_{\mathfrak{a}}$ for a flag ideal $\mathfrak{a}$ and $m \in \mathbb{Z}_{>0}$;
- $\mathrm{PL}(X)$ the set of differences of functions in $\mathrm{PL}^+(X)$, called *rational piecewise linear functions* ($\mathbb{Q}$*-PL functions* for short).

The sets $\mathrm{PL}^+_{\mathrm{hom}}(X)$ are stable under addition and max, while $\mathrm{PL}(X)$ is a $\mathbb{Q}$-vector space, stable under max, and is dense in $\mathrm{C}^0(X^{an})$.

As in [13, §3.1], we denote by $\mathrm{PL}(X)_{\mathbb{R}}$ the $\mathbb{R}$-vector space generated by $\mathrm{PL}(X)$. It is not stable under max anymore; to remedy this, we further introduce:

- the set $\mathrm{PL}^+(X)_{\mathbb{R}}$ of $\mathbb{R}_+$-linear combinations of functions in $\mathrm{PL}^+(X)$;
- the set $\mathbb{R}\mathrm{PL}^+(X)$ of finite maxima of functions in $\mathrm{PL}^+(X)_{\mathbb{R}}$;
- the set $\mathbb{R}\mathrm{PL}(X)$ of differences of functions in $\mathbb{R}\mathrm{PL}^+(X)$; we call its elements *real piecewise linear functions* ($\mathbb{R}$*-PL functions* for short).

As one immediately sees, the sets $\mathrm{PL}^+(X)_{\mathbb{R}}$ and $\mathbb{R}\mathrm{PL}^+(X)$ are convex cones in $\mathrm{C}^0(X^{an})$, and $\mathbb{R}\mathrm{PL}(X)$ is thus an $\mathbb{R}$-vector space. Further, $\mathbb{R}\mathrm{PL}^+(X)$, and hence $\mathbb{R}\mathrm{PL}(X)$, are clearly stable under max. Thus $\mathbb{R}\mathrm{PL}(X)$ is the smallest $\mathbb{R}$-linear subspace of $\mathrm{C}^0(X^{an})$ that is stable under max and contains $\mathrm{PL}(X)$.

Finally, introduce the convex cone $\mathrm{PL}^+_{\mathrm{hom}}(X)_{\mathbb{R}}$ of $\mathbb{R}_+$-linear combinations of functions in $\mathrm{PL}^+_{\mathrm{hom}}(X)$ (this is again not stable under max anymore). We then have:

Lemma 2. *A function $\varphi \in \mathrm{C}^0(X^{an})$ lies in $\mathbb{R}\mathrm{PL}^+(X)$ iff $\varphi = \max_i\{\psi_i + \lambda_i\}$ for a finite family $\psi_i \in \mathrm{PL}^+_{\mathrm{hom}}(X)_{\mathbb{R}}$ and $\lambda_i \in \mathbb{R}$.*

Proof. Since any function in $\mathbb{R}\mathrm{PL}^+(X)$ is a finite max of functions $\varphi \in \mathrm{PL}^+(X)_{\mathbb{R}}$, it suffices to show that φ is of the desired form. Write $\varphi = \sum_{i=1}^r t_i \varphi_i$ with $t_i \in \mathbb{R}_{>0}$ and $\varphi_i \in \mathrm{PL}^+(X)$, i.e. $\varphi_i = \max_j\{\psi_{ij} + \lambda_{ij}\}$ with $\psi_{ij} \in \mathrm{PL}^+_{\mathrm{hom}}(X)$ and $\lambda_{ij} \in \mathbb{Q}$. Then

$$\varphi = \max_{j_1,\dots,j_r} \sum_{i=1}^r t_i \left(\psi_{ij_i} + \lambda_{ij_i}\right).$$

Since each $\sum_i t_i \psi_{ij_i}$ lies in $\mathrm{PL}^+_{\mathrm{hom}}(X)_{\mathbb{R}}$, this shows that φ is of the desired form.

Conversely, assume $\varphi = \max_i\{\psi_i + \lambda_i\}$ for a finite family $\psi_i \in \mathrm{PL}^+_{\mathrm{hom}}(X)_{\mathbb{R}}$ and $\lambda_i \in \mathbb{R}$. For each i, write $\psi_i = \sum_j t_{ij}\psi_{ij}$ with $\psi_{ij} \in \mathrm{PL}^+_{\mathrm{hom}}(X) \le 0$. Pick $v \in X^{an}$ and i such that $\varphi(v) = \psi_i(v) + \lambda_i$. Since φ is bounded, we can find $c \in \mathbb{Q}$ such that $\psi_{ij}(v) \ge c$ for all j. This shows that $\varphi = \max_i \varphi_i$ with $\varphi_i := \sum_j t_{ij} \max\{\psi_{ij}, c\} + \lambda_i$. For all i, j, $\max\{\psi_{ij}, c\}$ lies in $\mathrm{PL}^+(X)$, thus $\varphi_i \in \mathrm{PL}^+(X)_{\mathbb{R}}$, and hence $\varphi \in \mathbb{R}\mathrm{PL}^+(X)$. □

1.4. *Homogeneous functions vs. b-divisors*

We use [7, §1] and [13, §6.4] as references for what follows. Recall that

- a *(real) b-divisor over X* is a collection $B = (B_Y)$ of $\mathbb{R}$-divisors on all (smooth) birational models $Y \to X$, compatible under push-forward as cycles, i.e. an element of the $\mathbb{R}$-vector space
$$\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}} := \varprojlim_Y \mathrm{Z}^1(Y)_{\mathbb{R}};$$
- a b-divisor $B = (B_Y)$ is *effective* if B_Y is effective for all Y; if B, B' are b-divisors, then we write $B \le B'$ iff $B' - B$ is effective;
- a b-divisor $B \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is said to be $\mathbb{R}$-*Cartier* if there exists a model Y, called a *determination* of B, such that $B_{Y'}$ is the pullback of B_Y for all higher birational models Y'; thus the space of $\mathbb{R}$-Cartier b-divisors is given by
$$\mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{R}} := \varinjlim_Y \mathrm{Z}^1(Y)_{\mathbb{R}}.$$

Example 3. Any $\mathbb{R}$-divisor D on a model $Y \to X$ determines an $\mathbb{R}$-Cartier b-divisor $\overline{D} \in \mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{R}}$, obtained by pulling back D to all higher models, and any $\mathbb{R}$-Cartier b-divisor is of this form.

For any $B \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ and $v \in X^{\mathrm{div}}$, we define $v(B) \in \mathbb{R}$ as follows: pick a prime divisor E on a birational model $Y \to X$ and $t \in \mathbb{Q}_{\ge 0}$ such that $v = t\,\mathrm{ord}_E$, and set
$$v(B) := t\,\mathrm{ord}_E(B_Y).$$
This is independent of the choices made, and the function $\psi_B\colon X^{\mathrm{div}} \to \mathbb{R}$ defined by
$$\psi_B(v) := v(B)$$
is homogeneous (with respect to the scaling action of $\mathbb{Q}_{>0}$).

Definition 4. *We say that a homogeneous function $\psi\colon X^{\mathrm{div}} \to \mathbb{R}$ is of* divisorial type *if $\psi(\mathrm{ord}_E) = 0$ for all but finitely many prime divisors $E \subset X$.*

The next result is straightforward:

Lemma 5. *The map $B \mapsto \psi_B$ sets up a vector space isomorphism between $\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ and the space of homogeneous functions of divisorial type on X^{div}. Moreover, $B \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is effective iff $\psi_B \ge 0$.*

We endow $\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ with the topology of pointwise convergence on X^{div}. If Ω is a topological space, then a map $f\colon \Omega \to \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is thus continuous iff $v \circ f\colon \Omega \to \mathbb{R}$ is continuous for all $v \in X^{\mathrm{div}}$. We will also say that $f\colon \Omega \to \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is lsc (resp. usc) iff $v \circ f\colon \Omega \to \mathbb{R}$ is lsc (resp. usc) for all $v \in X^{\mathrm{div}}$.

If Ω is a convex subset of a real vector space, then we say that $f\colon \Omega \to \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is convex if $v \circ f$ is convex for all $v \in X^{\mathrm{div}}$. This amounts to $f((1-t)x_0 + tx_1) \le (1-t)f(x_0) + tf(x_1)$ for $x_0, x_1 \in \Omega$, $0 \le t \le 1$. We say that f is concave if $-f$ is convex.

Finally, if $\Omega \subset \mathbb{R}$ is an interval, then $f\colon \Omega \to \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is increasing (resp. decreasing) if $v \circ f$ is increasing (resp. decreasing) for each $v \in X^{\mathrm{div}}$.

Next we will generalize [13, Theorem 6.32] to real coefficients.

Definition 6. *We denote by $\mathrm{Car}^+_{\mathrm{b}}(X)_{\mathbb{R}}$ the convex cone of divisors $B \in \mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{R}}$ that are antieffective and relatively semiample over X. We also set $\mathrm{Car}^+_{\mathrm{b}}(X)_{\mathbb{Q}} := \mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{Q}} \cap \mathrm{Car}^+_{\mathrm{b}}(X)_{\mathbb{R}}$.*

Proposition 7. *The map $B \mapsto \psi_B$ induces an isomorphism between $\mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{R}}$ and the $\mathbb{R}$-vector space generated by (the restrictions to X^{div} of) all functions $\log|\mathfrak{b}|$ with $\mathfrak{b} \subset \mathcal{O}_X$ a nonzero ideal. This isomorphism restricts to a bijection*
$$\mathrm{Car}^+_{\mathrm{b}}(X)_{\mathbb{R}} \xrightarrow{\sim} \mathrm{PL}^+_{\mathrm{hom}}(X)_{\mathbb{R}}.$$

Proof. The first point is a consequence of [13, Theorem 6.32], which also yields

$$\mathrm{Car}^+_{\mathrm{b}}(X)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{PL}^+_{\mathrm{hom}}(X).$$

Since the right-hand side generates the convex cone $\mathrm{PL}^+_{\mathrm{hom}}(X)_{\mathbb{R}}$, it suffices to show that the convex cone of antieffective and relatively semiample divisors in $\mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{R}}$ is generated by antieffective and semiample divisors in $\mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{Q}}$. By definition of a relatively semiample $\mathbb{R}$-Cartier b-divisor, we have $B = \sum_i t_i B_i$ with $t_i > 0$ and $B_i \in \mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{Q}}$ relatively semiample. By the Negativity Lemma (see [7, Proposition 2.12]), $B'_i := B_i - \overline{B_{i,X}}$ is antieffective, and still relatively semiample. Denoting by $B_X = -\sum_\alpha c_\alpha E_\alpha$ the irreducible decomposition of the antieffective $\mathbb{R}$-divisor B_X, we infer

$$B = \sum_i t_i B'_i + \sum_\alpha c_\alpha (-\overline{E_\alpha})$$

where $-\overline{E_\alpha} \in \mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{Q}}$ is antieffective and relatively semiample. The result follows. □

1.5. *Numerical b-divisor classes*

The space of *numerical b-divisor classes* is defined as

$$\mathrm{N}^1_{\mathrm{b}}(X) := \varprojlim_Y \mathrm{N}^1(Y),$$

equipped with the inverse limit topology (each finite dimensional $\mathbb{R}$-vector space $\mathrm{N}^1(Y)$ being endowed with its canonical topology).

Any b-divisor defines a numerical b-divisor class. This yields a natural quotient map

$$\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}} \longrightarrow \mathrm{N}^1_{\mathrm{b}}(X) \quad B \longmapsto [B].$$

One should be wary of the fact this map is *not* continuous with respect to the topology of pointwise convergence of $\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$. However, we observe:

Lemma 8. *For any finite set $\mathcal{E}$ of prime divisors on X, the quotient map $B \mapsto [B]$ is continuous on the subspace $\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R},\mathcal{E}}$ of b-divisors B such that B_X is supported by $\mathcal{E}$.*

Proof. For any model $Y \to X$, each B_Y with $B \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R},\mathcal{E}}$ lives in the finite dimensional vector space generated by the strict transforms of the elements of $\mathcal{E}$ and the π-exceptional prime divisors. Thus $B \mapsto [B_Y] \in \mathrm{N}^1(Y)$ is continuous on $\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R},\mathcal{E}}$, and the result follows. □

The set of numerical classes of $\mathbb{R}$-Cartier b-divisors can be identified with the direct limit

$$\varinjlim_Y \mathrm{N}^1(Y) \subset \mathrm{N}^1_{\mathrm{b}}(X).$$

In particular, any numerical class $\theta \in \mathrm{N}^1(X)$ defines a numerical b-divisor class $\overline{\theta} = (\theta_Y)_Y \in \mathrm{N}^1_{\mathrm{b}}(X)$, where θ_Y is the pullback of θ to Y.

Definition 9. *The cone of* nef b-divisor classes

$$\mathrm{Nef}_{\mathrm{b}}(X) \subset \mathrm{N}^1_{\mathrm{b}}(X)$$

is defined as the closed convex cone generated by all classes of nef $\mathbb{R}$-Cartier b-divisors.

Here an $\mathbb{R}$-Cartier b-divisor B is said to be nef if B_Y is nef for some (hence any) determination Y of B.

The following characterization is essentially formal (see [7, Lemma 2.10]).

Lemma 10. *A b-divisor $B \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is nef iff B_Y is movable for all birational models $Y \to X$. In other words,* $\mathrm{Nef}_{\mathrm{b}}(X) = \varprojlim_Y \mathrm{Mov}(Y)$.

We finally record the following version of the Negativity Lemma (see [7, Proposition 2.12]).

Lemma 11. *If $B \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is nef, then $B \le \overline{B_Y}$ for any birational model $Y \to X$.*

1.6. *Plurisubharmonic functions*

We use [13, §4] as a reference. Given a $\mathbb{Q}$-line bundle $L \in \operatorname{Pic}(X)_{\mathbb{Q}}$ and a numerical class $\theta \in \mathrm{N}^1(X)$, we denote by

- $\mathcal{H}^{\mathrm{gf}}(L) = \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L)$ the set of *generically finite Fubini–Study* functions for L, i.e. functions $\varphi\colon X^{\mathrm{an}} \to \mathbb{R} \cup \{-\infty\}$ of the form
$$\varphi = m^{-1} \max_i \{\log|s_i| + \lambda_i\},$$
where $m \in \mathbb{Z}_{>0}$ is sufficiently divisible, (s_i) is a (nonempty) finite set of nonzero sections of mL, and $\lambda_i \in \mathbb{Q}$;
- $\mathcal{H}_{\mathrm{hom}}(L) \subset \mathcal{H}^{\mathrm{gf}}(L)$ the set of *homogeneous Fubini–Study functions*, for which the λ_i can be chosen to be 0;
- $\mathrm{PSH}(\theta)$ the set of θ*-psh* functions $\varphi\colon X^{\mathrm{an}} \to \mathbb{R} \cup \{-\infty\}$, $\varphi \not\equiv -\infty$, obtained as limits of decreasing nets (φ_i) of generically finite Fubini–Study functions φ_i for $\mathbb{Q}$-line bundles L_i such that $c_1(L_i) \to \theta$ in $\mathrm{N}^1(X)$. We also write $\mathrm{PSH}(L) := \mathrm{PSH}(c_1(L))$;
- $\mathrm{CPSH}(\theta) \subset \mathrm{PSH}(\theta)$ the subset of continuous θ-psh functions;
- $\mathrm{PSH}_{\mathrm{hom}}(\theta) \subset \mathrm{PSH}(\theta)$ the subset of homogeneous θ-psh functions, that is, functions $\varphi \in \mathrm{PSH}(\theta)$ such that $\varphi(tv) = t\varphi(v)$ for $v \in X^{\mathrm{an}}$ and $t \in \mathbb{R}_{>0}$.

All functions in $\mathrm{PSH}(\theta)$ are finite valued on the set $X^{\mathrm{div}} \subset X^{\mathrm{an}}$ of divisorial valuations, and we endow $\mathrm{PSH}(\theta)$ with the topology of pointwise convergence on X^{div}. For all $\varphi, \psi \in \mathrm{PSH}(\theta)$, we further have
$$\varphi \le \psi \text{ on } X^{\mathrm{div}} \Longleftrightarrow \varphi \le \psi \text{ on } X^{\mathrm{an}}.$$
In particular, the topology of $\mathrm{PSH}(\theta)$ is Hausdorff. The set of θ-psh functions is preserved by the action of $\mathbb{R}_{>0}$ given by $(t,\varphi) \mapsto t\cdot\varphi$, where $(t\cdot\varphi)(v) := t\varphi(t^{-1}v)$.

Lemma 12. *For any $\theta \in \mathrm{N}^1(X)$ we have:*

(i) $\mathrm{PSH}(\theta) \neq \emptyset \Rightarrow \theta \in \operatorname{Psef}(X)$*;*
(ii) $0 \in \mathrm{PSH}(\theta) \Leftrightarrow \theta \in \operatorname{Nef}(X)$*;*
(iii) $\theta \in \operatorname{Big}(X) \Rightarrow \mathrm{PSH}(\theta) \neq \emptyset$*.*

As we shall see in Proposition 27, (i) is in fact an equivalence, rendering (iii) redundant.

Proof. For (i) and (ii) see [13, (4.1), (4.3)]. If θ is big, we find a big $\mathbb{Q}$-line bundle L such that $\theta - c_1(L)$ is nef. Then $\mathrm{PSH}(\theta) \supset \mathrm{PSH}(L) \supset \mathcal{H}^{\mathrm{gf}}(L) \neq \emptyset$, which proves (iii). □

Example 13. For any effective $\mathbb{R}$-divisor D, $\psi_D := \psi_{\overline{D}}$ (see Lemma 5) satisfies $-\psi_D \in \mathrm{PSH}_{\mathrm{hom}}([D])$.

Our assumption that X is smooth and k is of characteristic zero implies that the *envelope property* holds for any class $\theta \in \mathrm{N}^1(X)$, see [16, Theorem A]. This means that if $(\varphi_\alpha)_\alpha$ is any family in $\mathrm{PSH}(\theta)$ that is uniformly bounded above, and $\varphi := \sup_\alpha \varphi_\alpha$, then the usc regularization $\varphi^\star$, given by $\varphi^\star(x) = \limsup_{y\to x}\varphi(y)$, is θ-psh.

The envelope property has many favorable consequences, as discussed in [13, §5]. For example, for any birational model $\pi\colon Y \to X$ and any $\theta \in \mathrm{N}^1(X)$ we have
$$\mathrm{PSH}(\pi^\star\theta) = \pi^\star \mathrm{PSH}(\theta), \tag{2}$$
see [13, Lemma 5.13].

1.7. *The homogeneous decomposition of a psh function*

We refer to [13, §6.3] for details on what follows. Fix $\theta \in \mathrm{N}^1(X)$. For any $\varphi \in \mathrm{PSH}(\theta)$ and $\lambda \le \sup\varphi$, setting

$$\widehat{\varphi}^{\lambda} := \inf_{t>0}\{t\cdot\varphi - t\lambda\} \tag{3}$$

defines a homogeneous θ-psh function $\widehat{\varphi}^{\lambda} \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$. The family $(\widehat{\varphi}^{\lambda})_{\lambda\le\sup\varphi}$ is further concave, decreasing, and continuous for the topology of $\mathrm{PSH}_{\mathrm{hom}}(\theta)$ (i.e. pointwise convergence on X^{div}), and it gives rise to the *homogeneous decomposition*

$$\varphi = \sup_{\lambda\le\sup\varphi}\{\widehat{\varphi}^{\lambda} + \lambda\}. \tag{4}$$

For $\lambda = \sup\varphi = \varphi(v_{\mathrm{triv}})$, the function $\widehat{\varphi}^{\max} := \widehat{\varphi}^{\sup\varphi}$ computes the directional derivatives of φ at v_{triv}, i.e.

$$\widehat{\varphi}^{\max}(v) = \lim_{t\to 0_+}\frac{\varphi(tv) - \varphi(v_{\mathrm{triv}})}{t} \tag{5}$$

for $v \in X^{\mathrm{an}}$. The limit exists as the function $t \mapsto \varphi(tv)$ on $(0,\infty)$ is convex and decreasing, see [13, Proposition 4.12].

Example 14. Assume $\varphi = \varphi_{\mathfrak{a}}$ for a flag ideal $\mathfrak{a} = \sum_{\lambda\in\mathbb{Z}} \mathfrak{a}_\lambda \varpi^{-\lambda}$ on $X \times \mathbb{A}^1$. Then $\widehat{\varphi}^{\max} = \log|\mathfrak{a}_{\lambda_{\max}}|$ where $\lambda_{\max} := \max\{\lambda \in \mathbb{Z} \mid \mathfrak{a}_\lambda \neq 0\}$ (see [13, Example 6.28]).

2. Psh functions and families of b-divisors

We work with a fixed numerical class $\theta \in \mathrm{N}^1(X)$.

2.1. *Homogeneous psh functions and b-divisors*

Recall that a function $\psi \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$ is uniquely determined by its values on X^{div}. We say that ψ is of divisorial type if its restriction to X^{div} is of divisorial type, that is, $\psi(\mathrm{ord}_E) = 0$ for all but finitely many prime divisors $E \subset X$.

Slightly generalizing [13, Theorem 6.40], we show:

Proposition 15. *The map $B \mapsto \psi_B$ in Section 1.4 sets up a 1–1 correspondence between:*

(i) *the set of b-divisors $B \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ such that $B \le 0$ and $\overline{\theta} + [B] \in \mathrm{N}^1_{\mathrm{b}}(X)$ is nef;*
(ii) *the set of θ-psh homogeneous functions $\psi \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$ of divisorial type.*

Proof. Pick B as in (i). On the one hand, $\psi_{\overline{B_X}} \in \mathrm{PSH}_{\mathrm{hom}}(-B_X)$, see Example 13. On the other hand, since $\overline{\theta} + [B] = \overline{(\theta + [B_X])} + ([B] - \overline{[B_X]})$ is nef, it follows from [13, Theorem 6.40] that $\psi_{B-\overline{B_X}} = \psi_B - \psi_{\overline{B_X}}$ lies in $\mathrm{PSH}_{\mathrm{hom}}(\theta + B_X)$. Thus

$$\psi_B \in \mathrm{PSH}(\theta + B_X) + \mathrm{PSH}(-B_X) \subset \mathrm{PSH}(\theta).$$

Conversely, pick ψ as in (ii), so that $\psi = \psi_B$ with $0 \ge B \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$. By [13, Corollary 6.17], we can write ψ as the pointwise limit of a decreasing net (ψ_i) such that $\psi_i \in \mathscr{H}_{\mathrm{hom}}(L_i)$ with $L_i \in \mathrm{Pic}(X)_{\mathbb{Q}}$ and $\lim_i c_1(L_i) = \theta$. Then $\psi_i = \psi_{B_i}$ for a unique Cartier b-divisor $0 \ge B_i \in \mathrm{Car}_{\mathrm{b}}(X)_{\mathbb{Q}}$ such that $\overline{L_i} + B_i$ is semiample (see [13, Lemma 6.34]), and hence $\overline{c_1(L_i)} + [B_i] \in \mathrm{N}^1_{\mathrm{b}}(X)$ is nef. Further, $B_i \searrow B$ in $\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$, and hence $[B_i] \to [B]$ in $\mathrm{N}^1_{\mathrm{b}}(X)$ (see Lemma 8). Since $\overline{c_1(L_i)} + [B_i]$ is nef for all i, we conclude, as desired, that $\overline{\theta} + [B]$ is nef. $\square$

2.2. *Rees valuations*

In order to formulate a version of Proposition 15 for general θ-psh functions, the following notion will be useful.

Definition 16. *Given any effective $\mathbb{R}$-divisor D on X with irreducible decomposition $D = \sum_\alpha c_\alpha E_\alpha$ on X, we denote by $\Gamma_D \subset X^{\mathrm{div}}_{\mathbb{R}}$ the set of* Rees valuations *of D, defined as the real divisorial valuations $v_\alpha := c_\alpha^{-1} \operatorname{ord}_{E_\alpha}$.*

Note that $v_\alpha(D) = 1$ for all α. We can now state a variant of [13, Theorem 6.21]:

Proposition 17. *Pick $\psi \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$, and an effective $\mathbb{R}$-divisor D on X. Then*

$$\max_{\Gamma_D} \psi \le -1 \Longleftrightarrow \psi + \psi_D \in \mathrm{PSH}_{\mathrm{hom}}(\theta - D).$$

Recall that $0 \ge -\psi_D \in \mathrm{PSH}_{\mathrm{hom}}([D])$.

Proof. If $\psi + \psi_D \in \mathrm{PSH}_{\mathrm{hom}}(\theta - D)$, then $\psi \le -\psi_D$, and hence $\max_\Gamma \psi \le -1$, since $\psi_D \equiv 1$ on Γ_D. Conversely, assume $\max_{\Gamma_D} \psi \le -1$. Consider first the case where $\theta = c_1(L)$ for a $\mathbb{Q}$-line bundle and $\psi \in \mathscr{H}_{\mathrm{hom}}(L)$. For any m sufficiently divisible we thus have $\psi = \frac{1}{m} \max_i \log|s_i|$ for a finite set of nonzero section $s_i \in \mathrm{H}^0(X, mL)$. Using the notation of Definition 16, we get for all i and all α

$$c_\alpha^{-1} \operatorname{ord}_{E_\alpha}(s_i) = -\log|s_i|(v_\alpha) \ge m,$$

and hence $\operatorname{ord}_{E_\alpha}(s_i) \ge \lceil m c_\alpha \rceil$. This implies $s_i = s_i' s_{D_m}$ for some $s_i' \in \mathrm{H}^0(X, m(L - D_m))$, where

$$D_m := m^{-1} \lceil mD \rceil = \sum_\alpha m^{-1} \lceil m c_\alpha \rceil E_\alpha$$

and $s_{D_m} \in \mathrm{H}^0(X, D_m)$ is the canonical section. Since $\psi_{D_m} = -\log|s_{D_m}|$, we infer

$$\psi + \psi_{D_m} = \frac{1}{m} \max_i \log|s_i'| \in \mathscr{H}_{\mathrm{hom}}(L - D_m) \subset \mathrm{PSH}_{\mathrm{hom}}(L - D_m).$$

When $m \to \infty$, ψ_{D_m} decreases to ψ_D, and $[D_m] \to [D]$ in $\mathrm{N}^1(X)$, and we infer $\psi + \psi_D \in \mathrm{PSH}_{\mathrm{hom}}(L - D)$.

In the general case, ψ can be written as the pointwise limit of a decreasing net $\psi_i \in \mathscr{H}_{\mathrm{hom}}(L_i)$, where $L_i \in \mathrm{Pic}(X)_{\mathbb{Q}}$ satisfies that $c_1(L_i) - \theta$ is nef and tends to 0 (see [13, Corollary 6.17]). Pick $t \in (0,1)$. For all i large enough and all α, we then have $c_\alpha^{-1} \psi_i(\operatorname{ord}_{E_\alpha}) \le -t$, and hence

$$\psi_i + t\psi_D \in \mathscr{H}_{\mathrm{hom}}(L_i - tD) \subset \mathrm{PSH}_{\mathrm{hom}}(L_i - tD)$$

by the previous step of the proof. Since $\psi_i + t\psi_D$ decreases to $\psi + t\psi_D$ and $L_i - tD \to \theta - tD$ in $\mathrm{N}^1(X)$, we infer $\psi + t\psi_D \in \mathrm{PSH}_{\mathrm{hom}}(\theta - tD)$ (see [13, Theorem 4.5]). Pick any $\omega \in \mathrm{Amp}(X)$. Then $\psi + t\psi_D \in \mathrm{PSH}_{\mathrm{hom}}(\theta - D + \omega)$ for all $t \in (0,1)$ close to 1, so by the envelope property (see [13, Theorem 5.11]), we get $\psi + \psi_D \in \mathrm{PSH}_{\mathrm{hom}}(\theta - D + \omega)$. As this is true for all $\omega \in \mathrm{Amp}(X)$, we conclude $\psi + \psi_D \in \mathrm{PSH}_{\mathrm{hom}}(\theta - D)$ (again see [13, Theorem 4.5]). □

2.3. *Psh functions and families of b-divisors*

We now extend Proposition 15 to general θ-psh functions. We say that $\varphi \in \mathrm{PSH}(\theta)$ is of divisorial type if the homogeneous psh function $\widehat{\varphi}^{\max} \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$ is of divisorial type, see Section 1.7. By (5), this is equivalent to $\varphi(\operatorname{ord}_E) = \sup \varphi$ for all but finitely many prime divisors $E \subset X$.

Theorem 18. *There is a 1–1 correspondence between:*

(i) *the set of θ-psh functions $\varphi \in \mathrm{PSH}(\theta)$ of divisorial type;*
(ii) *the set of continuous, concave, decreasing families $(B_\lambda)_{\lambda \le \lambda_{\max}}$ of b-divisors, for some $\lambda_{\max} \in \mathbb{R}$, such that $B_\lambda \le 0$ and $\overline{\theta} + [B_\lambda] \in \mathrm{N}^1_{\mathrm{b}}(X)$ is nef for all $\lambda \le \lambda_{\max}$.*

The correspondence is given by

$$\varphi = \sup_{\lambda \le \lambda_{\max}} \{\psi_{B_\lambda} + \lambda\}, \quad \psi_{B_\lambda} = \widehat{\varphi}^\lambda. \tag{6}$$

In particular, we have $\lambda_{\max} = \sup \varphi$ *and* $\widehat{\varphi}^{\max} = \psi_{B_{\lambda_{\max}}}$.

Proof. Pick a family $(B_\lambda)_{\lambda \le \lambda_{\max}}$ as in (ii). By Proposition 15, setting $\psi_\lambda := \psi_{B_\lambda}$ defines a continuous, concave and decreasing family $(\psi_\lambda)_{\lambda \le \lambda_{\max}}$ in $\mathrm{PSH}_{\mathrm{hom}}(\theta)$. Since θ has the envelope property, the usc upper envelope $\varphi := \sup^\star_{\lambda \le \lambda_{\max}}(\psi_\lambda + \lambda)$ lies in $\mathrm{PSH}(\theta)$. On X^{div}, φ coincides with $\sup_{\lambda \le \lambda_{\max}}(\psi_\lambda + \lambda)$ (see [13, Theorem 5.6]). By Legendre duality, we further have $\psi_\lambda = \widehat{\varphi}^\lambda$ for $\lambda < \lambda_{\max}$ (see [13, Theorem 6.24]), and hence also for $\lambda = \lambda_{\max}$, by continuity of both sides on $(-\infty, \lambda_{\max}]$.

Conversely, pick φ as in (i), so that $\widehat{\varphi}^{\max} \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$ is of divisorial type. For each $\lambda \le \sup \varphi$ we then have $0 \ge \widehat{\varphi}^\lambda \ge \widehat{\varphi}^{\max}$, which shows that $\widehat{\varphi}^\lambda \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$ is also of divisorial type. By Proposition 15, we thus have $\widehat{\varphi}^\lambda = \psi_{B_\lambda}$ for a b-divisor $B_\lambda \le 0$ such that $\overline{\theta} + [B_\lambda]$ is nef, and the family $(B_\lambda)_{\lambda \le \sup \varphi}$ is concave, decreasing and continuous, since so is $(\widehat{\varphi}^\lambda)_{\lambda \le \sup \varphi}$. □

Remark 19. Not every θ-psh function is of divisorial type. For example, assume $\dim X = 1$, and pick a sequence $(p_j)_{j \in \mathbb{N}}$ of closed points on X, with corresponding ideals $\mathfrak{m}_j \subset \mathcal{O}_X$, and a sequence ε_j in $\mathbb{R}_{>0}$ such that $\sum_j \varepsilon_j \le \deg \theta$. Then $\varphi := \sum_j \varepsilon_j \log|\mathfrak{m}_j| \in \mathrm{PSH}(\theta)$, and $-\varepsilon_j = \varphi(\mathrm{ord}_{p_j}) < \sup \varphi = 0$ for all j (see [13, Example 4.13]).

3. The center of a θ-psh function

In this section we introduce the notion of the center of a θ-psh function. This is a subset of X defined in terms of the locus on X^{an} where φ is smaller than its maximum.

3.1. *The center map*

For any $v \in X^{\mathrm{an}}$, we denote by $c_X(v) \in X$ its center, and by

$$Z_X(v) := \overline{\{c_X(v)\}} \subset X$$

the corresponding subvariety. The center map $c_X \colon X^{\mathrm{an}} \to X$ is surjective and anticontinuous, i.e. the preimage of a closed subset is open. In particular, any subvariety $Z \subset X$ is of the form $Z = Z_X(v)$ for some v; we can simply take $v = \mathrm{ord}_Z$.

More generally, for any subset $S \subset X^{\mathrm{an}}$ we set

$$Z_X(S) := \bigcup_{v \in S} Z_X(v). \tag{7}$$

This is smallest subset of X that contains $c_X(S)$ and is closed under specialization.

3.2. *The center of a θ-psh function*

We can now introduce

Definition 20. *We define the* center *on X of any θ-psh function $\varphi \in \mathrm{PSH}(\theta)$ as*

$$Z_X(\varphi) := Z_X(\{\varphi < \sup \varphi\}) \subset X.$$

Example 21. For any nonzero ideal $\mathfrak{b} \subset \mathcal{O}_X$, the function $\psi = \log|\mathfrak{b}|$ is θ-psh if θ is sufficiently ample, and then $Z_X(\varphi) = V(\mathfrak{b})$. More generally, if $\varphi = \sum_i t_i \log|\mathfrak{b}_i|$ with $t_i \in \mathbb{R}_{>0}$ and $\mathfrak{b}_i \subset \mathcal{O}_X$ a nonzero ideal, then $Z_X(\varphi) = \bigcup_i V(\mathfrak{b}_i)$.

Recall that to any θ-psh function $\varphi \in \mathrm{PSH}(\theta)$ we can associate a homogeneous θ-psh function $\widehat{\varphi}^{\max} \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$, see Section 1.7.

Lemma 22. *For any $\varphi \in \mathrm{PSH}(\theta)$ we have $\{\varphi < \sup\varphi\} = \{\widehat{\varphi}^{\max} < 0\}$. As a consequence, $Z_X(\varphi) = Z_X(\widehat{\varphi}^{\max})$. Moreover, the following conditions are equivalent:*

(i) *φ is of divisorial type;*
(ii) *$\widehat{\varphi}^{\max}$ is of divisorial type;*
(iii) *$Z_X(\varphi) = Z_X(\widehat{\varphi}^{\max})$ contains at most finitely many prime divisors $E \subset X$.*

Proof. Pick any $v \in X^{\mathrm{an}}$. By (5) and the fact that $t \mapsto \varphi(tv)$ is decreasing and convex, it follows that $\varphi(v) < \sup\varphi$ iff $\widehat{\varphi}^{\max}(v) < 0$. Thus $Z_X(\varphi) = Z_X(\widehat{\varphi}^{\max})$ since $\sup\widehat{\varphi}^{\max} = 0$.

Now the equivalence (i) $\Leftrightarrow$ (ii) is definitional, and (ii) $\Leftrightarrow$ (iii) is clear since a prime divisor $E \subset X$ belongs to $Z_X(\widehat{\varphi}^{\max})$ iff $\widehat{\varphi}^{\max}(\mathrm{ord}_E) < 0$. □

Together with Example 14, Lemma 22 implies

Corollary 23. *If $\varphi = \varphi_{\mathfrak{a}}$ for a flag ideal $\mathfrak{a} = \sum_{\lambda\in\mathbb{Z}} \mathfrak{a}_\lambda \varpi^{-\lambda}$ on $X \times \mathbb{A}^1$, then $Z_X(\varphi_{\mathfrak{a}}) = V(\mathfrak{a}_{\lambda_{\max}})$, where $\lambda_{\max} := \max\{\lambda \in \mathbb{Z} \mid \mathfrak{a}_\lambda \neq 0\}$.*

Theorem 24. *For any $\varphi \in \mathrm{PSH}(\theta)$, the center $Z_X(\varphi)$ is a strict subset of X, and an at most countable union of (strict) subvarieties. Moreover, we have $c_X^{-1}(Z_X(\varphi)) = \{\varphi < \sup\varphi\}$.*

Proof. Note that $Z_X(\varphi)$ does not contain the generic point of X, so $Z_X(\varphi) \neq X$. Also note that by Lemma 22 we may assume that φ is homogeneous.

If $\varphi \in \mathscr{H}_{\mathrm{hom}}(L)$ for a $\mathbb{Q}$-line bundle L, so that $\varphi = \frac{1}{m}\max_i \log|s_i|$ for a finite set of nonzero sections $s_i \in \mathrm{H}^0(X, mL)$, then $Z_X(\varphi) = \bigcap_i (s_i = 0)$, which is Zariski closed. In general, φ can be written as the limit of a decreasing sequence $\varphi_m \in \mathscr{H}_{\mathrm{hom}}(L_m)$ with $L_m \in \mathrm{Pic}(X)_{\mathbb{Q}}$ such that $c_1(L_m) \to \theta$ (see [13, Remark 6.18]). For any $v \in X^{\mathrm{div}}$ we then have

$$c_X(v) \in Z_X(\varphi) \Longleftrightarrow \varphi(v) < 0 \Longleftrightarrow \varphi_m(v) < 0 \text{ for some } m,$$

i.e. $Z_X(\varphi) = \bigcup_m Z_X(\varphi_m)$, an at most countable union of strict subvarieties.

Next pick $v \in X^{\mathrm{an}}$, and set $Z = Z_X(v)$. By [13, Proposition 4.12], $\varphi(tv) = t\varphi(v)$ converges to $\varphi(v_{Z,\mathrm{triv}}) = \sup_{Z^{\mathrm{an}}}\varphi$ as $t \to +\infty$, and hence $\varphi(v) < 0 \Leftrightarrow \varphi \equiv -\infty$ on Z^{an}. By definition of the center, if $c_X(v)$ lies in $Z_X(\varphi)$, then we can find $w \in X^{\mathrm{an}}$ such that $\varphi(w) < 0$ and $c_X(v) \in Z_X(w)$, i.e. $Z \subset Z_X(w)$. Then $\varphi \equiv -\infty$ on $Z_X(w)^{\mathrm{an}} \supset Z^{\mathrm{an}}$, which yields $\varphi(v) < 0$. Conversely, assume $\varphi(v) < 0$, and hence $\varphi \equiv -\infty$ on Z^{an}. We can find $w \in X^{\mathrm{div}}$ such that $Z = Z_X(w)$. Since $\varphi \equiv -\infty$ on $Z^{\mathrm{an}} = Z_X(w)^{\mathrm{an}}$, we get $\varphi(w) < 0$, and hence $c_X(v) \in Z_X(w) \subset Z_X(\varphi)$. □

For later use we record

Lemma 25. *If $\varphi_i \in \mathrm{PSH}(\theta_i)$, $i = 1, 2$, then $Z_X(\varphi_1 + \varphi_2) = Z_X(\varphi_1) \cup Z_X(\varphi_2)$.*

3.3. *Centers of PL functions*

The following result will play a crucial role in what follows.

Lemma 26. *If $\varphi \in \mathrm{PSH}(\theta)$ lies in $\mathbb{R}\mathrm{PL}^+(X)$ (resp. $\mathbb{R}\mathrm{PL}(X)$), then $Z_X(\varphi)$ is Zariski closed (resp. not Zariski dense) in X.*

Proof. Assume first $\varphi \in \mathbb{R}\mathrm{PL}^+(X)$, and write $\varphi = \max_i\{\psi_i + \lambda_i\}$ for a finite set $\psi_i \in \mathrm{PL}^+_{\mathrm{hom}}(X)_{\mathbb{R}}$ and $\lambda_i \in \mathbb{R}$. As in Example 14, we then have $\max_i \lambda_i = \sup\varphi$, and $\widehat{\varphi}^{\max} = \max_{\lambda_i = \sup\varphi} \psi_i$. This shows that

$$Z_X(\varphi) = Z_X(\widehat{\varphi}^{\max}) = \bigcap_{\lambda_i = \sup\varphi} Z_X(\psi_i)$$

is Zariski closed (see Example 21). Assume next $\varphi \in \mathbb{R}\mathrm{PL}(X)$ and write $\varphi = \varphi_1 - \varphi_2$ with $\varphi_1, \varphi_2 \in \mathbb{R}\mathrm{PL}^+(X)$. After replacing θ with a sufficiently ample class, we may assume that φ_1, φ_2 are θ-psh. By (5) we have $\widehat{\varphi}^{\max} = \widehat{\varphi}_1^{\max} - \widehat{\varphi}_2^{\max}$, and hence

$$Z_X(\varphi) = Z_X(\widehat{\varphi}^{\max}) \subset Z_X(\widehat{\varphi}_1^{\max}) \cup Z_X(\widehat{\varphi}_2^{\max}) = Z_X(\varphi_1) \cup Z_X(\varphi_2),$$

which cannot be Zariski dense, since $Z_X(\varphi_1)$ and $Z_X(\varphi_2)$ are both Zariski closed strict subsets by the first part of the proof. □

4. Extremal functions and minimal vanishing orders

Next we define a trivially valued analogue of an important construction in the complex analytic case.

4.1. *Extremal functions*

For any $\theta \in \mathrm{N}^1(X)$, we define the *extremal function* $V_\theta \colon X^{\mathrm{an}} \to [-\infty, 0]$ as the pointwise envelope

$$V_\theta := \sup\left\{\varphi \in \mathrm{PSH}(\theta) \mid \varphi \le 0\right\}. \tag{8}$$

Proposition 27. *For any $\theta \in \mathrm{N}^1(X)$ we have*

$$\begin{aligned} \theta \in \mathrm{Psef}(X) &\Longrightarrow V_\theta \in \mathrm{PSH}_{\mathrm{hom}}(\theta); \\ \theta \notin \mathrm{Psef}(X) &\Longrightarrow V_\theta \equiv -\infty; \\ \theta \in \mathrm{Nef}(X) &\Longleftrightarrow V_\theta \equiv 0. \end{aligned}$$

In particular, $\mathrm{PSH}(\theta)$ *is nonempty iff θ is pseudoeffective. For any $\omega \in \mathrm{Amp}(X)$, we further have*

$$V_{\theta+\varepsilon\omega} \searrow V_\theta \ \text{as}\ \varepsilon \searrow 0. \tag{9}$$

Proof. Since the action $(t, \varphi) \mapsto t \cdot \varphi$ of $\mathbb{R}_{>0}$ preserves the set of candidate functions φ in (8), V_θ is necessarily fixed by the action, and hence homogeneous. If θ is not psef, then $\mathrm{PSH}(\theta)$ is empty (see Lemma 12), and hence $V_\theta \equiv -\infty$. By Lemma 12, we also have $V_\theta \equiv 0$ iff θ is nef.

Next, assume $\theta \in \mathrm{Big}(X)$. Then $\mathrm{PSH}(\theta)$ is non-empty (see Lemma 12), and the envelope property implies that $V_\theta^\star$ is θ-psh and nonpositive. It is thus a candidate in (8), and hence $V_\theta^\star \le V_\theta$, which shows that $V_\theta^\star = V_\theta$ is θ-psh.

Assume now $\theta \in \mathrm{Psef}(X)$, and pick $\omega \in \mathrm{Amp}(X)$. For each $\varepsilon > 0$, the previous step yields $V_\varepsilon := V_{\theta+\varepsilon\omega} \in \mathrm{PSH}_{\mathrm{hom}}(\theta + \varepsilon\omega)$. For $0 < \varepsilon < \delta$ we further have $\mathrm{PSH}(\theta) \subset \mathrm{PSH}(\theta + \varepsilon\omega) \subset \mathrm{PSH}(\theta + \delta\omega)$, and hence $V_\delta \ge V_\varepsilon \ge V_\theta$. Set $V := \lim_\varepsilon V_\varepsilon$. For any $\delta > 0$ fixed, we have $V_\varepsilon \in \mathrm{PSH}_{\mathrm{hom}}(\theta + \delta\omega)$ for $\varepsilon \le \delta$, and $V_\varepsilon \searrow V$ as $\varepsilon \to 0$. Thus $V \in \mathrm{PSH}_{\mathrm{hom}}(\theta + \delta\omega)$ for all $\delta > 0$, and hence $V \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$. Since V is a candidate in (8), we get $V \le V_\theta$, and hence $V_\theta = V = \lim_\varepsilon V_\varepsilon$. This proves that V_θ is θ-psh, as well as (9). □

4.2. *Minimal vanishing orders*

For $\theta \in \mathrm{Psef}(X)$, the function $V_\theta \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$ is uniquely determined by its restriction to X^{div}, where it is furthermore finite valued. For any $v \in X^{\mathrm{div}}$ we set

$$v(\theta) := -V_\theta(v) = \inf\{-\varphi(v) \mid \varphi \in \mathrm{PSH}(\theta), \varphi \le 0\} \in \mathbb{R}_{\ge 0}. \tag{10}$$

Note that

$$v(\theta) = \sup_{\varepsilon > 0} v(\theta + \varepsilon\omega) \tag{11}$$

for any $\omega \in \mathrm{Amp}(X)$, by (9). As we next show, these invariants coincide with the minimal/asymptotic vanishing orders studied in [6, 22, 40].

Proposition 28. *Pick $v \in X^{\mathrm{div}}$. Then:*

(i) *the function $\theta \mapsto v(\theta)$ is homogeneous, convex and lsc on* $\mathrm{Psef}(X)$*; in particular, it is continuous on* $\mathrm{Big}(X)$*;*

(ii) *for any $\theta \in \mathrm{Psef}(X)$ we have*

$$v(\theta) \le \inf\left\{v(D) \mid D \equiv \theta \text{ effective } \mathbb{R}\text{-divisor}\right\}, \tag{12}$$

and equality holds when θ is big.

Note that equality in (12) fails in general for θ is not big, as there might not even exist any effective $\mathbb{R}$-divisor D in the class of θ.

Proof. Using $\mathrm{PSH}(\theta)+\mathrm{PSH}(\theta') \subset \mathrm{PSH}(\theta+\theta')$ and $\mathrm{PSH}(t\theta) = t\,\mathrm{PSH}(\theta)$ for $\theta,\theta' \in \mathrm{Psef}(X)$ and $t > 0$, it is straightforward to see that $\theta \mapsto v(\theta)$ is convex and homogeneous on $\mathrm{Psef}(X)$. Being also finite valued, it is automatically continuous on the interior $\mathrm{Big}(X)$. For any $\omega \in \mathrm{Amp}(X)$ and $\varepsilon > 0$, $\theta \mapsto v(\theta+\varepsilon\omega)$ is thus continuous on $\mathrm{Psef}(X)$, and (11) thus shows that $\theta \mapsto v(\theta)$ is lsc, which proves (i).

Next pick $\theta \in \mathrm{Psef}(X)$. For each effective $\mathbb{R}$-divisor $D \equiv \theta$, the function $-\psi_D \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$, see Example 13, is a competitor in (8). Thus $-v(D) = \psi_D(v) \le V_\theta(v) = -v(\theta)$, which proves the first half of (ii). Now assume θ is big, and denote by $v'(\theta)$ the right-hand side of (12). Both $v(\theta)$ and $v'(\theta)$ are (finite valued) convex function of $\theta \in \mathrm{Big}(X)$. They are therefore continuous, and it is thus enough to prove the equality $v(\theta) = v'(\theta)$ when $\theta = c_1(L)$ with $L \in \mathrm{Pic}(X)_{\mathbb{Q}}$ big. To this end, pick an ample $\mathbb{Q}$-line bundle A, and set $\omega := c_1(A)$. By [13, Theorem 4.15], for any $\varepsilon > 0$ we can find $\varphi \in \mathcal{H}^{\mathrm{gf}}(L+A)$ such that $\varphi \ge V_\theta$ and $\varphi(v_{\mathrm{triv}}) = \sup\varphi \le \varepsilon$. By definition, we have $\varphi = m^{-1}\max_i\{\log|s_i| + \lambda_i\}$ with m sufficiently divisible and a finite family of nonzero sections $s_i \in \mathrm{H}^0(X, m(L+A))$ and constants $\lambda_i \in \mathbb{Q}$. Then $\max_i \lambda_i = m\sup\varphi \le m\varepsilon$, and $m^{-1}v(s_i) = v(D_i)$ with $D_i := m^{-1}\mathrm{div}(s_i) \equiv \theta+\omega$, and hence $m^{-1}v(s_i) \ge v'(\theta+\omega)$. Thus

$$-v(\theta) = V_\theta(v) \le \varphi(v) = m^{-1}\max_i\{v(s_i)+\lambda_i\} \le -v'(\theta+\omega)+\varepsilon.$$

This shows $v'(\theta) \ge v(\theta) \ge v'(\theta+\omega)$, and hence $v'(\theta) = v(\theta)$, since $\lim_{\omega\to 0} v'(\theta+\omega) = v'(\theta)$ by continuity on the big cone. □

Remark 29. If $L \in \mathrm{Pic}(X)$ is big, then [22, Corollary 2.7] (or, alternatively, a small variant of the above argument) shows that $v(c_1(L))$ is also equal to the asymptotic vanishing order

$$\begin{aligned} v(\|L\|) &:= \lim_{m\to\infty} \tfrac{1}{m}\min\left\{v(s) \mid s \in \mathrm{H}^0(X, mL)\setminus\{0\}\right\} \\ &= \inf\left\{v(D) \mid D \sim_{\mathbb{Q}} L \text{ effective } \mathbb{Q}\text{-divisor}\right\}. \end{aligned}$$

Remark 30. Continuity of minimal vanishing orders beyond the big cone is a subtle issue. For any $v \in X^{\mathrm{div}}$, the function $\theta \mapsto v(\theta)$, being convex and lsc on $\mathrm{Psef}(X)$, is automatically continuous on any polyhedral subcone (cf. [27]). When $\dim X = 2$, it is in fact continuous on the whole of $\mathrm{Psef}(X)$, but this fails in general when $\dim X \ge 3$ (see respectively Proposition III.1.19 and Example IV.2.8 in [40]).

4.3. *The center of an extremal function*

The following fact plays a key role in what follows.

Theorem 31. *For any $\theta \in \mathrm{Psef}(X)$, the function $V_\theta \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$ is of divisorial type (see Definition 4). Further, its center $Z_X(V_\theta)$ coincides with the diminished base locus $\mathbb{B}_-(\theta)$ (see Section 1.1).*

The proof relies on the next result, which corresponds to [40, Corollary III.1.11] (see also [6, Theorem 3.12] in the analytic context).

Lemma 32. *Pick $\theta \in \mathrm{Psef}(X)$, and assume $E_1, \dots, E_r \subset X$ are distinct prime divisors such that $\mathrm{ord}_{E_i}(\theta) > 0$ for all i. Then $[E_1], \dots, [E_r]$ are linearly independent in $\mathrm{N}^1(X)$. In particular, $r \le \rho(X) = \dim \mathrm{N}^1(X)$.*

Proof. We reproduce the simple argument of [8, Theorem 3.5 (v)] for the convenience of the reader. By (11), after adding to θ a small enough ample class we assume that θ is big. Suppose $\sum_i c_i [E_i] = 0$ with $c_i \in \mathbb{R}$, so that $G := \sum_i c_i E_i$ is numerically equivalent to 0, and choose $0 < \varepsilon \ll 1$ such that $\mathrm{ord}_{E_i}(\theta) + \varepsilon c_i > 0$ for all i. Pick any effective $\mathbb{R}$-divisor $D \equiv \theta$ and set $D' := D + \varepsilon G$. Then D' is effective, since

$$\mathrm{ord}_{E_i}(D') = \mathrm{ord}_{E_i}(D) + \varepsilon c_i \ge \mathrm{ord}_{E_i}(\theta) + \varepsilon c_i > 0$$

for all i. Since $G \equiv 0$, we also have $D' \equiv \theta$, and (12) thus yields for each i

$$\mathrm{ord}_{E_i}(\theta) \le \mathrm{ord}_{E_i}(D') = \mathrm{ord}_{E_i}(D) + \varepsilon c_i.$$

Taking the infimum over D we get $\mathrm{ord}_{E_i}(\theta) \le \mathrm{ord}_{E_i}(\theta) + \varepsilon c_i$ (see Proposition 28 (ii)), i.e. $c_i \ge 0$ for all i. Thus $G \ge 0$, and hence $G = 0$, since $G \equiv 0$. This proves $c_i = 0$ for all i which shows, as desired, that the $[E_i]$ are linearly independent. □

Proof of Theorem 31. By (10), the first assertion means that there are only finitely many prime divisors $E \subset X$ such that $\mathrm{ord}_E(\theta) > 0$, and is thus a direct consequence of Lemma 32. Pick $v \in X^{\mathrm{div}}$. The second point is equivalent to $v(\theta) > 0 \Leftrightarrow c_X(v) \in \mathbb{B}_-(\theta)$. When θ is big, this is the content of [22, Theorem B]. In the general case, pick $\omega \in \mathrm{Amp}(X)$. Then $v(\theta) > 0$ iff $v(\theta + \varepsilon\omega) > 0$ for $0 < \varepsilon \ll 1$, by (11), while $c_X(v) \in \mathbb{B}_-(\theta)$ iff $c_X(v) \in \mathbb{B}_-(\theta + \varepsilon\omega)$ for $0 < \varepsilon \ll 1$, by (1). The result follows. □

For later use, we also note:

Lemma 33. *For any polyhedral subcone $C \subset \mathrm{Psef}(X)$, we have:*

(i) *$\theta \mapsto v(\theta)$ is continuous on C for all $v \in X^{\mathrm{div}}$;*
(ii) *the set of prime divisors $E \subset X$ such that $\mathrm{ord}_E(\theta) > 0$ for some $\theta \in C$ is finite.*

Proof. As mentioned in Remark 30, any convex, lsc function on a polyhedral cone is continuous (see [27]), and (i) follows. To see (ii), pick a finite set of generators (θ_i) of C. Each $\theta \in C$ can be written as $\theta = \sum_i t_i \theta_i$ with $t_i \ge 0$. By convexity and homogeneity of minimal vanishing orders, this implies $\mathrm{ord}_E(\theta) \le \sum_i t_i \,\mathrm{ord}_E(\theta_i)$, so that $\mathrm{ord}_E(\theta) > 0$ implies $\mathrm{ord}_E(\theta_i) > 0$ for some i. The result now follows from Lemma 32. □

5. Zariski decompositions

Next we study the close relationship between the extremal function in Section 4, and the various versions of the Zariski decomposition of a psef numerical class.

5.1. *The b-divisorial Zariski decomposition*

Pick $\theta \in \mathrm{N}^1(X)$ a psef class. By Theorem 31, the function $X^{\mathrm{div}} \ni v \mapsto v(\theta) = -V_\theta(v)$ is of divisorial type. We denote by

$$\mathrm{N}(\theta) \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$$

the corresponding effective b-divisor, which thus satisfies

$$\psi_{\mathrm{N}(\theta)}(v) = v(\mathrm{N}(\theta)) = v(\theta) = -V_\theta(v)$$

for all $v \in X^{\mathrm{div}}$.

Theorem 34. *For any $\theta \in \mathrm{Psef}(X)$, the b-divisor class*

$$\mathrm{P}(\theta) := \overline{\theta} - [\mathrm{N}(\theta)] \in \mathrm{N}^1_{\mathrm{b}}(X)$$

is nef, and $\mathrm{N}(\theta)$ is the smallest effective b-divisor with this property. Moreover,

$$\mathrm{N}(\theta) \ge \overline{\mathrm{N}(\theta)_Y} \tag{13}$$

for all birational models $Y \to X$.

We call $\overline{\theta} = \mathrm{P}(\theta) + [\mathrm{N}(\theta)]$ the *b-divisorial Zariski decomposition* of θ. At least when θ is big, this construction is basically equivalent to [33, Theorem D], and to the case $p = 1$ of [9, §2.2].

Note that the b-divisorial Zariski decomposition is birationally invariant:

Lemma 35. *For any $\theta \in \mathrm{Psef}(X)$ and any birational model $\pi\colon Y \to X$, we have*

$$\mathrm{N}(\pi^\star\theta) = \mathrm{N}(\theta) \quad \textit{and} \quad \mathrm{P}(\pi^\star\theta) = \mathrm{P}(\theta)$$

in $\mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}} = \mathrm{Z}^1_{\mathrm{b}}(Y)_{\mathbb{R}}$ and $\mathrm{N}^1_{\mathrm{b}}(X)_{\mathbb{R}} = \mathrm{N}^1_{\mathrm{b}}(Y)_{\mathbb{R}}$, respectively.

Proof. Since $\mathrm{PSH}(\pi^\star\theta) = \pi^\star\,\mathrm{PSH}(\theta)$, see (2), we have $V_{\pi^\star\theta} = \pi^\star V_\theta$, and the result follows. □

Proof of Theorem 34. Since $\psi_{-\mathrm{N}(\theta)} = V_\theta$ is θ-psh, Proposition 15 shows that $\overline{\theta} - [\mathrm{N}(\theta)]$ is nef, which yields the last point, by the Negativity Lemma (see Lemma 11). Conversely, if $E \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is effective with $\overline{\theta} - [E]$ nef, then $-\psi_E \in \mathrm{PSH}_{\mathrm{hom}}(\theta)$, again by Proposition 15. Thus $-\psi_E \le V_\theta = -\psi_{\mathrm{N}(\theta)}$, and hence $E \ge \mathrm{N}(\theta)$. □

As a consequence of Proposition 28, we get

Corollary 36. *The map $\mathrm{Psef}(X) \ni \theta \mapsto \mathrm{N}(\theta) \in \mathrm{Z}^1_{\mathrm{b}}(X)$ is homogeneous, lsc, and convex.*

5.2. *The divisorial Zariski decomposition*

For any $\theta \in \mathrm{Psef}(X)$, we denote by $\mathrm{N}_X(\theta) := \mathrm{N}(\theta)_X$ the incarnation of $\mathrm{N}(\theta) \in \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ on X, which thus satisfies

$$\mathrm{N}_X(\theta) = \sum_{E \subset X} \mathrm{ord}_E(\theta) E \tag{14}$$

with E ranging over all prime divisors of X, and $\mathrm{ord}_E(\theta) = 0$ for all but finitely many E.

For any effective $\mathbb{R}$-divisor D on X with numerical class $[D] \in \mathrm{Psef}(X)$, (12) yields

$$\mathrm{N}_X(D) := \mathrm{N}_X([D]) \le D. \tag{15}$$

More generally, the following variational characterization holds.

Theorem 37. *For any $\theta \in \mathrm{Psef}(X)$, the class*

$$\mathrm{P}_X(\theta) := \theta - [\mathrm{N}_X(\theta)] \in \mathrm{N}^1(X)$$

is movable, and $\mathrm{N}_X(\theta)$ is the smallest effective $\mathbb{R}$-divisor on X with this property.

Following [6], we call the decomposition

$$\theta = \mathrm{P}_X(\theta) + [\mathrm{N}_X(\theta)]$$

the *divisorial Zariski decomposition* of θ. It coincides with the *σ-decomposition* of [40].

Proof of Theorem 37. By definition, $\mathrm{P}_X(\theta)$ is the incarnation on X of $\overline{\theta} - [\mathrm{N}(\theta)]$. By Theorem 34, the latter class is nef, and $\mathrm{P}_X(\theta)$ is thus movable, by Lemma 10.

To prove the converse, assume first that θ is movable. We then need to show $\mathrm{N}_X(\theta) = 0$, i.e. $\mathrm{ord}_E(\theta) = 0$ for each $E \subset X$ prime (see (14)). By (12), this is clear if $\theta = c_1(L)$ for a big line bundle L with base locus of codimension at least 2. Since the movable cone $\mathrm{Mov}(X)$ is generated by the classes of such line bundles, the continuity of $\theta \mapsto \mathrm{ord}_E(\theta)$ on the big cone yields the result when θ is further big, and the case of an arbitrary movable class follows by (11).

Finally, consider any $\theta \in \mathrm{Psef}(X)$ and any effective $\mathbb{R}$-divisor D on X such that $\theta - [D]$ is movable. For any $E \subset X$ prime we then have $\mathrm{ord}_E(\theta - [D]) = 0$ by the previous step, and $\mathrm{ord}_E([D]) \le \mathrm{ord}_E(D)$ by (15)). Thus

$$\mathrm{ord}_E(\theta) \le \mathrm{ord}_E(\theta - [D]) + \mathrm{ord}_E(D) = \mathrm{ord}_E(D).$$

This shows $\mathrm{N}_X(\theta) \le D$, which concludes the proof. □

Remark 38. Theorem 37 implies the following converse of Lemma 10: a class $\theta \in \mathrm{N}^1(X)$ is movable iff $\theta = \alpha_X$ for a nef b-divisor class $\alpha \in \mathrm{Nef_b}(X)$.

Corollary 39. *Pick $\theta \in \mathrm{Psef}(X)$ and a prime divisor $E \subset X$. Then $(\theta - \mathrm{ord}_E(\theta)E)|_E \in \mathrm{N}^1(E)$ is pseudoeffective.*

Proof. We have $\theta - \mathrm{ord}_E(\theta)[E] = \mathrm{P}_X(\theta) + \sum_{F \ne E} \mathrm{ord}_F(\theta)[F]$, where F ranges over all prime divisors of X distinct from E. Since $\mathrm{P}_X(\theta)$ is movable, $\mathrm{P}_X(\theta)|_E$ is psef. On the other hand, $[F]|_E$ is psef for any $F \ne E$, and the result follows. □

Lemma 40. *For any $\theta \in \mathrm{Psef}(X)$ and any birational model $\pi \colon Y \to X$, the incarnation of $\mathrm{N}(\theta)$ on Y coincides with $\mathrm{N}_Y(\pi^\star\theta)$. Further, the following are equivalent:*

(i) *the b-divisor $\mathrm{N}(\theta)$ is $\mathbb{R}$-Cartier, and determined on Y;*
(ii) *$\mathrm{P}_Y(\pi^\star\theta)$ is nef.*

Proof. The first point follows from Lemma 35. If (i) holds then the nef b-divisor class $\overline{\theta} - \mathrm{N}(\theta)$ is $\mathbb{R}$-Cartier and determined on Y. Thus $(\overline{\theta} - \mathrm{N}(\theta))_Y = \pi^\star\theta - \mathrm{N}_Y(\pi^\star\theta) = \mathrm{P}_Y(\pi^\star\theta)$ is nef, and hence (i) ⇒ (ii).

Conversely, assume (ii). Then $\overline{\mathrm{N}(\theta)_Y} = \overline{\mathrm{N}_Y(\pi^\star\theta)}$ is an effective b-divisor, and the b-divisor class $\overline{\theta} - [\overline{\mathrm{N}(\theta)_Y}] = \overline{\mathrm{P}_Y(\pi^\star\theta)}$ is nef. By Theorem 34 this implies $\mathrm{N}(\theta) \le \overline{\mathrm{N}(\theta)_Y}$, while $\mathrm{N}(\theta) \ge \overline{\mathrm{N}(\theta)_Y}$ always holds (see (13)). This proves (ii) ⇒ (i). □

Since any movable class on a surface is nef, we get:

Corollary 41. *If $\dim X = 2$ then $\mathrm{N}(\theta) = \overline{\mathrm{N}_X(\theta)}$ for all $\theta \in \mathrm{Psef}(X)$.*

In contrast, see [40, Theorem IV.2.10] for an example of a big line bundle L on a 4-fold X such that the b-divisor $\mathrm{N}(L)$ is not $\mathbb{R}$-Cartier, i.e. $\mathrm{P}_Y(\pi^\star L)$ is not nef for any model $\pi \colon Y \to X$.

5.3. *Zariski exceptional divisors and faces*

This section revisits [6, §3.3].

Definition 42. *We say that:*

(i) *an effective $\mathbb{R}$-divisor D on X is* Zariski exceptional *if $\mathrm{N}_X(D) = D$, or equivalently, $\mathrm{P}_X([D]) = 0$;*
(ii) *a finite family (E_i) of prime divisors $E_i \subset X$ is* Zariski exceptional *if every effective $\mathbb{R}$-divisor supported in the E_i's is Zariski exceptional.*

We also define a Zariski exceptional face *F of $\mathrm{Psef}(X)$ as an extremal subcone such that $\mathrm{P}_X|_F \equiv 0$.*

Here a closed subcone $C \subset \mathrm{Psef}(X)$ is extremal iff $\alpha, \beta \in \mathrm{Psef}(X)$, $\alpha + \beta \in C$ implies $\alpha, \beta \in C$.
We first note:

Lemma 43. *An effective $\mathbb{R}$-divisor D on X is Zariski exceptional iff $\mathrm{N}(D) = \overline{D}$.*

Proof. Assume $\mathrm{N}_X(D) = D$. Then $\mathrm{N}(D) \le \overline{D}$, by Theorem 34, and $\mathrm{N}(D) \ge \overline{\mathrm{N}_X(D)} = \overline{D}$ (see (13)). The result follows. □

The above notions are related as follows:

Theorem 44. *The following properties hold:*

(i) *if $E \subset X$ is a prime divisor, then E is either movable (in which case $E|_E$ is psef), or it is Zariski exceptional;*
(ii) *the set of Zariski exceptional families of prime divisors on X is at most countable;*
(iii) *for any $\theta \in \mathrm{Psef}(X)$, the irreducible components of $\mathrm{N}_X(\theta)$ form a Zariski exceptional family; in particular, $\mathrm{N}_X(\theta)$ is Zariski exceptional;*
(iv) *each Zariski exceptional family (E_i) is linearly independent in $\mathrm{N}^1(X)$, and generates a Zariski exceptional face $F := \sum_i \mathbb{R}_{\ge 0}[E_i]$ of $\mathrm{Psef}(X)$;*
(v) *conversely, each Zariski exceptional face F of $\mathrm{Psef}(X)$ arises as in* (iv).

Proof. Assume $E \subset X$ is a prime divisor. Then $\mathrm{N}_X(E) \le E$ (see (15)), and hence $\mathrm{N}_X(E) = cE$ with $c \in [0,1]$. If $c = 1$, then E is Zariski exceptional. Otherwise,

$$E = (1-c)^{-1}(E - \mathrm{N}_X(E)) \equiv (1-c)^{-1}\,\mathrm{P}_X(E)$$

is movable (and $c = 0$). This proves (i).

To see (ii), note that a Zariski exceptional prime divisor satisfies $E = \mathrm{N}_X(E)$, and hence is uniquely determined by its numerical class $[E] \in \mathrm{N}^1(X)_{\mathbb{Q}}$. As a consequence, the set of Zariski exceptional primes is at most countable, and hence so is the set of Zariski exceptional families.

Pick $\theta \in \mathrm{Psef}(X)$. We first claim that $D := \mathrm{N}_X(\theta)$ is Zariski exceptional. Since $\mathrm{P}_X(\theta) = \theta - [D]$ and $\mathrm{P}_X(D) = [D - \mathrm{N}_X(D)]$ are both movable, $\theta - [\mathrm{N}_X(D)]$ is movable as well. Theorem 37 thus yields $\mathrm{N}_X(D) \ge \mathrm{N}_X(\theta) = D$, which proves the claim in view of (15). Denote by $D = \sum_{i=1}^r c_i E_i$ the irreducible decomposition of D, and set $f_i(x) := \mathrm{ord}_{E_i}(\sum_j x_j E_j)$ for $1 \le i \le r$. This defines a convex function $f_i \colon \mathbb{R}^r_{\ge 0} \to \mathbb{R}_{\ge 0}$ which satisfies $f_i(x) \le x_i$ for all x, by (15). Since equality holds at the interior point $x = c \in \mathbb{R}^r_{>0}$, we necessarily have $f_i(x) = x_i$ for all $x \in \mathbb{R}^r_{\ge 0}$, which proves (iii).

Next pick a Zariski exceptional family (E_i). By Lemma 32, the $[E_i]$ are linearly independent in $\mathrm{N}^1(X)$. By definition, we have $\mathrm{P}_X \equiv 0$ on $F := \sum_i \mathbb{R}_{\ge 0}[E_i]$. To see that F is an extremal face of $\mathrm{Psef}(X)$, pick $D := \sum_i c_i E_i$ with $c_i \ge 0$, and assume $[D] = \alpha + \beta$ with $\alpha, \beta \in \mathrm{Psef}(X)$. We need to show that both α and β lie in F. By Definition 42 we have $D = \mathrm{N}_X(D) \le \mathrm{N}_X(\alpha) + \mathrm{N}_X(\beta)$, and hence

$$[\mathrm{N}_X(\alpha)] + [\mathrm{N}_X(\beta)] \le \mathrm{P}_X(\alpha) + \mathrm{P}_X(\beta) + [\mathrm{N}_X(\alpha)] + [\mathrm{N}_X(\beta)] = \alpha + \beta = [D] \le [\mathrm{N}_X(\alpha)] + \mathrm{N}_X(\beta)], \quad (16)$$

with respect to the psef order on $\mathrm{N}^1(X)$. Since $\mathrm{Psef}(X)$ is strict, we infer $\mathrm{P}_X(\alpha) = \mathrm{P}_X(\beta) = 0$ and $[D] = [\mathrm{N}_X(\alpha)] + [\mathrm{N}_X(\beta)]$. Since $\mathrm{N}_X(\alpha) + \mathrm{N}_X(\beta) - D$ is effective, it follows that $\mathrm{N}_X(\alpha) + \mathrm{N}_X(\beta) = D$. This implies that $\mathrm{N}_X(\alpha)$ and $\mathrm{N}_X(\beta)$ are supported in the E_i's, which proves, as desired, that $\alpha = [\mathrm{N}_X(\alpha)]$ and $\beta = [\mathrm{N}_X(\beta)]$ both lie in F. Thus (iv) holds.

Conversely, assume that $F \subset \mathrm{Psef}(X)$ is a Zariski exceptional face, and pick a class θ in its relative interior $\mathring{F}$. By (iii), the components (E_i) of $\mathrm{N}_X(\theta)$ form a Zariski exceptional family, which thus generates a Zariski exceptional face $F' := \sum_i \mathbb{R}_{\ge 0}[E_i]$. Since F and F' are both extremal faces containing θ in their relative interior, we conclude $F = F'$, which proves (v). □

As a result, Zariski exceptional families are in 1–1 correspondence with Zariski exceptional faces, which are rational simplicial cones generated by Zariski exceptional primes.

For surfaces, the notions above admit the following interpretation: see e.g. Theorems 5.4 and 4.8 in [6]:

Theorem 45. *Assume* $\dim X = 2$*. Then:*

(i) *a finite family* (E_i) *of prime divisors on* X *is Zariski exceptional iff the intersection matrix* $(E_i \cdot E_j)$ *is negative definite;*

(ii) *for any* $\theta \in \mathrm{Psef}(X)$*,* $\theta = \mathrm{P}_X(\theta) + [\mathrm{N}_X(\theta)]$ *coincides with the classical Zariski decomposition, i.e.* $\mathrm{P}_X(\theta)$ *is nef,* $\mathrm{N}_X(\theta)$ *is Zariski exceptional, and* $\mathrm{P}_X(\theta) \cdot \mathrm{N}_X(\theta) = 0$.

5.4. *Piecewise linear Zariski decompositions*

We introduce the following terminology:

Definition 46. *Given any convex subcone* $C \subset \mathrm{Psef}(X)$*, we say that* the Zariski decomposition is piecewise linear *(*PL *for short) on* C *if the map* $\mathrm{N}\colon C \to \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ *extends to a PL map* $\mathrm{N}^1(X) \to \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$*, i.e. a map that is linear on each cone of some finite fan decomposition of* $\mathrm{N}^1(X)$*. If the fan and the linear maps on its cones can further be chosen rational, then we say that* the Zariski decomposition is $\mathbb{Q}$-PL *on* C.

Lemma 47. *Let* $C \subset \mathrm{Psef}(X)$ *be a convex cone, and assume that* C *is written as the union of finitely many convex subcones* C_i*. Then the Zariski decomposition is PL (resp.* $\mathbb{Q}$*-PL) on* C *iff it is PL (resp.* $\mathbb{Q}$*-PL) on each* C_i*.*

Proof. The "only if" part is clear. Conversely, assume the Zariski decomposition is PL (resp. $\mathbb{Q}$-PL) on each C_i. After further subdividing each C_i according to a fan decomposition of $\mathrm{N}^1(X)$, we may assume that there exists a linear (resp. rational linear) map $L_i\colon \mathrm{N}^1(X) \to \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ that coincides with N on C_i. If C_i has nonempty interior in C, then $L_i|_{\mathrm{Vect}\, C}$ is uniquely determined as the derivative of N at any interior point of C_i, and we have $\mathrm{N} \ge L_i$ on C by convexity of N, see Corollary 36. Set $F := \max_i L_i$, where the maximum is over all C_i with nonempty interior in C. Then $F\colon \mathrm{N}^1(X) \to \mathrm{Z}^1_{\mathrm{b}}(X)_{\mathbb{R}}$ is PL (resp. $\mathbb{Q}$-PL), $\mathrm{N} \ge F$ on C, and equality holds outside the union A of all C_i with empty interior in C. Since A has zero measure, its complement is dense in C. Since $\mathrm{N} - F$ is lsc, see Corollary 36, we infer $\mathrm{N} \le F$ on C, which proves the "if" part. □

As a consequence of [22, Theorem 4.1] and its proof (especially Proposition 4.7) we have:

Example 48. If X is a Mori dream space (e.g. of log Fano type), then:

- for each $\theta \in \mathrm{Psef}(X)$, the b-divisor $\mathrm{N}(\theta)$ is $\mathbb{R}$-Cartier;
- $\mathrm{Psef}(X)$ is a rational polyhedral cone;
- the Zariski decomposition is $\mathbb{Q}$-PL on $\mathrm{Psef}(X)$.

The next result is closely related to the theory of Zariski chambers studied in [2].

Proposition 49. *If* $\dim X = 2$*, then the Zariski decomposition is* $\mathbb{Q}$*-PL on any convex cone* $C \subset \mathrm{Psef}(X)$ *with the property that the set of prime divisors* $E \subset X$ *with* $\mathrm{ord}_E(\theta) > 0$ *for some* $\theta \in C$ *is finite.*

By Lemma 33 (ii), the finiteness condition on C is satisfied as soon as C is polyhedral.

Proof. For each Zariski exceptional face F of $\mathrm{Psef}(X)$ with relative interior $\mathring{F}$, set $Z_F := \mathrm{N}_X^{-1}(\mathring{F})$. Thus $\theta \in \mathrm{Psef}(X)$ lies in Z_F iff the irreducible decomposition of $\mathrm{N}_X(\alpha)$ are precisely the generators of F. By Theorem 45 (ii), Z_F is a convex subcone of $\mathrm{Psef}(X)$ (whose intersection with $\mathrm{Big}(X)$ is a Zariski chamber in the sense of [2]); further, $\mathrm{N}_X|_{Z_F} : Z_F \to \mathring{F}$ is the restriction of the orthogonal projection onto $\mathrm{Vect}\, F$, which is a rational linear map. By Corollary 41, the Zariski decomposition is thus $\mathbb{Q}$-PL on Z_F. Finally, the finiteness assumption guarantees that C meets only finitely many Z_F's, and the result is thus a consequence of Lemma 47. □

We conclude this section with a higher-dimensional situation in which Zariski decompositions can be analyzed. Assuming again that $\dim X$ is arbitrary, consider next a 2-dimensional cone $C \subset \mathrm{N}^1(X)$ generated by two classes $\theta, \alpha \in \mathrm{N}^1(X)$ such that $\theta \in \mathrm{Nef}(X)$ and $\alpha \notin \mathrm{Psef}(X)$. Set

$$C_{\mathrm{nef}} := C \cap \mathrm{Nef}(X) \subset C_{\mathrm{psef}} := C \cap \mathrm{Psef}(X) \subset C,$$

and introduce the thresholds

$$\lambda_{\mathrm{nef}} := \sup\{\lambda \geq 0 \mid \theta + \lambda\alpha \in \mathrm{Nef}(X)\}, \quad \lambda_{\mathrm{psef}} := \sup\{\lambda \geq 0 \mid \theta + \lambda\alpha \in \mathrm{Psef}(X)\},$$

so that C_{nef} (resp. C_{psef}) is generated by θ and $\theta_{\mathrm{nef}} := \theta + \lambda_{\mathrm{nef}}\alpha$ (resp. $\theta_{\mathrm{psef}} := \theta + \lambda_{\mathrm{psef}}\alpha$).

The next result is basically contained in [41, §6.5].

Proposition 50. *With the above notation, suppose that C contains the class of a prime divisor $S \subset X$ such that $\mathrm{Nef}(S) = \mathrm{Psef}(S)$ and $S|_S$ is not nef. Then:*

(i) $\theta_{\mathrm{psef}} = t[S]$ *with* $t > 0$*;*
(ii) $\lambda_{\mathrm{nef}} = \lambda^S_{\mathrm{nef}} := \sup\{\lambda \geq 0 \mid (\theta + \lambda\alpha)|_S \in \mathrm{Nef}(S)\}$*;*
(iii) *the Zariski decomposition is PL on C_{psef}, with*

$$\mathrm{N} \equiv 0 \textit{ on } C_{\mathrm{nef}}, \quad \mathrm{N}(a\theta_{\mathrm{nef}} + b[S]) = b\bar{S} \textit{ for all } a, b \geq 0.$$

Proof. The assumptions imply that $S|_S$ is not psef. By Theorem 44 (i), S is thus Zariski exceptional, and $[S]$ generates an extremal ray of $\mathrm{Psef}(X)$. This ray is also extremal in C_{psef}, which proves (i).

Next, note that $\lambda_{\mathrm{nef}} \leq \lambda^S_{\mathrm{nef}} \leq \lambda_{\mathrm{psef}}$, by (i). Pick a curve $\gamma \subset X$. We need to show $(\theta + \lambda^S_{\mathrm{nef}}\alpha)\cdot\gamma \geq 0$. This is clear if $\gamma \subset S$ (since $(\theta + \lambda^S_{\mathrm{nef}}\alpha)|_S$ is nef), or if $\alpha\cdot\gamma \geq 0$ (since $\theta\cdot\gamma \geq 0$ and $\lambda^S_{\mathrm{nef}} \geq 0$). Otherwise, we have $S\cdot\gamma \geq 0$ and $\alpha\cdot\gamma \leq 0$, and we get again $(\theta + \lambda^S_{\mathrm{nef}}\alpha)\cdot\gamma \geq 0$ since

$$\theta + \lambda^S_{\mathrm{nef}}\alpha \equiv \theta_{\mathrm{psef}} + (\lambda^S_{\mathrm{nef}} - \lambda_{\mathrm{psef}})\alpha = t[S] + (\lambda^S_{\mathrm{nef}} - \lambda_{\mathrm{psef}})\alpha$$

with $\lambda^S_{\mathrm{nef}} - \lambda_{\mathrm{psef}} \leq 0$. This proves (ii).

For (iii), note that $\mathrm{N} \equiv 0$ on $\mathrm{Nef}(X) \supset C_{\mathrm{nef}}$ (see Theorem 34). Further, $\mathrm{N}([S]) = \bar{S}$ (see Lemma 43), and hence $\mathrm{N}(a\theta_{\mathrm{nef}} + b[S]) \leq b\bar{S}$ for $a, b \geq 0$. In particular, $c := \mathrm{ord}_S(a\theta_{\mathrm{nef}} + b[S]) \leq b$. On the other hand, (13) yields

$$\mathrm{N}(a\theta_{\mathrm{nef}} + b[S]) \geq \overline{\mathrm{N}(a\theta_{\mathrm{nef}} + b[S])} \geq c\bar{S},$$

and it thus remains to see $c = b$. By Corollary 39, $((a\theta_{\mathrm{nef}} + b[S]) - c[S])|_S$ lies in $\mathrm{Psef}(S) = \mathrm{Nef}(S)$. By (ii), we infer $a\theta_{\mathrm{nef}} + (b - c)[S] \in C_{\mathrm{nef}}$, and hence $b - c = 0$, since $C_{\mathrm{nef}} = \mathbb{R}_{\geq 0}\theta + \mathbb{R}_{\geq 0}\theta_{\mathrm{nef}}$ intersects $\mathbb{R}_{\geq 0}\theta_{\mathrm{nef}} + \mathbb{R}_{\geq 0}[S]$ only along $\mathbb{R}_{\geq 0}\theta_{\mathrm{nef}}$. □

6. Green's functions and Zariski decompositions

In this section we fix an ample class $\omega \in \mathrm{Amp}(X)$.

6.1. *Green's functions and equilibrium measures*

A subset $\Sigma \subset X^{\mathrm{an}}$ is *pluripolar* if $\Sigma \subset \{\varphi = -\infty\}$ for some $\varphi \in \mathrm{PSH}(\omega)$. By [13, Theorem 4.5], Σ is nonpluripolar iff

$$\mathrm{T}(\Sigma) := \sup_{\varphi \in \mathrm{PSH}(\omega)} (\sup \varphi - \sup_\Sigma \varphi) \in [0, +\infty]$$

is finite. The invariant $\mathrm{T}(\Sigma)$, which plays an important role in [5, 14], is modeled on the Alexander–Taylor capacity (which corresponds to $e^{-\mathrm{T}(\Sigma)}$) in complex analysis.

Definition 51. *For any subset $\Sigma \subset X^{\mathrm{an}}$ we set*

$$\varphi_\Sigma = \varphi_{\omega,\Sigma} := \sup\{\varphi \in \mathrm{PSH}(\omega) \mid \varphi|_\Sigma \leq 0\}. \tag{17}$$

Note that $\varphi_\Sigma(v_{\mathrm{triv}}) = \sup\varphi_\Sigma = \mathrm{T}(\Sigma)$, and hence

$$\varphi_\Sigma \in \mathrm{PL}(X) \Longrightarrow \mathrm{T}(\Sigma) \in \mathbb{Q}. \tag{18}$$

Theorem 52. *For any compact subset $\Sigma \subset X^{\mathrm{an}}$, the following holds:*

(i) $\varphi_\Sigma = \sup\{\varphi \in \mathrm{CPSH}(\omega) \mid \varphi|_\Sigma \le 0\}$*; in particular, φ_Σ is lsc;*
(ii) *if Σ is pluripolar then $\varphi_\Sigma^\star \equiv +\infty$;*
(iii) *if Σ is nonpluripolar, then $\varphi_\Sigma^\star$ is ω-psh and nonnegative; further, $\mu_\Sigma := \mathrm{MA}(\varphi_\Sigma^\star)$ is supported in Σ, $\int \varphi_\Sigma^\star\,\mu_\Sigma = 0$, and μ_Σ is characterized as the unique minimizer of the energy $\|\mu\|$ over all Radon probability measures μ with support in Σ.*

Since the energy of a Radon probability measure μ only appears in this statement, we simply recall here that it is defined as

$$\|\mu\| = \sup_{\varphi\in\mathcal{E}^1(\omega)}\left(\mathrm{E}(\varphi) - \int \varphi\,\mu\right) \in [0,+\infty], \tag{19}$$

and refer to [13, §9.1] for more details.

Definition 53. *Assuming Σ is nonpluripolar, we call μ_Σ its* equilibrium measure*, and $\varphi_\Sigma^\star$ its* Green's function.

The latter is characterized as the normalized potential of μ_Σ (in the terminology of [15, §1.6]), i.e. the unique $\varphi \in \mathcal{E}^1(\omega)$ such that $\mathrm{MA}(\varphi) = \mu_\Sigma$ and $\int \varphi\,\mu_\Sigma = 0$.

Proof of Theorem 52. Denote by φ'_Σ the right-hand side in (i), which obviously satisfies $\varphi'_\Sigma \le \varphi_\Sigma$. Pick $\varphi \in \mathrm{PSH}(\omega)$ with $\varphi|_\Sigma \le 0$, and write φ as the limit of a decreasing net (φ_i) in $\mathrm{CPSH}(\omega)$. For any $\varepsilon > 0$, a Dini type argument shows that $\varphi_i < \varepsilon$ on Σ for i large enough. Thus $\varphi_i \le \varphi'_\Sigma + \varepsilon$, and hence $\varphi \le \varphi'_\Sigma + \varepsilon$. This shows $\varphi_\Sigma \le \varphi'_\Sigma$, which proves (i).

Next, (ii) and the first half of (iii) follow from [13, Lemma 13.15]. Since the negligible set $\{\varphi_\Sigma < \varphi_\Sigma^\star\}$ is pluripolar (see [13, Theorem 13.17]), it has zero measure for any measure μ of finite energy [13, Lemma 9.2]. If μ has support in Σ, this yields $\int \varphi_\Sigma^\star\,\mu = \int \varphi_\Sigma\,\mu = 0$. By (19) we infer $\|\mu\| \ge \mathrm{E}(\varphi_\Sigma^\star) = \|\mu_\Sigma\|$. This proves that μ_Σ minimizes the energy, while uniqueness follows from the strict convexity of the energy [13, Proposition 10.10]. □

Further mimicking classical terminology in the complex analytic setting, we introduce:

Definition 54. *We say that a compact subset $\Sigma \subset X^{\mathrm{an}}$ is* regular *if $\varphi_\Sigma \in \mathrm{CPSH}(\omega)$.*

In particular, Σ is then nonpluripolar (see Theorem 52).

Lemma 55. *For any compact subset $\Sigma \subset X^{\mathrm{an}}$, the following hold:*

(i) *Σ is regular iff $\varphi_\Sigma^\star \le 0$ on Σ;*
(ii) *the regularity of Σ is independent of $\omega \in \mathrm{Amp}(X)$;*
(iii) *if $\Sigma \subset X^{\mathrm{lin}}$ then Σ is regular.*

Proof. If Σ is regular, then $\varphi_\Sigma^\star = \varphi_\Sigma$ vanishes on Σ. Conversely, assume $\varphi_\Sigma^\star \le 0$ on Σ. By (ii) and (iii) of Theorem 52, Σ is necessarily nonpluripolar, and $\varphi_\Sigma^\star$ is ω-psh. It is thus a competitor in (17), which implies that $\varphi_\Sigma = \varphi_\Sigma^\star$ is ω-psh, and also continuous by Theorem 52 (i).

Assume Σ is regular for ω, and pick $\omega' \in \mathrm{Amp}(X)$. Then $t\omega - \omega'$ is nef for $t \gg 1$, and hence $\mathrm{PSH}(\omega') \subset t\,\mathrm{PSH}(\omega)$. This implies $\varphi_{\omega',\Sigma} \le t\varphi_{\omega,\Sigma}$, and hence $\varphi^\star_{\omega',\Sigma} \le t\varphi_{\omega,\Sigma}$. In particular, $\varphi^\star_{\omega',\Sigma}|_\Sigma \le 0$, which proves that Σ is regular for ω', by (i).

Finally, assume $\Sigma \subset X^{\mathrm{lin}}$. Since $\{\varphi_\Sigma < \varphi_\Sigma^\star\}$ is pluripolar (see [13, Theorem 13.17]), it is disjoint from X^{lin}. As a result, $\varphi_\Sigma^\star \in \mathrm{PSH}(\omega)$ vanishes on Σ, and it again follows from (i) that Σ is regular. □

6.2. *The Green's function of a real divisorial set*

In what follows, we consider a *real divisorial set*, by which we mean a finite set $\Sigma \subset X^{\mathrm{div}}_{\mathbb{R}}$ of real divisorial valuations. By Lemma 55 (iii), $\Sigma \subset X^{\mathrm{lin}}$ is regular, i.e. $\varphi_\Sigma \in \mathrm{CPSH}(\omega)$. When $\Sigma = \{v\}$ for a single $v \in X^{\mathrm{div}}_{\mathbb{R}}$, we simply write $\varphi_v := \varphi_\Sigma$.

Example 56. Assume $\omega = c_1(L)$ with $L \in \mathrm{Pic}(X)_{\mathbb{Q}}$ ample and $v \in X^{\mathrm{div}}$. Then v is *dreamy* (with respect to L) in the sense of K.Fujita iff $\varphi_v \in \mathcal{H}(L)$; see [14, §1.7, Appendix A].

If $v_{\mathrm{triv}} \in \Sigma$, then $\varphi_\Sigma \equiv 0$, and we henceforth assume $v_{\mathrm{triv}} \notin \Sigma$. Pick a smooth birational model $\pi\colon Y \to X$ which extracts each $v \in \Sigma$, i.e. $v = t_v \operatorname{ord}_{E_v}$ for a prime divisor $E_v \subset Y$ and $t_v \in \mathbb{R}_{>0}$. We then introduce the effective $\mathbb{R}$-divisor on Y

$$D := \sum_\alpha t_\alpha^{-1} E_\alpha,$$

whose set of Rees valuations Γ_D coincides with Σ (see Definition 16).

Theorem 57. *With the above notation, the following holds:*

(i) $\sup \varphi_\Sigma = \mathrm{T}(\Sigma)$ *coincides with the pseudoeffective threshold*

$$\lambda_{\mathrm{psef}} := \max\left\{\lambda \ge 0 \mid \pi^\star\omega - \lambda D \in \mathrm{Psef}(Y)\right\};$$

(ii) $\varphi_\Sigma \in \mathrm{CPSH}(\omega)$ *is of divisorial type, and the associated family of b-divisors* $(B_\lambda)_{\lambda \le \lambda_{\mathrm{psef}}}$ *(see Theorem 18) is given by*

$$-B_\lambda = \begin{cases} \mathrm{N}(\pi^\star\omega - \lambda D) + \lambda \overline{D} & \text{for } \lambda \in [0, \lambda_{\mathrm{psef}}] \\ 0 & \text{for } \lambda \le 0. \end{cases}$$

Proof. Pick $\lambda \in \mathbb{R}$. For any $\psi \in \mathrm{PSH}(\omega)$, we have $\psi + \lambda \le \varphi_\Sigma \Leftrightarrow \psi|_\Sigma \le -\lambda$, and hence

$$\widehat{\varphi}^\lambda_\Sigma = \sup\{\psi \in \mathrm{PSH}_{\mathrm{hom}}(\omega) \mid \psi|_\Sigma \le -\lambda\}.$$

When $\lambda \le 0$ this yields $\widehat{\varphi}^\lambda_\Sigma = 0$. Now assume $\lambda > 0$. Using Proposition 17 and $\mathrm{PSH}_{\mathrm{hom}}(\pi^\star\omega) = \pi^\star \mathrm{PSH}_{\mathrm{hom}}(\omega)$, we get

$$\pi^\star \widehat{\varphi}^\lambda_\Sigma = \sup\{\tau \in \mathrm{PSH}_{\mathrm{hom}}(\pi^\star\omega - \lambda D)\} - \lambda \psi_D = V_{\pi^\star\omega - \lambda D} - \lambda\psi_D. \tag{20}$$

Now the left-hand side is not identically $-\infty$ iff $\lambda \le \sup\varphi$, while for the right-hand side this holds iff $\lambda \le \lambda_{\mathrm{psef}}$, by Proposition 27. This proves (i), and also (ii), by Theorem 31. ☐

Corollary 58. *The center of* φ_Σ *satisfies*

$$Z_X(\varphi_\Sigma) = \pi\left(\mathbb{B}_-(\pi^\star\omega - \lambda_{\mathrm{psef}} D)\right) \cup Z_X(\Sigma).$$

In particular, $Z_X(\varphi_\Sigma)$ *is Zariski dense in* X *iff* $\mathbb{B}_-(\pi^\star\omega - \lambda_{\mathrm{psef}} D)$ *is Zariski dense in* Y.

Proof. By Lemma 22, we have

$$Z_X(\varphi_\Sigma) = Z_X(\widehat{\varphi}^{\max}_\Sigma) = \pi(Z_Y(\pi^* \widehat{\varphi}^{\max}_\Sigma)).$$

It follows from Theorem 57 and its proof that

$$\pi^\star \widehat{\varphi}^{\max}_\Sigma = V_{\pi^\star\omega - \lambda_{\mathrm{psef}} D} - \lambda_{\mathrm{psef}} \psi_D.$$

Now $Z_Y(V_{\pi^\star\omega - \lambda_{\mathrm{psef}} D}) = \mathbb{B}_-(\pi^\star\omega - \lambda_{\mathrm{psef}} D)$ by Theorem 31, whereas we see from Example 21 that $Z_Y(-\lambda_{\mathrm{psef}}\psi_D) = Z_Y(\Sigma)$, so we conclude using Lemma 25. ☐

6.3. *Dimension one and two*

In this section we consider the case $\dim X \le 2$.

Proposition 59. *If* $\dim X = 1$, *then for any real divisorial set* $\Sigma \subset X^{\mathrm{div}}_{\mathbb{R}}$, *we have* $\varphi_\Sigma \in \mathbb{R}\mathrm{PL}^+(X)$. *If* ω *is rational and* $\Sigma \subset X^{\mathrm{div}}$, *then we further have* $\varphi_\Sigma \in \mathrm{PL}^+(X)$.

Proof. We may assume $v_{\mathrm{triv}} \notin \Sigma$, or else $\varphi_\Sigma \equiv 0$. Thus assume $\Sigma = \{v_i\}_{i\in I}$, where $v_i = t_i \operatorname{ord}_{p_i}$, $t_i \in \mathbb{R}_{>0}$, and $p_i \in X$ is a closed point. We may assume $p_i \ne p_j$ for $i \ne j$, or else $\varphi_\Sigma = \varphi_{\Sigma'}$ for $\Sigma' = \{v_i\}_{i\in I'}$, where $I' \subset I$ is defined by $i \in I'$ iff for all $j \ne i$, either $p_j \ne p_i$ or $t_j > t_i$. Under these assumptions,

$$\varphi_\Sigma = A \max\left\{1 + \sum_i t_i^{-1} \log|\mathfrak{m}_{p_i}|, 0\right\},$$

where $A > 0$ satisfies $A \sum_i t_i^{-1} = \deg\omega$, see [13, Example 3.19]. Thus $\varphi_\Sigma \in \mathbb{R}\mathrm{PL}^+(X)$. Further, if $\Sigma \subset X^{\mathrm{div}}$, then $t_i \in \mathbb{Q}_{>0}$ for all i, so if ω is rational, then $A \in \mathbb{Q}_{>0}$, and hence $\varphi_\Sigma \in \mathrm{PL}^+(X)$. □

Theorem 60. *If* $\dim X = 2$, *then for any real divisorial set* $\Sigma \subset X^{\mathrm{div}}_{\mathbb{R}}$, *we have* $\varphi_\Sigma \in \mathbb{R}\mathrm{PL}^+(X)$. *If* ω *is rational and* $\Sigma \subset X^{\mathrm{div}}$, *then we further have*

$$\varphi_\Sigma \in \mathrm{PL}(X) \Longleftrightarrow \varphi_\Sigma \in \mathrm{PL}^+(X) \Longleftrightarrow \mathrm{T}(\Sigma) \in \mathbb{Q}. \tag{21}$$

We will see in Example 63 that $\mathrm{T}(\Sigma)$ can be irrational.

Lemma 61. *Assume* $\dim X \le 2$, *and pick* $B \in \mathrm{Car_b}(X)_{\mathbb{R}}$. *Then* B *is relatively nef iff it is relatively semiample.*

Proof. Assume B is relative nef, and pick a determination $\pi\colon Y \to X$ of B. The relatively nef cone of $\mathrm{N}^1(Y/X)$ is dual to the cone generated by the (finite) set of π-exceptional prime divisors, and is thus a rational polyhedral cone. As a consequence, we can write $B_Y = \sum_i t_i D_i$ with $t_i > 0$ and $D_i \in \mathrm{Div}(Y)_{\mathbb{Q}}$ relatively nef. By [38, Theorem 12.1 (ii)], each D_i is relatively semiample, and the result follows. □

Proof of Theorem 60. Use the notation of Theorem 57. By Proposition 49, the Zariski decomposition is $\mathbb{Q}$-PL on the cone

$$C = (\mathbb{R}_+\pi^\star\omega + \mathbb{R}_+[-D]) \cap \mathrm{Psef}(Y) = \mathbb{R}_+\pi^\star\omega + \mathbb{R}_+(\pi^\star\omega - \lambda_{\mathrm{psef}}[D]).$$

We can thus find $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_N = \lambda_{\mathrm{psef}}$ such that

$$\lambda \longmapsto B_\lambda = -(\mathrm{N}(\pi^\star\omega - \lambda[D]) + \lambda\overline{D})$$

is affine linear on $[\lambda_i, \lambda_{i+1}]$ for $1 \le i < N$. Setting $B_i := B_{\lambda_i}$, it follows that

$$\varphi_\Sigma = \sup_{\lambda\in[0,\lambda_{\mathrm{psef}}]} \{\psi_{B_\lambda} + \lambda\} = \max_{1\le i\le N} \{\psi_{B_i} + \lambda_i\}.$$

Since $\overline{\omega} + [B_i]$ is nef, the antieffective divisor B_i is relatively nef, and hence relatively semiample (see Lemma 61). By Proposition 7, we infer $\psi_{B_i} \in \mathrm{PL}^+_{\mathrm{hom}}(X)_{\mathbb{R}}$, and hence $\varphi_\Sigma \in \mathbb{R}\mathrm{PL}^+(X)$.

Now assume ω and $\mathrm{T}(\Sigma) = \lambda_{\mathrm{psef}}$ are both rational, and that $\Sigma \subset X^{\mathrm{div}}$. Then D is rational as well, and C is thus a rational polyhedral cone. Since the Zariski decomposition on C is the restriction of a $\mathbb{Q}$-PL map on $\mathrm{N}^1(Y)$, this implies that the λ_i above can be chosen rational. Using again that the Zariski decomposition is $\mathbb{Q}$-PL on C, we infer that B_i is a $\mathbb{Q}$-divisor, hence $\psi_{B_i} \in \mathrm{PL}^+_{\mathrm{hom}}(X)$, which shows $\varphi_\Sigma \in \mathrm{PL}^+(X)$. The rest follows from (18). □

7. Examples of Green's functions

We now exhibit examples of Green's functions with various types of behavior. These examples serve as the underpinnings of Theorems A and B of the introduction.

7.1. *Divisors on abelian varieties*

As a direct application of Theorem 57, we show:

Proposition 62. *Assume* $\mathrm{Nef}(X) = \mathrm{Psef}(X)$. *Consider a real divisorial set* $\Sigma = \{v_\alpha\} \subset X^{\mathrm{div}}_{\mathbb{R}}$ *with* $v_\alpha = t_\alpha \operatorname{ord}_{E_\alpha}$ *for* $E_\alpha \subset X$ *prime and* $t_\alpha > 0$, *and set* $D := \sum_\alpha t_\alpha^{-1} E_\alpha$. *Then*

$$\mathrm{T}(\Sigma) = \lambda_{\mathrm{psef}} = \sup\{\lambda \geq 0 \mid \omega - \lambda D \in \mathrm{Psef}(X)\}$$

and

$$\varphi_\Sigma = \mathrm{T}(\Sigma) \max\left\{0, 1 - \psi_D\right\}.$$

In particular, $\varphi_\Sigma \in \mathbb{R}\mathrm{PL}^+(X)$. *If we further assume* $\Sigma \subset X^{\mathrm{div}}$, *then*

$$\varphi_\Sigma \in \mathrm{PL}(X) \Longleftrightarrow \varphi_\Sigma \in \mathrm{PL}^+(X) \Longleftrightarrow \mathrm{T}(\Sigma) \in \mathbb{Q}. \tag{22}$$

Proof. Using the notation of Theorem 57, we have $\mathrm{N}(\omega - \lambda D) = 0$ for $\lambda \leq \lambda_{\mathrm{psef}} = \mathrm{T}(\Sigma)$. Thus $\widehat{\varphi}^\lambda_\Sigma = -\lambda\psi_D$, and hence

$$\varphi_\Sigma = \sup_{0 \leq \lambda \leq \lambda_{\mathrm{psef}}} \{\lambda - \lambda\psi_D\} = \lambda_{\mathrm{psef}} \max\left\{0, 1 - \psi_D\right\}.$$

Since $-\psi_D = \sum_\alpha t_\alpha^{-1} \log|\mathcal{O}_X(-E_\alpha)|$ lies in $\mathrm{PL}^+(X)_{\mathbb{R}}$, it follows that $\varphi_\Sigma \in \mathbb{R}\mathrm{PL}^+(X)$. If $\Sigma \subset X^{\mathrm{div}}$, then D is a $\mathbb{Q}$-divisor, and hence $-\psi_D \in \mathrm{PL}^+_{\mathrm{hom}}(X)$. If we further assume $\mathrm{T}(\Sigma) \in \mathbb{Q}$, we get $\varphi_\Sigma \in \mathrm{PL}^+(X)$, and the remaining implication follows from (18). □

Example 63. Suppose X is an abelian surface, $\omega = c_1(L)$ with $L \in \mathrm{Pic}(X)_{\mathbb{Q}}$ ample, and $v = \operatorname{ord}_E$ with $E \subset X$ a prime divisor. Then $\mathrm{Nef}(X) = \mathrm{Psef}(X)$, and $\mathrm{T}(v) = \lambda_{\mathrm{psef}}$ is the smallest root of the quadratic equation $(L - \lambda E)^2 = 0$, see [34, Remark 1.5.6]. If X has Picard number $\rho(X) \geq 2$, then λ_{psef} is irrational for a typical choice of L and E, and hence $\varphi_v \notin \mathrm{PL}(X)$. (Compare [34, Example 2.3.8]). In particular, v is not dreamy (with respect to L) in the sense of Fujita, see Example 56.

7.2. *The Cutkosky example*

Building on a construction of Cutkosky [21] and Proposition 50 (itself based on [41, §6.5]), we provide an example of a divisorial valuation on $\mathbb{P}^3$ for which (21) fails. This relies on the following general result.

Proposition 64. *Consider a flag of smooth subvarieties* $Z \subset S \subset X$ *with* $\operatorname{codim} S = 1$, $\operatorname{codim} Z = 2$ *and ideals* $\mathfrak{b}_S \subset \mathfrak{b}_Z \subset \mathcal{O}_X$, *and assume that*

(i) $S \equiv \omega$;
(ii) $\mathrm{Nef}(S) = \mathrm{Psef}(S)$;
(iii) $\omega|_S - Z$ *is not nef on* S, *i.e.* $\lambda^S_{\mathrm{nef}} := \sup\{\lambda \geq 0 \mid \omega|_S - \lambda[Z] \in \mathrm{Nef}(S)\} < 1$.

The Green's function of $v := \operatorname{ord}_Z \in X^{\mathrm{div}}$ *is then given by*

$$\varphi_v = \max\left\{0, \lambda^S_{\mathrm{nef}}(\log|\mathfrak{b}_Z| + 1), \log|\mathfrak{b}_S| + 1\right\}.$$

In particular, $\mathrm{T}(v) = 1$, $\varphi_v \in \mathbb{R}\mathrm{PL}^+(X)$, *and*

$$\varphi_v \in \mathrm{PL}(X) \Longleftrightarrow \varphi_v \in \mathrm{PL}^+(X) \Longleftrightarrow \lambda^S_{\mathrm{nef}} \in \mathbb{Q}.$$

Proof. Let $\pi\colon Y \to X$ be the blowup along Z, with exceptional divisor E, and denote by $S' = \pi^\star S - E$ the strict transform of S. Since Z has codimension 1 on S, π maps S' isomorphically onto S, and takes $S'|_{S'} = \pi^\star S|_{S'} - E|_{S'}$ to $S|_S - Z \equiv \omega|_S - [Z]$. By (ii) and (iii), we thus have $\mathrm{Nef}(S') = \mathrm{Psef}(S')$, and $S'|_{S'}$ is not nef.

Consider the cone $C \subset \mathrm{N}^1(Y)$ generated by $\theta := \pi^\star\omega \in \mathrm{Nef}(Y)$ and $\alpha := -[E] \notin \mathrm{Psef}(Y)$. Since C contains the class of S', it follows from Proposition 50 that

$$1 = \lambda_{\mathrm{psef}} := \sup\{\lambda \geq 0 \mid \pi^\star\omega - \lambda[E] \in \mathrm{Psef}(Y)\}$$

and $\lambda \mapsto \mathrm{N}(\pi^\star\omega - \lambda E)$ vanishes on $[0, \lambda^S_{\mathrm{nef}}]$, and is affine linear on $[\lambda^S_{\mathrm{nef}}, 1]$, with value S' at $\lambda = 1$. By Theorem 57, the concave family $(B_\lambda)_{\lambda\leq 1}$ of b-divisors associated to φ_v is affine linear on $(-\infty, 0]$, $[0, \lambda^S_{\mathrm{nef}}]$ and $[\lambda^S_{\mathrm{nef}}, 1]$, with value

$$B_\lambda = 0, \quad \lambda^S_{\mathrm{nef}}\overline{E} \quad \text{and} \quad \overline{S' + E} = \overline{S}$$

at $\lambda = 0$, λ^S_{nef} and 1, respectively. By (6), the result follows, since $-\psi_{\overline{E}} = \log|\mathfrak{b}_Z|$ and $-\psi_{\overline{S}} = \log|\mathfrak{b}_S|$. □

Example 65. Assume $k = \mathbb{C}$, and set $(X, L) = (\mathbb{P}^3, \mathcal{O}(4))$. By [21], there exists a smooth quartic surface $S \subset X$ without (-2)-curves, and hence such that $\mathrm{Nef}(S) = \mathrm{Psef}(S)$, containing a smooth curve Z such that λ^S_{nef} is irrational and less than 1. By Proposition 64, we infer $\mathrm{T}(v) = 1$ and $\varphi_v \in \mathbb{R}\mathrm{PL}^+(X) \setminus \mathrm{PL}(X)$ (in contrast with (21)).

7.3. *The Lesieutre example*

Based on an example by Lesieutre [35], we now exhibit a Green's function that is not $\mathbb{R}$-PL. This forms the basis for Theorem B in the introduction.

Proposition 66. *Suppose that X admits a class $\theta \in \mathrm{Psef}(X)$ whose diminished base locus $\mathbb{B}_-(\theta)$ is Zariski dense. Then there exist $\omega \in \mathrm{Amp}(X)$ and $v \in X^{\mathrm{div}}$ such that $Z_X(\varphi_{\omega,v})$ is Zariski dense in X. In particular, $\varphi_{\omega,v} \notin \mathbb{R}\mathrm{PL}(X)$.*

Proof. Note first that θ cannot be big. Otherwise, there would exist an effective $\mathbb{R}$-divisor $D \equiv \theta$, and hence $\mathbb{B}_-(\theta)$ would be contained in $\operatorname{supp} D$. Pick an ample prime divisor E on X, choose $c \in \mathbb{Q}_{>0}$ large enough such that $\omega := \theta + c[E]$ is ample, and set $v := c^{-1}\operatorname{ord}_E \in X^{\mathrm{div}}$. Since ω is ample and $\omega - c[E] = \theta$ lies on the boundary of $\mathrm{Psef}(X)$, the threshold $\lambda_{\mathrm{psef}} = \sup\{\lambda \geq 0 \mid \omega - \lambda[E] \in \mathrm{Psef}(X)\}$ is equal to c. Thus $\mathbb{B}_-(\omega - \lambda_{\mathrm{psef}}[E])$ is Zariski dense, and hence so is $Z_X(\varphi_{\omega,v})$, by Corollary 58. The last point follows from Lemma 26. □

Example 67. By [35, Theorem 1.1], the assumptions in Proposition 66 are satisfied when $k = \mathbb{C}$ and X is the blowup of $\mathbb{P}^3$ at nine sufficiently general points.

If θ in Proposition 66 is rational, then the proof shows that ω can be taken rational as well, i.e. $\omega = c_1(L)$ for an ample $\mathbb{Q}$-line bundle. While no such rational example appears to be known at present, we can nevertheless exploit the structure of Lesieutre's example to get:

Proposition 68. *Set $(X, L) := (\mathbb{P}^3, \mathcal{O}(1))$. Then there exists a finite set $\Sigma \subset X^{\mathrm{div}}_{\mathbb{R}}$ such that $Z_X(\varphi_{L,\Sigma})$ is Zariski dense in X, and hence $\varphi_{L,\Sigma} \notin \mathbb{R}\mathrm{PL}(X)$.*

Proof. Let $\pi\colon Y \to X$ be the blowup at nine sufficiently general points, and denote by $\sum_{i=1}^9 E_i$ the exceptional divisor. By [35, Remark 4.5, Lemma 5.2], we can pick $D = \sum_i c_i E_i$ with $c_i \in \mathbb{R}_{>0}$ such that the diminished base locus of $\pi^\star L - D$ is Zariski dense. As above, this implies that this class lies on the boundary of the psef cone (it even generates an extremal ray, see [35, Lemma 5.1]), and the psef threshold

$$\lambda_{\mathrm{psef}} = \sup\{\lambda \geq 0 \mid \pi^\star L - \lambda D \in \mathrm{Psef}(Y)\}$$

is thus equal to 1. The result now follows from Corollary 58, with $\Sigma = \{c_i^{-1}\operatorname{ord}_{E_i}\}_{1\leq i\leq 9}$. □

It is natural to ask:

Question 69. *Can an example as in Proposition 68 be found with $\Sigma \subset X^{\mathrm{div}}$?*

8. The non-trivially valued case

In this section, we work over the non-Archimedean field $K = k((\varpi))$ of formal Laurent series, with valuation ring $K^\circ := k[[\varpi]]$. We use [10] as our main reference.

Thus X now denotes a smooth projective variety of dimension n over K. (In Section 9, it will be obtained as the base change of a smooth projective k-variety). Working "additively", we view the elements of the analytification X^{an} as valuations $x\colon K(Y)^\times \to \mathbb{R}$ for subvarieties $Y \subset X$, restricting to the given valuation on K.

8.1. *Models*

We define a *model* of X to be a normal, flat, projective K°-scheme $\mathscr{X}$ together with the data of an isomorphism $\mathscr{X}_K \simeq X$. The *special fiber* of $\mathscr{X}$ is the projective k-scheme $\mathscr{X}_0 := \mathscr{X} \times_{\operatorname{Spec} K} \operatorname{Spec} k$. Each $x \in X^{\mathrm{an}}$ can be viewed as a semivaluation on $\mathscr{X}$, whose center is denoted by $\mathrm{red}_{\mathscr{X}}(x) \in \mathscr{X}_0$. This defines a *reduction map* $\mathrm{red}_{\mathscr{X}}\colon X^{\mathrm{an}} \to \mathscr{X}_0$, which is surjective and anticontinuous (i.e. the preimage of an open set is closed). For each $x \in X^{\mathrm{an}}$ we also set

$$Z_{\mathscr{X}}(x) := \overline{\{\mathrm{red}_{\mathscr{X}}(x)\}} \subset \mathscr{X}_0.$$

The preimage under $\mathrm{red}_{\mathscr{X}}$ of the set of generic points of $\mathscr{X}_0$ is finite. We denote it by $\Gamma_{\mathscr{X}} \subset X^{\mathrm{an}}$, and call its elements the *Shilov points* of $\mathscr{X}$. As $\mathscr{X}$ is normal, each irreducible component E of $\mathscr{X}_0$ defines a *divisorial valuation* $x_E \in X_K^{\mathrm{an}}$ given by

$$x_E := b_E^{-1}\,\mathrm{ord}_E,\ b_E := \mathrm{ord}_E(\varpi);$$

it is the unique preimage under $\mathrm{red}_{\mathscr{X}}$ of the generic point of E, and the Shilov points of $\mathscr{X}$ are exactly these valuations x_E.

One says that another model $\mathscr{X}'$ *dominates* $\mathscr{X}$ if the canonical birational map $\mathscr{X}' \dashrightarrow \mathscr{X}$ extends to a morphism (necessarily unique, by separatedness). In that case, $\mathrm{red}_{\mathscr{X}}$ is the composition of $\mathrm{red}_{\mathscr{X}'}$ with the induced projective morphism $\mathscr{X}'_0 \to \mathscr{X}_0$. The set of models forms a filtered poset with respect to domination. The set

$$X^{\mathrm{div}} = \bigcup_{\mathscr{X}} \Gamma_{\mathscr{X}}$$

of all divisorial valuations is a dense subset of X^{an}.

8.2. *Piecewise linear functions*

A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on a model $\mathscr{X}$ of X is *vertical* if it is supported in $\mathscr{X}_0$; it then defines a continuous function on X^{an} called a *model function*. The $\mathbb{Q}$-vector space $\mathrm{PL}(X)$ of such functions is stable under max, and dense in $\mathrm{C}^0(X^{\mathrm{an}})$.

Definition 70. *We define the space* $\mathbb{R}\mathrm{PL}(X)$ *of* real piecewise linear functions *on* X^{an} *(*$\mathbb{R}$-PL functions *for short) as the smallest* $\mathbb{R}$*-linear subspace of* $\mathrm{C}^0(X^{\mathrm{an}})$ *that is stable under max (and hence also min) and contains* $\mathrm{PL}(X)$.

Fix a model $\mathscr{X}$. An ideal $\mathfrak{a} \subset \mathscr{O}_{\mathscr{X}}$ is *vertical* if its zero locus $V(\mathfrak{a})$ is contained in $\mathscr{X}_0$. This defines a nonpositive function $\log|\mathfrak{a}| \in \mathrm{PL}(X)$, determined by minus the exceptional divisor of the blowup of $\mathscr{X}$ along $\mathfrak{a}$, and such that

$$\log|\mathfrak{a}|(x) < 0 \Longleftrightarrow Z_{\mathscr{X}}(x) \subset V(\mathfrak{a}). \tag{23}$$

Functions of the form $\log|\mathfrak{a}|$ for a vertical ideal $\mathfrak{a} \subset \mathscr{O}_{\mathscr{X}}$ span the $\mathbb{Q}$-vector space $\mathrm{PL}(X)$ (see [10, Proposition 2.2]). As in Section 1.3, it follows that any function in $\mathbb{R}\mathrm{PL}(X)$ can be written as a difference of finite maxima of $\mathbb{R}_+$-linear combinations of functions of the form $\log|\mathfrak{a}|$.

8.3. *Dual complexes and retractions*

We use [10, 39] as references.

An *snc model* is a regular model $\mathcal{X}$ such that the Cartier divisor $\mathcal{X}_0$ has simple normal crossing support. Denote by $\mathcal{X}_0 = \sum_{i\in I} b_i E_i$ its irreducible decomposition. A *stratum* of $\mathcal{X}_0$ is defined as a non-empty irreducible component of $E_J := \bigcap_{j\in J} E_j$ for some $J \subset I$. By resolution of singularities, the set of snc models is cofinal in the poset of all models.

The *dual complex* $\Delta_{\mathcal{X}}$ of an snc model $\mathcal{X}$ is defined as the dual intersection complex of $\mathcal{X}_0$. Its faces are in 1–1 correspondence with the strata of $\mathcal{X}_0$, and further come with a natural integral affine structure. In particular, the vertices of $\Delta_{\mathcal{X}}$ are in 1–1 correspondence with the E_i's, and admit a natural realization in X^{an} as the set $\Gamma_{\mathcal{X}}$ of Shilov points x_{E_i}.

This extends to a canonical embedding $\Delta_{\mathcal{X}} \hookrightarrow X^{\mathrm{an}}$ onto the set of monomial points with respect to $\sum_i E_i$. The reduction $\mathrm{red}_{\mathcal{X}}(x) \in \mathcal{X}_0$ of a point $x \in \Delta_{\mathcal{X}} \subset X^{\mathrm{an}}$ is the generic point of the stratum of $\mathcal{X}_0$ associated with the unique simplex of $\Delta_{\mathcal{X}}$ containing x in its relative interior. In particular, $Z_{\mathcal{X}}(x)$ is a stratum of $\mathcal{X}_0$. This embedding is further compatible with the PL structures, in the sense that the $\mathbb{Q}$-vector space $\mathrm{PL}(\Delta_{\mathcal{X}})$ of piecewise rational affine functions on $\Delta_{\mathcal{X}}$ is precisely the image of $\mathrm{PL}(X)$ under restriction.

If another snc model $\mathcal{X}'$ dominates $\mathcal{X}$, then $\Delta_{\mathcal{X}}$ is contained in $\Delta_{\mathcal{X}'}$, and $\mathrm{PL}(\Delta_{\mathcal{X}'})$ restricts to $\mathrm{PL}(\Delta_{\mathcal{X}})$. Furthermore, the set

$$X^{\mathrm{qm}} := \bigcup_{\mathcal{X}} \Delta_{\mathcal{X}} \subset X^{\mathrm{an}}$$

of *quasimonomial valuations* coincides with the set of Abhyankar points of X, see [10, Remark 3.8] and [29, Proposition 3.7], while the subset of rational points $\bigcup_{\mathcal{X}} \Delta_{\mathcal{X}}(\mathbb{Q})$ coincides with the set X^{div} of divisorial valuations. For later use, we also note:

Lemma 71. *If $\mathcal{X}$ is an snc model, then the image $\mathrm{red}_{\mathcal{X}'}(\Delta_{\mathcal{X}}) \subset \mathcal{X}'_0$ of the dual complex of $\mathcal{X}$ under the reduction map of any other model $\mathcal{X}'$ is finite.*

Proof. Pick an snc model $\mathcal{X}''$ that dominates both $\mathcal{X}$ and $\mathcal{X}'$. Then $\Delta_{\mathcal{X}}$ is contained in $\Delta_{\mathcal{X}''}$, and $\mathrm{red}_{\mathcal{X}'}(\Delta_{\mathcal{X}})$ is thus contained in the image of $\mathrm{red}_{\mathcal{X}''}(\Delta_{\mathcal{X}''})$ under the induced morphism $\mathcal{X}''_0 \to \mathcal{X}_0$. After replacing both $\mathcal{X}$ and $\mathcal{X}'$ with $\mathcal{X}''$, we may thus assume without loss that $\mathcal{X} = \mathcal{X}'$. For any $x \in \Delta_{\mathcal{X}}$, $\mathrm{red}_{\mathcal{X}}(x)$ is then the generic point of some stratum of $\mathcal{X}_0$, and $\mathrm{red}_{\mathcal{X}}(\Delta_{\mathcal{X}})$ is thus a finite set. $\square$

Dually, each snc model $\mathcal{X}$ comes with a canonical *retraction* $p_{\mathcal{X}} \colon X^{\mathrm{an}} \to \Delta_{\mathcal{X}}$ that takes $x \in X^{\mathrm{an}}$ to the unique monomial valuation $y = p_{\mathcal{X}}(x)$ such that

- $Z_{\mathcal{X}}(y)$ is the minimal stratum containing $Z_{\mathcal{X}}(x)$;
- x and y take the same values on the E_i's.

This induces a homeomorphism $X^{\mathrm{an}} \xrightarrow{\sim} \varprojlim_{\mathcal{X}} \Delta_{\mathcal{X}}$, which is compatible with the PL structures in the sense that

$$\mathrm{PL}(X) = \bigcup_{\mathcal{X}} p_{\mathcal{X}}^{\star}\,\mathrm{PL}(\Delta_{\mathcal{X}}). \tag{24}$$

This implies

$$\mathbb{R}\mathrm{PL}(X) = \bigcup_{\mathcal{X}} p_{\mathcal{X}}^{\star}\,\mathbb{R}\mathrm{PL}(\Delta_{\mathcal{X}}), \tag{25}$$

where $\mathbb{R}\mathrm{PL}(\Delta_{\mathcal{X}})$ is the space $\mathbb{R}$-PL functions on $\Delta_{\mathcal{X}}$, i.e. functions that are real affine linear on a sufficiently fine decomposition of each face into real simplices.

8.4. *Psh functions and Monge–Ampère measures*

We use [10, 11, 26] as references.

A *closed* $(1,1)$*-form* $\theta \in \mathcal{Z}^{1,1}(X)$ in the sense of [10, §4.2] is represented by a relative numerical equivalence class on some model $\mathcal{X}$, called a *determination* of θ. It induces a numerical class $[\theta] \in \mathrm{N}^1(X)$. We say that θ is *semipositive*, written $\theta \geq 0$, if θ is determined by a nef numerical class on some model. In that case, $[\theta]$ is nef as well.

To each tuple $\theta_1, \dots, \theta_n$ in $\mathcal{Z}^{1,1}(X)$ is associated a signed Radon measure $\theta_1 \wedge \cdots \wedge \theta_n$ on X^{an} of total mass $[\theta_1] \cdot \ldots \cdot [\theta_n]$, with finite support in X^{div}. More precisely, if all θ_i are determined by a normal model $\mathcal{X}$, then $\theta_1 \wedge \cdots \wedge \theta_n$ has support in $\Gamma_{\mathcal{X}}$ (see [11, §2.7]).

Each $\varphi \in \mathrm{PL}(X)$ is determined by a vertical $\mathbb{Q}$-Cartier divisor D on some model $\mathcal{X}$, whose numerical class defines a closed $(1,1)$-form $\mathrm{dd}^c\varphi \in \mathcal{Z}^{1,1}(X)$. We say that φ is θ*-psh* for a given $\theta \in \mathcal{Z}^{1,1}(X)$ if $\theta + \mathrm{dd}^c\varphi \geq 0$.

From now on, we fix a semipositive form $\omega \in \mathcal{Z}^{1,1}(X)$ such that $[\omega]$ is ample. A function $\varphi\colon X^{\mathrm{an}} \to \mathbb{R} \cup \{-\infty\}$ is ω*-plurisubharmonic* (ω*-psh* for short) if $\varphi \not\equiv -\infty$ and φ can be written as the pointwise limit of a decreasing net of ω-psh PL functions. The space $\mathrm{PSH}(\omega)$ is closed under max and under decreasing limits.

By Dini's lemma, the space $\mathrm{CPSH}(\omega)$ of continuous ω-psh functions coincides with the closure in $\mathrm{C}^0(X)$ (with respect to uniform convergence) of the space of ω-psh PL functions.

Each $\varphi \in \mathrm{PSH}(\omega)$ satisfies the "maximum principle"

$$\sup_X \varphi = \max_{\Gamma_{\mathcal{X}}} \varphi \tag{26}$$

for any model $\mathcal{X}$ determining ω (see [26, Proposition 4.22]). For snc models, [10, §7.1] more precisely yields:

Lemma 72. *Pick* $\varphi \in \mathrm{PSH}(\omega)$ *and an snc model* $\mathcal{X}$ *on which* ω *is determined. Then:*

(i) *the restriction of* φ *to any face of* $\Delta_{\mathcal{X}}$ *is continuous and convex;*
(ii) *the net* $(\varphi \circ p_{\mathcal{X}})_{\mathcal{X}}$ *is decreasing and converges pointwise to* φ.

Remark 73. The definition of $\mathrm{PSH}(\omega)$ given here differs from the one in [10], but Theorem 8.7 in loc. cit. implies that the two definitions are equivalent.

To each continuous ω-psh function φ (or, more generally, any ω-psh function of finite energy) is associated its *Monge–Ampère measure* $\mathrm{MA}(\varphi) = \mathrm{MA}_\omega(\varphi)$, a Radon probability measure on X uniquely determined by the following properties:

- if φ is PL, then $\mathrm{MA}(\varphi) = V^{-1}(\omega + \mathrm{dd}^c\varphi)^n$ with $V := [\omega]^n$;
- $\varphi \mapsto \mathrm{MA}(\varphi)$ is continuous along decreasing nets.

By the main result of [11], any Radon probability measure μ with support in the dual complex $\Delta_{\mathcal{X}}$ of some snc model can be written as $\mu = \mathrm{MA}(\varphi)$ for some $\varphi \in \mathrm{CPSH}(\omega)$, unique up to an additive constant.

8.5. *Green's functions*

As in the trivially valued case, we can consider the Green's function associated to a nonpluripolar set $\Sigma \subset X^{\mathrm{an}}$. Here we will only consider the following case. Suppose $x \in X^{\mathrm{div}}$ is a divisorial point, and define

$$\varphi_x := \varphi_{\omega,x} := \sup\{\varphi \in \mathrm{PSH}(\omega) \mid \varphi(x) \leq 0\}.$$

It follows from [11, §8.4] that $\varphi_x \in \mathrm{CPSH}(\omega)$ satisfies $\mathrm{MA}(\varphi_x) = \delta_x$ and $\varphi_x(x) = 0$.

Proposition 74. *If* $\dim X = 1$ *and* $[\omega]$ *is a rational class, then* $\varphi_x \in \mathrm{PL}(X)$.

Proof. This follows from Proposition 3.3.7 in [42], and can also be deduced from properties of the intersection form on $\mathcal{X}_0$ for any snc model $\mathcal{X}$, as in [23, Theorem 7.17]. □

This proves part (i) of Theorem A in the introduction. We will prove (ii) in Section 9.5.

8.6. *Invariance under retraction*

It will be convenient to introduce the following terminology:

Definition 75. *We say that a function φ on X^{an} is* invariant under retraction *if $\varphi = \varphi \circ p_{\mathcal{X}}$ for some (and hence any sufficiently high) snc model $\mathcal{X}$ of X.*

Example 76. By (24) and (25), a function $\varphi \in \mathrm{C}^0(X^{\mathrm{an}})$ lies in $\mathrm{PL}(X)$ (resp. $\mathbb{R}\mathrm{PL}(X)$) iff φ is invariant under retraction and restricts to a $\mathbb{Q}$-PL (resp. $\mathbb{R}$-PL) function on the dual complex associated to any (equivalently, any sufficiently high) snc model.

Remark 77. The condition $\varphi = \varphi \circ p_{\mathcal{X}}$ in Definition 75 is stronger than the "comparison property" of [36, Definition 3.11], which merely requires $\varphi = \varphi \circ p_{\mathcal{X}}$ to hold on the preimage under $p_{\mathcal{X}}$ of the n-dimensional open faces of some dual complex $\Delta_{\mathcal{X}}$, i.e. the preimage of the 0-dimensional strata of $\mathcal{X}_0$ under the reduction map.

Proposition 78. *If $\varphi \in \mathrm{PSH}(\omega)$ is invariant under retraction, then $\varphi \in \mathrm{CPSH}(\omega)$, and $\mathrm{MA}(\varphi)$ is supported in some dual complex.*

The first point is a direct consequence of Lemma 72, while the second one is a special case of the following more precise result. Recall first that the *ω-psh envelope* of $f \in \mathrm{C}^0(X^{\mathrm{an}})$ is defined as

$$\mathrm{P}(f) = \mathrm{P}_\omega(f) := \sup\{\varphi \in \mathrm{PSH}(\omega) \mid \varphi \le f\}.$$

By [10], it lies in $\mathrm{CPSH}(\omega)$.

Theorem 79. *For any $\varphi \in \mathrm{CPSH}(\omega)$ and any snc model $\mathcal{X}$ on which ω is determined, the following properties are equivalent:*

(i) $\mathrm{MA}(\varphi)$ *is supported in* $\Delta_{\mathcal{X}}$*;*
(ii) $\varphi = \mathrm{P}(\varphi \circ p_{\mathcal{X}})$.

Proof. For any $\psi \in \mathrm{PSH}(\omega)$, we have $\psi \le \psi \circ p_{\mathcal{X}}$ (see Lemma 72 (ii)), and hence

$$\mathrm{P}(\varphi \circ p_{\mathcal{X}}) = \sup\left\{\psi \in \mathrm{PSH}(\omega) \mid \psi \le \varphi \text{ on } \Delta_{\mathcal{X}}\right\}. \tag{27}$$

Assume (i). By the domination principle (see [11, Lemma 8.4]), any $\psi \in \mathrm{PSH}(\omega)$ such that $\psi \le \varphi$ on $\operatorname{supp} \mathrm{MA}(\varphi) \subset \Delta_{\mathcal{X}}$ satisfies $\psi \le \varphi$ on X^{an}. In view of (27) this yields (ii). Conversely, assume (ii). For any finite set of rational points $\Sigma \subset \Delta_{\mathcal{X}}(\mathbb{Q}) \subset X^{\mathrm{div}}$, consider the envelope

$$\varphi_\Sigma := \sup\{\psi \in \mathrm{PSH}(\omega) \mid \psi \le \varphi \text{ on } \Sigma\}.$$

Then φ_Σ lies in $\mathrm{CPSH}(\omega)$, and $\mathrm{MA}(\varphi_\Sigma)$ is supported in Σ (see [11, Lemma 8.5]). The net (φ_Σ), indexed by the filtered poset of finite subsets $\Sigma \subset \Delta_{\mathcal{X}}(\mathbb{Q})$, is clearly decreasing, and bounded below by φ. Its limit $\psi := \lim_\Sigma \varphi_\Sigma$ is thus ω-psh, and we claim that it coincides with φ. Indeed, we have $\psi \le \varphi$ on $\bigcup_\Sigma \Sigma = \Delta_{\mathcal{X}}(\mathbb{Q})$, and hence on $\Delta_{\mathcal{X}}$, where both ψ and φ are continuous. By (27), this yields $\psi \le \mathrm{P}(\varphi \circ p_{\mathcal{X}}) = \varphi$. By continuity of the Monge–Ampère operator along decreasing nets, we infer $\mathrm{MA}(\varphi_\Sigma) \to \mathrm{MA}(\varphi)$ weakly on X, which yields (i) since each $\mathrm{MA}(\varphi_\Sigma)$ is supported in $\Delta_{\mathcal{X}}$. □

In view of Proposition 78 and Example 76, it is natural to conversely ask:

Question 80. *If the Monge–Ampère measure $\mathrm{MA}_\omega(\varphi)$ of $\varphi \in \mathrm{CPSH}(\omega)$ is supported in some dual complex, is φ invariant under retraction?*

This question appears as [25, Question 2], and is equivalent to asking whether $\varphi \circ p_{\mathscr{X}}$ is ω-psh for some high enough model $\mathscr{X}$, by Theorem 79. In Example 99 below (see also Theorem A) we show that the answer is negative. In this example, the support of $\mathrm{MA}_\omega(\varphi)$ is even a finite set. One can nevertheless ask:

Question 81. *Assume that $\varphi \in \mathrm{CPSH}(\omega)$ is such that the support of the Monge–Ampère measure $\mathrm{MA}_\omega(\varphi)$ is a finite set contained in some dual complex.*

(i) *is φ $\mathbb{R}$-PL on each dual complex?*
(ii) *if ω is rational, is φ $\mathbb{Q}$-PL on each dual complex?*

Example 99 below provides a negative answer to (ii). Indeed the function φ in this example is $\mathbb{R}$-PL but not $\mathbb{Q}$-PL, and by (24), (25), this implies that φ fails to be $\mathbb{Q}$-PL on some dual complex $\Delta_{\mathscr{X}}$. The answer to (i) is also likely negative in general, as suggested by Nakayama's counterexample to the existence of Zariski decompositions on certain toric bundles over an abelian suface [40, p. IV.2.10].

Question 82. *Suppose X is a toric variety, and let $\varphi \in \mathrm{CPSH}(\omega)$ be a torus invariant ω-psh function such that $\mathrm{MA}_\omega(\varphi)$ is supported on a compact subset of $N_{\mathbb{R}} \subset X^{\mathrm{an}}$. Is φ invariant under retraction?*

Question 83. *If $\varphi \in \mathrm{CPSH}(\omega)$ is invariant under retraction, is the same true for $\varphi|_{Z^{\mathrm{an}}}$, if $Z \subset X$ is a smooth subvariety?*

8.7. *The center of a plurisubharmonic function*

We end this section by a version of Theorem 24 in our present context. In analogy with (7), for any subset $S \subset X^{\mathrm{an}}$ and any model $\mathscr{X}$ we set

$$Z_{\mathscr{X}}(S) := \bigcup_{x \in S} Z_{\mathscr{X}}(x).$$

This is thus the smallest subset of $\mathscr{X}_0$ that is invariant under specialization and contains the image $\mathrm{red}_{\mathscr{X}}(S)$ of S under the reduction map $\mathrm{red}_{\mathscr{X}} \colon X^{\mathrm{an}} \to \mathscr{X}_0$. For any higher model $\mathscr{X}'$, the induced proper morphism $\mathscr{X}'_0 \to \mathscr{X}_0$ maps $Z_{\mathscr{X}'}(S)$ onto $Z_{\mathscr{X}}(S)$.

We say that $S \subset X^{\mathrm{an}}$ is *invariant under retraction* if $p_{\mathscr{X}}^{-1}(S) = S$ for some (and hence any sufficiently high) snc model $\mathscr{X}$.

Lemma 84. *If $S \subset X^{\mathrm{an}}$ is invariant under retraction, then $Z_{\mathscr{X}}(S)$ is Zariski closed for any model $\mathscr{X}$.*

Proof. Pick an snc model $\mathscr{X}'$ dominating $\mathscr{X}$ such that $S = p_{\mathscr{X}'}^{-1}(S)$. Since $Z_{\mathscr{X}}(S)$ is the image of $Z_{\mathscr{X}'}(S)$ under the proper morphism $\mathscr{X}'_0 \to \mathscr{X}_0$, we may replace $\mathscr{X}$ with $\mathscr{X}'$ and assume without loss that $\mathscr{X} = \mathscr{X}'$. The set $Z_{\mathscr{X}}(S)$ obviously contains $Z_{\mathscr{X}}(S \cap \Delta_{\mathscr{X}})$, which is Zariski closed since $Z_{\mathscr{X}}(y)$ is a stratum of $\mathscr{X}_0$ for any $y \in \Delta_{\mathscr{X}}$. Conversely, pick $x \in S$, and set $y := p_{\mathscr{X}}(x) \in \Delta_{\mathscr{X}}$. Then $y \in p_{\mathscr{X}}^{-1}(S) = S$, and $Z_{\mathscr{X}}(x) \subset Z_{\mathscr{X}}(y)$ since it follows from the definition of $p_{\mathscr{X}}$ that $\mathrm{red}_{\mathscr{X}}(x)$ is a specialization of $\mathrm{red}_{\mathscr{X}}(y)$. This shows, as desired, that $Z_{\mathscr{X}}(S) = Z_{\mathscr{X}}(S \cap \Delta_{\mathscr{X}})$ is Zariski closed. □

Definition 85. *Given $\varphi \in \mathrm{PSH}(\omega)$ and a model $\mathscr{X}$, we define the* center of φ on $\mathscr{X}$ *as*

$$Z_{\mathscr{X}}(\varphi) := Z_{\mathscr{X}}(\{\varphi < \sup \varphi\}) = \bigcup \{Z_{\mathscr{X}}(x) \mid x \in X, \varphi(x) < \sup \varphi\}.$$

Example 86. If $\varphi = \log|\mathfrak{a}|$ for a vertical ideal $\mathfrak{a} \subset \mathcal{O}_{\mathscr{X}}$, then $Z_{\mathscr{X}}(\varphi) = V(\mathfrak{a})$.

Theorem 87. *For any $\varphi \in \mathrm{PSH}(\omega)$ and any model $\mathscr{X}$, the following holds:*

(i) *$Z_{\mathscr{X}}(\varphi)$ is an at most countable union of subvarieties of $\mathscr{X}_0$;*
(ii) *if φ is invariant under retraction, then $Z_{\mathscr{X}}(\varphi)$ is Zariski closed;*
(iii) *$Z_{\mathscr{X}}(\varphi) = \mathrm{red}_{\mathscr{X}}(\{\varphi < \sup \varphi\})$;*
(iv) *$Z_{\mathscr{X}}(\varphi)$ is a strict subset of $\mathscr{X}_0$ as soon as $\mathscr{X}$ determines ω.*

Question 88. *Is it true that* $\{\varphi < \sup\varphi\} = \mathrm{red}_{\mathscr{X}}^{-1}(Z_{\mathscr{X}}(\varphi))$ *as in Theorem 24?*

Proof. By [11, Proposition 4.7], φ can be written as the pointwise limit of a decreasing sequence $(\varphi_m)_{m\in\mathbb{N}}$ of ω-psh PL functions. Since each φ_m is in particular invariant under retraction (see Example 76), Lemma 84 implies that $Z_{\mathscr{X}}\{(\varphi_m < \sup\varphi\})$ is Zariski closed for each m. On the other hand, since $\varphi_m \searrow \varphi$ pointwise on X, we have $\{\varphi < \sup\varphi\} = \bigcup_m \{\varphi_m < \sup\varphi\}$, and hence $Z_{\mathscr{X}}(\varphi) = \bigcup_m Z_{\mathscr{X}}(\{\varphi_m < \sup\varphi\})$. This proves (i), while (ii) is a direct consequence of Lemma 84.

Pick $x \in X^{\mathrm{an}}$ such that $\varphi(x) < \sup\varphi$. To prove (iii), we need to show that any $\xi \in Z_{\mathscr{X}}(x)$ lies in $\mathrm{red}_{\mathscr{X}}(\{\varphi < \sup\varphi\})$. By Lemma 72, we can find a high enough snc model $\mathscr{X}'$ such that $x' := p_{\mathscr{X}'}(x)$ satisfies $\varphi(x') < \sup\varphi$. By properness of $\mathscr{X}'_0 \to \mathscr{X}_0$, $Z_{\mathscr{X}}(x)$ is the image of $Z_{\mathscr{X}'}(x)$, which is itself contained in $Z_{\mathscr{X}'}(x')$. After replacing $\mathscr{X}$ with $\mathscr{X}'$ and x with x', we may thus assume without loss that $\mathscr{X}$ is snc and x lies in $\Delta_{\mathscr{X}}$. Pick $y \in X^{\mathrm{an}}$ with $\mathrm{red}_{\mathscr{X}}(y) = \xi$ (which exists by surjectivity of the reduction map, see [24, Lemma 4.12]). Set $z := p_{\mathscr{X}}(y)$, and denote by σ the unique face of $\Delta_{\mathscr{X}}$ that contains z in its relative interior, the corresponding stratum of $\mathscr{X}_0$ being the smallest one containing ξ. Since the latter point lies on the stratum $Z_{\mathscr{X}}(x)$, it follows that σ contains x (possibly on its boundary). Since φ is convex and continuous on σ (see Lemma 72), it can only achieve its supremum at the interior point z if it is constant on σ. As $x \in \sigma$ satisfies $\varphi(x) < \sup\varphi$, it follows that $\varphi(z) < \sup\varphi$ as well. Since $z = p_{\mathscr{X}}(y)$, this implies $\varphi(y) \le \varphi(z) < \sup\varphi$ (again by Lemma 72). Thus $\xi = \mathrm{red}_{\mathscr{X}}(y) \in \mathrm{red}_{\mathscr{X}}(\{\varphi < \sup\varphi\})$, which proves (iii).

Finally, assume that $\mathscr{X}$ determines ω. By (26), we can find an irreducible component E of $\mathscr{X}_0$ whose corresponding Shilov point $x_E \in \Gamma_{\mathscr{X}}$ satisfies $\varphi(x_E) = \sup\varphi$. Since x_E is the only point of X^{an} whose reduction on $\mathscr{X}_0$ is the generic point of E, it follows that the latter does not belong to $Z_{\mathscr{X}}(\varphi)$, which is thus a strict subset of $\mathscr{X}_0$. □

9. The isotrivial case

We now consider the *isotrivial* case, in which the variety over $K = k((\varpi))$ is the base change X_K of a smooth projective variety X over the (trivially valued) field k.

9.1. *Ground field extension*

We have a natural projection

$$\pi\colon X_K^{\mathrm{an}} \longrightarrow X^{\mathrm{an}},$$

while Gauss extension provides a continuous section

$$\sigma\colon X^{\mathrm{an}} \hookrightarrow X_K^{\mathrm{an}}$$

onto the set of $k^\times$-invariant points (see [12, Proposition 1.6]). By [12, Corollary 1.5], we further have:

Lemma 89. *If* $v \in X^{\mathrm{an}}$ *is divisorial (resp real divisorial) then* $\sigma(v) \in X_K^{\mathrm{an}}$ *is divisorial (resp. quasimonomial).*

The base change of X to the valuation ring $K^\circ := k[[\varpi]]$ defines the *trivial model*

$$\mathscr{X}_{\mathrm{triv}} := X_{K^\circ}$$

of X_K, whose special fiber $\mathscr{X}_{\mathrm{triv},0}$ will be identified with X. More generally, each *test configuration* $\mathscr{X} \to \mathbb{A}^1 = \operatorname{Spec} k[\varpi]$ for X induces via base change under $k[\varpi] \to k[[\varpi]] = K^\circ$ a $k^\times$-invariant model of X_K, that shares the same vertical ideals and vertical divisors as $\mathscr{X}$, and will simply be denoted by $\mathscr{X}$, for simplicity.

9.2. *Psh functions*

For any $\theta \in \mathrm{N}^1(X)$, we denote by $\pi^\star\theta \in \mathcal{Z}^{1,1}(X_K)$ the induced closed $(1,1)$-form, determined by the relative numerical class induced by θ on the trivial model. If $\omega \in \mathrm{Amp}(X)$, then $[\pi^\star\omega] \in \mathrm{N}^1(X_K)$ coincides with the base change of ω, and hence is ample.

Theorem 90. *Pick $\omega \in \mathrm{Amp}(X)$ and $\varphi \in \mathrm{PSH}(\omega)$. Then:*

(i) $\pi^\star\varphi \in \mathrm{PSH}(\pi^\star\omega)$;
(ii) *if φ further lies in* $\mathrm{CPSH}(\omega)$, *then* $\mathrm{MA}_{\pi^\star\omega}(\pi^\star\varphi) = \sigma_\star \mathrm{MA}_\omega(\varphi)$.

Lemma 91. *For any $\varphi \in \mathrm{PL}(X)$ and $\theta \in \mathrm{N}^1(X)$, the following holds:*

(i) $\pi^\star\varphi \in \mathrm{PL}(X_K)$;
(ii) $(\pi^\star\theta + \mathrm{dd}^c\pi^\star\varphi)^n = \sigma_\star(\theta + \mathrm{dd}^c\varphi)^n$;
(iii) *φ is θ-psh iff $\pi^\star\varphi$ is $\pi^\star\theta$-psh.*

Proof. The function φ is determined by a vertical $\mathbb{Q}$-Cartier divisor D on a test configuration $\mathcal{X}$, that may be taken to dominate the trivial one (see [13, Theorem 2.7]). The induced vertical divisor on the induced model of X_K then determines $\pi^\star\varphi$. This proves (i), and also (ii), by comparing [11, (2.2)] and [13, (3.6)]. Finally, denote by $\theta_{\mathcal{X}}$ the pullback of θ to $\mathrm{N}^1(\mathcal{X}/\mathbb{A}^1)$. Then φ is θ-psh iff $(\theta_{\mathcal{X}} + [D])|_{\mathcal{X}_0}$ is nef, which is also equivalent to $\pi^\star\varphi$ being $\pi^\star\theta$-psh. This proves (iii). □

Proof of Theorem 90. Write φ as the limit on X^{an} of a decreasing net of ω-psh PL functions φ_i. By Lemma 91, $\pi^\star\varphi_i$ is PL and $\pi^\star\omega$-psh. Since it decreases pointwise on X_K^{an} to $\pi^\star\varphi$, the latter is $\pi^\star\omega$-psh, which proves (i). For each i, Lemma 91 (ii) further implies $\mathrm{MA}_{\pi^\star\omega}(\pi^\star\varphi_i) = \sigma_\star \mathrm{MA}_\omega(\varphi_i)$. If φ is continuous, then $\mathrm{MA}_\omega(\varphi)$ and $\mathrm{MA}_{\pi^\star\omega}(\pi^\star\varphi)$ are both defined, and are the limits of $\mathrm{MA}_\omega(\varphi_i)$ and $\mathrm{MA}_{\pi^\star\omega}(\pi^\star\varphi_i)$, respectively. This proves (ii). □

9.3. *PL structures*

As a direct consequence of Lemma 91, the projection $\pi\colon X_K^{\mathrm{an}} \to X^{\mathrm{an}}$ is compatible with the PL structures:

Corollary 92. *We have $\pi^\star\,\mathrm{PL}(X) \subset \mathrm{PL}(X_K)$ and $\pi^\star\,\mathbb{R}\mathrm{PL}(X) \subset \mathbb{R}\mathrm{PL}(X_K)$.*

As we next show, this is also the case for Gauss extension.

Theorem 93. *We have $\sigma^\star\,\mathrm{PL}(X_K) = \mathrm{PL}(X)$ and $\sigma^\star\,\mathbb{R}\mathrm{PL}(X_K) = \mathbb{R}\mathrm{PL}(X)$.*

Any vertical ideal $\mathfrak{a}$ on $\mathcal{X}_{\mathrm{triv}}$, being trivial outside the central fiber, can be viewed as a vertical ideal on $X \times \mathbb{A}^1$, and $\widetilde{\mathfrak{a}} := \mathbb{G}_{\mathrm{m}} \cdot \mathfrak{a}$ is then the smallest flag ideal containing $\mathfrak{a}$.

Lemma 94. *With the above notation we have $\sigma^\star \log|\mathfrak{a}| = \varphi_{\widetilde{\mathfrak{a}}}$.*

Proof. Pick an ample line bundle L on X, and denote by $\mathcal{L}_{\mathrm{triv}}$ the trivial model of L_K, i.e. the pullback of L to the trivial model $\mathcal{X}_{\mathrm{triv}} = X_{K^\circ}$. After replacing L with a large enough multiple, we may assume $\mathcal{L}_{\mathrm{triv}} \otimes \mathfrak{a}$ is generated by finitely many sections $s_i \in \mathrm{H}^0(\mathcal{X}_{\mathrm{triv}}, \mathcal{L}_{\mathrm{triv}})$. Then $\log|\mathfrak{a}| = \max_i \log|s_i|$, where $|s_i|$ denotes the pointwise length of s_i in the model metric induced by $\mathcal{L}_{\mathrm{triv}}$. For each i write $s_i = \sum_{\lambda\in\mathbb{Z}} s_{i,\lambda}\varpi^\lambda$ where $s_{i,\lambda} \in \mathrm{H}^0(X, L)$, and denote by $\mathfrak{b}_\lambda \subset \mathcal{O}_X$ the ideal locally generated by $(s_{i,\lambda})_i$. Then $\widetilde{\mathfrak{a}} = \sum_{\lambda\in\mathbb{Z}} \mathfrak{b}_\lambda\varpi^\lambda$. By definition of Gauss extension, we have for any $v \in X^{\mathrm{an}}$

$$\log|s_i|(\sigma(v)) = \max_{\lambda\in\mathbb{Z}}\{\log|s_{i,\lambda}| + \lambda\}.$$

Thus $\sigma^\star \log|\mathfrak{a}| = \max_{\lambda\in\mathbb{Z}}\{\psi_\lambda - \lambda\}$ with $\psi_\lambda := \max_i \log|s_{i,\lambda}| = \log|\mathfrak{b}_\lambda|$, and hence $\sigma^\star \log|\mathfrak{a}| = \max_\lambda\{\log|\mathfrak{b}_\lambda| - \lambda\} = \varphi_{\widetilde{\mathfrak{a}}}$. □

Proof of Theorem 93. By Corollary 92 we have $\pi^\star \operatorname{PL}(X) \subset \operatorname{PL}(X_K)$. Since $\operatorname{PL}(X_K)$ is generated by functions of the form $\log|\mathfrak{a}|$ for a vertical ideal $\mathfrak{a} \subset \mathcal{O}_{\mathcal{X}_{\mathrm{triv}}}$, Lemma 94 yields $\sigma^\star \operatorname{PL}(X_K) \subset \operatorname{PL}(X)$, and hence also $\sigma^\star \mathbb{R}\operatorname{PL}(X_K) \subset \mathbb{R}\operatorname{PL}(X)$. This completes the proof, since $\sigma^\star\pi^\star = \mathrm{id}$. □

9.4. *Centers*

Next we study the relationships between the two center maps $Z_X \colon X^{\mathrm{an}} \to X$ and $Z_{\mathcal{X}_{\mathrm{triv}}} \colon X_K^{\mathrm{an}} \to \mathcal{X}_{\mathrm{triv},0} = X$.

Lemma 95. *For all $x \in X_K^{\mathrm{an}}$ and $v \in X^{\mathrm{an}}$ we have*

$$Z_{\mathcal{X}_{\mathrm{triv}}}(x) \subset Z_X(\pi(x)), \quad Z_X(v) = Z_{\mathcal{X}_{\mathrm{triv}}}(\sigma(v)).$$

Proof. Denote by $\mathfrak{b} \subset \mathcal{O}_X$ the ideal of the subvariety $Z_X(\pi(x))$. Then $\mathfrak{a} := \mathfrak{b} + (\varpi)$ is a vertical ideal on $\mathcal{X}_{\mathrm{triv}}$ such that $V(\mathfrak{a}) = V(\mathfrak{b}) = Z_X(\pi(x))$ under the identification $\mathcal{X}_{\mathrm{triv},0} = X$. Further,

$$\log|\mathfrak{a}|(x) = \max\{\log|\mathfrak{b}|(\pi(x)), -1\} < 0,$$

and hence $Z_{\mathcal{X}_{\mathrm{triv}}}(x) \subset V(\mathfrak{a}) = Z_X(\pi(x))$, see (23).

Applying this to $x = \sigma(v)$ yields $Z_{\mathcal{X}_{\mathrm{triv}}}(\sigma(v)) \subset Z_X(v)$. To prove the converse inclusion, denote by $\mathfrak{a} \subset \mathcal{O}_{\mathcal{X}_{\mathrm{triv}}}$ the ideal of $Z_{\mathcal{X}_{\mathrm{triv}}}(\sigma(v))$. Since $\sigma(v)$ is $k^\times$-invariant, $\mathfrak{a} = \sum_{\lambda \in \mathbb{Z}} \mathfrak{a}_\lambda \varpi^{-\lambda}$ is (induced by) a flag ideal. Further, $\varphi_{\mathfrak{a}}(v) = \log|\mathfrak{a}|(\sigma(v)) < 0$, and hence $Z_X(v) \subset Z_X(\varphi_{\mathfrak{a}})$. By Example 14 we have $Z_X(\varphi_{\mathfrak{a}}) = V(\mathfrak{a}_0)$. The latter is also equal to the zero locus of $\mathfrak{a}_0 + (\varpi)$ on $\mathcal{X}_{\mathrm{triv}}$, which is contained in $V(\mathfrak{a}) = Z_{\mathcal{X}_{\mathrm{triv}}}(\sigma(v))$ since $\mathfrak{a} \subset \mathfrak{a}_0 + (\varpi)$. Thus $Z_X(v) \subset Z_{\mathcal{X}_{\mathrm{triv}}}(\sigma(v))$, which concludes the proof. □

As a consequence we get:

Proposition 96. *If $\omega \in \operatorname{Amp}(X)$ and $\varphi \in \operatorname{PSH}(\omega)$, then $Z_{\mathcal{X}_{\mathrm{triv}}}(\pi^\star\varphi) = Z_X(\varphi)$.*

Proof. Pick $v \in X^{\mathrm{an}}$ such that $\varphi(v) < \sup\varphi$, and set $x := \sigma(v)$. Then $\pi^\star\varphi(x) = \varphi(v)$ and $\sup \pi^\star\varphi = \sup\varphi$, so x lies in $\{\pi^\star\varphi < \sup\pi^\star\varphi\}$, and hence $Z_X(v) = Z_{\mathcal{X}_{\mathrm{triv}}}(x) \subset Z_{\mathcal{X}_{\mathrm{triv}}}(\pi^\star\varphi)$ by Lemma 95. This implies $Z_X(\varphi) \subset Z_{\mathcal{X}_{\mathrm{triv}}}(\pi^\star\varphi)$. Conversely, assume $x \in X_K^{\mathrm{an}}$ satisfies $\pi^\star\varphi(x) < \sup\pi^\star\varphi$. Then $v := \pi(x)$ lies in $\{\varphi < \sup\varphi\}$, and hence $Z_X(v) \subset Z_X(\varphi)$. In view of Lemma 95, this implies $Z_{\mathcal{X}_{\mathrm{triv}}}(x) \subset Z_X(\varphi)$, and hence $Z_{\mathcal{X}_{\mathrm{triv}}}(\pi^\star\varphi) \subset Z_X(\varphi)$. □

Combining Proposition 96 and Theorem 87, we obtain

Corollary 97. *Let $\varphi \in \operatorname{PSH}(\omega)$, where $\omega \in \operatorname{Amp}(X)$, and suppose that $\pi^\star\varphi \in \operatorname{PSH}(\pi^\star\omega)$ is invariant under retraction. Then $Z_X(\varphi) \subset X$ is a Zariski closed proper subset of X.*

9.5. *Examples*

We are now ready to prove Theorems A and B in the introduction, and also provide additional examples. As in the previous section, X denotes a smooth projective variety over k. Pick a class $\omega \in \operatorname{Amp}(X)$, a $k^\times$-invariant divisorial point $x \in X_K^{\mathrm{div}}$, and denote as in Section 8.5 by $\varphi_x \in \operatorname{CPSH}(\pi^\star\omega)$ the Green's function associated to x; this is the unique solution to the Monge–Ampère equation

$$\operatorname{MA}_{\pi^\star\omega}(\varphi_x) = \delta_x \quad \text{and} \quad \varphi_x(x) = 0.$$

By Lemma 89, we have $x = \sigma(v)$ with $v := \pi(x) \in X^{\mathrm{div}}$. If $\varphi_v \in \operatorname{CPSH}(\omega)$ denotes the Green's function of $\{v\}$, see Section 6.1, then we have

$$\varphi_x = \pi^\star\varphi_v.$$

Indeed, $\pi^\star\varphi_v(x) = \varphi_v(v) = 0$, and by Theorem 90, we have $\operatorname{MA}_{\pi^\star\omega}(\pi^\star\varphi_v) = \sigma_\star\delta_v = \delta_x$.

Our goal is to investigate the regularity of φ_x.

Corollary 98. *If* $\dim X = 1$*, then* $\varphi_x \in \mathrm{PL}(X_K)$*. If* $\dim X = 2$*, then* $\varphi_x \in \mathbb{R}\mathrm{PL}(X_K)$*.*

Proof. The first statement follows from Proposition 74. Now suppose $\dim X = 2$. By Theorem 60, $\varphi_v \in \mathbb{R}\mathrm{PL}(X)$, so that $\varphi_x \in \mathbb{R}\mathrm{PL}(X_K)$, see Corollary 92. □

However, even when ω is rational, φ_x is in general not $\mathbb{Q}$-PL:

Example 99. Example 63 gives an example of an abelian surface X, a rational class $\omega \in \mathrm{Amp}(X)$, and a divisorial valuation $v \in X^{\mathrm{div}}$ such that $\varphi_v \in \mathbb{R}\mathrm{PL}(X) \setminus \mathrm{PL}(X)$. If $x = \sigma(v)$, then $\varphi_x = \pi^\star \varphi_v \in \mathbb{R}\mathrm{PL}(X_K) \setminus \mathrm{PL}(X_K)$, by Theorem 93.

Example 100. Similarly, Example 65 gives an example of a divisorial valuation $v \in \mathbb{P}^{3,\mathrm{div}}$ such that if we set $\omega = c_1(\mathscr{O}(4))$, then $\varphi_v := \varphi_{\omega,v} \in \mathbb{R}\mathrm{PL}(X) \setminus \mathrm{PL}(X)$. If $x = \sigma(v)$, then $\varphi_x = \pi^\star \varphi_v \in \mathbb{R}\mathrm{PL}(X_K) \setminus \mathrm{PL}(X_K)$, by Theorem 93.

Examples 99 and 100 establish Theorem A (ii). They also provide a negative answer to Question 81 (ii). Indeed, a function $\varphi \in \mathrm{C}^0(X_K^{\mathrm{an}})$ lies in $\mathbb{R}\mathrm{PL}(X_K)$ (resp. $\mathrm{PL}(X_K)$) iff φ is invariant under retraction and restricts to an $\mathbb{R}$-PL (resp. $\mathbb{Q}$-PL) function on each dual complex, see Example 76.

As the next example shows, if $\dim X = 3$, then φ_x need not be $\mathbb{R}$-PL. In fact, it may not even be invariant under retraction.

Example 101. Example 67 shows that we may have $\dim X = 3$ and $Z_X(\varphi_v)$ Zariski dense in X, and it follows from Corollary 97 that φ_x cannot be invariant under retraction.

It could, however, a priori be the case that the restriction φ_x to any dual complex is $\mathbb{R}$-PL, see Question 81 (i).

In Example 101, based on Lesieutre's work, the class ω is irrational. We do not know of an example for which the class ω is rational. However, the following example provides a proof of Theorem B in the introduction.

Example 102. Set $X = \mathbb{P}^3_k$ and $\omega := c_1(\mathscr{O}(1)) \in \mathrm{N}^1(X)$. By Proposition 68, there exists $\psi \in \mathrm{CPSH}(\omega)$ such that $\mathrm{MA}_\omega(\psi)$ is supported in a finite subset $\Sigma \subset X^{\mathrm{div}}_{\mathbb{R}}$, and $Z_X(\psi)$ is Zariski dense in X. Theorem 90 then shows that $\varphi := \pi^\star \psi$ lies in $\mathrm{CPSH}(\pi^\star \omega)$, $\mathrm{MA}_{\pi^\star\omega}(\varphi) = \sigma_\star \mathrm{MA}_\omega(\psi)$ has finite support in some dual complex (see Lemma 89), while Corollary 97 shows that φ cannot be invariant under retraction.

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Complex algebraic geometry, in memory of Jean-Pierre Demailly /
Géométrie algébrique complexe, en mémoire de Jean-Pierre Demailly

Frobenius integrability of certain p-forms on singular spaces

Intégrabilité au sens de Frobenius pour certaines p-formes sur des espaces singuliers

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In memory of Jean-Pierre Demailly, our great academic teacher

Abstract. Demailly proved that on a smooth compact Kähler manifold the distribution defined by a holomorphic p-form with values in an anti-pseudoeffective line bundle is always integrable. We generalise his result to compact Kähler spaces with klt singularities.

Résumé. Demailly a montré que la distribution définie par une p-forme holomorphe à valeurs dans un fibré en droites est toujours intégrable si la variété est kählerienne compacte et le dual du fibré en droites est pseudoeffectif. Nous généralisons son résultat à des espaces kähleriennes compactes à singularités klt.

Keywords. holomorphic p-forms, klt spaces, foliations.

Mots-clés. p-forme holomorphe, espace à singularités klt, feuilletages.

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1. Introduction

Let X be a compact Kähler manifold, and let $u \in H^0(X, \Omega_X^p)$ be a holomorphic p-form on X. As a consequence of the Kähler identity for the Laplacians $\Delta_d = 2\Delta_\partial$ one obtains that the holomorphic form is d-closed, i.e. $du = 0$. Twenty years ago Jean-Pierre Demailly used a very clever "integration by parts" to generalise this statement to forms with values in certain line bundles:

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Theorem 1 ([3, Main Thm.]). *Let X be a compact Kähler manifold. Let L be a pseudo-effective holomorphic line bundle on X. Let*

$$u \in H^0(X, \Omega_X^p \otimes L^\star)$$

be a non-zero holomorphic section, and let $S_u \subset T_X$ be the saturated subsheaf given by vector fields ξ such that the contraction $i_\xi u$ vanishes. Then S_u is integrable, i.e. it defines a (possibly singular) holomorphic foliation on X.

Moreover, let h be a possibly singular metric such that $i\Theta_h(L) \geqslant 0$ on X in the sense of currents. Then one has $D'_{h^\star} u = 0$ and (L, h) has flat curvature along the leaves. Here $D'_{h^\star}$ is the $(1,0)$-part of the Chern connection with respect to the dual metric $h^\star$ on $L^\star$.

Demailly's main motivation for this result was to prove that if a compact Kähler manifold admits a contact structure, then the canonical bundle K_X is never pseudoeffective [3, Cor. 2]. Moreover Theorem 1 has turned out to be a very efficient tool for the study of foliations with vanishing first Chern class [10, 14, 17]. In view of the increased interest in foliations on singular spaces (cf. e.g. [2, 5]) it seems worthwhile to look at Demailly's arguments in this setting. In this paper we extend his result to singular spaces with klt (resp. log-canonical) singularities (see Section 2 for the definitions), i.e. the most general classes of singularities appearing in the minimal model program. Our main result is:

Theorem 2. *Let Y be a normal compact Kähler space. Let $\mathscr{A}$ be a rank one reflexive sheaf such that the reflexive power $\mathscr{A}^{[m]}$ is locally free and pseudoeffective for some $m \in \mathbb{N}$. Let*

$$u \in H^0(Y, (\Omega_Y^p \otimes \mathscr{A}^\star)^{\star\star})$$

be a non-zero holomorphic section. Let $S_u \subset T_Y$ be the saturated subsheaf given by vector fields ξ such that the contraction $i_\xi u$ vanishes. Assume one of the following:

(1) *Y has klt singularities; or*
(2) *Y has log-canonical singularities and $p = 1$.*

Then S_u is integrable, i.e. it defines a (possibly singular) foliation on Y.

For applications in foliation theory it is interesting to verify if $\mathscr{A}$ has flat curvature along the leaves of S_u. Since $\mathscr{A}$ is not locally free the precise formulation would be a bit awkward, but flatness holds for the corresponding line bundle (L, h) on a resolution of singularities (see Propositions 8, 10 and Remark 6).

Our basic strategy is similar to the proof of Theorem 1, except that we have to carry out the computation on a resolution of singularities $\pi : X \to Y$. If $\mathscr{A}$ is not locally free this leads to some well-known difficulties, for example the saturation of $\pi^\star \mathscr{A}$ in Ω_X^p is not always pseudoeffective [9, 16]. Therefore we consider forms with logarithmic poles along the exceptional divisor E of the resolution π, in particular we obtain that the saturation in $\Omega_X^p(\log E)$ is pseudoeffective, cf. Corollary 13.

This leads us to the following problem:

Question 3. *Let (X, ω_X) be a compact Kähler manifold, and let $E = \sum E_i$ be a snc divisor. Let (L, h) be a holomorphic line bundle on X where h is a possibly singular metric such that $i\Theta_h(L) \geqslant 0$ on X in the sense of currents. Let $(L^\star, h^\star)$ be the dual metric.*

Let $u \in H^0(X, \Omega_X^p(\log E) \otimes L^\star)$. Can we prove that S_u is a holomorphic foliation and $D'_{h^\star} u = 0$ on $X \setminus E$?

If $p = 1$, the problem is totally solved in [19, Thm. 5][1]. It is still open when $p \geqslant 2$. We give a positive answer to this question when the metric h is smooth (Proposition 5). Our main technical result (Proposition 8) gives a positive answer making an assumption on the singularity of h along

[1]We thank Stéphane Druel and Daniel Greb for bringing this reference to our attention.

certain irreducible components E_i. This integrability condition can be verified for a resolution of singularities $X \to Y$ of a klt space, thereby establishing the first part of Theorem 2. When $p = 1$, by using the techniques in our article, we can also give an alternative proof of [19, Thm. 5], cf. Proposition 10. This implies the second part of Theorem 2.

Patrick Graf indicated an alternative path of proof for the second part of Theorem 2: by [7, Thm. 1.4][2] a holomorphic 1-form on the smooth locus of a log-canonical space extends to a resolution, even without admitting logarithmic poles. Therefore we can copy the proof of Theorem 2 and verify the technical condition of Proposition 8. Note that [7, Thm. 1.6] gives an example of a 2-form on a log-canonical 3-fold that does not extend to a resolution unless we admit logarithmic poles. Therefore this approach does not allow to generalise the second part of Theorem 2 to forms in $(\Omega_Y^p \otimes \mathscr{A}^\star)^{\star\star}$ with $p \geqslant 2$.

As a first application, we can consider singular contact spaces, cp. [1, 18]: a normal compact Kähler space of dimension $2n+1$ with log-canonical singularities has a contact structure if there exists a reflexive subsheaf $\mathscr{F} \subset T_X$ of rank $2n$ such that on the smooth locus $X_{\mathrm{nons}} \subset X$,

- the inclusion $\mathscr{F} \subset T_X$ is an injective morphism of vector bundles; and
- the map
$$\wedge^2 \mathscr{F} \longrightarrow T_X/\mathscr{F}$$
induced by the Lie bracket is surjective. In particular $\mathscr{F} \subset T_X$ is not integrable.

If we set $L := (T_X/\mathscr{F})^{**}$, we obtain as in the smooth case that $\omega_X \simeq L^{[-(n+1)]}$. In particular some reflexive power of L is locally free.

Corollary 4. *Let X be a normal compact Kähler space with log-canonical singularities which admits a contact structure. Then the canonical sheaf ω_X is not pseudoeffective.*

Indeed ω_X is pseudoeffective if and only if L^* is pseudoeffective. Yet then we can apply Theorem 2 to the section of $(\Omega_X \otimes L)^{**}$ defined by the inclusion $L^* \to \Omega_X$ and obtain that its kernel $\mathscr{F} \subset T_X$ is integrable, a contradiction.

Since it is not clear if a singular contact space admits a resolution by a contact manifold, the corollary does not reduce to Demailly's theorem [3, Cor. 2].

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2. Notation and terminology

For general definitions in complex and algebraic geometry we refer to [4, 11], for the terminology of singularities of the MMP we refer to [13]. Manifolds and normal complex spaces will always be supposed to be irreducible.

For the convenience of the reader, let us recall the definition of klt (resp. log-canonical) singularities (cf. [13, Def. 2.34] for more details): let Y be a normal complex space such that some reflexive power $\omega_Y^{[m]}$ of the canonical sheaf ω_Y is locally free. Let $\mu : X \to Y$ be a resolution of

[2] The statement is formulated for algebraic varieties, but in view of [12] should hold for analytic spaces.

singularities such that the exceptional locus is a simple normal crossings divisor. Then we can write

$$\omega_X^m \simeq \mu^* \omega_Y^{[m]} \otimes \mathscr{O}_Y\Big(\sum b_i E_i\Big)$$

where the $E_i \subset Y$ are μ-exceptional prime divisors. The space Y has klt (resp. log-canonical) singularities if $\frac{b_i}{m} > -1$ (resp. $\frac{b_i}{m} \geqslant -1$) for all i.

Given a normal complex space Y, we denote by $\Omega_Y^{[p]} := (\Omega_Y^p)^{\star\star}$ the sheaf of holomorphic reflexive p-forms. If Y has klt singularities we know by [12, Thm. 1.1] that this coincides with the sheaf of holomorphic p-forms that extend to a resolution of singularities $f: X \to Y$, i.e. we have $f_* \Omega_X^p \simeq \Omega_Y^{[p]}$.

For a reflexive sheaf $\mathscr{F}$ on Y, we denote by $\mathscr{F}^{[m]} := (\mathscr{F}^{\otimes m})^{\star\star}$ the m-th reflexive power. Given a surjective morphism $\varphi: X \to Y$ we denote by $\varphi^{[\star]}\mathscr{F}$ the reflexive pull-back $(\varphi^\star \mathscr{F})^{\star\star}$.

3. Twisted logarithmic forms

Proposition 5. *Let X be a compact Kähler manifold, and let $E = \sum E_i$ be a snc divisor. Let (L,h) be a holomorphic line bundle on X where h is a smooth metric such that $i\Theta_h(L) \geqslant 0$. Let $u \in H^0(X, \Omega_X^p(\log E) \otimes L^\star)$ and $(L^\star, h^\star)$ be the dual metric on (L,h). Then $D'_{h^\star} u = 0$ on X and $i\Theta_h(L) \wedge u \wedge \overline{u} = 0$.*

Proof. If L is a trivial line bundle, it is done by [15]. We generalize it to the twisted setting by the following argument.

Step 1. Since h is a smooth metric, we know that $D'_{h^\star} u \in C^\infty(X, \Omega_X^{p+1}(\log E) \otimes L^\star)$. We show in this step that $D'_{h^\star} u \in C^\infty(X, \Omega_X^{p+1} \otimes L^\star)$.

We consider the residue of u and $D'_{h^\star} u$ on E_i. First of all, by a direct calculation, we have

$$\mathrm{Res}_{E_i}(D'_{h^\star} u) = -D'_{h^\star} \mathrm{Res}_{E_i}(u) \qquad \text{on } E_i. \tag{1}$$

In fact, let Ω be a neighborhood of a generic point of E_i. We suppose that E_i is defined by $z_1 = 0$ and $h = e^{-\varphi}$ on Ω. Then we can write

$$u = \frac{\mathrm{d}z_1}{z_1} \wedge f + g$$

for two smooth forms f, g on Ω.

For the RHS of (1), since $\mathrm{Res}_{E_i}(u) = f$ and we obtain

$$-D'_{h^\star} \mathrm{Res}_{E_i}(u) = -(\partial f + \partial\varphi \wedge f)|_{E_i}.$$

For the LHS of (1), we have

$$\mathrm{Res}_{E_i}(D'_{h^\star} u) = \mathrm{Res}_{E_i}\left(D'_{h^\star}\left(\frac{\mathrm{d}z_1}{z_1} \wedge f\right)\right) = \mathrm{Res}_{E_i}\left(-\frac{\mathrm{d}z_1}{z_1} \wedge \partial f + \partial\varphi \wedge \frac{\mathrm{d}z_1}{z_1} \wedge f\right) = -(\partial f + \partial\varphi \wedge f)|_{E_i}.$$

Then we obtain (1).

Note that $\mathrm{Res}_{E_i}(u) \in H^0(E_i, \Omega_{E_i}^{p-1}(\log(E - E_i)) \otimes L^\star)$. By induction on dimension, we know that $\mathrm{Res}_{E_i}(u)$ is $D'_{h^\star}$-closed on E_i. Then (1) implies that $\mathrm{Res}_{E_i}(D'_{h^\star} u) = 0$. Therefore the form $D'_{h^\star} u$ is a smooth form on the total space X.

Step 2. Let $N \in \mathbb{N}^\star$ and let $\Xi_N(x)$ be a smooth function which equals to 1 on $[0,N]$, equals to 0 on $[N+1, \infty]$ and $0 \leqslant \Xi'_N(x) \leqslant 1$. Let s_E be the canonical section of E. We consider the integration

$$\int_X \Xi_N(\log(-\log|s_E|))\{D'_{h^\star} u, D'_{h^\star} u\} \wedge \omega_X^{n-p-1}. \tag{2}$$

Here $|s_E|$ denotes the norm of s_E with respect to a fixed smooth metric on E such that $|s_E| < 1$ everywhere.

By integration by parts, (2) equals to

$$= \int_X \{D'_{h^\star}(\Xi_N(\log(-\log|s_E|))u), D'_{h^\star}u\} \wedge \omega_X^{n-p-1} - \int_X \{\partial(\Xi_N(\log(-\log|s_E|))) \wedge u, D'_{h^\star}u\} \wedge \omega_X^{n-p-1}$$

$$= -\int_X (-1)^p \Xi_N(\log(-\log|s_E|))\{u, \bar\partial(D'_{h^\star}u)\} \wedge \omega_X^{n-p-1} - \int_X \{\partial(\Xi_N(\log(-\log|s_E|))) \wedge u, D'_{h^\star}u\} \wedge \omega_X^{n-p-1}$$

$$= -\int_X i\Theta_h(L)\Xi_N \cdot \{u,u\} \wedge \omega_X^{n-p-1} - \int_X \left\{ \frac{\Xi'_N \cdot \partial \log|s_E| \wedge u}{\log|s_E|}, D'_{h^\star}u \right\} \wedge \omega_X^{n-p-1}. \tag{3}$$

Since $i\Theta_h(L) \geqslant 0$, the first term of (3) is semi-negative. For the second term of (3), by Step 1, we know that $D'_{h^\star}u$ is smooth on X. Together with $\frac{ds_{E_i}}{s_{E_i}\log|s_{E_i}|} \wedge \frac{ds_{E_i}}{s_{E_i}} = 0$, we know that the second term of (3) is controlled by

$$\int_{N \leqslant \log(-\log|s_E|) \leqslant N+1} \frac{1}{\prod_i |s_{E_i}|} \omega_X^n,$$

which converges to zero when $N \to 0$.

As a consequence, when $N \to +\infty$, the upper limit of (3) will not be strictly positive. Since (2) is always positive, we obtain

$$\lim_{N \to +\infty} \int_X \Xi_N(\log(-\log|s_E|))\{D'_{h^\star}u, D'_{h^\star}u\} \wedge \omega_X^{n-p} = 0. \tag{4}$$

Therefore $D'_{h^\star}u = 0$ on X. □

Remark 6. For the convenience of the reader let us recall why $D'_{h^\star}u = 0$ implies that (L,h) has flat curvature along the generic leaf, following [3, Main thm]. Let $x \in X$ be a general point and fix a holomorphic base e_L of L near x. Then the metric h is written locally as $h = e^{-\varphi}$. In these local coordinates, $D'_{h^\star}u = 0$ means that $\partial\varphi \wedge u = -\partial u$. By taking the $\bar\partial$, we obtain $dd^c\varphi \wedge u = 0$. Now we suppose that the leaves of the foliation near the generic point x is given by

$$z_1 = c_1, z_2 = c_2, \ldots, z_r = c_r$$

where the c_i are constants. Then u depends only on $dz_1, \ldots, dz_r$ near x. Therefore the condition $dd^c\varphi \wedge u = 0$ implies that $\frac{\partial^2\varphi}{\partial z_j \partial \bar z_k} = 0$ for $j,k > r$. In other words, (L,h) is flat along the generic leaf.

Remark 7. By a standard argument, it is easy to generalize the above proposition to the case when the metric (L,h) is of analytic singularity. However, it is unclear whether we can generalize it to the case of arbitrary singularity cf. Question 3.

In the rest of the section, we will confirm Question 3 in two special cases.

Proposition 8. *Let (X,ω_X) be a compact Kähler manifold, and let $E = \sum_{i=1}^r E_i$ be a snc divisor. Let (L,h) be a holomorphic line bundle on X where h is a possibly singular metric such that $i\Theta_h(L) \geqslant 0$ on X in the sense of currents. Let $(L^\star, h^\star)$ be the dual metric. Let $u \in H^0(X, \Omega_X^p(\log E) \otimes L^\star)$. We assume that $\mathrm{Res}_{E_i}(u) \neq 0$ for every $1 \leqslant i \leqslant k$ and $\mathrm{Res}_{E_i}(u) = 0$ for every $k < i \leqslant r$.*

We write $h = e^{-\varphi} \cdot h_0$, where φ is a quasi-psh function on X and h_0 is a smooth metric on L. If the weight function φ satisfies:

$$\varphi \leqslant -2\sum_{i=1}^k \ln(-\ln|s_{E_i}|) + C, \tag{5}$$

where s_{E_i} is the canonical section of E_i, then $D'_{h^\star}u = 0$ and $i\Theta_h(L) \wedge u \wedge \bar u = 0$ on $X \setminus E$, where $D'_{h^\star}$ is the connection with respect to $h^\star$.

Remark 9. Note that if the Lelong number of φ along E_i is strictly positive for every $i \leqslant k$, then φ satisfies the condition (5).

Proof. The proof is divided into two steps.

Step 1. Let $N \in \mathbb{N}^\star$ and let $\Xi_N(x)$ be a smooth function which equals to 1 on $[0, N]$, equals to 0 on $[N+1, \infty]$ and $0 \leqslant \Xi'_N(x) \leqslant 1$. We consider the integration

$$\int_X \Xi_N^2(\log(\log(-\log|s_E|)))\{D'_{h^\star} u, D'_{h^\star} u\} \wedge \omega_X^{n-2}. \tag{6}$$

Since $D'_{h^\star} u$ is L^2 in the support of $\Xi_N(\log(\log(-\log|s_E|)))$, we can still do the integration by parts as in [3]. In particular, (6) equals to

$$= \int_X \{D'_{h^\star}(\Xi_N^2(\log(\log(-\log|s_E|)))u), D'_{h^\star} u\} \wedge \omega_X^{n-2} - \int_X \{\partial(\Xi_N^2(\log(\log(-\log|s_E|))) \wedge u, D'_{h^\star} u\} \wedge \omega_X^{n-2}$$

$$= -\int_X i\Theta_h(L)\Xi_N^2(\log(-\log|s_E|))\{u, u\} \wedge \omega_X^{n-2} - \int_X \left\{ \frac{2 \cdot \Xi'_N \cdot \partial \log|s_E| \wedge u}{\log(-\log|s_E|)\log|s_E|}, \Xi_N \cdot D'_{h^\star} u \right\} \wedge \omega_X^{n-2}. \tag{7}$$

Since $i\Theta_h(L) \geqslant 0$, the first term of (7) is semi-negative. For the second term of (7), by using Cauchy inequality, we get

$$\left| \int_X \left\{ \frac{\Xi'_N \cdot \partial \log|s_E| \wedge u}{\log(-\log|s_E|)\log|s_E|}, \Xi_N \cdot D'_{h^\star} u \right\} \wedge \omega_X^{n-2} \right|^2$$
$$\leqslant \int_X \Xi_N^2 \{D'_{h^\star} u, D'_{h^\star} u\} \wedge \omega_X^{n-2} \cdot \int_X \left\{ \frac{\Xi'_N \cdot \partial \log|s_E| \wedge u}{\log(-\log|s_E|)\log|s_E|}, \frac{\Xi'_N \cdot \partial \log|s_E| \wedge u}{\log(-\log|s_E|)\log|s_E|} \right\} \wedge \omega_X^{n-2}.$$

As a consequence, we obtain

$$\int_X \Xi_N^2 \cdot \{D'_{h^\star} u, D'_{h^\star} u\} \wedge \omega_X^{n-2} \leqslant \int_X \left\{ \frac{\Xi'_N \cdot \partial \log|s_E| \wedge u}{\log(-\log|s_E|)\log|s_E|}, \frac{\Xi'_N \cdot \partial \log|s_E| \wedge u}{\log(-\log|s_E|)\log|s_E|} \right\} \wedge \omega_X^{n-2} \tag{8}$$

Step 2. In this step, we would like to show the RHS of (8) tends to zero when $N \to +\infty$.

Since $\frac{ds_{E_i}}{s_{E_i}} \wedge \frac{ds_{E_i}}{s_{E_i}} = 0$, the assumption (5) implies that $\{\partial \log|s_E| \wedge u, \partial \log|s_E| \wedge u\} \wedge \omega_X^{n-2}$ is upper bounded by

$$C' \frac{\omega_X^n}{\prod_{i=1}^k |s_{E_i}|^2 \log^2|s_{E_i}|} \cdot \left(\sum_{i=k+1}^r \frac{1}{|s_{E_i}|^2} \right)$$

for some constant C'. Then the RHS of (8) is controlled by

$$C' \sum_{i=k+1}^r \int_X \frac{(\Xi'_N)^2 \omega_X^n}{\prod_{i=1}^k |s_{E_i}|^2 \log^2|s_{E_i}|} \cdot \frac{1}{|s_{E_i}|^2 \log^2|s_{E_i}|}. \tag{9}$$

which converges to zero when $N \to 0$. As a consequence, the RHS of (8) tends to zero when $N \to +\infty$. Therefore $D'_{h^\star} u = 0$ on $X \setminus E$. □

By using the argument in Proposition 8, we can give an alternative proof of [19, Thm. 5]:

Proposition 10. *Let X be a compact Kähler manifold, and let $E = \sum E_i$ be a snc divisor. Let (L, h) be a holomorphic line bundle on X where h is a possible singular metric such that $i\Theta_h(L) \geqslant 0$. Let $u \in H^0(X, \Omega_X^1(\log E) \otimes L^\star)$ and $(L^\star, h^\star)$ be the dual metric on (L, h). Then $D'_{h^\star} u = 0$ and $i\Theta_h(L) \wedge u \wedge \overline{u} = 0$ on $X \setminus E$.*

Proof. We follow the notations in Proposition 8. By the step 1 of Proposition (8), we know that

$$\int_X \Xi_N^2 \cdot \{D'_{h^\star} u, D'_{h^\star} u\} \wedge \omega_X^{n-2} \leqslant \int_X \left\{ \frac{\Xi'_N \cdot \partial \log|s_E| \wedge u}{\log(-\log|s_E|)\log|s_E|}, \frac{\Xi'_N \cdot \partial \log|s_E| \wedge u}{\log(-\log|s_E|)\log|s_E|} \right\} \wedge \omega_X^{n-2} \tag{10}$$

In order to prove the proposition, it is sufficient to show the RHS of (10) tends to zero when $N \to +\infty$.

Since $\frac{ds_{E_i}}{s_{E_i}} \wedge \frac{ds_{E_i}}{s_{E_i}} = 0$ and u is a 1-form, $\{\partial \log|s_E| \wedge u, \partial \log|s_E| \wedge u\} \wedge \omega_X^{n-2}$ is upper bounded by

$$C \cdot \sum_{i \neq j} \frac{\omega_X^n}{|s_{E_i} s_{E_j}|^2}.$$

Then the RHS (10) is controlled by

$$C\sum_{i\neq j}\int_X \frac{(\Xi'_N)^2\omega_X^n}{\log^2(-\log|s_E|)\log^2|s_E|\cdot|s_{E_i}s_{E_j}|^2}. \tag{11}$$

Note that the integral

$$\int_{0\leqslant r_1,r_2\leqslant 1}\frac{\mathrm{d}r_1\wedge\mathrm{d}r_2}{\log^2(-\log|r_1r_2|)\log^2|r_1r_2|\cdot r_1r_2}<+\infty.$$

Therefore (11) converges to zero when $N\to 0$. As a consequence, the RHS of (10) tends to zero when $N\to+\infty$. Therefore $D'_{h^\star}u=0$ on $X\setminus E$. □

4. Lifting subsheaves to the resolution

Let Y be a normal complex space with klt singularities, and let $\nu: Y'\to Y$ be a proper surjective morphism from a normal complex space Y'. Since klt singularities are rational [13, Thm. 5.22], by [12, Thm. 1.10] there exists for every $p\in\mathbb{N}$ a cotangent map

$$\mathrm{d}\nu:\nu^\star\Omega_Y^{[p]}\longrightarrow\Omega_{Y'}^{[p]} \tag{12}$$

If Y has log-canonical singularities we can still combine the proof of [8, Thm. 4.3] with [12, Thm. 1.5] to obtain[3] that there exists for every $p\in\mathbb{N}$ a cotangent map

$$\mathrm{d}\nu:\nu^\star\Omega_Y^{[p]}\longrightarrow\Omega_{Y'}^{[p]}(\log\Delta) \tag{13}$$

where $\Delta\subset Y'$ is the largest reduced Weil divisor contained in ν^{-1}(non-klt locus).

The following statement is well-known to experts and essentially a rewriting of the proof of [8, Thm. 7.2]. We include it for the convenience of the reader:

Lemma 11. *Let Y be a normal complex space with log-canonical singularities, and let $\mathscr{A}\subset\Omega_Y^{[p]}$ be a reflexive subsheaf of rank one that is $\mathbb{Q}$-Cartier, i.e. there exists a $m\in\mathbb{N}$ such that $\mathscr{A}^{[m]}$ is locally free.*

Let $\pi: X\to Y$ be a log resolution, and let E be the exceptional divisor. Let $\mathscr{C}\subset\Omega_X^p(\log E)$ be the saturation of the image of the morphism

$$\pi^\star\mathscr{A}\longrightarrow\pi^\star\Omega_Y^{[p]}\xrightarrow{\mathrm{d}\pi}\Omega_X^p(\log E).$$

Then there exists a non-zero morphism $\pi^\star\mathscr{A}^{[m]}\to\mathscr{C}^{\otimes m}$.

Remark. The morphism $\pi^\star\mathscr{A}^{[m]}\to\mathscr{C}^{\otimes m}$ is an isomorphism in the complement of the exceptional divisor E. Thus, up to multiplication by a holomorphic function that is a pull-back from Y, the morphism is unique.

If Y has klt singularities, we could use (12) and consider $\mathscr{C}'\subset\Omega_X^p$, the saturation of the image of the morphism

$$\pi^\star\mathscr{A}\longrightarrow\pi^\star\Omega_Y^{[p]}\xrightarrow{\mathrm{d}\pi}\Omega_X^p,$$

but in general there will be no morphism $\pi^\star\mathscr{A}^{[m]}\to(\mathscr{C}')^{\otimes m}$. However, in the course of the proof of Lemma 11 we will prove the following remark that will be useful for the proof of Proposition 14:

Remark 12. If Y is klt, let $\widetilde{\gamma}:\widetilde{Z}\to X$ be the cover induced by a (local) index-one cover $\gamma: Z\to Y$ of $\mathscr{A}$ (cf. Diagram (14)). Then $\pi_Z^\star\gamma^\star\mathscr{A}^{[m]}$ is a subsheaf of $S^{[m]}\Omega_{\widetilde{Z}}^{[p]}$.

[3] Note that [12, Thm. 1.10] holds for any morphism, while we only need the simpler case where the morphism is surjective.

For the proof let us recall the notion of index one covers [13, Def. 5.19]: given a normal complex space Y and a reflexive sheaf $\mathscr{A}$ such that some reflexive power $\mathscr{A}^{[m]}$ is trivial, there exists a quasi-étale morphism $\gamma : Z \to Y$ from a normal complex space Z such that the reflexive pull-back $\gamma^{[\star]}\mathscr{A}$ is isomorphic to $\mathscr{O}_Z$.

Proof of Lemma 11. The locally free sheaves coincide in the complement of the exceptional locus $E = \bigcup_i E_i$, so we can write $\mathscr{C}^{\otimes m} \simeq \pi^\star \mathscr{A}^{[m]} \otimes \mathscr{O}_X(\sum a_i E_i)$ with uniquely determined $a_i \in \mathbb{Z}$. We are done if we show that $a_i \geqslant 0$ for all i. This property can be checked locally on the base Y.

Therefore we can replace Y by a Stein neighborhood such that there exists an index-one cover $\gamma : Z \to Y$, and let $\tilde{\gamma} : \tilde{Z} \to X$ be the induced finite map from the normalisation $\tilde{Z}$ of $X \times_Y Z$. We denote by $\pi_Z : \tilde{Z} \to Z$ the bimeromorphic morphism induced by π and summarize the construction in a commutative diagram:

$$\begin{tikzcd} \tilde{Z} \arrow[r, "\tilde{\gamma}"] \arrow[d, "\pi_Z"'] & X \arrow[d, "\pi"] \\ Z \arrow[r, "\gamma"] & Y \end{tikzcd} \tag{14}$$

The morphism $\gamma : Z \to Y$ is an index-one cover for $\mathscr{A}$, so γ is étale in codimension one and $\gamma^{[\star]}\mathscr{A} =: \mathscr{B}$ is locally free. In particular Z still has log-canonical singularities [13, Prop. 5.20 (4)]. Denote the exceptional locus of π_Z by E_Z and observe that E_Z is equal to the support of $\tilde{\gamma}^\star E$. In particular E_Z contains the preimage of the non-klt locus of Z, so (13) gives a natural map

$$d\pi_Z : \pi_Z^\star \Omega_Z^{[p]} \longrightarrow \Omega_{\tilde{Z}}^{[p]}(\log E_Z)$$

Since $\mathscr{A} \subset \Omega_Y^{[p]}$ and γ is étale in codimension one we have an inclusion $\mathscr{B} \subset \Omega_Z^{[p]} \simeq \gamma^{[\star]}\Omega_Y^{[p]}$ and hence an induced map

$$\pi_Z^\star \mathscr{B} \longrightarrow \pi_Z^\star \Omega_Z^{[p]} \longrightarrow \Omega_{\tilde{Z}}^{[p]}(\log E_Z).$$

Since $\mathscr{B}$ is locally free, this induces an inclusion

$$\pi_Z^\star \mathscr{B}^{\otimes m} \simeq (\pi_Z^\star \mathscr{B})^{\otimes m} \longrightarrow S^{[m]}\Omega_{\tilde{Z}}^{[p]}(\log E_Z). \tag{15}$$

By assumption $A^{[m]}$ is locally free, so its (non-reflexive !) pull-back $\gamma^\star \mathscr{A}^{[m]}$ is still locally free. Thus $B^{\otimes m} \simeq \gamma^\star A^{[m]}$ since they are both reflexive and coincide in codimension one. Thus we have constructed a morphism

$$\pi_Z^\star \gamma^\star A^{[m]} \longrightarrow S^{[m]}\Omega_{\tilde{Z}}^{[p]}(\log E_Z).$$

We interrupt the proof of the lemma for the *Proof of Remark 12.*

If Y is klt, the index one cover Z also has klt singularities [13, Prop. 5.20 (4)]. Thus we can replace the pull-back with logarithmic poles (13) by the usual pull-back (12) to obtain

$$d\pi_Z : \pi_Z^\star \Omega_Z^{[p]} \longrightarrow \Omega_{\tilde{Z}}^{[p]}$$

As above the inclusion $\gamma^{[\star]}\mathscr{A} \simeq \mathscr{B} \subset \Omega_Z^{[p]} \simeq \gamma^{[\star]}\Omega_Y^{[p]}$ then gives the inclusion

$$\pi_Z^\star \gamma^\star \mathscr{A}^{[m]} \simeq \pi_Z^\star \mathscr{B}^{\otimes m} \simeq (\pi_Z^\star \mathscr{B})^{\otimes m} \longrightarrow S^{[m]}\Omega_{\tilde{Z}}^{[p]}.$$

This proves Remark 12, we now proceed with the proof of Lemma 11.

Since X is smooth, the saturated subsheaf $\mathscr{C} \subset \Omega_X^p(\log E)$ is locally free and a subbundle in codimension one. Thus

$$\mathscr{C}^{\otimes m} \subset S^m \Omega_X^p(\log E) \tag{16}$$

is locally free and a subbundle in codimension one, hence a saturated subsheaf. The finite morphism $\tilde{\gamma}$ is étale in the complement of E and $\Omega_X^p(\log E)$ is locally free, so the tangent map gives an isomorphism

$$\tilde{\gamma}^\star \Omega_X^p(\log E) \simeq \Omega_{\tilde{Z}}^{[p]}(\log E_Z). \tag{17}$$

and hence an isomorphism

$$\widetilde{\gamma}^\star S^m\Omega_X^p(\log E) \simeq S^{[m]}\Omega_{\widetilde{Z}}^{[p]}(\log E_Z).$$

Composing the inclusion (16) with this isomorphism we obtain that

$$\widetilde{\gamma}^\star \mathscr{C}^{\otimes m} \longrightarrow S^{[m]}\Omega_{\widetilde{Z}}^{[p]}(\log E_Z)$$

is a saturated subsheaf.

Since Y is Stein and $\mathscr{A}^{[m]}$ is invertible we can choose for every point $y \in Y$ a section $\sigma \in H^0(Y, \mathscr{A}^{[m]})$ that does not vanish in y. In particular σ generates $\mathscr{A}^{[m]}$ as an $\mathscr{O}_Y$-module near the point y. Thus it induces a section

$$\pi_Z^\star\gamma^\star\sigma \in H^0(\widetilde{Z}, S^{[m]}\Omega_{\widetilde{Z}}^{[p]}(\log E_Z))$$

that generates the image of $\pi_Z^\star\gamma^\star\mathscr{A}^{[m]}$. The pull-back $\pi^\star\sigma$ defines a meromorphic section of $\mathscr{C}^{\otimes m}$ that has poles at most along E, thus $\widetilde{\gamma}^\star\pi^\star\sigma$ defines a meromorphic section of $\widetilde{\gamma}^\star\mathscr{C}^{\otimes m}$ that has poles at most along E_Z. Since $\widetilde{\gamma}^\star\mathscr{C}^{\otimes m}$ is saturated in $S^{[m]}\Omega_{\widetilde{Z}}^{[p]}(\log E_Z)$ and

$$\pi_Z^\star\gamma^\star\sigma = \widetilde{\gamma}^\star\pi^\star\sigma \in H^0(\widetilde{Z}, S^{[m]}\Omega_{\widetilde{Z}}^{[p]}(\log E_Z))$$

has no poles, we see that

$$\widetilde{\gamma}^\star\pi^\star\sigma \in H^0(\widetilde{Z}, \widetilde{\gamma}^\star\mathscr{C}^{\otimes m}).$$

Thus the local generator of the subsheaf $\pi_Z^\star\gamma^\star\mathscr{A}^{[m]}$ lies in $\widetilde{\gamma}^\star\mathscr{C}^{\otimes m}$ and we have an inclusion

$$\widetilde{\gamma}^\star\pi^\star\mathscr{A}^{[m]} \simeq \pi_Z^\star\gamma^\star\mathscr{A}^{[m]} \hookrightarrow \widetilde{\gamma}^\star\mathscr{C}^{\otimes m}.$$

Thus we see that

$$\widetilde{\gamma}^\star\mathscr{O}_X(\sum a_i E_i) \simeq \widetilde{\gamma}^\star(\mathscr{C}^{\otimes m} \otimes \pi^\star\mathscr{A}^{[-m]})$$

is represented by an effective divisor with support in the exceptional locus of π_Z. Since $\widetilde{\gamma}^\star(\sum a_i E_i)$ is linearly equivalent to an effective, exceptional divisor and has also support in the exceptional locus of π_Z, it is effective. Thus we have shown that $a_i \geqslant 0$ for all i. $\square$

As in immediate application we obtain a variant of [8, Thm. 7.2], [6, Cor. 1.3] for pseudoeffective line bundles.

Corollary 13. *Let Y be a normal compact complex space with log-canonical singularities, and let $\mathscr{A} \subset \Omega_Y^{[p]}$ be a reflexive subsheaf of rank one that is $\mathbb{Q}$-Cartier, i.e. there exists a $m \in \mathbb{N}$ such that $\mathscr{A}^{[m]}$ is locally free. Let $\mathscr{C} \subset \Omega_X^p(\log E)$ be the saturation of $\pi^\star\mathscr{A}$. If $\mathscr{A}^{[m]}$ is pseudoeffective, then $\mathscr{C}$ is pseudoeffective.*

Proof. Since pseudoeffectivity of a line bundle is invariant under taking tensor powers, it is sufficient to show that $\mathscr{C}^{\otimes m}$ is pseudoeffective. Yet this follows from the non-zero morphism $\pi^*\mathscr{A}^{[m]} \to \mathscr{C}^{\otimes m}$ constructed in Lemma 11. $\square$

We need the following proposition.

Proposition 14. *In the situation of Lemma 11, write*

$$\mathscr{C}^{\otimes m} = \pi^\star\mathscr{A}^{[m]} \otimes \mathscr{O}_X\Big(\sum a_i E_i\Big), \tag{18}$$

where $a_i \geqslant 0$ and $E = \sum E_i$ is the exceptional locus.

Assume that Y has klt singularities, and let E_i be an irreducible component of the exceptional locus. Let $\mathrm{Res}_{E_i}(\mathscr{C})$ be the residue of the image of $\mathscr{C}$ in $\Omega_X^p(\log E)$. If $\mathrm{Res}_{E_i}(\mathscr{C}) \neq 0$, then $a_i > 0$.

Proof. The claim is local on Y, so we will use the construction from the proof of Lemma 11 summarized in the commutative diagram (14).

Fix a prime divisor $\widetilde{E}_i \subset \widetilde{Z}$ that maps onto $E_i \subset X$, and choose a general point $\tilde{x} \in \widetilde{E}_i \cap \widetilde{Z}_{\text{nons}}$ such that $\widetilde{E}_i$ (resp. E_i) is smooth in $\tilde{x}$ (resp. smooth in $x := \widetilde{\gamma}(\tilde{x})$). Since $\tilde{x}$ is general, the finite morphism $\widetilde{\gamma}$ has constant rank in an analytic neighborhood of $\widetilde{\gamma}$, hence we can find local coordinates on $\widetilde{Z}$ and X such that

$$E_i = \{z_1 = 0\}$$

and $\widetilde{\gamma}$ is given locally by

$$\widetilde{\gamma} : (t, z_2, ..z_n) \longrightarrow (t^d, z_2, .., z_n).$$

The exterior power $\Omega_X^p(\log E)_x$ is generated by $\left\{\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_J, \mathrm{d}z_I\right\}$ where $J \subset \{2, \ldots, n\}$ has length $p-1$ and $I \subset \{2, \ldots, n\}$ has length p. Thus we obtain a basis $\{e_1, \ldots, e_k\}$ of $S^m\Omega_X(\log E)_x$ by taking products of length m, where each e_i is of type:

$$e_i = \left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_1}\right) \otimes \left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_2}\right) \otimes \cdots \otimes \left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_q}\right) \otimes \mathrm{d}z_{I_1} \otimes \cdots \otimes \mathrm{d}z_{I_{m-q}}.$$

In our local coordinates the pull-back becomes

$$\widetilde{\gamma}^\star(e_i) = \left(\frac{\mathrm{d}t}{t} \wedge \mathrm{d}z_{J_1}\right) \otimes \left(\frac{\mathrm{d}t}{t} \wedge \mathrm{d}z_{J_2}\right) \otimes \cdots \otimes \left(\frac{\mathrm{d}t}{t} \wedge \mathrm{d}z_{J_q}\right) \otimes \mathrm{d}z_{I_1} \otimes \cdots \otimes \mathrm{d}z_{I_{m-q}}.$$

In particular, the pull back $\{\widetilde{\gamma}^\star(e_i)\}_{i=1}^k$ is a basis of $S^m\Omega_{\widetilde{Z}}(\log E_Z)$ at $\tilde{x}$.

Let σ be a generator of $\mathscr{A}^{[m]}$ at $\pi(x) \in Y$. Then $\pi^\star\sigma \in \pi^\star\mathscr{A}^{[m]} \subset S^m\Omega_X(\log E)$ is a local generator near x. We can write

$$\pi^\star\sigma = \sum f_i e_i,$$

where f_i are holomorphic functions near x. Now recall that by Remark 12

$$\pi_Z^\star\mathscr{B}^{\otimes m} \simeq \pi_Z^\star\gamma^\star\mathscr{A}^{[m]} \simeq \widetilde{\gamma}^\star\pi^\star\mathscr{A}^{[m]}$$

is a subsheaf of $S^{[m]}\Omega_{\widetilde{Z}}^{[p]}$. In particular, since $\widetilde{Z}$ is smooth in $\tilde{x}$, we have

$$(\widetilde{\gamma} \circ \pi)^\star\sigma \in (S^m\Omega_{\widetilde{Z}}^p)_{\tilde{x}}.$$

As a consequence, $f_i(x) = 0$ when e_i is of type

$$e_i = \left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_1}\right) \otimes \left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_2}\right) \otimes \cdots \otimes \left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_m}\right),$$

since this generator of $(S^m\Omega_{\widetilde{Z}}^p(\log E_Z))_{\tilde{x}}$ is not contained in $(S^m\Omega_{\widetilde{Z}}^p)_{\tilde{x}}$.

Now we can prove the proposition. Near a general point $x \in E_i$, we suppose that $\mathscr{C}_x \subset (\Omega_{\widetilde{Z}}^p)_{\tilde{x}}$ is generated by

$$\sum g_i \cdot \left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_i}\right) + \sum h_i \cdot \mathrm{d}z_{I_i},$$

where g_i, h_i are holomorphic functions. Thanks to Lemma 11, we have

$$F \cdot \left(\sum g_i\left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_i}\right) + \sum h_i \mathrm{d}z_{I_i}\right)^{\otimes m} = \left(\sum f_i e_i\right),$$

where F is a holomorphic function near x. If $\mathrm{Res}_{E_i}(\mathscr{C}) \neq 0$, we know that there is one i_0 such that $g_{i_0}(x) \neq 0$. Set

$$e_{i_0} := \left(\frac{\mathrm{d}z_1}{z_1} \wedge \mathrm{d}z_{J_{i_0}}\right)^{\otimes m}.$$

Then $F \cdot g_{i_0}^m = f_{i_0}$. By the above paragraph, we know that $f_{i_0}(x) = 0$. Then $F(x) = 0$. The proposition is thus proved. □

We are now in the position to verify the technical condition in Proposition 8:

Theorem 15. *In the setting of Theorem 2, let $\pi : X \to Y$ be a log-resolution and denote by E the exceptional locus. Let $L \subset \Omega_X^p(\log E)$ be the saturation of $\pi^\star \mathscr{A}$, and let $\widetilde{u} \in H^0(X, \Omega_X^p(\log E) \otimes L^\star)$ the corresponding section. Then there exists a metric h_1 on L such that we have $D'_{h_1^\star} \widetilde{u} = 0$ on $X \setminus E$*

Proof. By Lemma 11, we know that

$$c_1(L) = \frac{1}{m}\pi^\star c_1(\mathscr{A}^{[m]}) + \sum_{i\in I} a_i E_i + \sum_{i\in I'} a_i E_i, \tag{19}$$

such that all the coefficients $a_i \geqslant 0$ and the $i \in I$ correspond to the exceptional divisors E_i such that $\mathrm{Res}_{E_i}(\mathscr{C}) \neq 0$ and $i \in I'$ corresponds to $\mathrm{Res}_{E_i}(\mathscr{C}) = 0$. By Proposition 14 we have $a_i > 0$ when $i \in I$. Let h_0 be a possibly singular metric on $\pi^\star \mathscr{A}^{[m]}$ such that $i\Theta_{h_0}(\pi^\star \mathscr{A}^{[m]}) \geqslant 0$. By (19) this induces a metric h_1 on L. Thanks to Proposition 8, the theorem is proved. □

5. Proof of the main result

The setup for the proof of Theorem 2 is as follows: the non-zero section u determines an injective morphism of sheaves

$$\mathscr{A} \hookrightarrow \Omega_Y^{[p]}.$$

Let $\pi : X \to Y$ be a log-resolution of Y, and denote by E the exceptional locus. Since Y is log-canonical, we have the tangent map (13)

$$\mathrm{d}\pi : \pi^\star \Omega_Y^{[p]} \longrightarrow \Omega_X^p(\log E),$$

and we denote by $L \subset \Omega_X^p(\log E)$ the saturation of $\pi^\star \mathscr{A}$. By Lemma 11 there exists a morphism $\pi^\star \mathscr{A}^{[m]} \to L^{\otimes m}$, so L is a pseudoeffective line bundle on X. The inclusion $L \subset \Omega_X^p(\log E)$ corresponds to a non-zero holomorphic section

$$\widetilde{u} \in H^0(X, \Omega_X^p(\log E) \otimes L^\star)$$

which coincides with u on $X \setminus E \simeq Y_{\mathrm{nons}}$. In particular the subsheaf $S_{\widetilde{u}} \subset T_X$ defined by contraction with $\widetilde{u}$ coincides with $S_u \subset T_Y$ on a Zariski open set. Thus we are left to show the integrability of $S_{\widetilde{u}} \subset T_X$ on $X \setminus E$. By the formula for the exterior derivative of p-forms (cf. [3, p. 97]) the integrability of $S_{\widetilde{u}}$ follows if we find a metric h on L such that $D'_{h^\star} \widetilde{u} = 0$ on $X \setminus E$.

Assume that we are in the first case of Theorem 2: Since Y is klt, the existence of the metric h is guaranteed by Theorem 15.

Assume that we are in the second case of Theorem 2: Since $p = 1$ we know by Proposition 10 that any singular metric with positive curvature current will suffice. Since L is pseudoeffective, such a metric exists. □

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Complex algebraic geometry, in memory of Jean-Pierre Demailly /
Géométrie algébrique complexe, en mémoire de Jean-Pierre Demailly

Equality in the Miyaoka–Yau inequality and uniformization of non-positively curved klt pairs

Cas d'égalité de l'inégalité de Miyaoka–Yau et uniformisation des paires klt à courbure négative

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In memory of Jean-Pierre Demailly

Abstract. Let (X, Δ) be a compact Kähler klt pair, where $K_X + \Delta$ is ample or numerically trivial, and Δ has standard coefficients. We show that if equality holds in the orbifold Miyaoka–Yau inequality for (X, Δ), then its orbifold universal cover is either the unit ball (ample case) or the affine space (numerically trivial case).

Résumé. Soit (X, Δ) une paire klt compacte kählérienne pour laquelle $K_X + \Delta$ est ample ou numériquement trivial, et Δ à coefficients standard. Nous démontrons que, si l'inégalité de Miyaoka–Yau orbifold pour (X, Δ) est une égalité, alors le revêtement universel orbifold de la paire est soit la boule (cas ample), soit l'espace affine (cas numériquement trivial).

Keywords. Miyaoka–Yau inequality, orbifold uniformization, klt pairs.

Mots-clés. inégalité de Miyaoka–Yau, uniformisation orbifold, paires klt.

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1. Introduction

Let X be an n-dimensional compact Kähler manifold and let us assume that either

(I) K_X is ample (and X is thus projective), or
(II) K_X is numerically trivial (equivalently, $c_1(X) = 0$ in $H^2(X,\mathbb{R})$).

As a consequence of the existence of a Kähler–Einstein metric ω_{KE} on X (proved by Aubin [4] and Yau [43]), the Chern classes of X satisfy the *Miyaoka–Yau inequality*

$$\big(2(n+1)\,c_2(X) - n\,c_1^2(X)\big)\cdot\alpha^{n-2} \geq 0. \tag{MY}$$

where in case (I), we set $\alpha = [K_X]$, while in case (II), α can be an arbitrary Kähler class. Furthermore, in case of equality, the universal cover $\pi\colon \widetilde{X} \to X$ is (biholomorphic to)

(I) the n-dimensional unit ball $\mathbb{B}^n = \big\{(z_1,\dots,z_n)\in\mathbb{C}^n \;\big|\; |z_1|^2+\cdots+|z_n|^2 < 1\big\}$,
(II) the n-dimensional affine space $\mathbb{C}^n$.

We can reformulate the above conclusion by saying that

(I) $X = \mathbb{B}^n/\Gamma$ with $\Gamma \subset \mathrm{PU}(1,n) = \mathrm{Aut}(\mathbb{B}^n)$,
(II) $X = \mathbb{C}^n/\Gamma$ with $\Gamma \subset \mathbb{C}^n \rtimes \mathrm{U}(n) = \mathrm{Aut}(\mathbb{C}^n, \pi^*\omega_{KE})$,

where in both cases, the action of Γ on $\widetilde{X}$ is *fixed point-free*. Not surprisingly, there is a beautiful exposition of this circle of ideas by Jean-Pierre Demailly [18].

It seems natural to investigate the general case of quotients by cocompact lattices $\Gamma \subset \mathrm{Aut}(\widetilde{X})$ (with $\widetilde{X} = \mathbb{B}^n$ or $\mathbb{C}^n$ endowed with the Bergman metric or the flat metric, respectively), the action being of course assumed to be properly discontinuous. The corresponding quotients are then naturally endowed with an orbifold structure that can be encoded in the datum of a $\mathbb{Q}$-divisor with standard coefficients (see Setup 1 below). To sum up, it is natural to consider pairs (X,Δ) when dealing with these quotients.

The question of uniformizing *spaces* (as opposed to pairs) in the cases (I) and (II) has been considered in the framework of klt singularities. To quote a few relevant papers: [15, 24, 27, 28, 29, 30, 38]. This article grew out of an attempt to understand the general situation with an orbifold structure in codimension one.

Unfortunately, the parallels between cases (I) and (II) cannot be pursued throughout this introductory section since the difficulties (when dealing with the inequality (MY) in the singular setting) are not of the same nature. The following three facts illustrate this point:

- In case (I), the variety X is necessarily projective, but the codimension one part of the orbifold structure cannot be easily eliminated. Therefore we have to use orbifold techniques in the proof.
- In case (II), we also need to consider (non-algebraic) compact Kähler spaces, but we can get rid of the codimension one part of the orbifold structure via a cyclic covering (see Proposition 12). This enables us to assume that $\Delta = 0$ for most of the argument.
- In case (I), the Bergman metric is invariant under the full automorphism group of $\mathbb{B}^n$, but this is not true of the flat metric in case (II). Therefore (2) below does not have an analog in Corollary 7, although a conjecture due to Iitaka [32] (or rather an orbifold version thereof) predicts that this should in fact be true.

Due to this break in symmetry, we split the discussion according to the sign of the canonical bundle.

The canonically polarized case

Let us recall the singular version of the inequality (MY) as proven by the third-named author together with B. Taji [31]. When dealing with case (I), we work in the following setting:

Setup 1. Let (X, Δ) be an n-dimensional klt pair, where X is a projective variety and Δ has standard coefficients, i.e. $\Delta = \sum_{i \in I} \big(1 - \frac{1}{m_i}\big) \Delta_i$ with integers $m_i \geq 2$ and the Δ_i irreducible and pairwise distinct.

Theorem 2 ($\subset$ [31, Thm. B]). *Let (X, Δ) be as in Setup 1, and assume that $K_X + \Delta$ is big and nef. Assume additionally that every irreducible component Δ_i of Δ is $\mathbb{Q}$-Cartier. Then the following inequality holds:*

$$\big(2(n+1)\,\widetilde{c}_2(X,\Delta) - n\,\widetilde{c}_1^2(X,\Delta)\big) \cdot [K_X + \Delta]^{n-2} \geq 0. \tag{2}$$

Here, $\widetilde{c}_2(X,\Delta)$ and $\widetilde{c}_1^2(X,\Delta)$ denote the appropriate orbifold Chern classes of the pair (X,Δ), as defined e.g. in [31, Notation 3.7]. □

Remark. In the above theorem, the assumption that the Δ_i be $\mathbb{Q}$-Cartier is not necessary, and establishing this is one of the (minor) contributions of this paper, cf. Theorem 36. While this may seem like an innocuous technical issue at first sight, eliminating the $\mathbb{Q}$-Cartier assumption will become crucial below when deducing Corollary 4 from Theorem A, see Remark 38.

As in the smooth case, it is interesting to characterize geometrically those pairs that achieve equality in (2). In the case where $\Delta = 0$, this has been achieved in [29, Thm. 1.2] and [30, Thm. 1.5]: equality holds if and only if there is a finite quasi-étale Galois cover $Y \to X$ such that the universal cover of Y is the unit ball. An expectation concerning the general case was formulated in [29, §10.2]. Our first main result confirms this expectation.

Theorem A (Uniformization of canonical models). *Let (X, Δ) be as in Setup 1. Assume that $K_X + \Delta$ is ample and that equality holds in (2). Then the orbifold universal cover $\pi\colon \widetilde{X}_\Delta \to X$ of (X,Δ) is the unit ball (cf. Definition 24). More precisely, $(\widetilde{X}_\Delta, \widetilde{\Delta}) \cong (\mathbb{B}^n, \emptyset)$.*

In fact, a suitable converse of the above theorem also holds, and we obtain the following corollary.

Corollary 3 (Characterization of ball quotients). *Let (X, Δ) be as in Setup 1. The following are equivalent:*

1. *$K_X + \Delta$ is ample, and equality holds in (2).*
2. *The orbifold universal cover of (X, Δ) is the unit ball $\mathbb{B}^n$.*
3. *(X,Δ) admits a finite orbi-étale Galois cover $f\colon Y \to X$ (cf. Definition 8), where Y is a projective manifold whose universal cover is the unit ball.*

In the spirit of [30, Thm. 1.5], we can also prove the following uniformization statement for minimal pairs of log general type.

Corollary 4 (Uniformization of minimal models). *Let (X,Δ) be as in Setup 1. Assume that $K_X + \Delta$ is big and nef and that equality holds in (2). Then the canonical model $(X,\Delta)_{\mathrm{can}} =: (X_{\mathrm{can}}, \Delta_{\mathrm{can}})$ of the pair (X, Δ) is a ball quotient in the sense of Theorem A.*

The flat case

As mentioned earlier, Kähler quotients of $\mathbb{C}^n$ by cocompact groups of isometries are in general not projective, so we have to consider the following framework.

Setup 5. Let (X, Δ) be an n-dimensional klt pair, where X is a compact Kähler space and Δ has standard coefficients, i.e. $\Delta = \sum_{i \in I} \big(1 - \frac{1}{m_i}\big) \Delta_i$ with integers $m_i \geq 2$ and the Δ_i irreducible and pairwise distinct.

In this more general Kähler setting, the methods of [31] cannot be used to prove a singular analogue of the Miyaoka–Yau inequality. Instead, we rely on the Decomposition Theorem from [5] to deduce the following singular version of the inequality (MY) in case (II).

Theorem 6 (Singular Miyaoka–Yau inequality). *Let* (X,Δ) *be as in Setup 5 and assume that* $c_1(K_X+\Delta) = 0 \in H^2(X,\mathbb{R})$. *Let* $\alpha \in H^2(X,\mathbb{R})$ *be any Kähler class. We then have:*

$$\widetilde{c}_2(X,\Delta)\cdot\alpha^{n-2} \geq 0. \tag{3}$$

As before, we are particularly interested in what happens if equality is achieved.

Theorem B (Uniformization in the flat case). *Let* (X,Δ) *be as in Setup 5. Assume that* $c_1(K_X+\Delta) = 0 \in H^2(X,\mathbb{R})$ *and that equality holds in* (3) *for* some *Kähler class* α. *Then the orbifold universal cover* $\pi\colon \widetilde{X}_\Delta \to X$ *of* (X,Δ) *is the affine space (cf. Definition 24). More precisely,* $(\widetilde{X}_\Delta,\widetilde{\Delta}) \cong (\mathbb{C}^n,\emptyset)$.

As above, we can formulate a converse and get the following corollary.

Corollary 7 (Characterization of torus quotients). *Let* (X,Δ) *be as in Setup 5. The following are equivalent:*

(1) $c_1(K_X+\Delta) = 0 \in H^2(X,\mathbb{R})$, *and equality holds in* (3) *for some Kähler class* α.
(2) (X,Δ) *admits a finite orbi-étale Galois cover* $f\colon T \to X$ *(cf. Definition 8), where* T *is a complex torus.*

The previous statements are thus generalizations of [38, Thm. 1.2] (itself elaborating on [27, Thm. 1.17]). The generalization is threefold:

- Here X is a compact Kähler space, not necessarily projective.
- The class α is transcendental, a priori not an ample class.
- Ramification is allowed in codimension one; i.e. we work with klt pairs rather than klt spaces.

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2. Generalities on orbifolds

In this section, we consider Kawamata log terminal (klt) pairs (X,Δ) consisting of a normal algebraic variety or complex space X of dimension n and a $\mathbb{Q}$-divisor $\Delta = \sum_{i\in I}\left(1-\frac{1}{m_i}\right)\Delta_i$ on X, with $m_i \geq 2$.

2.1. *Orbi-structures and orbi-sheaves*

Most of the definitions and basic properties given below can be found in e.g. [31, §2] in the slightly more general setting of *dlt* pairs with standard coefficients, at least if X is algebraic. Working exclusively with klt pairs will simplify the exposition.

Definition 8 (Adapted morphisms). *Let* $f\colon Y \to X$ *be a finite surjective Galois morphism from a normal variety or complex space* Y. *One says that* f *is:*

- adapted *to* (X,Δ) *if for all* $i \in I$, *there exists* $a_i \in \mathbb{Z}^{\geq 1}$ *and a reduced divisor* Δ_i' *on* Y *such that* $f^*\Delta_i = a_i m_i \Delta_i'$,
- strictly adapted *to* (X,Δ) *if it is adapted and if* $a_i = 1$ *for all* $i \in I$,
- orbi-étale *if it is strictly adapted and the divisorial component of the branch locus of* f *is contained in* $\operatorname{supp}(\Delta)$. *Equivalently, if* f *is étale over* $X_{\mathrm{reg}} \setminus \operatorname{supp}(\Delta)$.

Remark. If X is compact, then a map $f\colon Y \to X$ as above is orbi-étale if and only if $K_Y = f^*(K_X + \Delta)$.

Definition 9 (Orbi-structures). *An* orbi-structure *for the pair* (X,Δ) *consists of a compatible collection of triples* $\mathscr{C} = \{(U_\alpha, f_\alpha, X_\alpha)\}_{\alpha\in J}$*, where* $(U_\alpha)_{\alpha\in J}$ *is a covering of* X *by étale-open subsets, and for each* $\alpha \in J$, $f_\alpha\colon X_\alpha \to U_\alpha$ *is an adapted morphism from a normal complex space* X_α *with respect to the pair structure on* U_α *induced by* (X,Δ)*. The compatibility condition means that for all* $\alpha,\beta \in J$*, the projection map* $g_{\alpha\beta}\colon X_{\alpha\beta} \to X_\alpha$ *is quasi-étale, where* $X_{\alpha\beta}$ *is the normalization of* $X_\alpha \times_X X_\beta$*.*

An orbi-structure $\mathscr{C} = \{(U_\alpha, f_\alpha, X_\alpha)\}_{\alpha\in J}$ *is called* strict *(resp.* orbi-étale*) if for each* $\alpha \in J$*, the morphism* f_α *is strictly adapted (resp. orbi-étale). It is called* smooth *if for each* $\alpha \in J$*, the variety* X_α *is smooth. In this case, the maps* $g_{\alpha\beta}$ *are étale by purity of branch locus.*

Definition 10 (Quotient singularities). *A pair* (X,Δ) *is said to have* quotient singularities *if locally analytically on* X*, there exists an orbi-étale morphism* $f\colon Y \to X$*, where* Y *is smooth. The maximal open subset of* X *where this condition is satisfied will also be referred to as the* orbifold locus *of* (X,Δ) *and will be denoted by* $X^\circ \subset X$ *or* $X^{\mathrm{orb}} \subset X$*.*

Remark. With the above terminology, a pair (X,Δ) admits a smooth orbi-étale orbi-structure if and only if it has quotient singularities. This is because the compatibility condition is automatically satisfied.

The following technical result will be useful in the sequel: a pair with quotient singularities whose underlying space is compact Kähler is a Kähler orbifold. The log smooth case had been already observed in [14, Prop. 2.1]. Slightly more generally, we have the following.

Lemma 11 (Existence of orbifold Kähler metrics). *Let* (Z,Δ) *be a pair with quotient singularities and such that* Z *is a Kähler space. Then for any relatively compact open subset* $X \Subset Z$*, there exists an orbifold Kähler metric* ω *adapted to* $(X,\Delta|_X)$ *in the sense that* ω *is a Kähler metric on* $X_{\mathrm{reg}} \setminus \operatorname{supp}\Delta$ *which pulls back to a smooth Kähler metric on the smooth local covers.*

Proof. One can find an open neighborhood X' of $\overline{X} \subset Z$ admitting a finite covering $X' = \bigcup_{\alpha\in I} X'_\alpha$ such that there exist smooth orbi-étale covers $p_\alpha\colon Y'_\alpha \to X'_\alpha$. We set $X_\alpha := X'_\alpha \cap X$ and $Y_\alpha := p_\alpha^{-1}(X_\alpha)$. We pick a Kähler metric ω_Z on Z, as well as potentials ϕ_α on X'_α such that $dd^c p_\alpha^*\phi_\alpha$ is a Kähler metric on Y'_α; the functions ϕ_α are solely continuous on X_α but $p_\alpha^*\phi_\alpha$ is smooth on Y'_α. We can assume that $|\phi_\alpha| \le 1$ on X_α. Finally, let $(\chi_\alpha)_{\alpha\in I}$ be some partition of unity subordinate to the covering $(X_\alpha)_{\alpha\in I}$ and set $\phi := \sum \chi_\alpha\phi_\alpha$. We set $N := |I|$ and pick a constant $C > 0$ such that

$$\|dd^c\chi_\alpha\|^2_{\omega_Z} + \|d\chi_\alpha\|^2_{\omega_Z} \le C, \tag{4}$$

holds for any $\alpha \in I$ and we claim that the current

$$\omega := M\omega_Z + dd^c\phi$$

is an orbifold Kähler metric on X for $M \gg 1$. Clearly, ω is smooth as an orbifold differential form, as one can see directly by using the compatibility of the covers. Let $x \in X$ and let $J := \{\alpha \in I, x \in X_\alpha\} = \{\alpha_1, \dots, \alpha_s\}$. We set $X_J := \bigcap_{\alpha\in J} X_\alpha$ and choose a connected component Y_J of the normalization of $p_{\alpha_1}^{-1}(X_J) \times_{X_J} \cdots \times_{X_J} p_{\alpha_s}^{-1}(X_J)$. The space Y_J is a smooth manifold endowed with an orbi-étale map $p_J\colon Y_J \to X_J$ induced by the p_{α_i}, $i = 1, \dots, s$.

We have $1 = \sum_{\alpha\in I}\chi_\alpha(x) = \sum_{\alpha\in J}\chi_\alpha(x)$, hence there exists $\beta \in J$ such that $\chi_\beta(x) \ge \frac{1}{N}$. Since $p_J^*(dd^c\phi_\beta|_{X_J})$ is a Kähler metric on Y_J (which extends slightly beyond), we infer that there exists $\delta > 0$ such that

$$\forall\, \alpha \in J, \quad dd^c\phi_\beta \ge \delta\, d\phi_\alpha \wedge d^c\phi_\alpha \quad \text{on } X_J.$$

Next, we have the following inequality for any $\varepsilon > 0$:

$$\pm(d\phi_\alpha \wedge d^c\chi_\alpha + d\chi_\alpha \wedge d^c\phi_\alpha) \le \varepsilon\, d\phi_\alpha \wedge d^c\phi_\alpha + \varepsilon^{-1} d\chi_\alpha \wedge d^c\chi_\alpha.$$

Combining the above inequality with (4), we get for any $\varepsilon > 0$:

$$\begin{aligned}\omega &= M\omega_Z + \sum_{\alpha\in I}\chi_\alpha \mathrm{dd}^c\phi_\alpha + \sum_{\alpha\in I}\phi_\alpha \mathrm{dd}^c\chi_\alpha + \sum_{\alpha\in I}(\mathrm{d}\phi_\alpha\wedge \mathrm{d}^c\chi_\alpha + \mathrm{d}\chi_\alpha\wedge \mathrm{d}^c\phi_\alpha)\\ &\geq (M - NC(1+\varepsilon^{-1}))\omega_Z + \chi_\beta \mathrm{dd}^c\phi_\beta - \varepsilon\sum_{\alpha\in I}\mathrm{d}\phi_\alpha\wedge \mathrm{d}^c\phi_\alpha\end{aligned}$$

which yields, at the point x:

$$\omega \geq (M - NC(1+\varepsilon^{-1}))\omega_Z + \left(\frac{1}{N} - \frac{N\varepsilon}{\delta}\right)\mathrm{dd}^c\phi_\beta.$$

Therefore, if we choose $\varepsilon := \frac{\delta}{2N^2}$ and $M = 2NC(1+\varepsilon^{-1})$, then ω is an orbifold Kähler metric near x. Since x is arbitrary and the constants N, C, δ are uniform, the lemma is now proved. □

2.2. *Covering constructions*

In what follows, we present some variations on the well-known cyclic covering theme. The first one, Proposition 12, is a consequence of [42, Ex. 2.4.1] when X is quasi-projective so that K_X is well-defined as a (class of) Weil divisor, but one needs to argue slightly differently in the complex analytic case. The second one, Proposition 13, improves upon previous results such as [33, Prop. 2.9], [31, Ex. 2.11] and [17, Prop. 2.38]. The main observation is that given a pair (X,Δ), it is (for our purposes) unnecessary to assume that the components of Δ are $\mathbb{Q}$-Cartier as long as $K_X + \Delta$ is. As explained in Remark 38, this is crucial for proving Corollary 4.

Proposition 12 (Existence of orbi-étale covers). *Let (X,Δ) be a (not necessarily klt) pair with standard coefficients, where X is a normal complex space. Assume that there is a reflexive rank 1 sheaf $\mathscr{L}$ and an integer $N \geq 1$ such that $N\Delta$ is a $\mathbb{Z}$-divisor and*

$$\mathscr{O}_X(N\Delta) \cong \mathscr{L}^{[N]}.$$

Then there exists an orbi-étale morphism $f\colon Y \to X$. In particular:

If (X,Δ) is klt and there is an integer $N \geq 1$ such that $N\Delta$ is a $\mathbb{Z}$-divisor and $\omega_X^{[N]}(N\Delta) \cong \mathscr{O}_X$, then we can find an orbi-étale morphism $f\colon Y \to X$ such that $\omega_Y \cong \mathscr{O}_Y$ and Y has canonical singularities.

Proof. Let $\sigma \in \mathrm{H}^0\big(X, \mathscr{L}^{[N]}\big)$ be such that $\mathrm{div}(\sigma) = N\Delta$, and let us consider the cyclic covering $g\colon Z \to X$ induced by σ, cf. e.g. [36, Def. 2.52]. In the analytic setting, we can construct f in the following way. On $X_{\mathrm{reg}} \setminus \mathrm{supp}(\Delta)$, $\mathscr{L}|_{X_{\mathrm{reg}}\setminus\mathrm{supp}(\Delta)}$ is torsion and it gives rise to an étale cover $g^\circ\colon Z^\circ \to X_{\mathrm{reg}}\setminus\mathrm{supp}(\Delta)$ (the N^{th}-root of $\sigma|_{X_{\mathrm{reg}}\setminus\mathrm{supp}(\Delta)}$) that is moreover a Galois cover with cyclic Galois group. According to [19, Thm. 3.4], the map g° can be extended to a finite cover $f\colon Z \to X$ with the same Galois group.

We claim that g ramifies exactly at order m_i along Δ_i. It is enough to check the claim at a general point of Δ_i. Therefore, there is no loss of generality assuming that $(X,\Delta) = (U, (1-\frac{1}{m})D)$ where $U \subset \mathbb{C}^n$ ($n = \dim(X)$) is a ball, $D = (z_1 = 0)\cap U$, and that $\sigma|_U = z_1^{N(1-\frac{1}{m})}\sigma_{\mathscr{L},U}^{\otimes N}$ with $\sigma_{\mathscr{L},U}$ a trivializing section of $\mathscr{L}$ over U.

Write $N = km$, and let $V := \{(t,z)\in\mathbb{C}\times\mathbb{C}^n \,|\, t^N = z_1^{k(m-1)}\} \subset \mathbb{C}\times\mathbb{C}^n$ and let $\nu\colon V^\nu \to V$ be its normalization. One can actually write down exactly what V^ν is. Indeed, let ζ be a primitive k-th root of unity, and set $V_p := \{(t,z)\,|\, t^m = \zeta^p z_1^{m-1}\} \subset \mathbb{C}\times\mathbb{C}^n$ for $p = 0,\dots,k-1$. We have a decomposition $V = \bigcup_p V_p$ into irreducible components, and the normalization $\nu_p\colon V_p^\nu \to V_p$ is the affine space $V_p^\nu \cong \mathbb{C} \times \mathbb{C}^{n-1}$ with map $\nu_p(u,w) = (\xi u^{m-1}, u^m, w)$ where ξ is an m-th root of ζ^p. Now, set $V^\nu := \bigsqcup_p V_p^\nu$ and define $\nu\colon V^\nu \to V$ by $\nu|_{V_p^\nu} := \nu_p$. We have a diagram

$$\begin{tikzcd} V^\nu \arrow[r, "\nu"] \arrow[rr, bend left, dotted, "j"] & V \arrow[d, "\mathrm{pr}_{\mathbb{C}^n}"'] & Z \arrow[d, "g"] \\ & U \arrow[r, hook] & X \end{tikzcd}$$

where j is obtained by the universal property of normalization. In particular, j is finite and generically 1-to-1 between normal varieties, hence it is an open embedding. Moreover, if $(u,w) \in V_p^\nu$, we have $\mathrm{pr}_{\mathbb{C}^n} \circ \nu(u,w) = (u^m, w)$, hence the latter map ramifies at order m along D. It follows that g ramifies at order m along D.

Finally, one picks one irreducible component Y of Z and sets $f := g|_Y$. It yields the expected cover, which is Galois with group $G < \mathbb{Z}/n\mathbb{Z} \cong \mathrm{Gal}(Z \to X)$ defined as the stabilizer of Y.

As for the last part of the proposition, we can apply the above construction to $\mathscr{L} = \omega_X^{[-1]} := \omega_X^\vee$. This provides us with an orbi-étale morphism $f\colon Y \to X$. In particular, Y is klt and the computations made above show that $f^*(K_X + \Delta)$ is trivial over $X_{\mathrm{reg}} \setminus \Delta_{\mathrm{sg}}$. So we get that ω_Y is trivial as well and finally that Y has only canonical singularities. □

Proposition 13 (Existence of strictly adapted covers). *Let (X,Δ) be a projective pair with standard coefficients such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier (but not necessarily klt). Then there exists a very ample divisor L on X such that for general $H \in |L|$, there exists a cyclic Galois cover $f\colon Y \to X$ with the following properties:*

(1) *The morphism f is orbi-étale for $\big(X, \Delta + (1 - \frac{1}{N})H\big)$, where $N := \deg(f)$.*
(2) *The morphism f is strictly adapted for (X,Δ).*
(3) *If (X,Δ) is klt, then so are the pairs $\big(X, \Delta + (1-\frac{1}{N})H\big)$ and $(Y, \emptyset)$.*

Proof. Pick, once and for all, a representative K of K_X, that is, an integral (but not necessarily effective) Weil divisor K on X such that $K_X \sim K$. Choose a very ample divisor A on X and a positive integer N such that

$$L := N \cdot \big(A - (K + \Delta)\big)$$

is integral and very ample, and pick a general element $H \in |L|$. Consider the principal divisor

$$D := H - L = H + N \cdot (K + \Delta - A) \sim 0.$$

Let $f\colon Y \to X$ be the degree N cyclic cover associated to D, as in [42, §2.3]. (To be more precise, Y is an arbitrary irreducible component of the normalization of that cover.) We need to check properties (1)–(3).

By construction, the branch locus of f is contained in $\mathrm{supp}(D)$. Recall from [42] that writing $D = \sum_i d_i D_i$, the ramification order of f along each component of $f^{-1}(D_i)$ is given by $N/\mathrm{hcf}(d_i, N)$. Since K, A and H are $\mathbb{Z}$-divisors, where H is even reduced, this implies (1). Property (2) is an immediate consequence.

For (3), it is enough to show the first claim thanks to (1) and [36, Prop. 5.20]. To check the claim, we take a log resolution $\pi\colon \widetilde{X} \to X$ of (X,Δ) and write

$$K_{\widetilde{X}} + \Delta' = \pi^*(K_X + \Delta) + \sum a_i E_i$$

as usual, where Δ' is the strict transform of Δ. Since H is a general element of $|L|$, and $\pi^*|L|$ is basepoint-free, one can assume that $\pi^* H = \pi_*^{-1} H$ is smooth and intersects each stratum of the exceptional divisor of π and of Δ' smoothly. In particular, π is also a log resolution for the pair $\big(X, \Delta + (1-\frac{1}{N})H\big)$. Now, the identity

$$K_{\widetilde{X}} + \Delta' + \left(1 - \frac{1}{N}\right)\pi_*^{-1} H = \pi^*\left(K_X + \Delta + \left(1 - \frac{1}{N}\right) H\right) + \sum a_i E_i$$

shows that $\big(X, \Delta + (1-\frac{1}{N})H\big)$ is klt. $\square$

Remark. More generally, it can be observed that a pair (X,Δ) (with X a normal analytic space) admits strictly adapted covers if there exists a Cartier divisor D on X having no component in common with Δ and such that $m(K_X+\Delta) \sim D$ for some (sufficiently divisibe) integer $m \geq 1$. We can indeed apply Proposition 12 to the pair $(X \setminus D, \Delta|_{X\setminus D})$ and get an orbi-étale cover $Y^\circ \to X \setminus D$. Its completion over X is then adapted with respect to Δ and the extra-ramification is supported over the components of D.

The following result seems to have been known to experts for a long time. A proof of it was written down in [26] in the case where $\Delta = 0$, and the general case follows almost immediately from Proposition 12 as we will explain.

Lemma 14 (Klt pairs have quotient singularities in codimension two). *Let (X,Δ) be a klt pair with standard coefficients. Then there is a Zariski closed subset $Z \subset X_{\mathrm{sg}} \cup \operatorname{supp}\Delta$ with $\operatorname{codim}_X(Z) \geq 3$ such that for $X^\circ := X \setminus Z$, the pair $(X^\circ, \Delta|_{X^\circ})$ admits a smooth orbi-étale orbi-structure $\mathscr{C}^\circ$.*

Proof. Since $K_X+\Delta$ is a $\mathbb{Q}$-Cartier divisor, we can cover X by (affine or Stein) open subsets $U_\beta \subset X$, $\beta \in I$, such that $(K_X+\Delta)|_{U_\beta} \sim_{\mathbb{Q}} 0$. By Proposition 12, we can find a finite cyclic cover $g_\beta \colon U'_\beta \to U_\beta$ that branches exactly over the $\Delta_i|_{U_\beta}$ with multiplicity m_i. Moreover, U'_β has klt singularities, since $K_{U'_\beta} = g_\beta^*(K_{U_\beta} + \Delta|_{U_\beta})$. We can now use [26, Prop. 9.3] or [24, Lem. 5.8] to find a *smooth* orbi-étale orbi-structure $\{U'_{\beta\gamma}, f_{\beta\gamma}, X'_{\beta\gamma}\}_{\gamma\in J}$ on $U'_\beta \setminus Z_\beta$, for some closed subset $Z_\beta \subset U'_\beta$ of codimension at least three. Set $U_{\beta\gamma} = g_\beta(U'_{\beta\gamma})$, so that $\bigcup_\beta U_{\beta\gamma} \subset U_\beta$ is an open subset whose complement is of codimension at least three. In summary, we get the following diagram:

$$\begin{tikzcd} X'_{\beta\gamma} \arrow[r, "f_{\beta\gamma}"] \arrow[rr, bend left, "h_{\beta\gamma}"] & U'_{\beta\gamma} \arrow[r, "g_\beta"] \arrow[d, hook] & U_{\beta\gamma} \arrow[d, hook] & \\ & U'_\beta \arrow[r, "g_\beta"] & U_\beta \arrow[r, hook] & X \end{tikzcd} \tag{5}$$

Now $\big\{U_{\beta\gamma}, h_{\beta\gamma}, X'_{\beta\gamma}\big\}_{(\beta,\gamma)\in I\times J}$ is the sought-after smooth orbi-étale orbi-structure on $(X^\circ, \Delta|_{X^\circ})$, where the open subset $X^\circ := \bigcup_{(\beta,\gamma)\in I\times J} U_{\beta\gamma}$ has complement of codimension at least three. $\square$

Remark 15. In particular, a klt surface pair with standard coefficients admits a smooth orbi-étale orbi-structure, hence it has quotient singularities in the sense of Definition 10. This is of course well-known and follows from the cyclic cover construction recalled above and [36, Prop. 4.18].

Definition 16 (Orbi-sheaves). *An* orbi-sheaf *with respect to an orbi-structure $\mathscr{C} = \big\{(U_\alpha, f_\alpha, X_\alpha)\big\}_{\alpha\in J}$ on (X,Δ) is the datum of a collection $(\mathscr{E}_\alpha)_{\alpha\in J}$ of coherent sheaves on each X_α, together with isomorphisms $g_{\alpha\beta}^*\mathscr{E}_\alpha \cong g_{\beta\alpha}^*\mathscr{E}_\beta$ of $\mathscr{O}_{X_{\alpha\beta}}$-modules satisfying the natural compatibility conditions on triple overlaps.*

All the usual notions for sheaves (locally free, reflexive, subsheaves, morphisms etc.) can be carried over to this setting in the obvious way, cf. [31, §2.7]. Ditto for Higgs fields and Higgs sheaves, cf. [31, Def. 2.24].

Recall the following definition from [17, §3]:

Definition 17 (Adapted differentials). *Let $\gamma\colon Y \to X$ be a strictly adapted morphism for (X,Δ). Let $X^\circ \subset X$ and $\iota\colon Y^\circ \hookrightarrow Y$ be the maximal open subsets where γ is* good *in the sense of [17, Def. 3.5]. The sheaf of* adapted reflexive differentials *is defined as*

$$\Omega^{[1]}_{(X,\Delta,\gamma)} := \iota_*\left[\left(\operatorname{im}\left(\gamma^*\Omega^1_{X^\circ} \to \Omega^1_{Y^\circ}\right) \otimes \mathscr{O}_{Y^\circ}(\gamma^*\Delta)\right) \cap \Omega^1_{Y^\circ}\right].$$

Lemma 18. *The following properties hold:*

(1) *The sheaf $\Omega^{[1]}_{(X,\Delta,\gamma)}$ is a coherent reflexive subsheaf of $\Omega^{[1]}_Y$.*
(2) *If γ is orbi-étale for (X,Δ), then $\Omega^{[1]}_{(X,\Delta,\gamma)} = \Omega^{[1]}_Y$.*
(3) *Let $\gamma_2\colon Z \to Y$ be quasi-étale, where Z is normal. Then $\delta := \gamma\circ\gamma_2\colon Z \to X$ is strictly adapted for (X,Δ), and $\Omega^{[1]}_{(X,\Delta,\delta)} = \gamma_2^{[*]}\Omega^{[1]}_{(X,\Delta,\gamma)}$.* □

Definition 19 (Orbifold cotangent sheaf, cf. [31, Def. 2.23]). *Consider on (X,Δ) any strictly adapted orbi-structure $\mathscr{C} = \{(U_\alpha, f_\alpha, X_\alpha)\}_{\alpha\in J}$. Then the sheaves*

$$\left(\Omega^{[1]}_{(X,\Delta,f_\alpha)}\right)_{\alpha\in J}$$

induce a reflexive orbi-sheaf called the orbifold cotangent sheaf, *or* sheaf of reflexive differential forms, *which we denote by $\Omega^{[1]}_{\mathscr{C}}$. If the orbi-structure $\mathscr{C}$ is smooth and orbi-étale, then $\Omega^{[1]}_{\mathscr{C}}$ is locally free. Changing the (strictly adapted) orbifold structure yields compatible sheaves in the sense of [31, Def. 3.2], hence we will often denote this sheaf by $\Omega^{[1]}_{(X,\Delta)}$.*

The same construction can be carried out for any integer $p \geq 0$, yielding orbi-sheaves $\Omega^{[p]}_{(X,\Delta)}$. For $p = 0$, we obtain the structure sheaf $\mathscr{O}_{(X,\Delta)}$, *which is nothing but $\mathscr{O}_{X_\alpha}$ in each chart f_α.*

Lemma 20. *Let (X,Δ) be a projective klt pair with standard coefficients, and let X° be endowed with a smooth orbi-étale orbi-structure $\mathscr{C}$ as in Lemma 14. Let H be an ample line bundle on X and pick a complete intersection surface*

$$S = D_1 \cap \cdots \cap D_{n-2}$$

of $n-2$ general hypersurfaces $D_i \in |mH|$ for $m \gg 1$. Then $S \subset X^\circ$ and the restriction of $\mathscr{C}$ to $(S,\Delta|_S)$ induces a smooth orbi-étale orbi-structure on $(S,\Delta|_S)$. In particular, $(S,\Delta|_S)$ has quotient singularities.

Proof. We have $S \subset X^\circ$ for dimensional and genericity reasons. Next, if we express the structure $\mathscr{C}$ as $\mathscr{C} = \{(X_\alpha, f_\alpha, U_\alpha)\}$, set $S_\alpha := S \cap U_\alpha$, $T_\alpha := f_\alpha^{-1}(S_\alpha)$, $g_\alpha := f_\alpha|_{T_\alpha}$, and define $\mathscr{C}|_S := \{(T_\alpha, g_\alpha, S_\alpha)\}$. We claim that T_α is smooth, which would prove the lemma. Indeed, since f_α is quasi-finite (as the composition of an étale map with a finite map), one can find an open immersion $X_\alpha \hookrightarrow \overline{X_\alpha}$ and a finite extension $\overline{f_\alpha}\colon \overline{X_\alpha} \to X$ of f_α as follows:

$$\begin{tikzcd} T_\alpha \arrow[r, hook] \arrow[d, "g_\alpha"] & X_\alpha \arrow[r, hook] \arrow[d, "f_\alpha"] & \overline{X_\alpha} \arrow[d, "\overline{f_\alpha}"] \\ S_\alpha \arrow[r, hook] & U_\alpha \arrow[r, hook] & X \end{tikzcd}$$

Since $\overline{f_\alpha}^*|mH|$ is basepoint-free, Bertini's theorem guarantees that if $\overline{T_\alpha}$ is a general intersection of $(n-2)$ hypersurfaces in $\overline{f_\alpha}^*|mH|$, then $\overline{T_\alpha} \cap \overline{X_\alpha}^{\mathrm{reg}}$ is smooth. Since $X_\alpha \subset \overline{X_\alpha}^{\mathrm{reg}}$, this shows that T_α is smooth, hence the lemma. □

2.3. *The orbifold fundamental group*

Let (X,Δ) be a klt pair with standard coefficients as before, and set $X^* := X_{\mathrm{reg}} \setminus \operatorname{supp}\Delta$.

Definition 21 (Fundamental group). *The* (orbifold) fundamental group *of* (X, Δ) *is defined as*

$$\pi_1^{\mathrm{orb}}(X, \Delta) := \pi_1(X^*) \big/ \langle\!\langle \gamma_i^{m_i},\ i \in I \rangle\!\rangle.$$

Here, for each $i \in I$*, the element* γ_i *is a "loop around* Δ_i*", i.e. a loop in the normal circle bundle of* $(\Delta_i)_{\mathrm{reg}} \cap X_{\mathrm{reg}} \subset X_{\mathrm{reg}}$*, and* $\langle\!\langle \cdots \rangle\!\rangle$ *denotes the normal subgroup generated by a given subset.*

Note that if $D = \emptyset$, then $\pi_1^{\mathrm{orb}}(X, \emptyset) = \pi_1(X_{\mathrm{reg}})$ is in general different from $\pi_1(X)$.

Definition 22 (Covers branched at Δ, cf. [14, Def. 1.3]). *A cover of* X branched at most at Δ *is a holomorphic map* $\pi\colon Y \to X$*, where:*

(1) *Y is a normal connected complex space (not necessarily quasi-projective),*
(2) *π has discrete fibres and $\pi^{-1}(X^*) \to X^*$ is étale,*
(3) *at each irreducible component $\widetilde{\Delta}_{j,k} \subset \pi^{-1}(\Delta_j)$, the ramification index $r_{j,k}$ of π divides m_j,*
(4) *every $x \in X$ has a connected neighborhood $V \subset X$ such that every connected component U of $\pi^{-1}(V)$ meets the fibre $\pi^{-1}(x)$ in only one point, and $\pi|_U\colon U \to V$ is finite.*

We say that π *is* branched exactly at Δ *if in* (3)*, we have* $r_{j,k} = m_j$ *for all* j, k*.*

Note that if Y is quasi-projective and π is Galois, then saying that π is branched exactly at Δ is the same as saying that π is orbi-étale.

Theorem 23 (Covers and the fundamental group). *There exists a natural one-to-one correspondence between subgroups $G \subset \pi_1^{\mathrm{orb}}(X, \Delta)$ and covers $\pi\colon Y \to X$ branched at most at Δ. Furthermore:*

(1) *G is of finite index if and only if π is finite.*
(2) *G is a normal subgroup if and only if π is Galois.*
(3) *Let $Y_{1,2} \to X$ be two covers branched at most at Δ, with corresponding subgroups $G_{1,2} \subset \pi_1^{\mathrm{orb}}(X, \Delta)$. Then there is a factorization*

$$\begin{tikzcd} & Y_2 \arrow[d] \\ Y_1 \arrow[r] \arrow[ur, dashed, "\exists"] & X \end{tikzcd}$$

if and only if $G_1 \subset G_2$.

Proof. The proof is the same as in the snc case, cf. [14, Thm. 1.1], with one important difference: in order to extend (possibly non-finite) étale covers of X^* to branched covers of X, we would like to apply [19, Thm. 3.4]. In order to do this, we must invoke the finiteness of local orbifold fundamental groups of klt pairs, as proved in [11, Thm. 1]. (Note that [11] works in the algebraic category, but in view of [22, Thm. 1.7] and [16, Rem. 6.10] his result extends to complex spaces as well.) □

Definition 24 (Universal cover). *The (*orbifold*)* universal cover *of* (X, Δ) *is the cover* $\pi\colon \widetilde{X}_\Delta \to X$ *corresponding to the trivial subgroup* $\{1\} \subset \pi_1^{\mathrm{orb}}(X, \Delta)$ *under the correspondence from Theorem 23.*

Let $\widetilde{\Delta}$ be the divisor on $\widetilde{X}_\Delta$ which is supported on $\pi^{-1}(\operatorname{supp} \Delta)$ and satisfies

$$K_{\widetilde{X}_\Delta} + \widetilde{\Delta} = \pi^*(K_X + \Delta).$$

It is easy to see that the pair $(\widetilde{X}_\Delta, \widetilde{\Delta})$ is again klt with standard coefficients. Also, $\widetilde{\Delta} = 0$ if and only if π is branched exactly at Δ.

Definition 25 (Developable pairs). *We say that* (X, Δ) *is* developable *if in the above notation,* $\widetilde{X}_\Delta$ *is smooth and* $\widetilde{\Delta} = 0$.

Intuitively, being developable means that the universal cover is a manifold.

Example 26. Consider the klt pair (X,Δ), where $X=\mathbb{P}^1$ and

$$\Delta=\left(1-\frac{1}{n}\right)\cdot[0]+\left(1-\frac{1}{m}\right)\cdot[\infty]$$

with $n,m\geq 2$. Set $d=\gcd(n,m)$. Then $\pi_1^{\mathrm{orb}}(X,\Delta)=\mathbb{Z}/d\mathbb{Z}$, and the universal cover $\pi\colon \widetilde{X}_\Delta=\mathbb{P}^1\to\mathbb{P}^1$ is given by $[z_0:z_1]\mapsto[z_0^d:z_1^d]$. We have

$$\widetilde{\Delta}=\left(1-\frac{1}{n/d}\right)\cdot[0]+\left(1-\frac{1}{m/d}\right)\cdot[\infty].$$

In particular, (X,Δ) is developable if and only if $n=m$.

Corollary 27 (Galois closure). *Let $Y\to X$ be a finite cover branched at most at Δ. Then there is a finite cover $Y'\to Y$ such that the composition $Y'\to X$ is finite, Galois, and branched at most at Δ. If additionally $Y\to X$ is branched exactly at Δ, then the same is true of $Y'\to X$, and $Y'\to Y$ is quasi-étale.*

We call $Y'\to X$ the *Galois closure* of $Y\to X$.

Proof. Using the correspondence from Theorem 23, the statement boils down to the following: for a group G and a subgroup $H\subset G$ of finite index, there is a normal subgroup $N\trianglelefteq G$ of finite index such that $N\subset H$. But this is easy (and well-known): simply set

$$N:=\bigcap_{g\in G/H} gHg^{-1}.$$

The last statement is easily seen to be true by comparing the ramification indices of $Y\to X$ and $Y'\to X$ over the components Δ_i. □

3. Orbifold Chern classes of klt pairs

In this section, we recall the definition of the first and second orbifold Chern classes for klt pairs, in the spirit of [24]. We then explain how to compute them concretely in two cases: in the projective setting by a cutting-down argument (Section 3.3), and when we have an "orbi-resolution" at our disposal (Section 3.4).

3.1. *The general Kähler case*

Let us begin by recalling how to define Chern numbers associated with the first and second Chern classes. This is nothing but a slight generalization of [24, Def. 5.2] that takes into account the presence of a boundary. The construction relies on the Chern–Weil formalism in the orbifold setting. We will not recall the basic definitions and properties for the differential geometry of orbifolds (e.g. Hermitian metrics on orbifold bundles, orbifold Chern classes, orbifold de Rham cohomology, and so on). A good reference is [8, §2].

Let (X,Δ) as in Setup 5 and let $X^\circ\subset X$ be the largest open subset of X such that (X,Δ) admits a smooth orbi-étale orbi-structure $\mathscr{C}^\circ$, and set $Z:=X\setminus X^\circ$. As proved in Lemma 14, $\dim Z\leq n-3$. Next, let $\alpha\in \mathrm{H}^{2n-4}(X,\mathbb{R})$ where that cohomology space is understood as the cohomology of the locally constant sheaf $\underline{\mathbb{R}}_X$. For dimensional reasons, we have an isomorphism $\mathrm{H}_\mathrm{c}^{2n-4}(X^\circ,\mathbb{R})\xrightarrow{\sim}\mathrm{H}^{2n-4}(X,\mathbb{R})$. Next, the de Rham complex of orbifold differential forms on X° yields a de Rham–Weil isomorphism $\mathrm{H}^\bullet_{\mathrm{dR,c}}(X^\circ,\mathbb{R})\to\mathrm{H}^\bullet_\mathrm{c}(X^\circ,\mathbb{R})$, so that in the end we get a natural isomorphism

$$\psi:\mathrm{H}^{2n-4}_{\mathrm{dR,c}}\big(X^\circ,\mathbb{R}\big)\xrightarrow{\sim}\mathrm{H}^{2n-4}(X,\mathbb{R}).\tag{6}$$

Now, let $E\to X^\circ$ be an orbifold bundle for the pair (X°,Δ°). We can equip it with an orbifold Hermitian metric h and form the Chern classes $\mathrm{c}_i^{\mathrm{orb}}(E,h)$ which are orbifold differential forms of bidegree (i,i). We can use the isomorphism (6) to define real numbers when $i=2$. If $\alpha \in \mathrm{H}^{2n-4}(X,\mathbb{R})$, the class $\psi^{-1}(\alpha)$ can be represented by a compactly supported orbifold $(2n-4)$-form Ω on X°, so that $\mathrm{c}_2^{\mathrm{orb}}(E,h)\wedge\Omega$ is a compactly supported orbifold (n,n)-form on X°.

Definition 28. *The orbifold second Chern class $\widetilde{\mathrm{c}}_2(E)$ is the unique element in the dual space $\mathrm{H}^{2n-4}(X,\mathbb{R})^\vee$ which under $\psi^\vee$ corresponds to the Poincaré dual of the class $\mathrm{c}_2^{\mathrm{orb}}(E)\in \mathrm{H}^4_{\mathrm{dR}}(X^\circ,\mathbb{R})$, where the latter is taken with respect to (but independent of) the orbi-structure $\mathscr{C}^\circ$. The quantity*

$$\widetilde{\mathrm{c}}_2(E)\cdot\alpha := \int_{X^\circ} \mathrm{c}_2^{\mathrm{orb}}(E,h)\wedge\Omega$$

is thus a well defined real number for any class $\alpha\in\mathrm{H}^{2n-4}(X,\mathbb{R})$.

Let us apply the above construction to $\Omega^1_{(X^\circ,\Delta^\circ)}$ the orbifold bundle of differential forms. For the first Chern class, one can avoid the use of orbistructures and define it directly as a cohomology class as follows.

Definition 29. *For a klt pair (X,Δ), we set*

$$\widetilde{\mathrm{c}}_1(X,\Delta) := \frac{1}{m}\mathrm{c}_1\big(\big(\omega_X{}^{[m]}\otimes\mathscr{O}_X(m\Delta)\big)^{\vee\vee}\big)\in\mathrm{H}^2(X,\mathbb{R})$$

where $m\geq 1$ is an integer such that the reflexive rank 1 sheaf $\big(\omega_X{}^{[m]}\otimes\mathscr{O}_X(m\Delta)\big)^{\vee\vee}$ is a line bundle.

Now let us consider the case of the second Chern class.

Definition 30. *The orbifold second Chern class $\widetilde{\mathrm{c}}_2(X,\Delta)\in\mathrm{H}^{2n-4}(X,\mathbb{R})^\vee$ of the pair (X,Δ) is the second Chern class of the orbi-bundle $\Omega^1_{(X^\circ,\Delta^\circ)}$ on X° defined in Definition 19.*

Remark 31. As already observed in [24, p. 893], the object constructed in Definition 30 is naturally a homology class:

$$\widetilde{\mathrm{c}}_2(X,\Delta)\in\mathrm{H}_{2n-4}(X,\mathbb{R}).$$

3.2. *The projective case — Mumford's construction*

Let (X,Δ) be a projective dlt pair with standard coefficients such that each component Δ_i of Δ is $\mathbb{Q}$-Cartier. In [31, §3.1, p. 1458], the orbifold Chern classes $\widetilde{\mathrm{c}}_2(X,\Delta)$ and $\widetilde{\mathrm{c}}_1^2(X,\Delta)$ were defined as multilinear forms on $\mathrm{N}^1(X)_{\mathbb{Q}}$. Here we would like to observe that this procedure can also be carried out without the assumption that the Δ_i be $\mathbb{Q}$-Cartier. Our argument follows the proof of [29, Thm. 3.13] closely. We will restrict attention to the case of klt pairs, as we are only concerned with those in this paper.

So let (X,Δ) be an n-dimensional projective klt pair with standard coefficients. Applying Lemma 14, we obtain an open subset $X^\circ\subset X$ whose complement has codimension ≥ 3 and such that $(X^\circ,\Delta|_{X^\circ})$ admits a smooth orbi-étale orbi-structure $\mathscr{C}$. Consider the "big global cover" $\gamma\colon\widehat{X^\circ}\to X^\circ$ associated to $\mathscr{C}$, cf. [41, §§2–3], which up to shrinking X° may be assumed to be Cohen–Macaulay. The locally free orbi-sheaf $\Omega^{[1]}_{\mathscr{C}}$ from Definition 19 induces a genuine locally free sheaf $\mathscr{F}$ on $\widehat{X^\circ}$. The Chern classes of $\mathscr{F}$ induce classes $\mathrm{c}_i\big(\Omega^{[1]}_{\mathscr{C}}\big)\in\mathrm{A}_{n-i}(X^\circ)$. Since $\mathrm{A}_*(X^\circ)$ is equipped with a ring structure, we also have $\mathrm{c}_1^2\big(\Omega^{[1]}_{\mathscr{C}}\big)\in\mathrm{A}_{n-2}(X^\circ)$. For dimensional reasons, $\mathrm{A}_{n-i}(X)\xrightarrow{\sim}\mathrm{A}_{n-i}(X^\circ)$ is an isomorphism for $i\leq 2$. We obtain classes $\mathrm{c}_2\big(\Omega^{[1]}_{\mathscr{C}}\big)$ and $\mathrm{c}_1^2\big(\Omega^{[1]}_{\mathscr{C}}\big)\in\mathrm{A}_{n-2}(X)$, which are independent of the choice of $\mathscr{C}$ by [31, Prop. 3.5]. The orbifold Chern classes $\widetilde{\mathrm{c}}_2(X,\Delta)$ and $\widetilde{\mathrm{c}}_1^2(X,\Delta)$ are then given by cap product with Chern classes of line bundles on X:

$$\widetilde{\mathrm{c}}_2(X,\Delta)\cdot\mathscr{L}_1\cdots\mathscr{L}_{n-2} := \deg\big(\mathrm{c}_2\big(\Omega^{[1]}_{\mathscr{C}}\big)\cap\mathrm{c}_1(\mathscr{L}_1)\cap\cdots\cap\mathrm{c}_1(\mathscr{L}_{n-2})\big),$$
$$\widetilde{\mathrm{c}}_1^2(X,\Delta)\cdot\mathscr{L}_1\cdots\mathscr{L}_{n-2} := \deg\big(\mathrm{c}_1^2\big(\Omega^{[1]}_{\mathscr{C}}\big)\cap\mathrm{c}_1(\mathscr{L}_1)\cap\cdots\cap\mathrm{c}_1(\mathscr{L}_{n-2})\big),$$

and these maps factors via $\mathrm{N}^1(X)_{\mathbb{Q}}$.

3.3. *The projective case — cutting down*

If (X,Δ) is a projective klt pair with standard coefficients, then Lemma 14 allows one to generalize Mumford's construction of $\mathbb{Q}$-Chern classes [41] to this setting as explained above. The fact that the Chern–Weil construction from Definition 30 and Mumford's definition of $\mathbb{Q}$-Chern classes are equivalent is given in [24, Claim 6.5] in the case where $\Delta = 0$. It extends readily to the more general setting of klt pairs with standard coefficients.

Since ψ is an abstract isomorphism, it is in practice difficult to actually compute these numbers. There is, however, an important situation where things get much more explicit and that is when $\alpha = \mathrm{c}_1(L)^{n-2}$ where L is an ample line bundle on X (we could also have $(n-2)$ different ample line bundles, but let us stick to the former case for simplicity). By homogeneity of the intersection product, we can assume that L is very ample and induces an embedding $i : X \hookrightarrow \mathbb{P}^N$ such that $L \cong i^*\mathscr{O}_{\mathbb{P}^N}(1)$. We pick $(n-2)$ hyperplanes $H_1,\dots,H_{n-2}$ in general position. In particular, one has that $\sum H_i$ has simple normal crossings and $S \coloneqq H_1 \cap \dots \cap H_{n-2} \cap X \subset X^\circ$.

Lemma 32. *With the notation as above, the Chern number from Definition 28 can be computed with the following formula:*

$$\widetilde{\mathrm{c}}_2(E)\cdot \mathrm{c}_1(L)^{n-2} = \int_S \mathrm{c}_2^{\mathrm{orb}}(E,h)\big|_S. \tag{7}$$

Proof. To begin with, let us choose sections $s_i \in \mathrm{H}^0\big(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(1)\big)$ such that $H_i = \{s_i = 0\}$, and we equip $\mathscr{O}_{\mathbb{P}^N}(1)$ with the Fubini–Study metric. Next, we choose cut-off functions $\chi_i : \mathbb{P}^N \to [0,1]$ such that

$$\chi_i = \begin{cases} 0 & \text{on} \quad \{|s_i| \le \delta\} \\ 1 & \text{on} \quad \{|s_i| \ge 2\delta\} \end{cases}$$

for some $\delta > 0$ small enough so that

$$\bigcap_{i=1}^{n-2} \{|s_i| \le 2\delta\} \cap X \subset X^\circ.$$

For any $\varepsilon \in (0,1]$, one defines $\varphi_{i,\varepsilon} \coloneqq \chi_i \log|s_i|^2 + (1-\chi_i)\log(|s_i|^2+\varepsilon^2)$ and set $\omega_{i,\varepsilon} \coloneqq \omega_{\mathrm{FS}} + dd^c\varphi_{i,\varepsilon}$. Clearly, $\omega_{i,\varepsilon}$ is supported on $\{|s_i| \le 2\delta\}$ and $\omega_{i,\varepsilon} \to [H_i]$ as $\varepsilon \to 0$, both weakly as currents on $\mathbb{P}^N$ and locally smoothly away from H_i. We set $\Omega_\varepsilon \coloneqq \bigwedge_{i=1}^{n-2} \omega_{i,\varepsilon}$, which is supported on $\bigcap_{i=1}^{n-2}\{|s_i| \le 2\delta\}$.

The immersion $i : X^\circ \hookrightarrow \mathbb{P}^N$ induces a commutative diagram

$$\begin{tikzcd} \mathrm{H}^{2n-4}_{\mathrm{dR}}\big(\mathbb{P}^N,\mathbb{R}\big) \arrow[r, "\sim"] \arrow[d, "i^*"] & \mathrm{H}^{2n-4}\big(\mathbb{P}^N,\mathbb{R}\big) \arrow[d, "i^*"] \\ \mathrm{H}^{2n-4}_{\mathrm{dR}}(X^\circ,\mathbb{R}) \arrow[r, "\sim"] & \mathrm{H}^{2n-4}(X^\circ,\mathbb{R}). \end{tikzcd}$$

and by our choices the image $i_*[\Omega_\varepsilon]$ lands in the image of the natural map

$$\mathrm{H}^{2n-4}_{\mathrm{dR,c}}\big(X^\circ,\mathbb{R}\big) \to \mathrm{H}^{2n-4}_{\mathrm{dR}}\big(X^\circ,\mathbb{R}\big)$$

and satisfies $\psi(i_*[\Omega_\varepsilon]) = \mathrm{c}_1(\mathscr{O}_{\mathbb{P}^N}(1))^{n-2}|_X = \mathrm{c}_1(L)^{n-2}$. Therefore, we have for any $\varepsilon > 0$ the identity

$$\widetilde{\mathrm{c}}_2(E)\cdot \mathrm{c}_1(L)^{n-2} = \int_{X^\circ} \mathrm{c}_2^{\mathrm{orb}}(E,h) \wedge \Omega_\varepsilon. \tag{8}$$

Now, since $\sum H_i$ has simple normal crossings, an easy local computation shows that Ω_ε converges to the current of integration along the submanifold $W \coloneqq \bigcap_{i=1}^{n-2} H_i$, both weakly on $\mathbb{P}^N$ and locally smoothly away from W. Since the support of $\Omega_\varepsilon|_X$ is contained in a fixed compact subset of X°, ones sees that $\Omega_\varepsilon|_{X^\circ}$ converges weakly to $[S] = [W \cap X^\circ]$ in the sense of currents on the orbifold X°. Letting ε tend to 0 in (8), we finally get the formula (7). □

3.4. *Orbi-resolutions and Chern numbers*

When X is smooth in codimension two, one can compute Chern numbers on a resolution of singularities, cf. e.g. [15]. In the presence of singularities in codimension two, it is explained in loc. cit. that a resolution does not compute Chern numbers anymore in general. The substitute of a resolution in that setting is an *orbi-resolution* as defined below.

Definition 33 (Orbi-resolutions). *Let (X,Δ) be a pair, where X is a normal complex space, Δ has standard coefficients and let $X^\circ \subset X$ be the orbifold locus of (X,Δ). An orbi-resolution of (X,Δ) is a surjective, proper bimeromorphic map $\pi\colon \widehat{X} \to X$ from a normal complex space $\widehat{X}$ such that:*

(1) $\big(\widehat{X}, \widehat{\Delta} := \pi_*^{-1}(\Delta)\big)$ *has only quotient singularities, and*
(2) π *is isomorphic over X°.*

The existence of orbi-resolutions can be established[1] for quasi-projective varieties (with $\Delta = 0$), using deep results about stacks as Chenyang Xu has showed in [37, §3]. However, the construction proposed there is highly non-canonical (or non-functorial) and this makes it difficult to generalize it to the complex analytic setting, even assuming algebraic singularities.

One important application of the existence of orbi-resolutions is highlighted by the following lemma, which shows that we can use such partial resolutions to compute the orbifold second Chern class of (X,Δ) against a class in $\mathrm{H}^{2n-4}(X,\mathbb{R})$.

Lemma 34. *Let (X,Δ) be a pair as in Setup 5. Assume that (X,Δ) admits an orbi-resolution $\pi\colon (\widehat{X},\widehat{\Delta}) \to (X,\Delta)$ as in Definition 33. Given any $a \in \mathrm{H}^{2n-4}(X,\mathbb{R})$, one has the formula*

$$\widetilde{\mathrm{c}}_2(X,\Delta)\cdot a = \mathrm{c}_2^{\mathrm{orb}}\big(\widehat{X},\widehat{\Delta}\big)\cdot \psi(\pi^* a),$$

where on the right-hand side, $\mathrm{c}_2^{\mathrm{orb}}\big(\widehat{X},\widehat{\Delta}\big) \in \mathrm{H}^4_{\mathrm{dR}}\big(\widehat{X},\mathbb{R}\big)$ is the usual orbifold second Chern class of $(\widehat{X},\widehat{\Delta})$ and $\psi\colon \mathrm{H}^\bullet\big(\widehat{X},\mathbb{R}\big) \to \mathrm{H}^\bullet_{\mathrm{dR}}\big(\widehat{X},\mathbb{R}\big)$ is the orbifold de Rham–Weil isomorphism.

Proof. With the notation from Definition 33, let us denote $\widehat{X}\setminus E := \pi^{-1}(X^\circ)$ and $j\colon \widehat{X}\setminus E \to \widehat{X}$ the natural inclusion; for simplicity we set $k := 2n-4$ and skip the reference to $\mathbb{R}$ in the cohomology spaces below. Finally, we set $\pi_0 := \pi|_{\widehat{X}\setminus E}\colon \widehat{X}\setminus E \to X^\circ$.

We then have the following diagram

$$\begin{tikzcd}
 & & \mathrm{H}^k_{\mathrm{dR}}\big(\widehat{X}\big) \\
\mathrm{H}^k_{\mathrm{dR,c}}\big(\widehat{X}\setminus E\big) \arrow[r, "\phi"] \arrow[urr, "j_*^{\mathrm{dR}}"] & \mathrm{H}^k_{\mathrm{c}}\big(\widehat{X}\setminus E\big) \arrow[r, "j_*"] & \mathrm{H}^k\big(\widehat{X}\big) \arrow[u, "\psi"] \\
\mathrm{H}^k_{\mathrm{dR,c}}(X^\circ) \arrow[r, "\phi"] \arrow[u, "(\pi_0^{\mathrm{dR}})^*"] & \mathrm{H}^k_{\mathrm{c}}(X^\circ) \arrow[r, "i_*"] \arrow[u, "\pi_0^*"] & \mathrm{H}^k(X) \arrow[u, "\pi^*"]
\end{tikzcd}$$

where all arrows except for j_*, j_*^{dR} and π^* are isomorphisms. Now, one can pick an orbifold Hermitian metric $\widehat{h}$ on $T_{\widehat{X},\widehat{\Delta}}$ and descend it to an orbifold Hermitian metric h on T_{X° since π

[1] The proof of [37, Thm. 3] applies verbatim when $\Delta \neq 0$, but we will only use the existence of orbi-resolutions when $\Delta = 0$.

is an isomorphism $\widehat{X} \setminus E \to X^\circ$. Then, if as before α is an orbifold representative of $\phi^{-1}(i_*^{-1}(a))$ with compact support in X°, we have

$$\begin{aligned}\widetilde{c}_2(X,\Delta)\cdot a &= \int_{X^\circ} c_2^{\mathrm{orb}}\big(X^\circ, h\big)\wedge\alpha \\ &= \int_{\widehat{X}\setminus E} c_2^{\mathrm{orb}}\big(\widehat{X}, \widehat{h}\big)\wedge\pi^*\alpha \\ &= c_2^{\mathrm{orb}}\big(\widehat{X}, \widehat{\Delta}\big)\cdot[\pi^*\alpha]_{\mathrm{dR}} \\ &= c_2^{\mathrm{orb}}\big(\widehat{X}, \widehat{\Delta}\big)\cdot\psi(\pi^* a)\end{aligned}$$

since we have $\psi(\pi^* a) = (j_*)^{\mathrm{dR}}([\pi^*\alpha]_{\mathrm{dR}})$ from the commutativity of the diagram above. □

We conclude this paragraph with a remark on the non-orbifold locus. For the sake of clarity (and also since we will use only this case), we stick to the case $\Delta = 0$.

If X is a normal complex space that admits an orbi-resolution $\pi\colon \widehat{X} \to X$ in the sense of Definition 33, it is immediate that its non-orbifold locus $X \setminus X^{\mathrm{orb}}$ coincides with $\pi(E)$, where $E \subset \widehat{X}$ is the exceptional locus of π. In particular, the non-orbifold locus is an analytic subset of X. This latter statement is very natural and should be true regardless of the existence of orbi-resolutions. Unfortunately, we are neither able to prove it in the general analytic setting nor able to locate a suitable reference. We can, however, prove it under the additional assumption that the singularities of X are algebraic. This is sufficient for the application in Section 7.

Lemma 35 (Analyticity of the non-orbifold locus). *Let X be a normal complex space having only algebraic singularities (in the sense of [16, Def. 2.4]). Then its non-orbifold locus $Z := X \setminus X^{\mathrm{orb}}$ is a closed analytic subset.*

In particular, this applies if X is a compact klt Kähler space with $c_1(X) = 0$.

Proof. When X is algebraic, this is a straightforward consequence of [3, Cor. 2.6]. If $U \subset X$ is a euclidean open subset of X being isomorphic through a map $\varphi\colon U \xrightarrow{\sim} V$ to an open subset $V \subset Y$ of an algebraic variety, then we have $\varphi(Z \cap U) = V \setminus V^{\mathrm{orb}}$, and this is an analytic subset of V by the algebraic case. The subset $Z \cap U$ is then given by the vanishing of a family of holomorphic functions, i.e. it is analytic in U.

The last statement is a consequence of [5, Thm. B]: X can be realized as a member of a locally trivial family which also has projective fibers. The family being locally trivial (over a smooth connected base), all the fibers are locally isomorphic and such an X then has locally algebraic singularities (cf. [16, Ex. 2.5]). □

4. Uniformization of canonical models

In this section, we prove Theorem A. Let us first introduce notation. We set $A := K_X + \Delta$ and pick a complete intersection surface $S = D_1 \cap \cdots \cap D_{n-2}$ of $n-2$ general hypersurfaces $D_i \in |mA|$, where m is sufficiently large and divisible. The proof is divided into four steps.

Step 1: The orbi Higgs-sheaf $(\mathscr{E}_X, \vartheta_X)$

Using the notation introduced in the proof of Lemma 14, we can find a (a priori non-smooth) orbi-étale structure $\mathscr{C} = \{U_\alpha, g_\alpha, U'_\alpha\}$ with respect to (X,Δ) on the whole X. Then, one can define the reflexive orbi-Higgs sheaf $(\mathscr{E}_X, \vartheta_X)$ with respect to $\mathscr{C}$ as follows:

$$\vartheta_X\colon \mathscr{E}_X := \Omega^{[1]}_{(X,\Delta)} \oplus \mathscr{O}_{(X,\Delta)} \longrightarrow \mathscr{E}_X \otimes \Omega^{[1]}_{(X,\Delta)}, \tag{9}$$

where on each chart U'_α, we define $\vartheta_{U'_\alpha}(a,f) := (0,a)$ where (a,f) is a section of $\mathscr{E}_{U'_\alpha} := \Omega^{[1]}_{U'_\alpha} \oplus \mathscr{O}_{U'_\alpha}$. Cf. also Definition 19 and [31, §5.1, Step 2].

In order to compute Chern numbers involving $\mathscr{E}_X$, one needs to introduce a global cover $f\colon Y \to X$ and an actual reflexive sheaf $\mathscr{E}_Y$ on Y as we now explain. Thanks to Proposition 13, there exists a finite morphism $f\colon Y \to X$ that is strictly adapted for (X, Δ) and whose extra ramification in codimension one (i.e. away from $\operatorname{supp}(\Delta)$) is supported over a general element H of a very ample linear system on X. Let N be the ramification order along H; we have

$$K_Y = f^*\left(K_X + \Delta + \left(1 - \frac{1}{N}\right)H\right). \tag{10}$$

We set $D := \Delta + \left(1 - \frac{1}{N}\right)H$ and define $(X, D)_{\mathrm{orb}}$ to be the largest open subset of X where the pair (X, D) admits a smooth orbi-étale orbi-structure $\mathscr{C}^\circ$; we know that $\operatorname{codim}_X(X \setminus (X, D)_{\mathrm{orb}}) \geq 3$ by Lemma 14. One can be a bit more precise about the shape of $\mathscr{C}^\circ$, which will be useful later. Recall from the proof of Lemma 14 that if we set $K := I \times J$ and $\alpha := (\beta, \gamma) \in K$, then we have a diagram

$$\begin{tikzcd} X'_\alpha \arrow[r, "f_\alpha"] \arrow[rr, bend left, "h_\alpha"] & U'_\alpha \arrow[r, "g_\alpha"] \arrow[d, hook] & U_\alpha \arrow[r, hook] \arrow[d, hook] & X \arrow[d, "\mathrm{id}"] \\ & U'_\beta \arrow[r, "g_\beta"] & U_\beta \arrow[r, hook] & X \end{tikzcd}$$

where X'_α is smooth and f_α is quasi-étale. Note that one can "restrict" $\mathscr{E}_X$ to the orbifold locus $\bigcup_\alpha U_\alpha \subset X$ of (X, Δ) to get a *locally free* orbi-Higgs sheaf with respect to the smooth orbi-étale structure $\{U_\alpha, h_\alpha, X'_\alpha\}_{\alpha \in K}$ for the pair (X, Δ) in codimension two, given by $\mathscr{E}_{X'_\alpha} := f_\alpha^{[*]}(\mathscr{E}_{U'_\beta}|_{U'_\alpha}) \simeq \Omega^1_{X'_\alpha} \oplus \mathscr{O}_{X'_\alpha}$. In particular, one can define the Chern number $\widetilde{c}_2(\mathscr{E}_X) \cdot A^{n-2}$ as explained in Section 3.1.

By choosing H general, one can arrange that $h_\alpha^* H$ is smooth for all indices $\alpha \in K$ thanks to Bertini's theorem, so that a further Kawamata cover $\kappa_\alpha\colon X_\alpha \to X'_\alpha$ orbi-étale with respect to $(X'_\alpha, h_\alpha^*(1 - \frac{1}{N})H)$ yields the expected smooth orbi-étale orbi-structure $\mathscr{C}^\circ := \{U_\alpha, p_\alpha, X_\alpha\}_{\alpha \in K}$ for the pair (X, D) in codimension two where $p_\alpha = h_\alpha \circ \kappa_\alpha$. We end up with the following factorization:

$$\begin{tikzcd} X_\alpha \arrow[rr, "p_\alpha"] \arrow[dr, "\kappa_\alpha"'] & & U_\alpha \arrow[r, "\text{étale}"] & X \\ & X'_\alpha \arrow[ur, "h_\alpha"'] & & \end{tikzcd}$$

Next, set

$$Y^\circ := f^{-1}\big((X, D)_{\mathrm{orb}}\big) \cap (Y, \emptyset)_{\mathrm{orb}} \subset Y.$$

Since f is finite, and by Lemma 14 applied to $(Y, \emptyset)$, we have $\operatorname{codim}_Y(Y \setminus Y^\circ) \geq 3$. The map f restricts to $f^\circ\colon Y^\circ \to X^\circ := (X, D)_{\mathrm{orb}}$.

Finally, we set $T := f^{-1}(S)$. Since the linear system $|mA|$ (resp. $f^*|mA|$) is basepoint-free and S is general, we have $S \subset X^\circ$ (resp. $T \subset Y^\circ$). Also, recall from Lemma 20 that $(S, D|_S)$ has quotient singularities. The following diagram summarizes the situation:

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d, "f|_T"'] & Y^\circ \arrow[r, hook] \arrow[d, "f^\circ"] & Y \arrow[d, "f"] \\ S \arrow[r, hook] & X^\circ \arrow[r, hook] & X \end{tikzcd}$$

Moreover, the ramification formula $K_T = f^*(K_S + D|_S)$ shows that T is klt as well, i.e. it is a surface with quotient singularities.

Step 2: Computing Chern numbers for $\mathscr{E}_X$.

Set $\Delta^\circ := \Delta|_{X^\circ}$ and $D^\circ := D|_{X^\circ}$. Consider the locally free orbi-sheaf for the pair (X°, D°) with respect to the orbi-structure $\mathscr{C}^\circ$ constructed in Step 1 above, defined by

$$\mathscr{E}_{X_\alpha} = \Omega^{[1]}_{(X^\circ,\Delta^\circ,p_\alpha)} \oplus \mathscr{O}_{X_\alpha}. \tag{11}$$

Since $\left(X_\alpha, p_\alpha^{-1}(H)\right)$ is log smooth, the subsheaf $\Omega^{[1]}_{(X^\circ,\Delta^\circ,p_\alpha)} \subset \Omega^1_{X_\alpha}$ has a very explicit expression in terms of local coordinates. More precisely, if $(z_1,\dots,z_n)$ is a local chart such that $p_\alpha^{-1}(H) = \{z_1 = 0\}$ on that chart, then the bundle at play is the subbundle of $\Omega^1_{X_\alpha}$ generated by $z_1^{N-1}\mathrm{d}z_1, \mathrm{d}z_2, \dots, \mathrm{d}z_n$. In particular, it agrees with $\Omega^1_{X_\alpha}$ outside of $p_\alpha^{-1}(H)$.

Now set $\mathscr{E}_Y := \Omega^{[1]}_{(X,\Delta,f)} \oplus \mathscr{O}_Y \subset \Omega^{[1]}_Y \oplus \mathscr{O}_Y$, which we should think of as the reflexive pull back of $\mathscr{E}_X$ by f. We equip this sheaf with the usual Higgs field ϑ_Y, and denote by $\mathscr{E}_{Y^\circ}$ its restriction to Y°. Note that by (2), $\mathscr{E}_Y = \Omega^{[1]}_Y \oplus \mathscr{O}_Y$ holds on $Y \setminus f^{-1}(H)$. Let $\left\{(V_\beta, q_\beta, Y_\beta)\right\}_{\beta\in K}$ be a smooth orbi-étale (i.e. quasi-étale, in this case) orbi-structure for $(Y^\circ, \emptyset)$, which exists by (3) and Lemma 14 again, at least after shrinking Y°. Set $\mathscr{E}_{Y_\beta} := q_\beta^{[*]}\mathscr{E}_Y$ and consider the diagram

$$\begin{tikzcd} W_{\alpha\beta} \arrow[r, "r_{\alpha\beta}"] \arrow[dd, "g_{\alpha\beta}"'] & Y_\beta \arrow[d, "q_\beta"] \\ & Y^\circ \arrow[d, "f"] \\ X_\alpha \arrow[r, "p_\alpha"] & X^\circ \end{tikzcd} \tag{12}$$

where $W_{\alpha\beta}$ is the normalization of $X_\alpha \times_{X^\circ} Y_\beta$. Since p_α is orbi-étale with respect to D°, the map $r_{\alpha\beta}$ is étale over $X^\circ_{\mathrm{reg}} \setminus \mathrm{supp}(D^\circ)$. Moreover, since q_β is quasi-étale, it follows that $f \circ q_\beta$ and p_α ramify to the same order along each component of D. In other words, the smooth orbi-étale orbi-structures $\mathscr{C}^\circ$ and $\left\{\left(f(V_\beta), f\circ q_\beta, Y_\beta\right)\right\}$ are compatible. In particular, $g_{\alpha\beta}$ and $r_{\alpha\beta}$ are étale so that $W_{\alpha\beta}$ is smooth, and we have additionally $g^*_{\alpha\beta}\mathscr{E}_{X_\alpha} \cong r^*_{\alpha\beta}\mathscr{E}_{Y_\beta}$ by (3). Since $\mathscr{E}_{X_\alpha}$ is locally free, so is $\mathscr{E}_{Y_\beta}$, so that the reflexive sheaf $\mathscr{E}_{Y^\circ}$ is a genuine orbifold bundle on the orbifold Y°.

Let ω be an orbifold Kähler metric adapted to (X°, Δ°), as given by Lemma 11. It is defined on an arbitrarily large relatively compact open subset of X°. In particular, it is defined in a neighborhood of S and this will be enough for our purposes. Set $S^* := S_{\mathrm{reg}} \setminus \mathrm{supp}\, D$. By definition, one has

$$\widetilde{\mathrm{c}}_2\left(\Omega^{[1]}_{(X,\Delta)}\big|_S\right) = \int_{S_{\mathrm{reg}}\setminus\mathrm{supp}(\Delta)} \mathrm{c}_2(\Omega^1_{X_{\mathrm{reg}}}, \omega) = \int_{S^*} \mathrm{c}_2(\Omega^1_{X_{\mathrm{reg}}}, \omega)$$

and the last two integrals on the right are well-defined since ω pulls back to a smooth Kähler metric across points in $S_{\mathrm{sing}} \cup \mathrm{supp}(\Delta)$ via the finite maps h_α. The smooth form $p_\alpha^*\omega = f_\alpha^* h_\alpha^*\omega$ is semipositive, degenerate along $p_\alpha^{-1}(H)$. More precisely, if $p_\alpha^{-1}(H) \cap U = \{z_1 = 0\}$ for some coordinate chart $U \subset X_\alpha$, then

$$\begin{aligned} p_\alpha^*\omega|_U = a_{1\bar{1}}|z_1|^{2(N-1)} i\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \sum_{k=2}^n a_{1\bar{k}} z_1^{N-1}\mathrm{d}z_1 \wedge i\mathrm{d}\bar{z}_k \\ + \sum_{k=2}^n a_{k\bar{1}}\bar{z}_1^{N-1}\mathrm{d}z_k \wedge i\mathrm{d}\bar{z}_1 + \sum_{j,k=2}^n a_{j\bar{k}}\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k \end{aligned}$$

where $(a_{j\bar{k}})$ is smooth and definite positive. In particular, $p_\alpha^*\omega$ defines a smooth Hermitian metric on $\Omega^{[1]}_{(X^\circ,\Delta^\circ,p_\alpha)}$. Said otherwise, $g^*_{\alpha\beta}p^*_\alpha\omega$ induces a smooth Hermitian metric on $g^*_{\alpha\beta}\Omega^{[1]}_{(X^\circ,\Delta^\circ,p_\alpha)} \cong r^*_{\alpha\beta}\Omega^{[1]}_{(X^\circ,\Delta^\circ,f\circ q_\beta)}$. Hence, $q^*_\beta f^*\omega$ is a smooth Hermitian metric on the vector bundle $\Omega^{[1]}_{(X^\circ,\Delta^\circ,f\circ q_\beta)} = q_\beta^{[*]}\Omega^{[1]}_{(X^\circ,\Delta^\circ,f)}$, so that $f^*\omega$ induces an orbifold metric on the orbi-bundle $\Omega^{[1]}_{(X^\circ,\Delta^\circ,f)}$. By the definition of the Chern classes of orbifold vector bundles, we have

$$\begin{aligned}\widetilde{c}_2\Big(\Omega^{[1]}_{(X^\circ,\Delta^\circ,f)}\big|_T\Big) &= \int_{f^{-1}(S^*)} c_2(\Omega^1_{Y_{\mathrm{reg}}}, f^*\omega)\\ &= \deg(f|_T)\cdot\int_{S^*} c_2(\Omega^1_{X_{\mathrm{reg}}},\omega)\\ &= \deg(f)\cdot\widetilde{c}_2\Big(\Omega^{[1]}_{(X,\Delta)}\big|_S\Big)\end{aligned}$$

where the last identity follows from $\deg(f|_T) = \deg(f)$ since S is general. All in all, we find by Lemma 32

$$\widetilde{c}_2(\mathscr{E}_Y)\cdot(f^*A)^{n-2} = \deg(f)\,\widetilde{c}_2(\mathscr{E}_X)\cdot A^{n-2}. \tag{13}$$

The same arguments show the similar identity

$$\widetilde{c}_1^2(\mathscr{E}_Y)\cdot(f^*A)^{n-2} = \deg(f)\,\widetilde{c}_1^2(\mathscr{E}_X)\cdot A^{n-2}. \tag{14}$$

Step 3: (X,Δ) *has quotient singularities*

Consider on X the orbi-Higgs sheaf $(\mathscr{F}_X,\Theta_X) := \mathrm{End}(\mathscr{E}_X,\vartheta_X)$. It satisfies:

$$\widetilde{c}_1^2(\mathscr{F}_X)\cdot A^{n-2} = \widetilde{c}_2(\mathscr{F}_X)\cdot A^{n-2} = 0,$$

as follows from the assumption on the Chern classes of (X,Δ), i.e. the assumption that equality holds in (2). Combined with (13)–(14), the latter identity implies that the (genuine) Higgs sheaf $(\mathscr{F}_Y,\Theta_Y) := \mathrm{End}(\mathscr{E}_Y,\vartheta_Y)$ on Y satisfies

$$\widetilde{c}_1^2(\mathscr{F}_Y)\cdot(f^*A)^{n-2} = \widetilde{c}_2(\mathscr{F}_Y)\cdot(f^*A)^{n-2} = 0.$$

Moreover, by [31, §4.4, proof of Thm. C], the sheaf $\Omega^{[1]}_{(X,\Delta,f)}$ is (f^*A)-semistable. Recall that $c_1(\Omega^{[1]}_{(X,\Delta,f)}) = f^*A$ by [17, (3.11.5)]. It follows that $(\mathscr{E}_Y,\vartheta_Y)$ is (f^*A)-Higgs-stable, cf. the calculations in [29, proof of Cor. 7.2]. This in turn implies that the endomorphism sheaf $(\mathscr{F}_Y,\Theta_Y)$ is (f^*A)-Higgs-polystable. Indeed, the last assertion can be deduced from the usual smooth case by restricting to a general complete intersection curve and using the Mehta–Ramanathan theorem for Higgs sheaves [29, Thm. 5.22]. Cf. also [30, Lem. 4.7].

By the Simpson correspondence for klt spaces [30, Thm. 5.1], the Higgs sheaf $(\mathscr{F}_Y,\Theta_Y)\big|_{Y_{\mathrm{reg}}}$ is locally free and is induced by a tame, purely imaginary harmonic bundle. By [30, Prop. 3.17], the reflexive pull-back $g^{[*]}\mathscr{F}_Y$ of $\mathscr{F}_Y$ to a maximally quasi-étale cover $g\colon Z\to Y$ (whose existence is guaranteed by [27, Thm. 1.5]) is locally free.

Now, set $W := X\setminus H\subset X$ and $h := f\circ g\colon Z\to X$. On $h^{-1}(W)$, we have that

$$g^{[*]}\mathscr{E}_Y \cong g^{[*]}\big(\Omega^{[1]}_Y\oplus\mathscr{O}_Y\big)\cong\Omega^{[1]}_Z\oplus\mathscr{O}_Z.$$

It follows that $g^{[*]}\mathscr{F}_Y\cong\mathrm{End}\big(\Omega^{[1]}_Z\oplus\mathscr{O}_Z\big)$, which contains the tangent sheaf $\mathscr{T}_Z$ as a direct summand (again, only on $h^{-1}(W)$). Since direct summands of locally free sheaves are locally free by Nakayama's lemma, the resolution of the Lipman–Zariski Conjecture for klt spaces [20, 25, 26] implies that $h^{-1}(W)$ is smooth.

By construction, the map $h^{-1}(W)\to W$ is branched exactly at $\Delta|_W$. By Corollary 27, its Galois closure $\widetilde{W}\to W$ also has this property, and $\widetilde{W}$ is smooth, being a quasi-étale (hence étale) cover of the smooth space $h^{-1}(W)$. This shows that $(W,\Delta|_W)$ has quotient singularities. So far, we have only imposed that H is general in its (basepoint-free) linear system. We can therefore repeat the argument by choosing general elements $H_1,\dots,H_{n+1}\in|H|$ and conclude that (X,Δ) has quotient singularities. This means that (X,Δ) is a “complex orbifold” in the sense of [10, p. 109].

Step 4: (X,Δ) is a ball quotient

Since (X,Δ) is a complex orbifold with $K_X+\Delta$ ample, there is an orbifold Kähler–Einstein metric ω such that $\operatorname{Ric}\omega=-\omega$, cf. [10, Thm. 5.2.2]. Set $X^* := X_{\mathrm{reg}} \setminus \operatorname{supp}(\Delta)$, so that ω is a genuine Kähler metric on X^*. One can compute the orbifold Chern classes using ω, and, in particular, one has from the usual Chern form computations

$$\begin{aligned} 0 &= \big(2(n+1)\widetilde{c}_2(X,\Delta) - n\widetilde{c}_1^2(X,\Delta)\big)\cdot[K_X+\Delta]^{n-2} \\ &= \int_{X^*}\big(2(n+1)c_2(X,\omega) - nc_1^2(X,\omega)\big)\wedge\omega^{n-2} \\ &= C_n\int_{X^*}|\Theta^\circ(T_X,\omega)|^2_\omega\,\omega^n, \end{aligned}$$

where $C_n>0$ is a dimensional constant, while

$$\Theta^\circ(T_X,\omega) := \Theta(T_X,\omega) - \frac{1}{n}\operatorname{tr}_{\mathrm{End}}(\Theta(T_X,\omega))\cdot\mathrm{id}_{T_X}$$

is the trace-free Chern curvature tensor of (T_X,ω).

As a result, ω has constant negative bisectional curvature. This implies that ω has negative Riemannian sectional curvature on X^* by e.g. [23, §2.4.2]. (Note that one could also have said that (X^*,ω) is locally isometric to the complex hyperbolic space $(\mathbb{B}^n,\omega_{\mathrm{hyp}})$ by [9, Thm. 6] and conclude by the usual curvature properties of the complex hyperbolic metric.)

Let $\pi\colon \widetilde{X}_\Delta\to X$ be the orbifold universal cover of (X,Δ), cf. Definition 24. By the previous paragraph, (X,Δ,ω) is an orbifold of nonpositive Riemannian sectional curvature. It then follows from [12, Cor. 2.16 on p. 603] that (X,Δ) is developable. Now, $(\widetilde{X}_\Delta,\pi^*\omega)$ is a simply connected Kähler manifold with constant negative bisectional curvature, so it is holomorphically isometric to $(\mathbb{B}^n,\omega_{\mathrm{hyp}})$ by [34, Thm. 7.9]. In particular, $\widetilde{X}_\Delta\cong\mathbb{B}^n$, proving Theorem A. □

5. Characterization of ball quotients

In this section, we prove Corollary 3. We prove the implications (1) ⇒ (2) ⇒ (3) ⇒ (1) separately.

(1) ⇒ (2). This is Theorem A.

(2) ⇒ (3). Let $\pi\colon\mathbb{B}^n\to X$ be the orbifold universal cover of (X,Δ). (In particular, (X,Δ) is developable.) By (2), the map π is Galois, with Galois group $\Gamma\cong\pi_1^{\mathrm{orb}}(X,\Delta)$. Note that $\Gamma\subset\operatorname{Aut}(\mathbb{B}^n)=\mathrm{PU}(1,n)$ is a finitely generated linear group. Furthermore, the stabilizers of the action $\Gamma \circlearrowright \mathbb{B}^n$ are finite by (4). By Selberg's lemma [2], there is a finite index normal subgroup $\Gamma'\subset\Gamma$ which is torsion-free. This implies that Γ' acts freely on $\mathbb{B}^n$. We obtain the following factorization of π:

$$\mathbb{B}^n \longrightarrow \mathbb{B}^n/\Gamma' \xrightarrow{\ f\ } \mathbb{B}^n/\Gamma = X,$$

where f is the quotient by the action of the finite group $G:=\Gamma/\Gamma'$ on the projective manifold $Y:=\mathbb{B}^n/\Gamma'$. Since the first map is étale, it exhibits $\mathbb{B}^n$ as the universal cover of Y. Combining this with the fact that π is branched exactly at Δ, we infer that f is orbi-étale.

(3) ⇒ (1). Recall that K_Y is ample and that Y satisfies equality in the Miyaoka–Yau inequality, cf. e.g. [35, (8.8.3)]. As $f\colon Y\to X$ is orbi-étale, it follows that also $K_X+\Delta$ is ample and equality likewise holds in the Miyaoka–Yau inequality for (X,Δ). □

6. Uniformization of minimal models

This section has two (related) purposes: first, to remove the assumption about the irreducible components of Δ being $\mathbb{Q}$-Cartier from Theorem 2. And second, to prove Corollary 4.

6.1. *Orbifold Miyaoka–Yau inequality*

In Theorem 2, or more generally in [31, Thm. B], the assumption that the Δ_i be $\mathbb{Q}$-Cartier can be dropped without replacement. We give two proofs of this result, the first one relying on [7] and the second one on Proposition 13.

Theorem 36 (Miyaoka–Yau inequality). *Let (X,Δ) be an n-dimensional projective klt pair with standard coefficients, and assume that $K_X+\Delta$ is big and nef. Then the following inequality holds:*

$$\big(2(n+1)\,\widetilde{c}_2(X,\Delta)-n\,\widetilde{c}_1^2(X,\Delta)\big)\cdot[K_X+\Delta]^{n-2}\geq 0. \tag{15}$$

First proof. Consider a $\mathbb{Q}$-factorialization $f\colon X'\to X$, cf. [7, Cor. 1.4.3] applied with $\mathfrak{E}=\emptyset$. Set $\Delta':=f_*^{-1}\Delta$. The map f is small, meaning that $\mathrm{Exc}(f)\subset X'$ has codimension at least two. Therefore (X',Δ') reproduces all the assumptions made on (X,Δ), and in addition X' is $\mathbb{Q}$-factorial. In particular, $K_{X'}+\Delta'=f^*(K_X+\Delta)$ is big and nef. Furthermore, $f(\mathrm{Exc}(f))\subset X$ has codimension ≥ 3, therefore $f_*\big(\widetilde{c}_2\big(X',\Delta'\big)\big)=\widetilde{c}_2(X,\Delta)$ as homology classes, and likewise for $\widetilde{c}_1^2\big(X',\Delta'\big)$ (cf. Remark 31). By the projection formula, we obtain

$$\big(2(n+1)\,\widetilde{c}_2(X,\Delta)-n\,\widetilde{c}_1^2(X,\Delta)\big)\cdot[K_X+\Delta]^{n-2}=\big(2(n+1)\,\widetilde{c}_2\big(X',\Delta'\big)-n\,\widetilde{c}_1^2\big(X',\Delta'\big)\big)\cdot[K_{X'}+\Delta']^{n-2}.$$

The right-hand side is non-negative by [31, Thm. B]. □

Second proof. Observe that in [31], the assumption that the Δ_i be $\mathbb{Q}$-Cartier is only used in order to construct a strictly adapted morphism whose extra ramification is supported on a general very ample divisor (cf. Ex. 2.11 of that paper). However, using Proposition 13 we can construct such a cover even without that assumption. After that, the proof of [31, Thm. B] applies verbatim. □

6.2. *Uniformization of minimal models*

In order to prove Corollary 4, we use the strategy explained in [30, Step 1, p. 1086]. This means we first have to prove the following lemma.

Lemma 37. *In the setting of Corollary 4, the canonical model $(X_{\mathrm{can}},\Delta_{\mathrm{can}})$ also satisfies equality in* (2).

Assuming Lemma 37 for the moment, we then apply Theorem A on $(X_{\mathrm{can}},\Delta_{\mathrm{can}})$ to conclude. This finishes the proof of Corollary 4.

Remark 38. If we had proved Theorem A only in the setting of [31] (that is, assuming that the Δ_i are $\mathbb{Q}$-Cartier), then the above argument would break down. This is because the irreducible components of Δ_{can} may not be $\mathbb{Q}$-Cartier (even if the same is true of Δ).

Proof of Lemma 37. As in the statement of Corollary 4, let $(X_{\mathrm{can}},\Delta_{\mathrm{can}})$ denote the canonical model of the pair (X,Δ) and $\pi\colon(X,\Delta)\to(X_{\mathrm{can}},\Delta_{\mathrm{can}})$ the canonical morphism ($K_X+\Delta$ being big and nef, some multiple is basepoint-free and so π is a morphism). By construction, $K_{X_{\mathrm{can}}}+\Delta_{\mathrm{can}}$ is ample and π is crepant:

$$K_X+\Delta=\pi^*\big(K_{X_{\mathrm{can}}}+\Delta_{\mathrm{can}}\big). \tag{16}$$

The pair $(X_{\mathrm{can}},\Delta_{\mathrm{can}})$ still has klt singularities. From Theorem 2, we know that the inequality (2) holds for $(X_{\mathrm{can}},\Delta_{\mathrm{can}})$ and we are led to checking that:

$$\begin{aligned}&\big(2(n+1)\,\widetilde{c}_2(X,\Delta)-n\,\widetilde{c}_1^2(X,\Delta)\big)\cdot[K_X+\Delta]^{n-2}\\ &\qquad\geq\big(2(n+1)\,\widetilde{c}_2(X_{\mathrm{can}},\Delta_{\mathrm{can}})-n\,\widetilde{c}_1^2(X_{\mathrm{can}},\Delta_{\mathrm{can}})\big)\cdot[K_{X_{\mathrm{can}}}+\Delta_{\mathrm{can}}]^{n-2}.\end{aligned} \tag{17}$$

In view of (16), this amounts to showing

$$\widetilde{c}_2(X,\Delta)\cdot[K_X+\Delta]^{n-2}\geq\widetilde{c}_2(X_{\mathrm{can}},\Delta_{\mathrm{can}})\cdot[K_{X_{\mathrm{can}}}+\Delta_{\mathrm{can}}]^{n-2}. \tag{18}$$

At this point, let us consider a general surface $\Sigma \subset X_{\mathrm{can}}$ cut out by the linear system $|m(K_{X_{\mathrm{can}}} + \Delta_{\mathrm{can}})|$ (for $m \gg 1$ sufficiently divisible) and let us look at its preimage $S := \pi^{-1}(\Sigma) \subset X$ in X. The pairs[2] (S,Δ) and $(\Sigma,\Delta_{\mathrm{can}})$ are orbifold surfaces and contained in the orbifold loci of (X,Δ) and $(X_{\mathrm{can}},\Delta_{\mathrm{can}})$ respectively. Obviously, $(\Sigma,\Delta_{\mathrm{can}})$ is nothing but $(S,\Delta)_{\mathrm{can}}$ and we can apply [40, Thm. 4.2]. This yields

$$4\,\widetilde{\mathrm{c}}_2(\Sigma,\Delta_{\mathrm{can}}) - \widetilde{\mathrm{c}}_1^2(\Sigma,\Delta_{\mathrm{can}}) \le 4\,\widetilde{\mathrm{c}}_2(S,\Delta) - \widetilde{\mathrm{c}}_1^2(S,\Delta)\,.$$

The morphism $\pi|_S\colon (S,\Delta) \to (\Sigma,\Delta_{\mathrm{can}})$ being crepant, the above inequality reads as

$$\widetilde{\mathrm{c}}_2(\Sigma,\Delta_{\mathrm{can}}) \le \widetilde{\mathrm{c}}_2(S,\Delta)\,. \tag{19}$$

With the notation introduced, the inequality (18) boils down to the following:

$$\widetilde{\mathrm{c}}_2\big(\mathscr{T}_{(X,\Delta)}\big|_S\big) \ge \widetilde{\mathrm{c}}_2\big(\mathscr{T}_{(X_{\mathrm{can}},\Delta_{\mathrm{can}})}\big|_\Sigma\big)\,.$$

This last inequality can be checked as in [30, pp. 1086–1087] by considering the (orbifold) normal sequences

$$0 \longrightarrow \mathscr{T}_{(S,\Delta)} \longrightarrow \mathscr{T}_{(X,\Delta)}\big|_S \longrightarrow \mathscr{N}_{(S,\Delta)|(X,\Delta)} \longrightarrow 0, \tag{20}$$

$$0 \longrightarrow \mathscr{T}_{(\Sigma,\Delta_{\mathrm{can}})} \longrightarrow \mathscr{T}_{(X_{\mathrm{can}},\Delta_{\mathrm{can}})}\big|_\Sigma \longrightarrow \mathscr{N}_{(\Sigma,\Delta_{\mathrm{can}})|(X_{\mathrm{can}},\Delta_{\mathrm{can}})} \longrightarrow 0. \tag{21}$$

It is worth noting that both sequences (20) and (21) are exact sequences of orbifold vector bundles, since the surface S (resp. Σ) is contained in the orbifold locus of (X,Δ) (resp. $(X_{\mathrm{can}},\Delta_{\mathrm{can}})$) and the terms in the middle are thus genuine orbifold bundles. Now it is enough to remark that the normal bundles $\mathscr{N}_{(S,\Delta)|(X,\Delta)}$ and $\mathscr{N}_{(\Sigma,\Delta_{\mathrm{can}})|(X_{\mathrm{can}},\Delta_{\mathrm{can}})}$ satisfy

$$\mathscr{N}_{(S,\Delta)|(X,\Delta)} \cong \pi^*\big(\mathscr{N}_{(\Sigma,\Delta_{\mathrm{can}})|(X_{\mathrm{can}},\Delta_{\mathrm{can}})}\big)\,. \tag{22}$$

Together with (16) and (19), this finally proves that the inequality (18) holds true. This concludes the proof of Lemma 37. □

Remark. In general, the canonical morphism $\pi|_S\colon (S,\Delta) \to (\Sigma,\Delta_{\mathrm{can}})$ is *not* an orbifold morphism, but the normal bundles are actual locally free sheaves defined on S (resp. on Σ) and not only on the orbifold (S,Δ) (resp. $(\Sigma,\Delta_{\mathrm{can}})$). The Chern classes of $\mathscr{N}_{(\Sigma,\Delta_{\mathrm{can}})|(X_{\mathrm{can}},\Delta_{\mathrm{can}})}$ thus come from Σ and can be pulled back to S in the usual way.

7. Characterization of torus quotients

In this final section, we first establish the positivity of the orbifold second Chern class for Calabi–Yau and for irreducible holomorphic symplectic varieties. Using the Decomposition Theorem [5], we can then easily deduce Theorem 6 and Theorem B. Finally, we prove Corollary 7.

7.1. *Positivity of the second Chern class — the projective case*

If X is projective, then we know that it has an orbi-resolution in the sense of Definition 33, and we can use this to understand the orbifold second Chern class of X.

Proposition 39. *Let X be a projective irreducible Calabi–Yau (resp. irreducible holomorphic symplectic) variety of dimension n with klt singularities and let $\beta \in \mathrm{H}^2(X,\mathbb{R})$ be a Kähler class. Then we have*

$$\widetilde{\mathrm{c}}_2(X)\cdot\beta^{n-2} > 0.$$

[2] To avoid cumbersome notation, the restriction of the divisors Δ and Δ_{can} to S and Σ is not written out.

Proof. Let $\pi\colon \widehat{X} \to X$ be an orbi-resolution, whose existence is garanteed by [37] since X is projective. Let $\widehat{\beta}$ be a Kähler class on $\widehat{X}$ and let $\omega \in \beta$ (resp. $\widehat{\omega} \in \widehat{\beta}$) be a Kähler form. Recall that it follows easily from the Bochner principle [16, Thm. A] that T_X is stable with respect to β. This implies that $T_{\widehat{X}}$ is stable with respect to $\pi^*\beta$, hence $T_{\widehat{X}}$ is stable with respect to $\pi^*\beta + \varepsilon\widehat{\beta}$ for $\varepsilon > 0$ small enough, cf e.g. [15, Prop. 3.4]. In particular, as explained in [21, Thm. 4.2], there exists an orbifold Hermite–Einstein metrics h_ε on $T_{\widehat{X}}$ with respect to $\omega_\varepsilon := \pi^*\omega + \varepsilon\widehat{\omega}$. From Lemma 34, we have

$$\widetilde{\mathrm{c}}_2(X)\cdot\beta^{n-2} = \lim_{\varepsilon\to 0}\int_{\widehat{X}} \mathrm{c}_2^{\mathrm{orb}}(T_{\widehat{X}}, h_\varepsilon)\wedge\omega_\varepsilon^{n-2}.$$

The exact same arguments as in [15, Prop. 3.11] using orbifold forms instead of usual forms shows that the latter quantity is non-negative, and if it is zero, then we have $\widetilde{\mathrm{c}}_2(X)\cdot\gamma^{n-2} = 0$ for *any* Kähler class γ on X. We claim that this cannot happen. Indeed, since X is projective, this applies to classes of the form $\mathrm{c}_1(H)$ for an ample divisor H on X. Then [38] would imply that X is the quotient of an Abelian variety, clearly a contradiction. □

7.2. *Positivity of the second Chern class — the IHS case*

We will derive the general Kähler case from the projective one using a deformation argument, as in [15, Prop. 4.4].

Proposition 40. *Let X be an irreducible holomorphic symplectic variety of dimension n with klt singularities and let $\beta \in \mathrm{H}^2(X,\mathbb{R})$ be a Kähler class. Then we have*

$$\widetilde{\mathrm{c}}_2(X)\cdot\beta^{n-2} > 0.$$

Proof. We will first prove that there exists a constant $C_X \in \mathbb{R}$ such that

$$\widetilde{\mathrm{c}}_2(X)\cdot a = C_X\, q_X(a)^{\frac{n}{2}-1} \tag{23}$$

for any $a \in \mathrm{H}^2(X,\mathbb{R})$, where $q_X\colon \mathrm{H}^2(X,\mathbb{R}) \to \mathbb{C}$ is the Beauville–Bogomolov–Fujiki quadratic form. Moreover, we will see that C_X is constant when X moves in a locally trivial family.

The result follows from standard arguments (see e.g. [15, Prop. 4.4] and references therein) once one has proved that the formation of $\widetilde{\mathrm{c}}_2(X)\cdot a$ is invariant under parallel transport along a locally trivial deformation, which we now prove.

Let $\pi\colon \mathfrak{X} \to \mathbb{D}$ be a proper surjective map which is a locally trivial deformation of $X = \pi^{-1}(0)$. We denote by $\mathfrak{X}^{\mathrm{orb}}$ (resp. X_t^{orb}) the orbifold locus of $\mathfrak{X}$ (resp. X_t), which is a Zariski open subset of $\mathfrak{X}$ (resp. X_t) according to Lemma 35. Next, we set $Z := \mathfrak{X}\setminus\mathfrak{X}^{\mathrm{orb}}$ and $Z_t = Z\cap X_t$. The family being locally trivial, we infer that $\mathfrak{X}^{\mathrm{orb}}\cap X_t = X_t^{\mathrm{orb}}$ and thus that $Z_t = X_t \setminus X_t^{\mathrm{orb}}$.

Claim 41. *Up to shrinking $\mathbb{D}$, there exists a $\mathscr{C}^\infty$ diffeomorphism $F : \mathfrak{X} \to X_0\times\mathbb{D}$ commuting with the projection to $\mathbb{D}$ such that*

(i) *F preserves the orbifold locus, i.e. $F(X_t^{\mathrm{orb}}) = X_0^{\mathrm{orb}}\times\{t\}$.*
(ii) *$F|_{X_t^{\mathrm{orb}}}\colon X_t^{\mathrm{orb}} \to X_0^{\mathrm{orb}}$ is smooth in the orbifold sense.*

In this singular context, we mean that F is the restriction of a smooth map under local embeddings in $\mathbb{C}^N$ which induces an homeomorphism between $\mathfrak{X}$ and $X_0\times\mathbb{D}$.

Proof of Claim 41. Let us start with the existence of the diffeomorphism F. To do so, one can find a proper $\mathscr{C}^\infty$ embedding $\iota\colon \mathfrak{X} \hookrightarrow \mathbb{C}^N$ thanks to [1]. Next, extend π smoothly to a smooth map f with support in a neighborhood of $\iota(X)$. Since $\pi : \mathfrak{X} \to \mathbb{D}$ is locally trivial, one can stratify $\mathfrak{X}$ such that the restriction of π to each stratum is proper and smooth (in the analytic sense, i.e. it is a submersion). The existence of F then follows from Thom's first isotopy lemma, cf [39, Prop. 11.1].

In order to prove the two items in the claim, let us briefly recall the construction of F in loc. cit. while emphasizing on the important points for our purposes. Start with local holomorphic trivializations $g_\alpha : U_\alpha \to (U_\alpha \cap X_0) \times \mathbb{D}$ for a covering of analytic open sets $(U_\alpha)_{\alpha\in A}$ of $\mathfrak{X}$, and let $Z = \bigsqcup Z^{(k)}$ be the standard stratification of the analytic set $Z \subset \mathfrak{X}$. The maps g_α induces a local biholomorphism between $Z^{(k)}$ and $Z_0^{(k)} \times \mathbb{D}$ for all k; in particular the holomorphic vector fields $v_\alpha := g_\alpha^* \frac{\partial}{\partial t}$ satisfy

$$v_\alpha\big|_{Z^{(k)}} \in \mathrm{H}^0\Big(Z^{(k)}, \mathscr{T}_{Z^{(k)}}\Big)$$

Next, let (χ_α) be a partition of unity subordinate to the open cover $(U_\alpha)_{\alpha\in A}$. The $\mathscr{C}^\infty$ vector field $v := \sum \chi_\alpha v_\alpha$ still satisfies

$$v|_{Z^{(k)}} \in \mathscr{C}^\infty(Z^{(k)}, T_{Z^{(k)}}).$$

As showed in [39], its flow (F_t) is well-defined over $\pi^{-1}(\mathbb{D}_{1/2})$ for $|t| < 1/2$, and it preserves $Z^{(k)}$ for all k, hence it preserves Z as well. Equivalently, the flow of v preserves $\mathfrak{X}^{\mathrm{orb}}$, which proves (i).

Moreover, $v|_{\mathfrak{X}^{\mathrm{orb}}}$ is smooth in the orbifold sense (i.e. when pulled back to the local smooth covers), a property which need not be true for arbitrary vector fields. This is straightforward since the v_α satisfy this property (they lift to holomorphic vector fields on the quasi-étale local covers), and multiplying by smooth functions is harmless. In order to prove (ii), let $x_0 \in X_0^{\mathrm{orb}}$ be an arbitrary point and let $U \subset \mathfrak{X}^{\mathrm{orb}}$ be a small connected open neighborhood of x_0 admitting a smooth quasi-étale cover $p : \widehat{U} \to U$. We can find $U' \Subset U$ such that for $|t| \le s$ (with $s > 0$ small enough) the flow F_t is defined on U and satisfies $F_t(U') \subset U$. Remember that $\widehat{v} := p^* v|_{U_{\mathrm{reg}}}$ extends to a smooth vector field on $\widehat{U}$ which we still denote by $\widehat{v}$, and whose flow we denote by $\widehat{F}_t$. Since p is étale over U_{reg}, uniqueness of flow ensures that we have a commutative diagram

$$\begin{array}{ccc} p^{-1}(U') & \xrightarrow{\widehat{F}_t} & p^{-1}(F_t(U')) \\ \downarrow p & & \downarrow p \\ U' & \xrightarrow{F_t} & F_t(U'). \end{array}$$

Indeed, since p is a local diffeomorphism over U_{reg}, we get

$$F_t \circ p = p \circ \widehat{F}_t \text{ on } p^{-1}(U_{\mathrm{reg}}),$$

hence everywhere by continuity of the above maps. In summary, $F_t : U' \to F_t(U')$ is an homeomorphism which therefore lifts to the diffeomorphism $\widehat{F}_t$ between the manifolds $p^{-1}(U')$ and its image $p^{-1}(F_t(U'))$. That is, F_t induces an orbifold diffeomorphism between U' and $F_t(U')$. Item (ii) is now proved. □

Let us now consider the orbifold diffeomorphisms $F_t^{\mathrm{orb}} : X_t^{\mathrm{orb}} \to X_0^{\mathrm{orb}}$, and let h_0 be an orbifold Hermitian metric on $T_{X_0^{\mathrm{orb}}}$. Finally, let α_0 be a closed orbifold form with compact support on X_0^{orb} representing a class $a_0 \in \mathrm{H}^{2n-4}(X_0, \mathbb{R})$. We have

$$\begin{aligned} \widetilde{\mathrm{c}}_2(X_0) \cdot a_0 &= \int_{X_0^{\mathrm{orb}}} \mathrm{c}_2^{\mathrm{orb}}(X_0^{\mathrm{orb}}, h_0) \wedge \alpha_0 \\ &= \int_{X_t^{\mathrm{orb}}} (F_t^{\mathrm{orb}})^* \Big(\mathrm{c}_2^{\mathrm{orb}}(X_0^{\mathrm{orb}}, h_0) \wedge \alpha_0\Big) \\ &= \int_{X_t^{\mathrm{orb}}} \mathrm{c}_2^{\mathrm{orb}}(X_t^{\mathrm{orb}}, (F_t^{\mathrm{orb}})^* h_t) \wedge (F_t^{\mathrm{orb}})^* \alpha_0 \\ &= \widetilde{\mathrm{c}}_2(X_t) \cdot F_t^* a_0 \end{aligned}$$

where the last line comes from the fact that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^{2n-4}_{\mathrm{dR,c}}\big(X_t^{\mathrm{orb}}, \mathbb{C}\big) & \xrightarrow{\sim} & \mathrm{H}^{2n-4}(X_t, \mathbb{C}) \\ {\scriptstyle (F_t^{\mathrm{orb}})^*}\uparrow & & {\scriptstyle F_t^*}\uparrow \\ \mathrm{H}^{2n-4}_{\mathrm{dR,c}}\big(X_0^{\mathrm{orb}}, \mathbb{C}\big) & \xrightarrow{\sim} & \mathrm{H}^{2n-4}(X_0, \mathbb{C}) \end{array}$$

so that (23) is proved.

Finally, we must show that $C_X > 0$. Since C_X is invariant under locally trivial deformation, one can use [6, Cor. 1.3] and [5, Cor. 3.10] to deform X locally trivially to a projective IHS variety Y. Proposition 39 shows that $C_Y > 0$, which concludes the proof of the proposition. □

7.3. *Simultaneous proof of Theorem 6 and Theorem B*

Here we closely follow the arguments from [15, proof of Thm. 5.2].

Let (X,Δ) be as in Setup 5 and such that $\widetilde{c}_1(X,\Delta)=0$. We denote by $X^\circ := (X,\Delta)_{\mathrm{orb}}$ the open locus where the pair has quotient singularities, and set $\Delta^\circ := \Delta|_{X^\circ}$. It has been proved in [13, Cor. 1.18] that abundance holds for such a pair and in particular $K_X+\Delta$ is torsion. We can then apply Proposition 12 and infer the existence of an orbi-étale map $f\colon Y \to X$ such that

$$\mathscr{O}_Y \cong K_Y \cong f^*(K_X+\Delta).$$

Arguing as in the proof of formula (13), one has:

Lemma 42. *We have the identity*

$$\widetilde{c}_2(Y)\cdot f^*(\alpha)^{n-2} = \deg(f)\,\widetilde{c}_2(X,\Delta)\cdot\alpha^{n-2}. \tag{24}$$

Proof. Let a be an orbifold differential form of degree $2n-4$ with compact support in X° representing α^{n-2} and let h be an orbifold Hermitian metric on $\Omega^1_{(X^\circ,\Delta^\circ)}$. Consider the space $Y^\circ = f^{-1}(X^\circ)$; by taking a fiber product with local smooth charts of X°, it follows easily from purity of branch locus that Y° admits a smooth orbistructure and that f^*h induces an smooth Hermitian metric on Ω_{Y°. In particular, we have

$$\begin{aligned}
\widetilde{c}_2(Y)\cdot f^*(\alpha)^{n-2} &= \int_{Y^\circ} c_2(\Omega_{Y^\circ}, f^*h)\wedge f^*a\\
&= \int_{Y^\circ\setminus f^{-1}(\operatorname{supp}\Delta)} c_2(\Omega_{Y^\circ}, f^*h)\wedge f^*a\\
&= \deg(f)\int_{X^\circ\setminus\operatorname{supp}\Delta} c_2(\Omega_{(X^\circ,\Delta^\circ)}, h)\wedge a\\
&= \deg(f)\int_{X^\circ} c_2(\Omega_{(X^\circ,\Delta^\circ)}, h)\wedge a\\
&= \deg(f)\,\widetilde{c}_2(X,\Delta)\cdot\alpha^{n-2},
\end{aligned}$$

which proves the lemma. □

Both members of the equation (24) being simultaneously non-negative or zero (and $f^*(\alpha)$ still being a Kähler class on Y), we shall replace X with Y and assume from now on that there is no orbifold structure in codimension one, i.e. that $\Delta = 0$.

By [5, Thm. A], there exists a finite, Galois quasi-étale cover $f\colon X' \to X$ such that $X' \cong T\times\prod_{i\in I} Y_i \times \prod_{j\in J} Z_j$ where T is a torus, Y_i are CY varieties and Z_j are IHS varieties. By [24, Prop. 5.6], we have

$$\widetilde{c}_2\big(X'\big)\cdot f^*\beta^{n-2} = \deg(f)\,\widetilde{c}_2(X)\cdot\beta^{n-2},$$

while $f^*\beta$ is still a Kähler class by [24, Prop. 3.5]. All in all, there is no loss in generality assuming that $X = X'$ is split, which we do from now on.

Since $\mathrm{H}^1(Y_i,\mathbb{R}) = \mathrm{H}^1\big(Z_j,\mathbb{R}\big) = 0$, the Künneth decomposition on the space $\mathrm{H}^2(X,\mathbb{R})$ enables us to write

$$\beta = p_T^*\beta_T + \sum_{i\in I} p_{Y_i}^*\beta_{Y_i} + \sum_{j\in J} p_{Z_j}^*\beta_{Z_j}$$

where β_T, β_{Y_i} and β_{Z_j} are Kähler classes on T, Y_i and Z_j respectively. In particular, we get

$$\widetilde{\mathrm{c}}_2(X)\cdot\beta^{n-2} = \sum_{i\in I}\lambda_i\,\widetilde{\mathrm{c}}_2(Y_i)\cdot\beta_{Y_i}^{\dim(Y_i)-2} + \sum_{j\in J}\mu_j\,\widetilde{\mathrm{c}}_2\big(Z_j\big)\cdot\beta_{Z_j}^{\dim(Z_j)-2},$$

where $\lambda_i, \mu_j > 0$ are positive combinatorial coefficients. Proposition 39 and Proposition 40 imply that the above quantity is non-negative, and strictly positive unless $I = J = \emptyset$; i.e. unless $X = T$ is a torus. Theorem 6 and Theorem B are now proved. □

7.4. *Proof of Corollary 7*

To finish, we prove Corollary 7 by proving both implications separately, similar to Corollary 3.

(1) ⇒ (2). This is what we have just proved in the above lines.

(2) ⇒ (1). If $f\colon T\to X$ is a Galois orbi-étale map (for the pair (X,Δ)) from a complex torus, the section $(\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n)^{\otimes m}$ is G-invariant, where $G := \mathrm{Gal}(f)$ and $m := |G|$. This proves that $m(K_X+\Delta)\sim 0$ and thus that $\mathrm{c}_1(K_X+\Delta)=0$. Let ω_T be any Kähler metric on T and let us consider

$$\omega_f := \sum_{g\in G} g^*\omega_T.$$

It descends to an orbifold Kähler metric ω_X on (X,Δ) and, the map f being orbi-étale, we have:

$$\widetilde{\mathrm{c}}_2(X,\Delta)\cdot[\omega_X]^{n-2} = \frac{1}{\deg(f)}\,\widetilde{\mathrm{c}}_2(T)\cdot[\omega_f]^{n-2} = 0.$$

Since $[\omega_X]$ is a Kähler class, this ends the proof. □

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Complex algebraic geometry, in memory of Jean-Pierre Demailly /
Géométrie algébrique complexe, en mémoire de Jean-Pierre Demailly

Existence of Good Minimal Models for Kähler varieties of Maximal Albanese Dimension

Existence d'un bon modèle minimal pour les variétés kählériennes de dimension d'Albanese maximale

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Abstract. In this short article we show that if (X,B) is a compact Kähler klt pair of maximal Albanese dimension, then it has a good minimal model, i.e. there is a bimeromorphic contraction $\phi : X \dashrightarrow X'$ such that $K_{X'} + B'$ is semi-ample.

Résumé. Dans ce court article, nous montrons que si (X,B) est une paire kählérienne compacte klt de dimension d'Albanese maximale, (X,B) admet un bon modèle minimal, c'est-à-dire qu'il existe une contraction biméromorphe $\phi : X \dashrightarrow X'$ telle que $K_{X'} + B'$ est semi-ample.

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1. Introduction

The main result of this paper is the following

Theorem 1. *Let (X,B) be a compact Kähler klt pair of maximal Albanese dimension. Then (X,B) has a good minimal model.*

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This generalizes the main result of [6] from the projective case to the Kähler case. The main idea is to observe that replacing X by an appropriate resolution, then the Albanese morphism $X \to A$ is projective and so by [5] and [7] we may run the relative MMP over A. Thus we may assume that K_X+B is nef over A. If X is projective and K_X+B is not nef, then by the cone theorem, X must contain a K_X+B negative rational curve C. Since A contains no rational curves, then C is vertical over A, contradicting the fact that K_X+B is nef over A [6]. Unluckily the cone theorem is not known for Kähler varieties and so we pursue a different argument. It would be interesting to find an alternative proof based on the approach of [3].

2. Preliminaries

An *analytic variety* or simply a *variety* is a reduced irreducible complex space. Let X be a compact Kähler manifold and $\mathrm{Alb}(X)$ is the *Albanese torus* (not necessarily an Abelian variety). Then by $a: X \to \mathrm{Alb}(X)$ we will denote the *Albanese morphism*. This morphism can also be characterized via the following universal property: $a: X \to \mathrm{Alb}(X)$ is the Albanese morphism if for every morphism $b: X \to T$ to a complex torus T there is a unique morphism $\phi: \mathrm{Alb}(X) \to T$ such that $b = \phi \circ a$.

The Albanese dimension of X is defined as $\dim a(X)$. We say that X has maximal Albanese dimension if $\dim a(X) = \dim X$ or equivalently, the Albanese morphism $a: X \to \mathrm{Alb}(X)$ is *generically finite* onto its image. For the definition of *singular* Kähler space see [4] or [11].

A compact analytic variety X is said to be in *Fujiki's class* $\mathscr{C}$ if X is bimeromorphic to a compact Kähler manifold Y. In particular, there is a resolution of singularities $f: Y \to X$ such that Y is a compact Kähler manifold.

Definition 2. *Let X be a compact analytic variety in Fujiki's class $\mathscr{C}$. Assume that X has rational singularities. Choose a resolution of singularities $\mu: Y \to X$ such that Y is a Kähler manifold and let $a_Y: Y \to \mathrm{Alb}(Y)$ be the Albanese morphism of Y. Then from the proof of [12, Lemma 8.1] it follows that $a_Y \circ \mu^{-1}: X \dashrightarrow \mathrm{Alb}(Y)$ extends to a unique morphism $a: X \to \mathrm{Alb}(X) := \mathrm{Alb}(Y)$. We call this morphism the Albanese morphism of X. Observe that $a: X \to \mathrm{Alb}(X)$ satisfies the universal property stated above. The Albanese dimension of X is defined as above. Note that if X is a compact analytic variety with rational singularities, bimeromorphic to a complex torus A, then $A \cong \mathrm{Alb}(X)$ and $X \to A$ is a bimeromorphic morphism.*

The following result is well known, however, for a lack of an appropriate reference and for the convenience of the reader we give a complete proof here.

Lemma 3. *Let A be a complex torus and $X \subset A$ is an analytic subvariety. Then for any resolution of singularities $\mu: Y \to X$, $H^0(Y, \omega_Y) \neq \{0\}$.*

Proof. Let $\mu: Y \to X$ be a resolution of singularities of X. If $d = \dim X$, then the map $\mu^*\Omega^d_A \to \Omega^d_Y$ is generically surjective. Since Ω^d_A is a trivial vector bundle, it is globally generated and hence there is a non-zero section in the image of $\mu^*: H^0(\Omega^d_A) \to H^0(\Omega^d_Y)$. □

Corollary 4. *Let X be a compact analytic variety in Fujiki's class $\mathscr{C}$ with canonical singularities. If X has maximal Albanese dimension, then $\kappa(X) \geq 0$.*

Proof. First note that if $f: W \to X$ is a proper bimeromorphic morphism, then $\kappa(X) \geq 0$ if and only if $\kappa(W) \geq 0$, since X has canonical singularities. Now let $a: X \to \mathrm{Alb}(X)$ be the Albanese morphism, $Y := a(X)$, and $\pi: Z \to Y$ is a resolution of singularities of Y. Then $\kappa(Z) \geq 0$ by Lemma 3. Note that there is a generically finite meromorphic map $\phi: X \dashrightarrow Z$; resolving the graph of ϕ we may assume that X is smooth and $\phi: X \to Z$ is a morphism. Then $K_X = \phi^* K_Z + E$, where $E \geq 0$ is an effective divisor. Therefore $\kappa(X) \geq 0$, since $\kappa(Z) \geq 0$. □

2.1. *Fourier-Mukai transform*

Let T be a complex torus of dimension g and $\widehat{T} = \mathrm{Pic}^0(T)$ its dual torus. Let $p_T : T \times \widehat{T} \to T$ and $p_{\hat{T}} : T \times \widehat{T} \to \widehat{T}$ be the projections, and $\mathscr{P}$ the normalized Poincaré line bundle on $T \times \widehat{T}$ so that $\mathscr{P}|_{T\times\{0\}} \cong \mathscr{O}_T$ and $\mathscr{P}|_{\{0\}\times\hat{T}} \cong \mathscr{O}_{\hat{T}}$. Let $\widehat{S}$ be the functor from the category of $\mathscr{O}_T$-sheaves to the category of $\mathscr{O}_{\hat{T}}$-sheaves, defined by

$$\widehat{S}(\mathscr{F}) := p_{\hat{T},*}(p_T^*\mathscr{F} \otimes \mathscr{P}),$$

where $\mathscr{F}$ is a sheaf of $\mathscr{O}_T$-modules. Similarly, S is a functor from the category of $\mathscr{O}_{\hat{T}}$-sheaves to the category of $\mathscr{O}_T$-sheaves, defined as

$$S(\mathscr{G}) := p_{T,*}(p_{\hat{T}}^*\mathscr{G} \otimes \mathscr{P}),$$

where $\mathscr{G}$ is a sheaf of $\mathscr{O}_{\hat{T}}$-modules.

The corresponding derived functors are

$$\mathbf{R}\widehat{S}(\cdot) := \mathbf{R}p_{\hat{T},*}(p_T^*(\cdot) \otimes \mathscr{P}) \text{ and } \mathbf{R}S(\cdot) := \mathbf{R}p_{T,*}(p_{\hat{T}}^*(\cdot) \otimes \mathscr{P}).$$

Recall the following fundamental result of Mukai [13, Theorem 2.2, and (3.8)], [14, Theorem 13.1]

Theorem 5. *With notations and hypothesis as above, there are isomorphisms of functors (on the bounded derived category of coherent sheaves)*

$$\mathbf{R}\widehat{S} \circ \mathbf{R}S \cong (-1)_{\hat{T}}^*[-g], \qquad \mathbf{R}S \circ \mathbf{R}\widehat{S} \cong (-1)_T^*[-g],$$
$$\mathbf{\Delta}_T \circ \mathbf{R}S = ((-1_T)^* \circ \mathbf{R}S \circ \mathbf{\Delta}_{\hat{T}})[-g].$$

Recall that $\mathbf{\Delta}_T(\cdot) := \mathbf{R}\mathscr{H}om(\cdot, \mathscr{O}_T)[g]$ is the dualizing functor.

Definition 6. *Let A be a complex torus. For $a \in A$, let $t_a : A \to A$ be the usual translation morphism defined by a. A vector bundle $\mathscr{E}$ on A is called* homogeneous, *if $t_a^*\mathscr{E} \cong \mathscr{E}$ for all $a \in A$.*

Remark 7. Let A be a complex torus, $\widehat{A}$ the dual torus and $\dim A = \dim \widehat{A} = g$. Then from the proof of [13, Example 3.2] it follows that $R^g\widehat{S}$ gives an equivalence of categories

$$\mathbf{H}_A := \{\text{Homogeneous vector bundles on } A\},$$
$$\text{and} \quad \mathbf{C}^f_{\hat{A}} := \{\text{Coherent sheaves on } \widehat{A} \text{ supported at finitely many points}\}.$$

Note that in [13] the results are all stated for abelian varieties, however, we observe that in the proof of [13, Example 3.2] the main arguments follow from Theorem 5 and the isomorphisms in [13, (3.1), p. 158], both of which hold for complex tori. In particular, [13, Example 3.2] holds for complex tori.

We will need the following result on the rational singularity of (log) canonical models of klt pairs.

Proposition 8. *Let (X, B) be a klt pair, where X is a compact analytic variety in Fujiki's class $\mathscr{C}$. Assume that the Kodaira dimension $\kappa(X, K_X + B) \geq 0$. Then $R(X, K_X + B) := \oplus_{m\geq 0} H^0(X, m(K_X + B))$ is a finitely generated $\mathbb{C}$-algebra and*

$$\overline{Z} = \mathrm{Proj}\, R(X, K_X + B)$$

has rational singularities.

Proof. The finite generation of $R(X, K_X + B)$ follows from [5, Theorem 1.3] and [6, Theorem 5.1]. Let $f : X \dashrightarrow Z$ be the Iitaka fibration of $K_X + B$. Resolving Z, f and X, we may assume that X is a compact Kähler manifold, B has SNC support, Z is a smooth projective variety and f is a morphism. Then from the proof of [6, Theorem 5.1] it follows that there is a smooth projective variety Z' which is birational to Z and an effective $\mathbb{Q}$-divisor $B_{Z'} \geq 0$ such that $(Z', B_{Z'})$ is klt, $K_{Z'} + B_{Z'}$ is big and the following holds

$$R(X, K_X + B)^{(d)} \cong R(Z', K_{Z'} + B_{Z'})^{(d')},$$

where the superscripts d and d' represent the corresponding d and d'-Veronese subrings.

Thus $\overline{Z} = \operatorname{Proj} R(X, K_X + B) \cong \operatorname{Proj} R(Z', K_{Z'} + B_{Z'})$ is the log-canonical model of $(Z', B_{Z'})$. If $(Z'', B_{Z''})$ is a minimal model of $(Z', B_{Z'})$ as in [2, Theorem 1.2 (2)], then by the base-point free theorem, there is a birational morphism $\phi : Z'' \to \overline{Z}$ such that $K_{Z''} + B_{Z''} = \phi^*(K_{\overline{Z}} + B_{\overline{Z}})$, where $B_{\overline{Z}} := \phi_* B_{Z''} \geq 0$. Thus $(\overline{Z}, B_{\overline{Z}})$ is a klt pair, and hence $\overline{Z}$ has rational singularities. □

3. Main Theorem

In this section we will prove our main theorem. We begin with some preparation.

Definition 9. *Let X be a smooth compact analytic variety. Then the m-th* plurigenera *of X is defined as*

$$P_m(X) := \dim_{\mathbb{C}} H^0(X, \omega_X^m).$$

The next result is one of our main tools in the proof of the main theorem, it is also of independent interest. It follows immediately from the main results of [14].

Theorem 10. *Let X be a compact Kähler variety with terminal singularities. Assume that X has maximal Albanese dimension and $\kappa(X) = 0$. Then X is bimeromorphic to a torus. Additionally, if K_X is also nef, then X is isomorphic to a torus.*

Remark 11. Note that the above result holds if we simply assume that X is in Fujiki's class $\mathscr{C}$. Indeed, if $X' \to X$ is a resolution of singularities such that X' is Kähler, then $\kappa(X') = 0$ and so $X' \to \mathrm{Alb}(X')$ is bimeromorphic, and hence so is $X \to \mathrm{Alb}(X')$. Note also that if X is a complex manifold of maximal Albanese dimension, then X is automatically in Fujiki's class $\mathscr{C}$. To see this, consider the Stein factorization $X \to Y \to A$. Then $Y \to A$ is finite and so Y is also Kähler (see [15, Proposition 1.3.1 (v) and (vi), p. 24]). Let $X' \to X$ be a resolution of sungularities such that $X' \to Y$ is projective, then X' is Kähler and so X is in Fujiki's class $\mathscr{C}$.

Proof of Theorem 10. Since X is terminal, it has rational singularities, and thus by Definition 2 the Albanese morphism $a : X \to \mathrm{Alb}(X)$ exists. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities of X. Then $a \circ \pi : \widetilde{X} \to \mathrm{Alb}(X)$ is the Albanese morphism of $\widetilde{X}$. Moreover, since X has terminal singularities, $\kappa(\widetilde{X}) = \kappa(X) = 0$. Thus replacing X by $\widetilde{X}$, we may assume that X is a compact Kähler manifold. Let $d = \dim X$ and pick a general element $\Theta \in H^0(\Omega^d_A)$, where $A = \mathrm{Alb}(X)$. Then $0 \neq a^*\Theta \in H^0(\Omega^d_X)$ and so $P_1(X) > 0$. It follows that $P_k(X) = h^0(X, \omega_X^k) > 0$ for all $k > 0$. Since $\kappa(X) = 0$, we have $P_1(X) = P_2(X) = 1$. Thus by [14, Theorem 19.1], $X \to A$ is surjective, and hence $\dim X = \dim A = h^{1,0}(X)$. Thus by [14, Theorem B], X is bimeromorphic to a complex torus and so $a : X \to A$ is (surjective and) bimeromorphic.

Assume now that X has terminal singularities and K_X is nef. Let $a : X \to A$ be the Albanese morphism. By what we have seen above, this morphism is bimeromorphic. Thus $K_X \equiv a^*K_A + E \equiv E$, where $E \geq 0$ is an effective Cartier divisor such that $\mathrm{Supp}(E) = \mathrm{Ex}(a)$ (since A is smooth). By the negativity lemma (see [16, Lemma 1.3]) we have $E = 0$, and hence a is an isomorphism. □

Corollary 12. *Let (X, B) be a compact Kähler klt pair. Assume that X has maximal Albanese dimension and $\kappa(X, K_X + B) = 0$. Then X is bimeromorphic to a torus. Additionally, if $K_X + B \sim_{\mathbb{Q}} 0$, then X is isomorphic to a torus.*

Proof. Passing to a terminalization by running an appropriate MMP over X (using [5, Theorem 1.4]) we may assume that (X, B) has $\mathbb{Q}$-factorial terminal singularities. Now since $\kappa(X) \geq 0$ by Corollary 4, $\kappa(X, K_X + B) = 0$ implies that $\kappa(X, K_X) = 0$. Thus by Theorem 10, $a: X \to A := \mathrm{Alb}(X)$ is a surjective bimeromorphic morphism. Now assume that $K_X + B \sim_{\mathbb{Q}} 0$. Then $K_X + B = a^*K_A + E + B \sim_{\mathbb{Q}} B + E$, where $E \geq 0$ is an effective Cartier divisor such that $\mathrm{Supp}(E) = \mathrm{Ex}(a)$, since A is smooth. Thus $(B + E) \sim_{\mathbb{Q}} 0$, as $K_X + B \sim_{\mathbb{Q}} 0$, and hence $B = E = 0$ (as X is Kähler). In particular, $a: X \to A$ is an isomorphism. □

Now we are ready to prove our main theorem.

Proof of Theorem 1. Let $a: X \to A$ be the Albanese morphism. Since X has maximal Albanese dimension, a is generically finite over its image $a(X)$. By the relative Chow lemma (see [10, Corollary 2] and [4, Theorem 2.16]) there is a log resolution $\mu: X' \to X$ of (X, B) such that the Albanese morphism $a' = a \circ \mu: X' \to A$ is projective. Let $K_{X'} + B' = \mu^*(K_X + B) + F$, where $F \geq 0$ such that $\mathrm{Supp}(F) = \mathrm{Ex}(\mu)$, and (X', B') has klt singularities. Note that if (X', B') has a good minimal model $\psi: X' \dashrightarrow X^m$, then ψ contracts every component of F and the induced bimeromorphic map $X \dashrightarrow X^m$ is a good minimal model of (X, B) (see [9, Lemmas 2.5 and 2.4] and their proofs). Thus, we may replace (X, B) by (X', B') and assume that (X, B) is a log smooth pair and $X \to A$ is a projective morphism. From Corollary 4 it follows that $\kappa(X) \geq 0$. In particular, $\kappa(X, K_X + B) \geq 0$. Now we split the proof into two parts. In Step 1 we deal with the $\kappa(X, K_X + B) = 0$ case, and the remaining cases are dealt with in Step 2.

Step 1. Suppose that $\kappa(X, K_X + B) = 0$. Then by Theorem 10, the Albanese morphism $a: X \to A := \mathrm{Alb}(X)$ is bimeromorphic. Let D be an irreducible component of the unique effective divisor $G \in |m(K_X + B)|$ for $m > 0$ sufficiently divisible. We make the following claim.

Claim 13. *D is a-exceptional; in particular, G is a-exceptional.*

Proof. First passing to a higher model of X we may assume that D has SNC support. Consider the short exact sequence

$$0 \longrightarrow \omega_X \longrightarrow \omega_X(D) \longrightarrow \omega_D \longrightarrow 0.$$

Let $V^0(\omega_D) := \{P \in \mathrm{Pic}^0(A) \mid h^0(D, \omega_D \otimes a^*P) \neq 0\}$. If $\dim V^0(\omega_D) > 0$, then it contains a subvariety $K + P$, where P is torsion in $\mathrm{Pic}^0(A)$ and K is a subtorus of $\mathrm{Pic}^0(A)$ with $\dim K > 0$ (see [14, Corollary 17.1]). Since $a: X \to A$ is surjective and bimeromorphic, we have $H^i(X, a^*Q) = H^i(A, Q) = 0$ for any $\mathscr{O}_A \neq Q \in \mathrm{Pic}^0(A)$; in particular, $H^1(X, \omega_X \otimes a^*Q) = H^{n-1}(X, a^*Q^{-1})^\vee = 0$, where $n = \dim X$. Thus $H^0(X, \omega_X(D) \otimes a^*Q) \to H^0(D, \omega_D \otimes a^*Q)$ is surjective for all $\mathscr{O}_A \neq Q \in \mathrm{Pic}^0(A)$, and so $h^0(X, \omega_X(D) \otimes a^*Q) > 0$ for all $\mathscr{O}_A \neq Q \in P + K$. Since P is torsion, $\ell P = 0$ for some $\ell > 0$. Consider the morphism

$$|K_X + D + P + Q_1| \times \cdots \times |K_X + D + P + Q_\ell| \longrightarrow |\ell(K_X + D)|, \tag{1}$$

where $Q_i \in K$ such that $\sum_{i=1}^{\ell} Q_i = 0$.

Since $\dim K > 0$, for $\ell \geq 2$, the $Q_1, \ldots, Q_\ell$ vary in the subvariety $\mathscr{K} \subset K^{\times \ell}$ defined by the equation $\sum_{i=1}^{\ell} Q_i = 0$. Thus $\dim \mathscr{K} \geq \ell \cdot (\dim K) - 1 \geq \ell - 1 \geq 1$. Therefore $\dim |\ell(K_X + D)| > 0$, i.e. $h^0(X, \ell(K_X + D)) > 1$. Since D is contained in the support of G, we have $(r - \ell)G \geq \ell D$ for some $r > 0$. Then $h^0(X, rm(K_X + B)) \geq h^0(X, \ell(K_X + D)) > 1$, which is a contradiction. Therefore, $\dim V^0(\omega_D) \leq 0$. By [14, Theorem A], $a_*\omega_D$ is a GV sheaf so that $\mathbf{R}\widehat{S}\mathbf{\Delta}_A(a_*\omega_D) = \mathbf{R}^0\widehat{S}\mathbf{\Delta}_A(a_*\omega_D)$. If $\dim V^0(\omega_D) = 0$, then $\mathbf{R}^0\widehat{S}(\mathbf{\Delta}_A(a_*\omega_D))$ is an Artinian sheaf of modules on A, and hence by Theorem 5 and Remark 7

$$\mathbf{\Delta}_A(a_*\omega_D) = (-1_A)^*\mathbf{R}S(\mathbf{R}\widehat{S}\mathbf{\Delta}_A(a_*\omega_D))[g] = (-1_A)^*\mathbf{R}S(\mathbf{R}^0\widehat{S}\mathbf{\Delta}_A(a_*\omega_D))[g]$$

is a shift of a homogeneous vector bundle which we denote by $\mathscr{E}$ (see Remark 7). But then

$$a_*\omega_D = \mathbf{\Delta}_A(\mathbf{\Delta}_A(a_*\omega_D)) = \mathscr{E}^\vee$$

is also a homogeneous vector bundle and hence its support is either empty or entire A. The latter is clearly impossible, since $\operatorname{Supp}(a_*\omega_D) \neq A$, and hence $V^0(\omega_D) = \emptyset$. Thus by [14, Proposition 13.6 (b)], $a_*\omega_D = 0$; in particular D is a-exceptional. □

Now by [5, Theorem 1.4] and [7, Theorem 1.1] we can run the relative minimal model program over A and hence may assume that $K_X + B$ is nef over A. From our claim above we know that $K_X + B \sim_{\mathbb{Q}} E \geq 0$ for some effective a-exceptional divisor $E \geq 0$. Then by the negativity lemma we have $E = 0$; thus $\mathscr{O}_X(m(K_X + B)) \cong \mathscr{O}_X$ for sufficiently divisible $m > 0$, and hence we have a good minimal model.

Step 2. Suppose now that $\kappa(X, K_X + B) \geq 1$ and let $f : X \dashrightarrow Z$ be the Iitaka fibration. Note that the ring $R(X, K_X + B) := \oplus_{m\geq 0} H^0(X, \mathscr{O}_X(\lfloor m(K_X + B)\rfloor))$ is a finitely generated $\mathbb{C}$-algebra by [5, Theorem 1.3]. Define $\overline{Z} := \operatorname{Proj} R(X, K_X + B)$. Then $Z \dashrightarrow \overline{Z}$ is a birational map of projective varieties. Resolving the graph of $Z \dashrightarrow \overline{Z}$ we may assume that Z is a smooth projective variety and $\nu : Z \to \overline{Z}$ is a birational morphism. Then passing to a resolution of X we may assume that f is a morphism and (X, B) is a log smooth pair. Write $K_F + B_F = (K_X + B)|_F$, where F is a very general fiber of f, so that $\kappa(F, K_F + B_F) = 0$. Note that $a|_F$ is also generically finite (as F is a very general fiber of f) and thus F has maximal Albanese dimension. In particular, (F, B_F) has a good minimal model by Step 1. Let $\psi : F \dashrightarrow F'$ be this minimal model; then $K_{F'} + B_{F'} \sim_{\mathbb{Q}} 0$. Thus by Corollary 12, F' is a torus and $B_{F'} = 0$; in particular, $\psi : F \to F'$ is the Albanese morphism. Thus $a|_F : F \to A$ factors through $\psi : F \to F'$; let $\alpha : F' \to A$ be the induced morphism. Let $K := \alpha(F')$; then K is a torus, and α is étale over K, as F' and K are both homogeneous varieties. Now since A contains at most countably many subtori and F is a very general fiber, K is independent of the very general points $z \in Z$, and hence so is F'. Define $A' := A/K$, then A' is again a torus. Since the composite morphism $X \to A'$ contracts F and $\dim F = \dim K$, from the rigidity lemma (see [1, Lemma 4.1.13]) and dimension count it follows that there is a meromorphic map $Z \dashrightarrow A'$ generically finite onto its image. Since Z is smooth, we may assume that $Z \to A'$ is a morphism (see [12, Lemma 8.1]). Similarly, since $\overline{Z}$ has rational singularities by Proposition 8, again from [12, Lemma 8.1] it follows that $\overline{Z} \to A'$ is a morphism.

Since $\overline{Z} = \operatorname{Proj} R(X, K_X + B)$, we may choose an ample $\mathbb{Q}$-divisor $\overline{H}$ on $\overline{Z}$ such that if H_X is its pull-back to X, then $K_X + B \sim_{\mathbb{Q}} H_X + E$ and $|k(K_X + B)| = |kH_X| + kE$ for any sufficiently large and divisible integer $k > 0$, where $E \geq 0$ is effective (it suffices to pick k so that $k(K_X + B)$ and kH_X are Cartier and $R(X, K_X + B)$ is generated in degree k).

Now let $\overline{A} := \overline{Z} \times_{A'} A$. Observe that there is a unique morphism $\overline{a} : X \to \overline{A}$ determined by the universal property of fiber products. We claim that E is exceptional over $\overline{A}$. If not, then let D be a component of E which is not exceptional over $\overline{A}$. Let $h : X \to \overline{Z}$ be the composite morphism $X \to Z \to \overline{Z}$ and $W := h(D)$. Choose a sufficiently divisible and large positive integer $s > 0$ such that $s\overline{H}$ is very ample, $r(K_X + B)$ is Cartier, $rE \geq D$ and $|r(K_X + B)| = |rH_X| + rE$, where $r = (n+1)s$ and $n = \dim X$.

$$
\begin{tikzcd}
X \arrow[dr, "\overline{a}"'] \arrow[drr, bend left, "a"] \arrow[dd, bend right, "f"'] & & \\
& \overline{A} := \overline{Z} \times_{A'} A \arrow[r] \arrow[d] & A \arrow[d] \\
Z \arrow[r] & \overline{Z} \arrow[r] & A' := A/K
\end{tikzcd}
\tag{2}
$$

Claim 14. *$|K_D + (n+1)sH_D| \neq \emptyset$, where $H_D = H_X|_D$.*

Proof. Let $D_i = G_1 \cap \ldots \cap G_i$ be the intersection of general divisors $G_1, \ldots, G_m \in |sH_D|$, where $0 \le i \le m := \dim W$ and $D_0 := D$. Let $M := K_D + (n+1)sH_D$, then we have the short exact sequences

$$0 \longrightarrow \mathscr{O}_{D_i}(M - G_{i+1}) \longrightarrow \mathscr{O}_{D_i}(M) \longrightarrow \mathscr{O}_{D_{i+1}}(M) \longrightarrow 0.$$

Recall that $h: X \to \overline{Z}$ is the given morphism; let $h_i := h|_{D_i}$. Then

$$\begin{aligned}(M - G_{i+1})|_{D_i} &\sim (K_D + nsH_D)|_{D_i} \\ &\sim \left(K_D + \sum_{j=1}^{i} G_j + (n-i)sH_D\right)\Big|_{D_i} \\ &\sim K_{D_i} + (n-i)sH_{D_i} \\ &\sim K_{D_i} + h_i^*(n-i)s\overline{H},\end{aligned}$$

where $H_{D_i} := H_X|_{D_i}$. By [8, Theorem 3.1 (i)] the only associated subvarieties of

$$R^1 h_{i,*}\mathscr{O}_{D_i}(M - G_{i+1}) = R^1 h_{i,*}\mathscr{O}_{D_i}(K_{D_i}) \otimes \mathscr{O}_{\overline{Z}}((n-i)s\overline{H})$$

are $W_i := h(D_i) \subset \overline{Z}$, i.e. $R^1 h_{i,*}\mathscr{O}_{D_i}(M - G_{i+1})$ is a torsion free sheaf on W_i. Therefore, the induced homomorphism $h_{i,*}\mathscr{O}_{D_{i+1}}(M) \to R^1 h_{i,*}\mathscr{O}_{D_i}(M - G_{i+1})$ is zero and we have the following exact sequence

$$0 \longrightarrow h_{i,*}\mathscr{O}_{D_i}(M - G_{i+1}) \longrightarrow h_{i,*}\mathscr{O}_{D_i}(M) \longrightarrow h_{i,*}\mathscr{O}_{D_{i+1}}(M) \longrightarrow 0.$$

By [8, Theorem 3.1 (ii)] we have

$$H^1(\overline{Z}, h_{i,*}\mathscr{O}_{D_i}(M - G_{i+1})) = H^1(\overline{Z}, h_{i,*}\mathscr{O}_{D_i}(K_{D_i}) \otimes \mathscr{O}_{\overline{Z}}((n-i)s\overline{H})) = 0,$$

and thus we have the following surjections

$$H^0(D, \mathscr{O}_D(M)) \longrightarrow H^0(D_1, \mathscr{O}_{D_1}(M_{D_1})) \longrightarrow \cdots \longrightarrow H^0(D_m, \mathscr{O}_{D_m}(M_{D_m})) \longrightarrow H^0(G, \mathscr{O}_G(M|_G)), \quad (3)$$

where G is a connected (and hence irreducible, as D_m is smooth) component of D_m. Note that G is a general fiber of $D \to W$, since H_D is a pullback from W and $m = \dim W$.

Let $w := h(G) \in W \subset \overline{Z}$. Then $G \to \overline{G} := \overline{a}(G)$ is generically finite (as so is $D \to \overline{a}(D)$ by our assumption), and $\overline{G} \to a(G)$ is an isomorphism, since $\overline{A}_w \to K \subset A$ is an isomorphism, as $\overline{A}_w = (A \times_{A'} \overline{Z})_w = A \times_{A'} \{w\} \cong K$. In particular, G has maximal Albanese dimension, and hence $h^0(G, K_G) > 0$ by Lemma 3. Now since $M|_G \sim K_G$, from the surjections in (3) it follows that $|M| = |K_D + (n+1)sH_D| \neq \emptyset$, and hence the claim follows. □

Now consider the short exact sequence

$$0 \longrightarrow \omega_X(L) \longrightarrow \omega_X(L + D) \longrightarrow \omega_D(L) \longrightarrow 0,$$

where $L = rH_X$. Then by [8, Theorem 3.1 (i)], $R^1 h_* \omega_X(L) = R^1 h_* \omega_X \otimes \mathscr{O}_{\overline{Z}}(r\overline{H})$ is torsion free, and hence $h_*\omega_X(L + D) \to h_*\omega_D(L)$ is surjective. Again by [8, Theorem 3.1 (ii)], $H^1(\overline{Z}, h_*\omega_X(L)) = H^1(\overline{Z}, h_*\omega_X \otimes \mathscr{O}_{\overline{Z}}(r\overline{H})) = 0$, and so $H^0(X, \omega_X(L + D)) \to H^0(D, \omega_D(L))$ is surjective. Since $|K_D + L|_D| \neq 0$ by Claim 14, D is not contained in the base locus of $|K_X + L + D|$. Let $0 \le b := \mathrm{mult}_D(B) < 1$ and $e := \mathrm{mult}_D(E) > 0$. Then $\sigma E + B - D \ge 0$ and $\mathrm{mult}_D(\sigma E + B - D) = 0$ for $\sigma = \frac{1-b}{e} > 0$. We may assume that $\sigma \le r$ (as r is sufficiently large and divisible). Adding $rE + B - D$ to a general divisor $G \in |K_X + L + D|$ we get

$$\Gamma := rE + B - D + G \sim_{\mathbb{Q}} (r+1)(K_X + B) \sim_{\mathbb{Q}} (r+1)(H_X + E).$$

Then for any sufficiently divisible $m > 0$ we have

$$\mathrm{mult}_D(m\Gamma) = m(r - \sigma)\mathrm{mult}_D(E) < m(r+1)\mathrm{mult}_D(E),$$

which is a contradiction to the fact that $|k(K_X + B)| = |kH_X| + kE$ for sufficiently divisible $k = m(r+1) > 0$. Thus D is exceptional over $\overline{A}$.

Let $n = \dim X$. We will run a relative $(K_X + B + (2n+3)sH_X)$-MMP over A. Note that since $|(2n+3)sH_X|$ is a base-point free linear system on a smooth compact variety X, by Sard's theorem there is an effective $\mathbb{Q}$-divisor $H' \geq 0$ such that $(2n+3)sH_X \sim_{\mathbb{Q}} H'$ and $(X, B+H')$ has klt singularities. Thus $K_X + B + (2n+3)sH_X \sim_{\mathbb{Q}} K_X + B + H'$ and we can run a $(K_X + B + (2n+3)sH_X)$-MMP over A by [5, Theorem 1.4], and obtain $X \dashrightarrow X'$ so that $K_{X'} + B' + (2n+3)sH_{X'} \sim_{\mathbb{Q}} ((2n+3)s+1)H_{X'} + E'$ is nef over A. Note that if R is a $(K_X + B + (2n+3)sH_X)$-negative extremal ray over A, then it is also $(K_X + B)$-negative and so it is spanned by a rational curve C such that $0 > (K_X + B) \cdot C \geq -2n$ (see [5, Theorem 2.46]). But then C is vertical over $\overline{Z}$, otherwise $(K_X + B + (2n+3)sH_X) \cdot C > 0$, as H_X is the pullback of an ample divisor $\overline{H}$ on $\overline{Z}$, this is a contradiction. Thus it follows that every step of this MMP is also a step of an MMP over $\overline{Z}$, and hence there is an induced morphism $\mu : X' \to \overline{A} := \overline{Z} \times_{A'} A$. It follows that

$$K_{X'} + B' \sim_{\mathbb{Q}} \mu^* H_{\overline{A}} + E' \sim_{\mathbb{Q}, \overline{A}} E' \geq 0,$$

where $H_{\overline{A}}$ is the pullback of the ample divisor $\overline{H}$ by the projection $\overline{A} \to \overline{Z}$.

Then E' is nef and exceptional over $\overline{A}$, and hence by the negativity lemma, $E' = 0$. But then $K_{X'} + B' \sim_{\mathbb{Q}} \mu^* H_{\overline{A}}$ and since $H_{\overline{A}}$ is semi-ample, so is $K_{X'} + B'$. □

Corollary 15. *Let (X, B) be a compact Kähler klt pair of maximal Albanese dimension such that $a : X \to A := \mathrm{Alb}(X)$ is a projective morphism. Then we can run a $(K_X + B)$-Minimal Model Program which ends with a good minimal model.*

Proof. Note that since $a : X \to A$ is generically finite over image, $K_X + B$ is relatively big over $a(X)$. Thus by [5, Theorem 1.4] and [7, Theorem 1.8], we can run a $(K_X + B)$-Minimal Model Program over A. Notice that each step of this MMP is also a step of the $(K_X + B)$-MMP. Therefore, we may assume that $K_X + B$ is nef over A and we must check that it is indeed nef on X. Let $(\overline{X}, \overline{B})$ be a good minimal model of (X, B), which exists by Theorem 1. By what we have seen, $(\overline{X}, \overline{B})$ is also a minimal model over A. But then $\phi : (X, B) \dashrightarrow (\overline{X}, \overline{B})$ is an isomorphism in codimension 1. If $p : Y \to X$ and $q : Y \to \overline{X}$ is a common resolution, then $p^*(K_X + B) - q^*(K_{\overline{X}} + \overline{B})$ is exceptional over X (resp. $\overline{X}$) and nef over $\overline{X}$ (resp. anti-nef over X). From the negativity lemma, it follows that $p^*(K_X + B) = q^*(K_{\overline{X}} + \overline{B})$. In particular, $p^*(K_X + B)$ is semi-ample, and hence so is $K_X + B$. Thus (X, B) is a good minimal model. □

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Complex algebraic geometry, in memory of Jean-Pierre Demailly /
Géométrie algébrique complexe, en mémoire de Jean-Pierre Demailly

A decomposition theorem for $\mathbb{Q}$-Fano Kähler–Einstein varieties

Un théorème de décomposition pour les variétés $\mathbb{Q}$-Fano qui admettent une métrique de Kähler–Einstein

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It is an honor for us to dedicate this paper to the memory of Jean-Pierre Demailly, who always shared his tremendous insights and vision of complex geometry with kindness and generosity.

Abstract. Let X be a $\mathbb{Q}$-Fano variety admitting a Kähler–Einstein metric. We prove that up to a finite quasi-étale cover, X splits isometrically as a product of Kähler–Einstein $\mathbb{Q}$-Fano varieties whose tangent sheaf is stable with respect to the anticanonical polarization. This relies among other things on a very general splitting theorem for algebraically integrable foliations. We also prove that the canonical extension of T_X by $\mathscr{O}_X$ is polystable with respect to the anticanonical polarization.

Résumé. Soit X une variété $\mathbb{Q}$-Fano admettant une métrique de Kähler–Einstein. Nous montrons, qu'à un revêtement fini quasi-étale près, X est un produit de variétés $\mathbb{Q}$-Fano admettant une métrique de Kähler–Einstein dont le fibré tangent est stable relativement au diviseur anticanonique. La démonstration repose notamment sur un théorème de décompostion pour les feuilletages algébriquement intégrables. Nous montrons également que l'extension canonique de T_X par $\mathscr{O}_X$ est polystable à nouveau relativement au diviseur anticanonique.

Keywords. $\mathbb{Q}$-Fano varieties, singular Kähler–Einstein metrics, stable reflexive sheaves, algebraically integrable foliations.

Mots-clés. Variétés $\mathbb{Q}$-Fano, métriques de Kähler–Einstein singulières, faisceaux réflexifs stables, feuilletages algébriquement intégrables.

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1. Introduction

Let (X,ω) be a Fano Kähler–Einstein manifold, i.e. X is a projective manifold with $-K_X$ ample and admitting a Kähler metric ω solving $\mathrm{Ric}\,\omega = \omega$. It follows from the (easy direction of the) Kobayashi–Hitchin correspondence that the tangent bundle of X splits as a direct sum of parallel subbundles

$$T_X = \bigoplus_{i\in I} F_i \tag{1}$$

such that F_i is stable with respect to $-K_X$. Since X is simply connected, de Rham's splitting theorem asserts that one can integrate the foliations arising in decomposition (1) and obtain an isometric splitting

$$(X,\omega) \simeq \prod_{i\in I}(X_i,\omega_i)$$

into Kähler–Einstein Fano manifolds which is compatible with (1).

Over the last few decades, a lot of attention has been drawn to projective varieties with mild singularities, in relation to the progress of the Minimal Model Program (MMP). In that context, the notion of $\mathbb{Q}$-Fano variety (cf. Definition 1) has emerged and played a central role in birational geometry.

On the analytic side, singular Kähler–Einstein metrics have been introduced and constructed in various settings (see e.g. [2, 4, 18] and Definition 2). They induce genuine Kähler–Einstein metrics on the regular part of the variety but are in general incomplete, preventing the use of most useful results in differential geometry (like the de Rham splitting theorem mentionned above) to analyze their behavior. However, these objects are well-suited to study (poly)-stability properties of the tangent sheaf as it was observed by [25], relying on earlier results by [17].

In the Ricci-flat case, the holonomy of the singular metrics was computed in [20]. Moreover, [15] provided an algebraic integrability result for foliations as well as a splitting result in that setting. Building upon those results, Höring and Peternell [26] could eventually prove the singular version of the Beauville–Bogomolov decomposition theorem.

In the positive curvature case, some simplifications appear (for instance, the algebraic integrability of foliations can be related to stability properties by [5]) but new difficulties also arise: the singularities are klt rather than canonical and Gorenstein, and one cannot regularize the singular Kähler–Einstein metrics with an equally good control on the Ricci curvature. In this paper, our main contribution is to single out and overcome those difficulties in order to prove the following structure theorem for $\mathbb{Q}$-Fano varieties that admit a Kähler–Einstein metric.

Theorem A. *Let X be a $\mathbb{Q}$-Fano variety admitting a Kähler–Einstein metric ω. Then T_X is polystable with respect to $c_1(X)$. Moreover, there exists a quasi-étale cover $f\colon Y \to X$ such that $(Y, f^*\omega)$ decomposes isometrically as a product*

$$(Y, f^*\omega) \simeq \prod_{i\in I}(Y_i,\omega_i),$$

where Y_i is a $\mathbb{Q}$-Fano variety with stable tangent sheaf with respect to $c_1(Y_i)$ and ω_i is a Kähler–Einstein metric on Y_i.

Below are a few remarks about the result above.

- Theorem A shows that for all "practical aspects" the tangent sheaf of a $\mathbb{Q}$-Fano variety admitting a Kähler–Einstein metric can always be assumed to be stable. Moreover, it can be expressed in a purely algebraic way using the notion of K-stability, cf. Remark 4 (this is the case for Theorem B below as well).
- The quasi-étale cover above is needed to split X even when T_X is already split, as we see by taking e.g. $X = (\mathbb{P}^1\times\mathbb{P}^1)/\langle\iota\times\iota\rangle$ where $\iota:\mathbb{P}^1\to\mathbb{P}^1$ is the involution $\iota([u:v]) = [u:-v]$.

- It was proved very recently by Braun [6, Thm. 2] that the fundamental group of the regular locus of a $\mathbb{Q}$-Fano variety is finite. Relying on that result, one can refine Theorem A and obtain that the varieties Y_i satisfy the additional property: $\pi_1(Y_i^{\mathrm{reg}}) = \{1\}$.
- Semistability of T_X for a Kähler–Einstein $\mathbb{Q}$-Fano variety X was proved by Chi Li in [35, Prop. 3.7] in the case where X admits a resolution where all exceptional divisors have non-positive discrepancy, e.g. a crepant resolution.

Our second main result is the following generalisation of a theorem of Tian [37, Thm. 0.1], which is a way to express some "strong" polystability of T_X.

Theorem B. *Let X be a $\mathbb{Q}$-Fano variety admitting a Kähler–Einstein metric. Then the canonical extension of T_X by $\mathscr{O}_X$ is polystable with respect to $c_1(X)$.*

We refer to Section 3.1 for the construction of the canonical extension. As we explain further below, at the end of the introduction (see paragraph on the strategy of proof of Theorem B), the generalization from the smooth to the singular case requires some non-trivial new input on top of the analytic techniques already developed for the proof of the semistability/polystability of the *tangent sheaf* T_X, i.e. Theorem 6.

In another direction, the semistability of the canonical extension has been proved in [35, Thm. 1.4] for K-semistable *log smooth* log Fano pairs. It is very likely that the proof of the above theorem will carry over *mutatis mutandis* to the more general setting of log Fano pairs, but we will not pursue this direction in this paper.

Our last main result is a very general splitting theorem for algebraicallly integrable foliations, which plays a key role in the proof of Theorem A, but is certainly of independent interest.

Theorem C. *Let X be a normal projective variety, and let*

$$T_X = \bigoplus_{i\in I} \mathscr{F}_i$$

be a decomposition of T_X into involutive subsheaves with algebraic leaves. Suppose that there exists a $\mathbb{Q}$-divisor Δ such that (X,Δ) is klt. Then there exists a quasi-étale cover $f\colon Y \to X$ as well as a decomposition

$$Y \simeq \prod_{i\in I} Y_i$$

of Y into a product of normal projective varieties such that the decomposition $T_X = \bigoplus_{i\in I} \mathscr{F}_i$ lifts to the canonical decomposition

$$T_{\prod_{i\in I} Y_i} = \bigoplus_{i\in I} \mathrm{pr}_i^*\, T_{Y_i}.$$

Theorem C can be seen as the generalization of the splitting result in [15] where additional assumptions are made, both on the singularities of X and the positivity of $K_{\mathscr{F}_i}$. More precisely, in [15] X is assumed to have canonical singularities, and the $K_{\mathscr{F}_i}$ are assumed to be $\mathbb{Q}$-linearly trivial. We also refer to [16, Thm. 1.5] for a somewhat related result. In comparison to [15, Prop. 4.10], the range of applications of Theorem C is significantly broader.

Strategy of proof of the main results

Theorem A. The first step is the object of Theorem 6 where one proves that T_X is the direct sum of stable subsheaves that are parallel with respect to the Kähler–Einstein metric ω on X_{reg}. This is achieved by computing slopes of subsheaves using the metric induced by the Kähler–Einstein metric and using Griffiths' well-known formula for the curvature of a subbundle. However, the presence of singularities (for X and ω) makes it hard to carry out the analysis directly on X. One has to work on a resolution using approximate Kähler–Einstein metrics as in [25]. Yet an additional error term appears in the Fano case, requiring to introduce some new ideas to deal with it as explained on page 99, cf "term (**I**)".

Once Theorem 6 is at hand, one can appeal to Theorem C where the foliations are induced by the Kähler–Einstein metric as showed in the first step. Note that the algebraic integrability of these foliations follows from the deep results of [5]. An easy induction allows one to split X as a product of $\mathbb{Q}$-Fano varieties with stable tangent sheaf. The isometric splitting follows from a suitable characterization of singular Kähler–Einstein metrics, cf. Claim 28.

Theorem B. The proof of Theorem B takes up most of Section 3. It relies largely on the computations carried out in Section 2 to prove the polystability of T_X, but on top of those, several new ideas are needed to overcome the presence of singularities.

First, one needs to reduce the statement to one on a resolution in order to use analytic methods. Then we use again the technique of working with approximate Kähler–Einstein metrics, but in the current context this has the effect of modifying the canonical extension as well. As a result, we cannot evaluate directly the slope of a subsheaf of the canonical extension corresponding to the initial Kähler–Einstein metric. Dealing with this difficulty is our main contribution in this framework. The rest of the proof uses a combination of the original idea of Tian and the computations of Section 2.

Theorem C. The starting point is the observation that since each foliation $\mathscr{F}_i$ admits a complement inside T_X, $\mathscr{F}_i$ is automatically weakly regular. It turns out that weakly regular foliations have many nice properties. The important fact which is established here is that an algebraically integrable, weakly regular foliation on a $\mathbb{Q}$-factorial projective variety with klt singularities is induced by a surjective, equidimensional morphism $X \to Y$, cf. Theorem 17. When combined with suitable generalisations of other techniques and results in [16], this leads to the proof of Theorem C.

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2. Polystability of the tangent sheaf

2.1. *Set-up*

2.1.1. *Notation*

Definition 1. *Let X be a projective variety of dimension n. We say that X is a $\mathbb{Q}$-Fano variety if X has klt singularities and $-K_X$ is an ample $\mathbb{Q}$-line bundle.*

We also recall the definition of (twisted) singular Kähler–Einstein metric, cf. [2].

Definition 2. *Let X be a $\mathbb{Q}$-Fano variety, let $\vartheta \in c_1(X)$ be a smooth representative and let $\gamma \in [0,1)$. A twisted Kähler–Einstein metric relatively to the couple (ϑ, γ) is a closed, positive current $\omega_{\mathrm{KE},\gamma} \in c_1(X)$ with bounded potentials, which is smooth on X_{reg} and satisfies*

$$\operatorname{Ric} \omega_{\mathrm{KE},\gamma} = (1-\gamma)\omega_{\mathrm{KE},\gamma} + \gamma\vartheta$$

on that open set. When $\gamma = 0$, we write $\omega_{\mathrm{KE}} := \omega_{\mathrm{KE},0}$ and we call it a Kähler–Einstein metric.

Remark 3. By [2, Prop. 3.8], a smooth Kähler metric $\omega \in c_1(X_{\mathrm{reg}})$ on X_{reg} satisfying $\mathrm{Ric}\,\omega = \omega$ extends to a Kähler–Einstein metric in the sense of Definition 2 if and only if $\int_{X_{\mathrm{reg}}} \omega^n = c_1(X)^n$. In particular, if $f : Y \to X$ is a (finite) quasi-étale cover between $\mathbb{Q}$-Fano varieties and ω_{KE} is a Kähler–Einstein metric on X, then $f^*\omega_{\mathrm{KE}}$ is a Kähler–Einstein metric on Y.

Let $\omega_X \in c_1(X)$ be a fixed Kähler metric on X. We will systematically make either one of the following assumptions:

Assumption A. *For any $\gamma \in (0,1)$ small enough, there exists a twisted Kähler–Einstein metric $\omega_{\mathrm{KE},\gamma}$ on X relatively to (ω_X, γ).*

Assumption B. *There exists a Kähler–Einstein metric ω_{KE} on X.*

Remark 4. One can rephrase the Assumptions A-B using the algebraic notion of K-stability. It follows from [36] (building upon results of [8–10], [38], [34], [3] in the smooth case) that

- X satisfies Assumption A if and only if X is K-semistable.
- X satisfies Assumption B if X is uniformly K-stable, and the converse holds provided $\mathrm{Aut}^\circ(X) = \{1\}$.

Notation 5. Let $\pi : \widehat{X} \to X$ be a resolution of singularities of X with exceptional divisor $E = \sum_{k\in I} E_k$ and discrepancies $a_k > -1$ given by

$$K_{\widehat{X}} = \pi^* K_X + \sum a_k E_k.$$

There exist numbers $\varepsilon_k \in \mathbb{Q}_+$ such that the cohomology class $\pi^* c_1(X) - \sum \varepsilon_k c_1(E_k)$ contains a Kähler metric $\omega_{\widehat{X}}$. We fix them for the rest of the paper. Next, we pick sections $s_k \in H^0(\widehat{X}, \mathscr{O}_{\widehat{X}}(E_k))$ such that $E_k = (s_k = 0)$, smooth hermitian metrics h_k on $\mathscr{O}_{\widehat{X}}(E_k)$ with Chern curvature $\vartheta_k := i\Theta_{h_k}(E_k)$ and a volume form $\mathrm{d}V$ on $\widehat{X}$ such that $\mathrm{Ric}\,\mathrm{d}V = \pi^*\omega_X - \sum_{k\in I} a_k \vartheta_k$. We set

$$h_E := \prod_{k\in I} h_k \tag{2}$$

which defines a smooth metric on $\mathscr{O}_{\widehat{X}}(E)$.

2.1.2. *The twisted Kähler–Einstein metric and its regularizations*

In this section, we assume that either Assumption A or Assumption B is fulfilled so that there exists a (twisted) Kähler–Einstein metric $\omega_{\mathrm{KE},\gamma}$

- either for any $\gamma \in [0,1)$ such that $0 < \gamma \ll 1$
- or for $\gamma = 0$.

For the time being, the parameter γ is *fixed.*

We denote by $\pi^*\omega_{\mathrm{KE},\gamma} = \pi^*\omega_X + \mathrm{dd}^c\varphi$ the singular metric solving

$$(\pi^*\omega_X + \mathrm{dd}^c\varphi)^n = e^{-(1-\gamma)\varphi} f \mathrm{d}V$$

where $f = \prod_{i\in I} |s_i|^{2a_i} \in L^p(\mathrm{d}V)$ for some $p > 1$. It is known that φ is bounded (even continuous) on $\widehat{X}$ and smooth outside E, cf. [2]. Note that φ depends on γ, but as notation will get quite heavy later, we choose not to highlight that dependence.

Next, we choose a family $\psi_\varepsilon \in \mathscr{C}^\infty(\widehat{X})$ of quasi-psh functions on $\widehat{X}$ such that:

- One has $\psi_\varepsilon \to \varphi$ in $L^1(\widehat{X})$ and in $\mathscr{C}^\infty_{\mathrm{loc}}(\widehat{X} \setminus E)$.
- There exists $C > 0$ such that $\|\psi_\varepsilon\|_{L^\infty(\widehat{X})} \le C$.
- There exists a continuous function $\kappa : [0,1] \to \mathbb{R}_+$ with $\kappa(0) = 0$ such that $\pi^*\omega_X + \mathrm{dd}^c\psi_\varepsilon \ge -\kappa(\varepsilon)\omega_{\widehat{X}}$.

This is a standard application of Demailly's regularization results ([11]). The smooth convergence outside E claimed in the first item follows from the explicit expression of the function ψ_ε, see e.g. [13, (3.3)].

For $\varepsilon, t \geq 0$, one introduces the unique function $\varphi_{t,\varepsilon} \in L^\infty(X) \cap \mathrm{PSH}(\widehat{X}, \pi^*\omega_X + t\omega_{\widehat{X}})$ solving

$$\begin{cases} (\pi^*\omega_X + t\omega_{\widehat{X}} + dd^c\varphi_{t,\varepsilon})^n = f_\varepsilon e^{-(1-\gamma)\psi_\varepsilon} e^{-c_t} dV \\ \sup_{\widehat{X}} \varphi_{t,\varepsilon} = 0 \end{cases}$$

where

- $f_\varepsilon := e^{a_\varepsilon} \prod (|s_i|^2 + \varepsilon^2)^{a_i}$,
- a_ε is a normalizing constant such that $\int_{\widehat{X}} f_\varepsilon e^{-(1-\gamma)\psi_\varepsilon} dV = c_1(X)^n$; it converges to 1 when $\varepsilon \to 0$.
- c_t is defined by $\{\pi^*\omega_X + t\omega_{\widehat{X}}\}^n = e^{c_t} \cdot c_1(X)^n$.

The existence and uniqueness of $\varphi_{t,\varepsilon}$ follows from Yau's theorem [39] when $t, \varepsilon > 0$ (in which case $\varphi_{t,\varepsilon}$ is actually smooth) while the general case is treated in [18]. It follows from *ibid.* that there exists a constant $C > 0$ such that

$$\|\varphi_{t,\varepsilon}\|_{L^\infty(X)} \leq C \tag{3}$$

for any $t, \varepsilon \in [0,1]$. Moreover, any weak limit $\widehat{\varphi}$ of a sequence $(\varphi_{t_k,\varepsilon_k})$ is bounded and is a smooth limit outside E. Therefore, it solves the equation

$$(\pi^*\omega_X + dd^c\widehat{\varphi})^n = e^{-(1-\gamma)\varphi} f dV$$

on $\widehat{X}$. By the uniqueness result [18, Thm. A], we have $\widehat{\varphi} = \varphi$. That is

$$\varphi_{t,\varepsilon} \underset{t,\varepsilon\to 0}{\longrightarrow} \varphi \quad \text{in } L^1(\widehat{X}) \text{ and in } \mathscr{C}^\infty_{\mathrm{loc}}(\widehat{X} \setminus E). \tag{4}$$

One sets

$$\omega_{t,\varepsilon} := \pi^*\omega_X + t\omega_{\widehat{X}} + dd^c\varphi_{t,\varepsilon} \tag{5}$$

which solves the equation

$$\mathrm{Ric}\,\omega_{t,\varepsilon} = \pi^*\omega_X + (1-\gamma)dd^c\psi_\varepsilon - \Theta_\varepsilon \tag{6}$$

where

$$\Theta_\varepsilon = \Theta(E, h_E^\varepsilon) = \sum a_i \vartheta_{i,\varepsilon} \tag{7}$$

is the curvature of

$$h_E^\varepsilon = \prod_i (|s_i|^2 + \varepsilon^2)^{-1} h_i \tag{8}$$

and $\vartheta_{i,\varepsilon} = \vartheta_i + dd^c \log(|s_i|^2 + \varepsilon^2)$ converges to the current of integration along E_i when $\varepsilon \to 0$.

2.2. *Stability of T_X.*

Setup and notation as in Section 2.1.

Let $\mathscr{F} \subset T_{\widehat{X}}$ be a subsheaf of positive rank r. We can assume that $\mathscr{F}$ is saturated in $T_{\widehat{X}}$, i.e. $T_{\widehat{X}}/\mathscr{F}$ is torsion-free. This is because saturating a subsheaf increases its slope.

From now on, we choose small numbers $t, \varepsilon > 0$ which we will later let go to zero. The Kähler metric $\omega_{t,\varepsilon}$ defined in (5) induces an hermitian metric $h_{t,\varepsilon}$ on $T_{\widehat{X}}$ which in turn induces a hermitian metric h_F on $F := \mathscr{F}|_W$, where $W \subset \widehat{X}$ is the maximal locus where $\mathscr{F}$ is a subbundle of $T_{\widehat{X}}$. Then, it is classical (see e.g. [29, Rem. 8.5]) that one can compute the slope of $\mathscr{F}$ by integrating the trace of the first Chern form of (F, h_F) over W, i.e.

$$\int_W c_1(F, h_F) \wedge \omega_{t,\varepsilon}^{n-1} = c_1(\mathscr{F}) \cdot \{\omega_{t,\varepsilon}\}^{n-1}. \tag{9}$$

On W, we have the following standard identity (cf. e.g. [12, Thm. 14.5])

$$i\Theta(F, h_F) = \mathrm{pr}_F\left(i\Theta(T_{\widehat{X}}, h_{t,\varepsilon})|_F\right) + \beta_{t,\varepsilon} \wedge \beta_{t,\varepsilon}^*,$$

where $\beta \in \mathscr{C}^\infty_{0,1}(W, \mathrm{Hom}(T_{\widehat{X}}, F))$ (i.e. β is a smooth $(0,1)$-form on W with values in $\mathrm{Hom}(T_{\widehat{X}}, F)$) and β^* is its adjoint with respect to $h_{t,\varepsilon}$ and h_F. Therefore, we get

$$c_1(F, h_F) \wedge \omega_{t,\varepsilon}^{n-1} = \mathrm{tr}_{\mathrm{End}}\left(\mathrm{pr}_F\left(i\Theta(T_{\widehat{X}}, h_{t,\varepsilon})|_F\right)\right) \wedge \omega_{t,\varepsilon}^{n-1} + \mathrm{tr}_{\mathrm{End}}(\beta_{t,\varepsilon} \wedge \beta_{t,\varepsilon}^* \wedge \omega_{t,\varepsilon}^{n-1}). \tag{10}$$

By (9), the integral of the left-hand side over W, yields r times the slope of $\mathscr{F}$ with respect to $\{\pi^*\omega_X + t\omega_{\widehat{X}}\}$. As for the right-hand side, one can simplify the first term using the formula

$$n \cdot i\Theta(T_{\widehat{X}}, h_{t,\varepsilon}) \wedge \omega_{t,\varepsilon}^{n-1} = (\sharp\operatorname{Ric}\omega_{t,\varepsilon})\,\omega_{t,\varepsilon}^{n}. \tag{11}$$

Here we denote by $\sharp\operatorname{Ric}\omega_{t,\varepsilon}$ the endomorphism of $T_{\widehat{X}}$ induced by the Ricci curvature of $\omega_{t,\varepsilon}$.

The equation (6) is equivalent to

$$\operatorname{Ric}\omega_{t,\varepsilon} = (1-\gamma)\omega_{t,\varepsilon} + \gamma\pi^*\omega_X - t\omega_{\widehat{X}} + (1-\gamma)dd^c(\psi_\varepsilon - \varphi_{t,\varepsilon}) - \Theta_\varepsilon. \tag{12}$$

Using the formula above, one gets

$$\begin{aligned}\mu_{\omega_{t,\varepsilon}}(\mathscr{F}) \le (1-\gamma)\mu_{\omega_{t,\varepsilon}}(T_X) &+ \frac{1-\gamma}{nr}\underbrace{\int_{\widehat{X}} \operatorname{tr}_{\mathrm{End}}\operatorname{pr}_F(\sharp dd^c(\psi_\varepsilon - \varphi_{t,\varepsilon}))|_F\,\omega_{t,\varepsilon}^n}_{=:(\mathbf{I})}\\ &+ \underbrace{\frac{\gamma}{nr}\int_{\widehat{X}} \operatorname{tr}_{\mathrm{End}}\operatorname{pr}_F(\sharp\pi^*\omega_X)|_F\,\omega_{t,\varepsilon}^n}_{=:(\mathbf{II})} - \frac{1}{nr}\underbrace{\int_{\widehat{X}} \operatorname{tr}_{\mathrm{End}}\operatorname{pr}_F(\sharp\Theta_\varepsilon)|_F\,\omega_{t,\varepsilon}^n}_{=:(\mathbf{III})}\\ &+ \frac{1}{nr}\underbrace{\int_W \operatorname{tr}_{\mathrm{End}}(\beta_{t,\varepsilon}\wedge\beta_{t,\varepsilon}^*\wedge\omega_{t,\varepsilon}^{n-1})}_{=:(\mathbf{IV})}.\end{aligned}$$

We therefore have four terms to deal with. To deal with (**II**)–(**IV**), we will use the same computations as in [25], cf. explanations below. The main new term is (**I**), which we treat first.

The term (I). It arises from the fact that, say when $\gamma = 1$, we can not necessarily solve the perturbed equation $\operatorname{Ric}\omega_{t,\varepsilon} = \omega_{t,\varepsilon} - t\omega_{\widehat{X}} - \Theta_\varepsilon$ unlike in the case where K_X is ample or trivial. If all the discrepancies a_i were negative, one could likely still solve that equation using e.g. properness of Ding functional but we will not expand on that.

In order to deal with (**I**), one makes the following observations:

- Given $\delta > 0$, there exist $\eta = \eta(\delta) > 0$ and an open neighborhood U_δ of $E \subset \widehat{X}$ such that

$$\forall\, \varepsilon, t \le \eta, \quad \int_{U_\delta} (\omega_{\psi_\varepsilon} + \omega_{t,\varepsilon}) \wedge \omega_{t,\varepsilon}^{n-1} \le \delta, \tag{13}$$

where $\omega_{\psi_\varepsilon} = \pi^*\omega_X + t\omega_{\widehat{X}} + dd^c\psi_\varepsilon$. This inequality is a consequence of the Chern–Levine–Nirenberg inequality along with the bound of the potentials below

$$\exists\, C > 0, \forall\, \varepsilon, t, \quad \|\varphi_{t,\varepsilon}\|_{L^\infty(\widehat{X})} + \|\psi_\varepsilon\|_{L^\infty(\widehat{X})} \le C \tag{14}$$

that we infer from (3). Indeed, as explained in [25], one proceeds as follows. Let $(\Xi_\delta)_{\delta>0}$ be a family of functions defined on $\mathbb{R}_+$, such that $\Xi_\delta(x) = 0$ if $x \le \delta^{-1}$ and $\Xi_\delta(x) = 1$ if $x \ge 1 + \delta^{-1}$. Moreover we can assume that the derivative of Ξ_δ is bounded by a constant independent of δ. Then we evaluate the quantity

$$\int_{\widehat{X}} \Xi_\delta\left(\log\log\frac{1}{|s_E|^2}\right)(\omega_{\psi_\varepsilon} + \omega_{t,\varepsilon}) \wedge \omega_{t,\varepsilon}^{n-1} \tag{15}$$

and the proof of the classical Chern–Levine–Nirenberg (see e.g. [12, III.3 (3.3)]) inequality shows that the integral in (13) is smaller than

$$\int_{U_\delta} \omega_E^n \tag{16}$$

up to a constant which is independent of t, ε. In (16) we denote by ω_E a metric with Poincaré singularities along the divisor E, and by U_δ the support of the truncation function $\Xi_\delta\big(\log\log\frac{1}{|s_E|^2}\big)$. Here the main point is that the norm of the Hessian of the truncation function is uniformly bounded when measured with respect to ω_E. The conclusion follows.

The hermitian endormorphism $\sharp dd^c(\psi_\varepsilon - \varphi_{t,\varepsilon})$ is dominated (in absolute value) by the positive endomorphism

$$\sharp(\omega_{\psi_\varepsilon} + \omega_{t,\varepsilon})$$

whose endomorphism trace is nothing but $\mathrm{tr}_{\omega_{t,\varepsilon}}(\omega_{\psi_\varepsilon} + \omega_{t,\varepsilon})$. By (13), we are done with (**I**) on U_δ.

- The second observation is that given $K \Subset \widehat{X} \setminus E$, there exists $\eta = \eta(K) > 0$ such that

$$\forall\, \varepsilon, t \le \eta, \quad \|\psi_\varepsilon - \varphi_{t,\varepsilon}\|_{\mathscr{C}^2(K)} \le \delta. \tag{17}$$

This is a consequence of the fact that $(\varphi_{t,\varepsilon})$ and (ψ_ε) converge uniformly (in ε, t) to φ on K by stability of the Monge–Ampère operator, cf. e.g. [24, Thm. C], and have uniformly bounded $\mathscr{C}^p(K)$ norm for any p thanks to (14), Tsuji's trick and Evans–Krylov plus Schauder estimates.

Therefore, one has $\pm\sharp dd^c(\psi_\varepsilon - \varphi_{t,\varepsilon}) \le \delta\omega_{\widehat{X}}$ hence (**I**) is controlled on K by $\delta \int_K \omega_{\widehat{X}} \wedge \omega_{t,\varepsilon}^n \le C\delta$.

Conclusion. Let $F_{t,\varepsilon} := \mathrm{tr}_{\mathrm{End}}\, \mathrm{pr}_F(\sharp dd^c(\psi_\varepsilon - \varphi_{t,\varepsilon}))|_F\, \omega_{t,\varepsilon}^n$. One fixes $\delta > 0$. We get a neighborhood U_δ of E and a number $\eta' = \eta'(\delta) > 0$ such that $\int_{U_\delta} F_{t,\varepsilon} \le \delta$ for any $\varepsilon, t \le \eta'$. Applying the second observation to $K = \widehat{X} \setminus U_\delta$, we find $\eta'' = \eta''(\delta)$ such that $\int_{X\setminus U_\delta} F_{t,\varepsilon} \le C\delta$ for any $\varepsilon, t \le \eta''$. Choosing $\eta := \min\{\eta', \eta''\}$, we find that

$$\forall\, \varepsilon, t \le \eta, \quad \int_{\widehat{X}} F_{t,\varepsilon} \le C'\delta.$$

In short, the term (**I**) converges to zero when $\varepsilon, t \to 0$.

The term (II). As $\pi^*\omega_X \ge 0$, one has

$$\begin{aligned} \mathrm{tr}_{\mathrm{End}}\, \mathrm{pr}_F(\sharp\pi^*\omega_X)|_F\omega_{t,\varepsilon}^n &\le \mathrm{tr}_{\mathrm{End}}(\sharp\pi^*\omega_X)\omega_{t,\varepsilon}^n \\ &= \mathrm{tr}_{\omega_{t,\varepsilon}}(\pi^*\omega_X) = n\pi^*\omega_X \wedge \omega_{t,\varepsilon}^{n-1}. \end{aligned}$$

Integrating over X, one finds

$$(\mathbf{II}) \le \gamma r^{-1}(\pi^* c_1(X) \cdot \{\omega_{t,\varepsilon}\}^{n-1})$$

and the right-hand side converges to $\frac{\gamma n}{r}\mu(T_{\widehat{X}})$ when $t \to 0$, where the slope is taken with respect to $\pi^* c_1(X)$.

The term (III). As said above, the arguments to treat this term are borrowed from [25]. For the convenience of the reader, we will recall the important steps. To lighten notation, we will drop the index i. One can write $\Theta_\varepsilon = \frac{\varepsilon^2|D's|^2}{(|s|^2+\varepsilon^2)^2} + \frac{\varepsilon^2}{|s|^2+\varepsilon^2}\cdot\vartheta$. Let us set $g_\varepsilon := \frac{\varepsilon^2}{|s|^2+\varepsilon^2}$. Up to rescaling $\omega_{\widehat{X}}$, one can assume that $-\omega_{\widehat{X}} \le \vartheta \le \omega_{\widehat{X}}$ so that $\Theta_\varepsilon + g_\varepsilon\omega_{\widehat{X}} \ge 0$. Then one sees easily that

$$\begin{aligned} \mathrm{tr}_{\mathrm{End}}\, \mathrm{pr}_F(\sharp\Theta_\varepsilon)|_F\, \omega_{t,\varepsilon}^n &\le \mathrm{tr}_{\mathrm{End}}\Big(\sharp\Theta_\varepsilon + \sharp(g_\varepsilon\omega_X)\Big)\omega_{t,\varepsilon}^n \\ &= \Theta_\varepsilon \wedge \omega_{t,\varepsilon}^{n-1} + g_\varepsilon\omega_{\widehat{X}} \wedge \omega_{t,\varepsilon}^{n-1} \end{aligned}$$

and one obtains that the term (**III**) converges to zero when $\varepsilon, t \to 0$ since

- $\int_X \Theta_\varepsilon \wedge \omega_{t,\varepsilon}^{n-1} = c_1(E)\cdot\{\pi^*\omega_X + t\omega_{\widehat{X}}\}^{n-1}$ and E is exceptional,
- $\int_X g_\varepsilon\omega_{\widehat{X}} \wedge \omega_{t,\varepsilon}^{n-1} \to 0$ when $\varepsilon, t \to 0$ thanks to the smooth convergence to 0 outside E and the Chern–Levine–Nirenberg inequality combined with the bound (3) on the potentials, cf. first item in Part **(I)**.

The term (IV). Note that the term $\beta_{t,\varepsilon} \wedge \beta_{t,\varepsilon}^*$ is pointwise negative in the sense of Griffiths on W. In particular, the term (**IV**) is non-positive. Since (**I**) and (**III**) converge to zero, this shows that

$$\mu(\mathscr{F}) \le \big(1 + \gamma\big(\frac{n}{r} - 1\big)\big)\cdot\mu(T_{\widehat{X}}), \tag{18}$$

where the slope is taken with respect to $\pi^* c_1(X)$.

Working under Assumption A, one obtains the inequality (18) above for any $\gamma > 0$ small enough. In particular, this shows that under Assumption A, $T_{\widehat{X}}$ is semistable with respect to $\pi^* c_1(X)$.

From now on, we assume that the stronger Assumption B holds; i.e. one can choose $\gamma = 0$. Assume additionally that there exists a subsheaf $\mathscr{F} \subset T_{\widehat{X}}$ with the same slope as $T_{\widehat{X}}$ and let $\mathscr{F}^{\text{sat}}$ be its saturation in $T_{\widehat{X}}$; it is a subbundle in codimension one. As the slope has not increased by saturation, $\mathscr{F} = \mathscr{F}^{\text{sat}}$ in codimension one on $\widehat{X} \setminus E$. Therefore, if we set $W^\circ := W \cap (\widehat{X} \setminus E)$, then $W^\circ \subset \widehat{X} \setminus E$ has codimension at least two and by the above computation, one has

$$\lim_{\varepsilon, t \to 0} \int_{W^\circ} (\beta_{t,\varepsilon} \wedge \beta^*_{t,\varepsilon} \wedge \omega^{n-1}_{t,\varepsilon}) = 0.$$

We know by (4) that $\beta_{t,\varepsilon} \to \beta_\infty$ locally smoothly on W° when $\varepsilon, t \to 0$ where β_∞ is the second fundamental form induced by the hermitian metric h_{KE} induced by $\pi^*\omega_{\text{KE}}$ on $T_{\widehat{X}}|_{W^\circ}$ and on $\mathscr{F}|_{W^\circ}$ by restriction. By Fatou lemma, we have $\beta_\infty \equiv 0$ on W°, that is, we have a holomorphic decomposition $T_{\widehat{X}}|_{W^\circ} = \mathscr{F}|_{W^\circ} \oplus \mathscr{F}|^\perp_{W^\circ}$ where the orthogonal is taken with respect to h_{KE}.

We are now ready to prove

Theorem 6. *Let X be a $\mathbb{Q}$-Fano variety.*

(i) *If Assumption A is satisfied, then T_X is semistable with respect to $c_1(X)$.*

(ii) *If Assumption B is satisfied, then T_X is polystable with respect to $c_1(X)$. More precisely, we have:*

- *Any saturated subsheaf $\mathscr{F} \subset T_X$ with $\mu(\mathscr{F}) = \mu(T_X)$ is a direct summand of T_X and $\mathscr{F}|_{X_{\text{reg}}} \subset T_{X_{\text{reg}}}$ is a parallel subbundle with respect to ω_{KE}.*
- *There exists a decomposition*

$$T_X = \bigoplus_{i \in I} \mathscr{F}_i$$

such that $\mathscr{F}_i$ is stable with respect to $c_1(X)$, $\mathscr{F}_i|_{X_{\text{reg}}} \subset T_{X_{\text{reg}}}$ is a parallel subbundle with respect to ω_{KE}, and the decomposition $T_{X_{\text{reg}}} = \oplus_{i\in I} \mathscr{F}_i|_{X_{\text{reg}}}$ is orthogonal with respect to ω_{KE}.

Proof. Let $\mathscr{F} \subset T_X$ be a subsheaf and let $\alpha := c_1(X)$. The sheaf $\mathscr{F}$ induces a subsheaf $\mathscr{G}^\circ \subset T_{\widehat{X}}|_{\widehat{X}\setminus E}$ and we denote by $\mathscr{G} \subset T_{\widehat{X}}$ the saturation of $\mathscr{G}^\circ$ in $T_{\widehat{X}}$. By the arguments above (cf. inequality (18) and the comments below it), one has $\mu_{\pi^*\alpha}(\mathscr{G}) \le \mu_{\pi^*\alpha}(T_{\widehat{X}}) = c_1(X)^n/n = \mu_\alpha(T_X)$. Moreover, one has clearly $\mu_{\pi^*\alpha}(\mathscr{G}) = \mu_\alpha(\mathscr{F})$. This shows that T_X is semistable with respect to $c_1(X)$.

Now, assume that there exists a Kähler–Einstein metric ω_{KE}. If $\mathscr{F} \subset T_X$ satisfies $\mu_\alpha(\mathscr{F}) = 0$, then $\mu_{\pi^*\alpha}(\mathscr{G}) = 0$ and we have shown above that $\pi^*\omega_{\text{KE}}$ induces a splitting $T_{\widehat{X}}|_W = \mathscr{G}|_W \oplus (\mathscr{G}|_W)^\perp$ over a Zariski open subset $W \subset \widehat{X}\setminus E$ whose complement in $\widehat{X}\setminus E$ has codimension at least two. Set $V := \pi(W) \subset X_{\text{reg}}$ so that $\mathscr{F}|_V$ is a subbundle of T_X and we have a splitting $T_X|_V = \mathscr{F}|_V \oplus (\mathscr{F}|_V)^\perp$ induced by ω_{KE} and $\operatorname{codim}_X(X \setminus V) \ge 2$.

Let us denote by $j : V \hookrightarrow X$ the open immersion. As $\mathscr{F} \subset T_X$ is saturated, it is reflexive, hence $j_*(\mathscr{F}|_V) = \mathscr{F}$. Moreover, $(\mathscr{F}|_V)^\perp$ extends to a reflexive sheaf $\mathscr{F}^\perp := j_*((\mathscr{F}|_V)^\perp)$ on X satisfying $T_X = \mathscr{F} \oplus \mathscr{F}^\perp$ on the whole X. In particular, $\mathscr{F}$ is a direct summand of T_X and as such, it is subbundle of T_X over X_{reg}. By iterating this process and starting with $\mathscr{F}$ with minimal rank, one can decompose $T_X = \bigoplus_{i\in I} \mathscr{F}_i$ into reflexive sheaves which, over X_{reg}, are parallel (pairwise orthogonal) subbundles with respect to ω_{KE}. □

3. Polystability of the canonical extension

In this section, we keep using the setup and notation of Section 2.1.

3.1. *The canonical extension*

Let $\mathscr{E}$ be a coherent sheaf on X sitting in the exact sequence below

$$0 \longrightarrow \Omega^{[1]}_X \longrightarrow \mathscr{E} \longrightarrow \mathscr{O}_X \longrightarrow 0. \tag{19}$$

The sheaf $\mathscr{E}$ is automatically torsion-free and it is locally free on X_{reg}.

Remark 7. Let $U \subset X$ be a non-empty Zariski open subset. As an extension of $\mathscr{O}_X$ by $\Omega_X^{[1]}$, $\mathscr{E}|_U$ is uniquely determined by the image of $1 \in H^0(U,\mathscr{O}_X)$ in $H^1(U,\Omega_X^{[1]})$ under the connecting morphism in the long exact sequence arising from $H^0(U,-)$.

From now on, one assumes that the extension class of $\mathscr{E}$ is the image of $c_1(X)$ in $H^1(X,\Omega_X^1)$ under the canonical map

$$\mathrm{Pic}(X)\otimes\mathbb{Q} \simeq H^1(X,\mathscr{O}_X^*)\otimes\mathbb{Q} \to H^1(X,\Omega_X^1) \to H^1(X,\Omega_X^{[1]}).$$

This is legitimate since K_X is $\mathbb{Q}$-Cartier.

Definition 8. *The dual $\mathscr{E}^*$ of the sheaf $\mathscr{E}$ sitting in the exact sequence* (19) *with extension class $c_1(X)$ is called the canonical extension of T_X by $\mathscr{O}_X$.*

The exact sequence (19) is locally splittable since for any affine $U \subset X$, one has $h^1(U,\Omega_U^{[1]}) = 0$. In particular, when one dualizes (19), one see that the canonical extension of T_X by $\mathscr{O}_X$ sits in the short exact sequence below

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{E}^* \longrightarrow T_X \longrightarrow 0. \tag{20}$$

The goal of this section is to prove the following, cf. Theorem B.

Theorem 9. *Let X be a $\mathbb{Q}$-Fano variety. If Assumption A (resp. Assumption B) is satisfied, then the canonical extension $\mathscr{E}^*$ of T_X by $\mathscr{O}_X$ is semistable (resp. polystable) with respect to $c_1(X)$.*

The proof of Theorem 9 above is divided into three main steps corresponding to the next three sub-sections. First one can reduce the semistability statement above to a semistability property on the resolution $\widehat{X}$ thanks to Lemma 10, then we prove the said statement, cf. Theorem 11 and, finally, we prove polystability assuming the existence of a Kähler–Einstein metric.

3.2. *Reduction to a statement on the resolution*

Let $\widehat{\mathscr{E}}$ be the vector bundle on $\widehat{X}$ sitting in the exact sequence below

$$0 \longrightarrow \Omega^1_{\widehat{X}} \longrightarrow \widehat{\mathscr{E}} \longrightarrow \mathscr{O}_{\widehat{X}} \longrightarrow 0 \tag{21}$$

such that its extension class is $\pi^*c_1(X) \in H^1(\widehat{X},\Omega^1_{\widehat{X}})$. Its dual sits in the exact sequence

$$0 \longrightarrow \mathscr{O}_{\widehat{X}} \longrightarrow \widehat{\mathscr{E}}^* \longrightarrow T_{\widehat{X}} \longrightarrow 0. \tag{22}$$

Lemma 10. *If the vector bundle $\widehat{\mathscr{E}}^*$ is semistable with respect to $\pi^*c_1(X)$, then the torsion-free sheaf $\mathscr{E}^*$ is semistable with respect to $c_1(X)$.*

Although slope stability is usually defined with respect to an ample polarization, the same definition actually makes sense with respect to an arbitrary nef class like $\pi^*c_1(X)$, cf e.g. [22].

Proof. Set $\alpha := c_1(X)$. Let $X^\circ \subseteq X_{\mathrm{reg}}$ be an open set with complement of codimension at least 2 in X such that the restriction $\pi_{|\widehat{X}^\circ}$ of π to $\widehat{X}^\circ := \pi^{-1}(X^\circ)$ induces an isomorphism $\widehat{X}^\circ \simeq X^\circ$. By Remark 7 we have

$$(\pi^*\mathscr{E}^*)_{|\widehat{X}^\circ} \simeq \widehat{\mathscr{E}}^*_{|\widehat{X}^\circ}. \tag{23}$$

Let $\mathscr{F} \subseteq \mathscr{E}^*$ be a subsheaf and let $\widehat{\mathscr{F}} \subseteq \widehat{\mathscr{E}}^*$ be the saturated subsheaf of $\widehat{\mathscr{E}}^*$ whose restriction to $\widehat{X}^\circ$ is $(\pi^*\mathscr{F})_{|\widehat{X}^\circ}$. By the projection formula together with the fact that $X \setminus X^\circ$ has codimension at least 2 in X, we have

$$\mu_\alpha(\mathscr{F}) = \mu_{\pi^*\alpha}(\widehat{\mathscr{F}}) \quad \text{and} \quad \mu_\alpha(\mathscr{E}^*) = \mu_{\pi^*\alpha}(\widehat{\mathscr{E}}^*).$$

The lemma follows easily. $\square$

3.3. *Statement on the resolution*

In this section, we prove that the vector bundle $\widehat{\mathscr{E}}^*$ from Section 3.2 is semistable with respect to $\pi^* c_1(X)$, cf. Theorem 11 below. In order to streamline the notation, we set $\mathscr{V} := \widehat{\mathscr{E}}^*$ and in the following we will not distinguish between the locally free sheaf $\mathscr{V}$ and the associated vector bundle. Recall that $\mathscr{V}$ fits into the exact sequence of locally free sheaves

$$0 \longrightarrow \mathscr{O}_{\widehat{X}} \longrightarrow \mathscr{V} \to T_{\widehat{X}} \longrightarrow 0. \tag{24}$$

We denote by $\beta \in H^1(\widehat{X}, T^\star_{\widehat{X}})$ the second fundamental form.

Our result in this section is a singular version of Theorem 0.1 in [37].

Theorem 11. *Let X be a $\mathbb{Q}$-Fano variety satisfying Assumption A. Let $\mathscr{V}$ be the vector bundle on $\widehat{X}$ appearing in* (24)*, whose extension class β coincides with the inverse image of the first Chern class of X by the resolution $\pi : \widehat{X} \to X$. Then $\mathscr{V}$ is semistable with respect to $\pi^\star c_1(X)$.*

Proof. The strategy of proof is as follows. We would like to compute the slope of $\mathscr{F}$ using an hermitian metric on $\mathscr{V}$ induced by the (twisted) Kähler–Einstein metric, using an approximation process as in Section 2.2. As the natural metric in the extension class of $\mathscr{V}$ is singular, we introduce an algebraic 1-parameter family $(\mathscr{V}_z)_{z\in\mathbb{C}}$ that can be endowed with natural smooth hermitian metrics for suitable $z \in \mathbb{R}$ close to zero and such that we have sheaf injections $\mathscr{V} \hookrightarrow \mathscr{V}_t \otimes \mathscr{O}_{\widehat{X}}(E)$. We then proceed to compute slopes following the strategy of Section 2.2.

Step 1. Deformations of $\mathscr{V}$. We pick an arbitrary subsheaf $\mathscr{F} \subseteq \mathscr{V}$ of the vector bundle $\mathscr{V}$ sitting in the exact sequence below

$$0 \to \mathscr{O}_{\widehat{X}} \to \mathscr{V} \to T_{\widehat{X}} \to 0$$

and corresponding to the extension class

$$\alpha = (a_{ij}) \in \mathrm{Ext}^1(T_{\widehat{X}}, \mathscr{O}_{\widehat{X}}) \simeq H^1(\widehat{X}, \mathscr{H}om(T_{\widehat{X}}, \mathscr{O}_{\widehat{X}}))$$

relatively to a covering by open subsets (U_i). The bundle $\mathscr{V}$ can be obtained as follows: on U_i, it is the trivial extension, $\mathscr{V}_{|U_i} = \mathscr{O}_{\widehat{X}|U_i} \oplus T_{\widehat{X}|U_i}$ and the transition functions are given by

$$\begin{pmatrix} \mathrm{Id}_{\mathscr{O}_{\widehat{X}}}|_{U_{ij}} & a_{ij} \\ 0 & \mathrm{Id}_{T_{\widehat{X}}}|_{U_{ij}} \end{pmatrix}.$$

The subsheaf $\mathscr{F}$ is given by two morphisms of sheaves $p_i \colon \mathscr{F}_{|U_i} \to \mathscr{O}_{\widehat{X}|U_i}$ and $q_i \colon \mathscr{F}_{|U_i} \to T_{\widehat{X}|U_i}$ satisfying

$$\begin{cases} p_i|_{U_{ij}} = p_j|_{U_{ij}} + a_{ij} \circ (q_j|_{U_{ij}}), \\ q_i|_{U_{ij}} = q_j|_{U_{ij}}. \end{cases}$$

Recall that we have a reduced divisor $E = E_1 + \cdots + E_r$. Up to refining the covering (U_i), one can assume that E_k is given by the equation $f_{ki} = 0$ on U_i. The transition functions of $\mathscr{O}_{\widehat{X}}(E_k)$ are $g_{k,ij} = \frac{f_{kj}}{f_{ki}}$.

Now, given complex numbers $z_1, \ldots, z_r \in \mathbb{C}$, one considers the extension $\mathscr{V}_{z_1,\ldots,z_r}$ of $T_{\widehat{X}}$ by $\mathscr{O}_{\widehat{X}}$ whose class is

$$\alpha + z_1 \left[\frac{\mathrm{d}g_{1,ij}}{g_{1,ij}} \right] + \cdots + z_r \left[\frac{\mathrm{d}g_{r,ij}}{g_{r,ij}} \right] = \alpha + \sum_k z_k c_1(E_k).$$

Set $\mathscr{V}_{z_1,\ldots,z_r}(E) := \mathscr{V}_{z_1,\ldots,z_r} \otimes \mathscr{O}_{\widehat{X}}(E)$. Then, there is an injection of sheaves

$$\mathscr{F} \subseteq \mathscr{V}_{z_1,\ldots,z_s}(E)$$

extending $\mathscr{F} \subseteq \mathscr{V} \subseteq \mathscr{V}(E)$ for (z_k) in a Zariski open neighborhood of $0 \in \mathbb{C}^r$.

Indeed, consider the morphism $\mathscr{F}_{|U_i} \to \mathscr{V}_{z_1,\dots,z_s}(E)|_{U_i}$ given by $p_i + \sum_k z_k \frac{\mathrm{d}f_{ki}}{f_{ki}} \circ q_i$ on the first factor and q_i on the second. Those morphisms can be glued since one has

$$\frac{\mathrm{d}f_{ki}}{f_{ki}} = \frac{\mathrm{d}g_{k,ij}}{g_{k,ij}} + \frac{\mathrm{d}f_{kj}}{f_{kj}},$$

for any index k. The induced map $\mathscr{F} \to \mathscr{V}_{z_1,\dots,z_s}(E)$ is obviously injective for (z_k) in a Zariski open neighborhood of $0 \in \mathbb{C}^r$.

Now, recall that $\alpha = \pi^* c_1(X)$ and that the Kähler metric $\omega_{\widehat{X}}$ lives in the class $\alpha - \sum \varepsilon_k c_1(E_k)$ for some $\varepsilon_k > 0$, so that the approximate Kähler–Einstein metric $\omega_{t,\varepsilon}$ belongs to $(1+t)\alpha_t$, where

$$\alpha_t := \alpha - \frac{t}{1+t}\sum_k \varepsilon_k c_1(E_k).$$

For any $t \in \mathbb{R}$, we set

$$\mathscr{V}_t := V_{z_1,\dots,z_r} \quad \text{and} \quad \mathscr{V}_t(E) := \mathscr{V}_t \otimes \mathscr{O}_{\widehat{X}}(E)$$

where $z_k := -\frac{t}{1+t}\cdot \varepsilon_k$ for $1 \le k \le r$. This vector bundle $\mathscr{V}_t$ is the extension of $T_{\widehat{X}}$ by $\mathscr{O}_{\widehat{X}}$ with extension class α_t and $\mathscr{V}_t(E)$ comes equipped with a sheaf injection

$$\mathscr{F} \subseteq \mathscr{V}_t(E). \tag{25}$$

Moreover, it is clear from the definition of $\mathscr{V}_{z_1,\dots,z_r}$ that we have

$$c_1(\mathscr{V}_t(E)) = c_1(\mathscr{V}) + c_1(E) \tag{26}$$

for any $t \in \mathbb{R}$.

Step 2. Metric properties of $\mathscr{V}_t(E)$. First of all, we pick one number $\gamma > 0$ as in Assumption A. It will be fixed until the very end of the argument.

We seek to endow $\mathscr{V}_t(E)$ with a suitable smooth hermitian metric, at least when $t > 0$ is small enough. Given that $\mathscr{V}_t(E) = \mathscr{V}_t \otimes \mathscr{O}_{\widehat{X}}(E)$ and that we have already fixed a smooth hermitian metric h_E on $\mathscr{O}_{\widehat{X}}(E)$ in (2), it is enough to construct a hermitian metric on $\mathscr{V}_t$.

Now, we can endow the bundles $\mathscr{O}_{\widehat{X}}$ and $T_{\widehat{X}}$ with the trivial metric and the hermitian metric $h_{t,\varepsilon}$ induced by $\omega_{t,\varepsilon}$, respectively. Now, we set

$$\beta_t = \frac{1}{1+t}\omega_{t,\varepsilon} \in \alpha_t$$

which we view as an element of $\mathscr{C}^\infty_{0,1}(\widehat{X}, T^*_{\widehat{X}})$. Relatively to a fixed $\mathscr{C}^\infty$ splitting of $\mathscr{V}_t$, the direct sum metric $h_{\mathscr{V}_t}$ induced on $\mathscr{V}_t$ has a Chern connection $D_{\mathscr{V}_t}$ which has the following expression

$$D_{\mathscr{V}_t} = \begin{pmatrix} d & -\beta_t \\ \beta_t^* & D_{T_{\widehat{X}}} \end{pmatrix}$$

or equivalently

$$D_{\mathscr{V}_t}(s_1, s_2) = \Big(ds_1 - \beta_t \cdot s_2, \beta_t^\star \cdot s_1 + D_{T_{\widehat{X}}} s_2\Big) \tag{27}$$

where $D_{T_{\widehat{X}}}$ is the Chern connections induced by $h_{t,\varepsilon}$ on $T_{\widehat{X}}$. Of course, it depends strongly on the parameters t, ε. We denote by $\beta_t^\star \in \mathscr{C}^\infty_{1,0}(\widehat{X}, T_{\widehat{X}})$ the adjoint of $\beta_t \in \mathscr{C}^\infty_{0,1}(\widehat{X}, T^*_{\widehat{X}})$. Moreover, the Chern curvature of $D_{\mathscr{V}_t}$ is given by

$$\Theta(\mathscr{V}_t, h_{\mathscr{V}_t}) = \begin{pmatrix} -\beta_t \wedge \beta_t^* & D'_{T^*_{\widehat{X}}}\beta_t \\ \bar\partial \beta_t^* & \Theta(T_{\widehat{X}}, h_{t,\varepsilon}) - \beta_t^* \wedge \beta_t \end{pmatrix},$$

where $D'_{T^*_{\widehat{X}}}$ is the $(1,0)$-part of the Chern connection of $(T^\star_{\widehat{X}}, h^*_{t,\varepsilon})$.

We analyze next several quantities which are playing a role in the evaluation of the curvature of $\mathscr{V}_t$.

• *The factor* β_t. The form β_t is given by

$$\beta_t = \frac{1}{1+t}\sum \omega_{p\overline{q}}\left(\frac{\partial}{\partial z_p}\right)^{\star} \otimes \mathrm{d}z_{\overline{q}}, \tag{28}$$

where $\omega_{p\overline{q}}$ are the coefficients of $\omega_{t,\varepsilon}$ with respect to the coordinates $(z_i)_{i=1,\dots,n}$. Its adjoint is computed by the formula

$$\langle \beta_t \cdot v, w\rangle + \langle v, \beta_t^{\star} \cdot w\rangle = 0, \tag{29}$$

where the first bracket is the standard hermitian product in $\mathbb{C}$ and the second one is the one induced by $(T_{\widehat{X}}, h_{t,\varepsilon})$. We have

$$\beta_t^{\star} = -\frac{1}{1+t}\sum \frac{\partial}{\partial z_i} \otimes \mathrm{d}z_i. \tag{30}$$

We have the following formulas

$$D'_{T^*_{\widehat{X}}}\beta_t = 0, \qquad \overline{\partial}\beta_t^{\star} = 0. \tag{31}$$

The first equality holds since $\omega_{t,\varepsilon}$ is a Kähler metric while the second one is obvious from (30).

Moreover, we have

$$(1+t)^2 \cdot \beta_t \wedge \beta_t^{\star} \wedge \omega_{t,\varepsilon}^{n-1} = -\frac{1}{n}\omega_{t,\varepsilon}^n \tag{32}$$

as well as

$$(1+t)^2 \cdot \beta_t^{\star} \wedge \beta_t = \omega_{t,\varepsilon} \otimes \mathrm{Id}_{T_{\widehat{X}}}. \tag{33}$$

• *The curvature of* $\mathscr{V}_t$. If we replace β_t by $(1+t)\sqrt{\mu}\beta_t$ for some positive number μ, this does not affect the complex structure of the bundles at stake but only the metrics. Moreover, we see from the identities (31)-(32)-(33) that the curvature becomes

$$\Theta(\mathscr{V}_t, h_{\mathscr{V}_t}) \wedge \omega_{t,\varepsilon}^{n-1} = \begin{pmatrix} \frac{\mu}{n}\omega_{t,\varepsilon}^n & 0 \\ 0 & \Theta(T_{\widehat{X}}, h_{t,\varepsilon}) \wedge \omega_{t,\varepsilon}^{n-1} - \mu\omega_{t,\varepsilon}^n \otimes \mathrm{Id}_{T_{\widehat{X}}} \end{pmatrix}.$$

Now we choose μ so that $\frac{\mu}{n} = 1-\mu$, i.e. $\mu := \frac{n}{n+1}$. Recalling (11) and the expression of the Ricci curvature of $\omega_{t,\varepsilon}$ given in (12), we get that

$$\Theta(T_{\widehat{X}}, h_{t,\varepsilon}) \wedge \omega_{t,\varepsilon}^{n-1} - \mu\omega_{t,\varepsilon}^n \otimes \mathrm{Id}_{T_{\widehat{X}}} = \frac{1}{n+1}\omega_{t,\varepsilon}^n \otimes \mathrm{Id}_{T_{\widehat{X}}} + A_{t,\varepsilon,\gamma}\omega_{t,\varepsilon}^n,$$

where

$$A_{t,\varepsilon,\gamma} = -\gamma\,\mathrm{Id}_{T_{\widehat{X}}} + \sharp\left[\gamma\pi^*\omega_X - t\omega_{\widehat{X}} + (1-\gamma)\mathrm{dd}^c(\psi_\varepsilon - \varphi_{t,\varepsilon}) - \Theta_\varepsilon\right] \tag{34}$$

is such that the number

$$a_{t,\varepsilon,\gamma} := \frac{1}{n}\int_{\widehat{X}} \mathrm{tr}_{\mathrm{End}}\,\mathrm{pr}_F(A_{t,\varepsilon,\gamma})|_F\,\omega_{t,\varepsilon}^n$$

satisfies

$$\limsup_{\gamma\to 0}\limsup_{t\to 0}\limsup_{\varepsilon\to 0} a_{t,\varepsilon,\gamma} = 0 \tag{35}$$

thanks to the computations of Section 2.2.

• *The curvature of* $\mathscr{V}_t(E)$. Finally, we endow $\mathscr{V}_t(E)$ with the metric $h_{\mathscr{V}_t(E)} := h_{\mathscr{V}_t} \otimes h_E$. It satisfies

$$\Theta(\mathscr{V}_t(E), h_{\mathscr{V}_t(E)}) \wedge \omega_{t,\varepsilon}^{n-1} = \frac{1}{n+1}\omega_{t,\varepsilon}^n \otimes \mathrm{Id}_{\mathscr{V}_t} + A_{t,\varepsilon,\gamma}\omega_{t,\varepsilon}^n + (\Theta_E \wedge \omega_{t,\varepsilon}^{n-1}) \otimes \mathrm{Id}_{\mathscr{V}_t(E)}. \tag{36}$$

where $A_{t,\varepsilon,\gamma}$ is defined in (34) and satisfies (35).

Step 3. The slope inequality. Now, one wants to follow the strategy in Section 2.2 and compute the slope of $\mathscr{F}$ using the induced metric h_{F_t} from $(\mathscr{V}_t(E), h_{\mathscr{V}_t(E)})$ under the sheaf injection (25). The metric h_{F_t} is well-defined only on the locus $W \subset \widehat{X}$ where $F_t := \mathscr{F}|_W$ is a subbundle. As $\mathscr{F}$

may not be saturated in $\mathscr{V}_t(E)$, the complement of W may have codimension one. However, we have the formula

$$\begin{aligned}\mu_{\omega_{t,\varepsilon}}(\mathscr{F}) &= \frac{1}{r}\int_W c_1(F_t,h_{F_t})\wedge\omega_{t,\varepsilon}^{n-1} - c_1(D)\cdot\{\omega_{t,\varepsilon}\}^{n-1}\\ &\le \frac{1}{r}\int_W c_1(F_t,h_{F_t})\wedge\omega_{t,\varepsilon}^{n-1}\\ &\le \mu_{\omega_{t,\varepsilon}}(\mathscr{V}_t(E)) + a_{t,\varepsilon,\gamma} + c_1(E)\cdot\{\omega_{t,\varepsilon}\}^{n-1}\end{aligned}$$

where D is an effective divisor such that $\mathscr{O}_X(D) = \det((\mathscr{V}_t(E)/\mathscr{F})_{\mathrm{tor}})$. Since E is π-exceptional, the conclusion follows from the curvature formula (36) along with (35) and the two easy facts below

- $\mu_{\omega_{t,\varepsilon}}(\mathscr{F}) \to \mu_\alpha(\mathscr{F})$ when $t\to 0$,
- $\mu_{\omega_{t,\varepsilon}}(\mathscr{V}_t(E)) \to \mu_\alpha(\mathscr{V})$ when $t,\varepsilon\to 0$ since E is exceptional, cf. (26).

Theorem 11 is now proved. □

3.4. *Polystability*

In this paragraph, we work under the Assumption B and we aim to prove the second part of Theorem 9, i.e. that $\mathscr{E}^*$ is polystable with respect to $c_1(X)$.

By a standard inductive argument, it is enough to prove that if $\mathscr{F}\subset\mathscr{E}^*$ is any saturated subsheaf with $\mu_{c_1(X)}(\mathscr{F}) = \mu_{c_1(X)}(\mathscr{E}^*)$, then it is holomorphically complemented; i.e. there exists $\mathscr{G}\subset\mathscr{E}^*$ such that $\mathscr{E}^* = \mathscr{F}\oplus\mathscr{G}$.

Let $\mathscr{F}$ be such a subsheaf and let $\widehat{\mathscr{F}}\subset\mathscr{V}$ the induced sheaf on $\widehat{X}$, cf. Lemma 10; it satisfies $\mu_\alpha(\widehat{\mathscr{F}}) = \mu_\alpha(\mathscr{E}^*)$. The same arguments as in the end of Section 2.2 show the orthogonal complement $\widehat{\mathscr{G}}$ of $\widehat{\mathscr{F}}\subset\mathscr{V}_0(E)$ with respect to the well-defined hermitian metric $h_{\mathscr{V}_0(E)}$ on $\widehat{X}\setminus E$ is holomorphic. Note that $\mathscr{V}_0(E)\simeq\widehat{\mathscr{E}}^*$ on $\widehat{X}\setminus E$, hence $\pi_*(\mathscr{V}_0(E)|_{\widehat{X}\setminus E})\simeq\mathscr{E}^*$ by (23).

Now, define $\mathscr{G} := \pi_*\widehat{\mathscr{G}}$ on X_{reg}; this is a coherent subsheaf of $\mathscr{E}^*|_{X_{\mathrm{reg}}}$ by the observation above. We can extend it to a coherent saturated subsheaf $\mathscr{G}\subset\mathscr{E}^*$ across X_{sing}; in particular, $\mathscr{G}$ is reflexive. The injection $\mathscr{F}\oplus\mathscr{G}\hookrightarrow\mathscr{E}^*$ isomorphic over X_{reg}, hence everywhere by reflexivity of the sheaves involved. This concludes the proof of Theorem 9.

4. A splitting theorem

4.1. *Foliations*

In this section, we recollect some results about foliations that we will use later on for the reader's convenience. We refer to [16, §3 and 4] and the references therein for notions around foliations on normal varieties and their singularities.

Here we only recall the notion of weakly regular foliation. Let $\mathscr{F}$ be a foliation of positive rank r on a normal variety X. The r-th wedge product of the inclusion $\mathscr{F}\subseteq T_X$ gives a map

$$\mathscr{O}_X(-K_{\mathscr{F}})\hookrightarrow(\wedge^r T_X)^{**}.$$

We will refer to the dual map

$$\Omega_X^{[r]}\to\mathscr{O}_X(K_{\mathscr{F}})$$

as the *Pfaff field* associated to $\mathscr{F}$. The foliation $\mathscr{F}$ is called *weakly regular* if the induced map

$$(\Omega_X^r\otimes\mathscr{O}_X(-K_{\mathscr{F}}))^{**}\to\mathscr{O}_X$$

is surjective (see [16, §5.1]).

Examples of weakly regular foliations are provided by the following result (see [16, Lem. 5.8]).

Lemma 12. *Let X be a normal variety, and let $\mathscr{F}$ be a foliation on X. Suppose that there exists a distribution $\mathscr{G}$ on X such that $T_X = \mathscr{F} \oplus \mathscr{G}$. Then $\mathscr{F}$ is weakly regular.*

The following lemma says that a weakly regular foliation has mild singularities if its canonical divisor is Cartier and the ambient space has klt singularities (see [16, Lem. 5.9]).

Lemma 13. *Let X be a normal variety with klt singularities, and let $\mathscr{F}$ be a foliation on X. Suppose that $K_{\mathscr{F}}$ is Cartier. If $\mathscr{F}$ is weakly regular, then it has canonical singularities.*

Next, we recall the behaviour of weakly regular foliations with respect to finite covers (see [16, Prop. 5.13]).

Lemma 14. *Let X be a normal variety, let $\mathscr{F}$ be a foliation on X, and let $f\colon X_1 \to X$ be a finite cover. Suppose that each codimension 1 irreducible component of the branch locus of f is $\mathscr{F}$-invariant. Then $\mathscr{F}$ is weakly regular if and only if $f^{-1}\mathscr{F}$ is weakly regular.*

Finally, we recall the behaviour of foliations with canonical singularities with respect to finite covers and birational maps (see [16, Lem. 4.3]).

Lemma 15. *Let $f\colon X_1 \to X$ be a finite cover of normal varieties, and let $\mathscr{F}$ be a foliation on X with $K_{\mathscr{F}}$ $\mathbb{Q}$-Cartier. Suppose that each codimension 1 component of the branch locus of f is $\mathscr{F}$-invariant. If $\mathscr{F}$ has canonical singularities, then $f^{-1}\mathscr{F}$ has canonical singularities as well.*

Recall that $\mathbb{Q}$-divisors D_1 and D_2 are said to be $\mathbb{Q}$-linearly equivalent if there exists an integer $m > 0$ such that mD_1 and mD_2 are linearly equivalent. We write $D_1 \sim_{\mathbb{Q}} D_2$.

Lemma 16. *Let $q\colon Z \to X$ be a birational quasi-projective morphism of normal varieties, and let $\mathscr{F}$ be a foliation on X. Suppose that $K_{\mathscr{F}}$ is $\mathbb{Q}$-Cartier and that $K_{q^{-1}\mathscr{F}} \sim_{\mathbb{Q}} q^*K_{\mathscr{F}}$. If $\mathscr{F}$ has canonical singularities, then $q^{-1}\mathscr{F}$ has canonical singularities as well.*

Proof. By assumption, there exist a normal variety $\overline{Z} \supseteq Z$ and a projective birational morphism $\overline{q}\colon \overline{Z} \to X$ whose restriction to Z is q. The same argument used in the proof of [16, Lem. 4.2] shows that

$$a(E, \overline{Z}, \overline{q}^{-1}\mathscr{F}) = a(E, X, \mathscr{F})$$

for any exceptional prime divisor E over $\overline{Z}$ with non-empty center in Z. The lemma follows easily. □

4.2. *Weakly regular foliations with algebraic leaves*

This section contains a generalization of Theorem 6.1 in [16]. The following result is proved in [16] under the additional assumption that $\mathscr{F}$ has canonical singularities.

Theorem 17. *Let X be a normal projective variety with $\mathbb{Q}$-factorial klt singularities, and let $\mathscr{F}$ be a weakly regular foliation on X with algebraic leaves.*

(1) *Then $\mathscr{F}$ is induced by a surjective equidimensional morphism $p\colon X \to Y$ onto a normal projective variety Y.*
(2) *Moreover, there exists an open subset Y° with complement of codimension at least 2 in Y such that $p^{-1}(y)$ is irreducible for any $y \in Y^\circ$.*

Before we give the proof of Theorem 17, we need to prove a number of auxiliary statements. Throughout the present section, we will be working in the following setup.

Setup 18. Let X and Y be normal quasi-projective varieties, and let $p'\colon X \dashrightarrow Y$ be a dominant rational map with $r := \dim X - \dim Y > 0$. Let Z be the normalization of the graph of p', and let $p\colon Z \to Y$ and $q\colon Z \to X$ be the natural morphisms. Let $\mathscr{F}$ be the foliation induced by p'.

Proposition 19. *Let the setting and notation be as in 18, and assume that $K_{\mathscr{F}}$ is Cartier.*

(1) *Then the Pfaff field $\Omega_X^{[r]} \to \mathscr{O}_X(K_{\mathscr{F}})$ associated to $\mathscr{F}$ induces a map*

$$\Omega_Z^{[r]} \to q^*\mathscr{O}_X(K_{\mathscr{F}})$$

which factors through the Pfaff field $\Omega_Z^{[r]} \to \mathscr{O}_Z(K_{q^{-1}\mathscr{F}})$ associated to $q^{-1}\mathscr{F}$. In particular, there exists an effective q-exceptional Weil divisor B on Z such that

$$K_{q^{-1}\mathscr{F}} + B \sim_{\mathbb{Z}} q^*K_{\mathscr{F}}.$$

(2) *Moreover, if E is a q-exceptional prime divisor on Z such that $p(E) = Y$, then $E \subseteq \operatorname{Supp} B$.*

Proof. Let $Z_0 \subseteq Y \times X$ be the graph of p', and denote by $n\colon Z \to Z_0$ the normalization map. Consider the foliation

$$\mathscr{G} := \operatorname{pr}_X^*\mathscr{F} \subseteq \operatorname{pr}_X^* T_X \subseteq \operatorname{pr}_Y^* T_Y \oplus \operatorname{pr}_X^* T_X.$$

Let $\Omega_X^r \to \mathscr{O}_X(K_{\mathscr{F}})$ be the map induced by the Pfaff field $\Omega_X^{[r]} \to \mathscr{O}_X(K_{\mathscr{F}})$. By construction, Z_0 is invariant under $\mathscr{G}$, and hence, there is a factorization:

$$\begin{tikzcd}
\Omega^r_{Y\times X}|_{Z_0} \arrow[r] \arrow[d, two heads] & \operatorname{pr}_X^*\Omega_X^r|_{Z_0} \arrow[r] & (\operatorname{pr}_X^*\mathscr{O}_X(K_{\mathscr{F}}))|_{Z_0} \arrow[d, equal] \\
\Omega^r_{Z_0} \arrow[rr] & & \mathscr{O}_{Y\times X}(K_{\mathscr{G}})|_{Z_0}.
\end{tikzcd}$$

Notice that the foliation induced by $\mathscr{G}$ on Z is $q^{-1}\mathscr{F}$. By [1, Prop. 4.5], the map $\Omega^r_{Z_0} \to (\operatorname{pr}_X^*\mathscr{O}_X(K_{\mathscr{F}}))|_{Z_0}$ extends to a map

$$\Omega_Z^r \to n^*((\operatorname{pr}_X^*\mathscr{O}_X(K_{\mathscr{F}}))|_{Z_0}) \simeq q^*\mathscr{O}_X(K_{\mathscr{F}}),$$

which gives a morphism

$$\Omega_Z^{[r]} \to q^*\mathscr{O}_X(K_{\mathscr{F}}).$$

This map factors through the Pfaff field

$$\nu_Z\colon \Omega_Z^{[r]} \to \mathscr{O}_Z(K_{q^{-1}\mathscr{F}})$$

associated to $q^{-1}\mathscr{F}$ away from the closed set where ν_Z is not surjective, which has codimension at least 2 in Z. Hence, there exists an effective Weil divisor B on Z such that

$$K_{q^{-1}\mathscr{F}} + B \sim_{\mathbb{Z}} q^*K_{\mathscr{F}}.$$

Moreover, the morphism $\Omega_Z^{[r]} \to q^*\mathscr{O}_X(K_{\mathscr{F}})$ identifies with the composition

$$\Omega_Z^{[r]} \to \mathscr{O}_Z(K_{q^{-1}\mathscr{F}}) \to q^*\mathscr{O}_X(K_{\mathscr{F}})$$

since $q^*\mathscr{O}_X(K_{\mathscr{F}})$ is torsion-free. Note that B is obviously q-exceptional, proving the first item.

The second item follows from [16, Lem. 4.19] by induction on the rank of $\mathscr{F}$ as in the proof of Proposition 4.17 in [16]. Notice that the assumption that the birational morphism is projective in the statement of Lemma 4.19 in [16] is not necessary. □

Corollary 20. *Setting and notation as in Setup 18. Suppose that X has klt singularities. Suppose in addition that $K_{\mathscr{F}}$ is Cartier and that $\mathscr{F}$ is weakly regular.*

(1) *Then the foliation $q^{-1}\mathscr{F}$ is weakly regular and $K_{q^{-1}\mathscr{F}} \sim_{\mathbb{Z}} q^* K_{\mathscr{F}}$.*
(2) *Moreover, if E is a prime q-exceptional divisor on Z, then $p(E) \subsetneq Y$.*

Proof. By Proposition 19 (1), the Pfaff field

$$\Omega_X^{[r]} \to \mathscr{O}_X(K_{\mathscr{F}})$$

associated to $\mathscr{F}$ induces a map

$$\Omega_Z^{[r]} \to q^*\mathscr{O}_X(K_{\mathscr{F}})$$

which factors through the Pfaff field $\Omega_Z^{[r]} \to \mathscr{O}_Z(K_{q^{-1}\mathscr{F}})$ associated to $q^{-1}\mathscr{F}$. On the other hand, by [28, Thm. 1.3], there exists a morphism of sheaves

$$q^*\Omega_X^{[r]} \to \Omega_Z^{[r]}$$

that agrees with the usual pull-back morphism of Kähler differentials wherever this makes sense. One then readily checks that we obtain a commutative diagram as follows:

$$\begin{tikzcd} q^*\Omega_X^{[r]} \arrow[r, two heads] \arrow[d] & q^*\mathscr{O}_X(K_{\mathscr{F}}) \arrow[d, equal] \\ \Omega_Z^{[r]} \arrow[r] & q^*\mathscr{O}_X(K_{\mathscr{F}}). \end{tikzcd}$$

This implies that the map $\Omega_Z^{[r]} \to q^*\mathscr{O}_X(K_{\mathscr{F}})$ is surjective. Consequently, this map identifies with the Pfaff field associated to $q^{-1}\mathscr{F}$, proving item (2).

Finally, item (2) is an immediate consequence of item 1 together with Proposition 19 (2). □

As we will see, Theorem 17 is an easy consequence of Lemma 21 and Lemma 22 below.

Lemma 21. *Setting and notation as in 18. Suppose that X has klt singularities and that $\mathscr{F}$ is weakly regular. Then there exists an open subset Y° with complement of codimension at least* 2 *in Y such that, for any $y \in Y^\circ$, either $p^{-1}(y)$ is empty or any connected component of $p^{-1}(y)$ is irreducible.*

Proof. We argue by contradiction and assume that there exists a prime divisor $D \subset Y$ such that, for a general point $y \in D$, $p^{-1}(y)$ is non-empty and some connected component of $p^{-1}(y)$ is reducible. Let $S \subseteq p^{-1}(D)$ be a subvariety of maximal dimension and dominating D such that for a general point $z \in S$ there is at least two irreducible components of $p^{-1}(p(z))$ passing through z. We will show in Step 2 that S has codimension 2 in Z.

Step 1. Construction. Shrinking Y if necessary, we may assume without loss of generality that p is equidimensional. Replacing X by an open neighborhood of the generic point of $q(S)$, we may also assume that there exists a positive integer m such that

$$\mathscr{O}_X(mK_{\mathscr{F}}) \simeq \mathscr{O}_X.$$

Let $f\colon X_1 \to X$ be the associated cyclic cover, which is quasi-étale (see [33, Def. 2.52]), and let Z_1 be the normalization of the product $Z \times_X X_1$. The induced morphism $g\colon Z_1 \to Z$ is then a finite cover.

By [14, Lem. 4.2], there exists a finite cover $Y_2 \to Y$ with Y_2 normal and connected such that the following holds. If Z_2 denotes the normalization of the product $Y_2 \times_Y Z_1$, then the natural morphism $p_2\colon Z_2 \to Y_2$ has reduced fibers over codimension 1 points in Y_2. We may also assume that $Y_2 \to Y$ is a Galois cover. We obtain a commutative diagram as follows:

$$\begin{tikzcd} Z_2 \arrow[r, "g_1"] \arrow[ddd, "p_2"'] & Z_1 \arrow[r, "q_1"] \arrow[d, "g"] \arrow[ddd, bend right, "p_1"'] & X_1 \arrow[d, "f"] \\ & Z \arrow[r, "q"] \arrow[dd, "p"] & X \\ & & \\ Y_2 \arrow[r] & Y. & \end{tikzcd}$$

Notice that $g\circ g_1\colon Z_2 \to Z$ is a finite Galois cover.

Step 2. Away from a closed subset of codimension at least 3, Z has quotient singularities and the foliation induced by p on Z is weakly regular. Moreover, S has codimension 2 in Z. Notice that X_1 has klt singularities by [30, Prop. 3.16], and that the foliation $\mathscr{F}_{X_1} := f^{-1}\mathscr{F}$ is weakly regular by Lemma 14. Observe now that the foliation $\mathscr{F}_{Z_1} := q_1^{-1}\mathscr{F}_{X_1}$ is given by p_1 and that Z_1 identifies with the normalization of the graph of the rational map $p_1 \circ q_1^{-1}$. Therefore, $\mathscr{F}_{Z_1}$ is weakly regular and

$$K_{\mathscr{F}_{Z_1}} \sim_{\mathbb{Z}} q_1^* K_{\mathscr{F}_{X_1}}$$

by Corollary 20 (1). On the other hand, $\mathscr{F}_{X_1}$ has canonical singularities (see Lemma 13). Applying Lemma 16, we conclude that $\mathscr{F}_{Z_1}$ has canonical singularities as well. This in turn implies that the foliation $\mathscr{F}_{Z_2} := g_2^{-1}\mathscr{F}_{Z_1}$ has also canonical singularities (see Lemma 15). From [14, Lem. 5.4], we conclude that Z_2 has canonical singularities over a big open set contained in Y_2, using the fact that p_2 has reduced fibers over codimension 1 points by construction. In particular, Z_2 has canonical singularities in codimension 2.

Since $g\circ g_1\colon Z_2 \to Z$ is a finite Galois cover, there exists an effective $\mathbb{Q}$-divisor Δ on Z such that

$$K_{Z_2} \sim_{\mathbb{Q}} (g\circ g_1)^*(K_Z + \Delta).$$

Moreover, away from a closed subset of codimension at least 3, $K_Z + \Delta$ is $\mathbb{Q}$-Cartier by [16, Lem. 2.6]), and the pair (Z,Δ) is klt by [30, Prop. 3.16] so that it has Cohen–Macaulay singularities. Then Harstshorne's connectedness theorem implies that S has codimension 2 in Z.

By construction, any irreducible codimension 1 component of the ramification locus of g is q_1-exceptional, and hence invariant under $\mathscr{F}_{Z_1}$ by Corollary 20 (2). It follows from Lemma 14 that $\mathscr{F}_Z := q^{-1}\mathscr{F}$ is weakly regular in codimension 2.

Step 3. End of proof. Let $z \in S$ be a general point. Recall from [21, Prop. 9.3] that z has an analytic neighborhood $U \subseteq Z$ that is biholomorphic to an analytic neighborhood of the origin in a variety of the form $\mathbb{C}^{\dim Z}/G$, where G is a finite subgroup of $\mathrm{GL}(\dim Z,\mathbb{C})$ that does not contain any quasi-reflections. In particular, if W denotes the inverse image of U in the affine space $\mathbb{C}^{\dim Z}$, then the quotient map

$$g_U\colon W \to W/G \simeq U$$

is étale outside of the singular set.

By Lemma 14 again, $\mathscr{F}_Z$ induces a regular foliation on W. Let F_1 and F_2 be irreducible components of $p^{-1}(p(z))$ passing through z with $F_1 \neq F_2$. Note that

$$g_U^{-1}(F_1\cap U)\cap g_U^{-1}(F_2\cap U) \neq \emptyset.$$

By general choice of z, F_1 and F_2 are not contained in the singular locus of $\mathscr{F}_Z$, and hence both $g_U^{-1}(F_1\cap U)$ and $g_U^{-1}(F_2\cap U)$ are a disjoint union of leaves. But then, any leaf passing through some point of $g_U^{-1}(F_1\cap U)\cap g_U^{-1}(F_2\cap U)$ is a connected component of both $g_U^{-1}(F_1\cap U)$ and

$g_U^{-1}(F_2 \cap U)$. This in turn implies that $F_1 = F_2$, yielding a contradiction. This finishes the proof of the lemma. $\square$

Lemma 22. *Setting and notation as in 18. Suppose that X has klt singularities and that $\mathscr{F}$ is weakly regular. Let E be a prime q-exceptional divisor on Z such that* $\dim p\,(E) \geq \dim Y - 1$.

(1) *Then* $\dim p\,(E) = \dim Y - 1$. *In particular, E is invariant under the foliation on Z induced by p.*
(2) *Moreover, if z is a general point in E, then there exists a curve $T \subseteq E$ passing through z with* $\dim p(T) = 1$ *such that $q(E_{p(t_1)}(t_1)) = q(E_{p(t_2)}(t_2))$ for general points t_1 and t_2 in T, where $E_{p(t)}(t)$ denotes the irreducible component of $E_{p(t)} \subseteq p^{-1}(p(t))$ passing through $t \in T \subset E$.*

Proof. For the reader's convenience, the proof is subdivided into a number of steps.

Step 1. Reduction to the case where $K_{\mathscr{F}}$ is Cartier and proof of (1). Replacing X by an open neighborhood of the generic point of $q(E)$, we may assume without loss of generality that there exists a positive integer m such that

$$\mathscr{O}_X(mK_{\mathscr{F}}) \simeq \mathscr{O}_X.$$

Let $f\colon X_1 \to X$ be the associated cyclic cover, which is quasi-étale (see [33, Def. 2.52]), and let Z_1 be the normalization of the product $Z \times_X X_1$. The induced morphism $g\colon Z_1 \to Z$ is then a finite cover. We obtain a commutative diagram as follows:

$$\begin{tikzcd}
Z_1 \arrow[r, "q_1"] \arrow[d, "g"] \arrow[dd, bend right=60, "p_1"'] & X_1 \arrow[d, "f"] \\
Z \arrow[r, "q"] \arrow[d, "p"] & X \\
Y. &
\end{tikzcd}$$

Notice that X_1 has klt singularities by [30, Prop. 3.16], and that the foliation $\mathscr{F}_{X_1} := f^{-1}\mathscr{F}$ is weakly regular by Lemma 14. Observe now that the foliation $\mathscr{F}_{Z_1} := q_1^{-1}\mathscr{F}_{X_1}$ is given by p_1 and that Z_1 identifies with the normalization of the graph of the rational map $p_1 \circ q_1^{-1}$. By item 1 in Corollary 20, $\mathscr{F}_{Z_1}$ is weakly regular. Let E_1 be a prime divisor on Z_1 such that $g(E_1) = E$. Notice that E_1 is q_1-exceptional and that $\dim p\,(E) = \dim p_1(E_1)$. Thus, replacing X by X_1, we may assume without loss of generality that

$$K_{\mathscr{F}} \sim_{\mathbb{Z}} 0.$$

Then, by Corollary 20 (2), we must have $p(E) \subsetneq Y$. It follows that $p(E)$ is a prime divisor on Y since $\dim p\,(E) \geq \dim Y - 1$ by assumption. In particular, E is invariant under the foliation $\mathscr{F}_Z := q^{-1}\mathscr{F}$.

Step 2. The foliation induced by $\mathscr{F}$ on $q(E)$. Set $B := q(E)$, and let $E^\circ \subseteq E \cap Z_{\mathrm{reg}}$ be a non-empty open set. We obtain a commutative diagram as follows:

$$\begin{tikzcd}
E^\circ \arrow[r, "a"] \arrow[d, hook] \arrow[dd, bend right, "j"'] & B \arrow[d, equal] \\
E \arrow[r, two heads] \arrow[d, hook] & B \arrow[d, hook, "i"] \\
Z \arrow[r, "q"] \arrow[d, "p"'] & X \\
Y. &
\end{tikzcd}$$

Shrinking X, if necessary, we may assume without loss of generality that B is smooth. By [28, Thm. 1.3 and Prop. 6.1], there is a factorization

$$\begin{tikzcd}
\Omega^r_X|_B \arrow[r] \arrow[rr, bend left, two heads, "di"] & \Omega^{[r]}_X|_B \arrow[r, "d_{\mathrm{refl}}\, i"'] & \Omega^r_B.
\end{tikzcd}$$

This implies that the map $\Omega^{[r]}_X|_B \to \Omega^r_B$ is surjective.

Claim 23. The foliation $\mathscr{F}_{E^\circ}$ on E° induced by $\mathscr{F}_Z$ is projectable under a.

Proof of Claim 23. Let

$$\nu_X\colon \Omega^{[r]}_X \twoheadrightarrow \mathscr{O}_X(K_{\mathscr{F}}) \quad \text{and} \quad \nu_Z\colon \Omega^{[r]}_Z \to \mathscr{O}_Z(K_{\mathscr{F}_Z})$$

be the Pfaff fields associated to $\mathscr{F}$ and $\mathscr{F}_Z$ respectively. Since E° is invariant by $\mathscr{F}_Z$, there is a factorization

$$\begin{tikzcd}
\Omega^r_Z|_{E^\circ} \arrow[r] \arrow[d, two heads] & \Omega^{[r]}_Z|_{E^\circ} \arrow[r, "\nu_Z|_{E^\circ}"] \arrow[d, "d_{\mathrm{refl}}\, j"'] & \mathscr{O}_Z(K_{\mathscr{F}_Z})|_{E^\circ} \arrow[d, equal] \\
\Omega^r_{E^\circ} \arrow[r, equal] & \Omega^r_{E^\circ} \arrow[r] & \mathscr{O}_Z(K_{\mathscr{F}_Z})|_{E^\circ}.
\end{tikzcd}$$

Recall from the proof of Corollary 1 that there is a commutative diagram

$$\begin{tikzcd}
q^*\Omega^{[r]}_X \arrow[r, two heads, "q^*\nu_X"] \arrow[d, "d_{\mathrm{refl}}\, q"'] & q^*\mathscr{O}_X(K_{\mathscr{F}}) \\
\Omega^{[r]}_Z \arrow[r, two heads, "\nu_Z"] & \mathscr{O}_Z(K_{\mathscr{F}_Z}). \arrow[u, "\imath"]
\end{tikzcd}$$

Finally, by [28, Prop. 6.1], the diagram

$$\begin{tikzcd}
(q^*\Omega^{[r]}_X)|_{E^\circ} \simeq a^*(\Omega^{[r]}_X|_B) \arrow[r, "a^* d_{\mathrm{refl}}\, i"] \arrow[d, "d_{\mathrm{refl}}\, q|_{E^\circ}"'] & a^*\Omega^r_B \arrow[d] \\
\Omega^{[r]}_Z|_{E^\circ} \arrow[r, "d_{\mathrm{refl}}\, j"'] & \Omega^r_{E^\circ}
\end{tikzcd}$$

is commutative as well. Therefore, we have a commutative diagramm as follows:

$$\begin{tikzcd} (q^*\Omega_X^{[r]})|_{E^\circ} \simeq a^*(\Omega_X^{[r]}|_B) \arrow[r, two heads, "a^* d_{\mathrm{refl}} i"] \arrow[dd, two heads, "(q^*\nu_X)|_{E^\circ}"'] & a^*\Omega_B^r \arrow[d] \\ & \Omega_{E^\circ}^r \arrow[d] \\ (q^*\mathscr{O}_X(K_{\mathscr{F}}))|_{E^\circ} & \mathscr{O}_Z(K_{\mathscr{F}_Z})|_{E^\circ}. \arrow[l, "\sim"] \end{tikzcd}$$

This in turn implies that there is a factorization

$$\begin{tikzcd} \Omega_X^{[r]}|_B \arrow[r, two heads, "d_{\mathrm{refl}} i"] \arrow[d, two heads, "\nu_X|_B"] & \Omega_B^r \arrow[d] \\ \mathscr{O}_X(K_{\mathscr{F}})|_B \arrow[r, equal] & \mathscr{O}_X(K_{\mathscr{F}})|_B \end{tikzcd}$$

whose pull-back to E° gives the diagram above. It follows that the map

$$\Omega_B^r \twoheadrightarrow \mathscr{O}_X(K_{\mathscr{F}})|_B$$

is the Pfaff field associated to a weakly regular foliation $\mathscr{F}_B$ of rank r on B such that $da(\mathscr{F}_{E^\circ}) = \mathscr{F}_B$. This completes the proof of the claim. □

Then item (2) is an immediate consequence of Claim 23 above. □

We are now ready to prove Theorem 17.

Proof of Theorem 17. Let $p\colon Z \to Y$ be the family of leaves, and let $q\colon Z \to X$ be the natural morphism. Since p has connected fibers by construction, Lemma 21 applied to $p \circ q^{-1}$ implies that p has irreducible fibers over a big open set contained in Y. Hence, to prove Theorem 17, it suffices to show that $\operatorname{Exc} q$ is empty.

We argue by contradiction and assume that $\operatorname{Exc} q \neq \emptyset$. Let E be an irreducible component of $\operatorname{Exc} q$. Then E has codimension 1 since X is $\mathbb{Q}$-factorial by assumption. Recall from Lemma 21 that $p^{-1}(y)$ is irreducible for a general point y in $p(E)$. Therefore, by Lemma 22, we must have $E = p^{-1}(p(E))$. Moreover, if y is a general point in $p(E)$, then there exists a curve $T \subseteq p(E)$ passing through y such that $q(p^{-1}(t_1)) = q(p^{-1}(t_2))$ for general points t_1 and t_2 in T. Now, there exists a positive integer t such that the cycle theoretic fiber $p^{[-1]}(y)$ is $t[p^{-1}(y)]$ for a general point y in $p(E)$. It follows that the restriction of the map $Y \to \operatorname{Chow}(X)$ to $p(E)$ has positive dimensional fibers, yielding a contradiction. This finishes the proof of the theorem. □

Remark 24. In the setup of Theorem 17, let $p\colon Z \to Y$ be the family of leaves, and let $q\colon Z \to X$ be the natural morphism. If X is only assumed to have klt singularities, then the same argument used in the proof of the theorem shows that q is a small birational map. We have

$$K_{Z/Y} - R(p) \sim_{\mathbb{Q}} q^* K_{\mathscr{F}},$$

where $R(p)$ denotes the ramification divisor of p. In particular, if F denotes the normalization of the closure of a general leaf of $\mathscr{F}$, then

$$K_{\mathscr{F}}|_F \sim_{\mathbb{Q}} K_F.$$

4.3. *A splitting theorem*

The following theorem, advertised in the introduction as Theorem C, is the main result of this section.

Theorem 25. *Let X be a normal projective variety, and let*

$$T_X = \bigoplus_{i\in I} \mathscr{F}_i$$

be a decomposition of T_X into involutive subsheaves with algebraic leaves. Suppose that there exists a $\mathbb{Q}$-divisor Δ such that (X,Δ) is klt. Then there exists a quasi-étale cover $f\colon Y \to X$ as well as a decomposition

$$Y \simeq \prod_{i\in I} Y_i$$

of Y into a product of normal projective varieties such that the decomposition $T_X = \bigoplus_{i\in I}\mathscr{F}_i$ lifts to the canonical decomposition

$$T_{\prod_{i\in I} Y_i} = \bigoplus_{i\in I} \mathrm{pr}_i^* \, T_{Y_i}.$$

Proof. To prove the theorem, it is obviously enough to consider the case where $I = \{1,2\}$. Set $\tau(i) = 3-i$ for each $i \in \{1,2\}$.

Step 1. Reduction to the case where X is $\mathbb{Q}$-factorial with klt singularities. Let $\pi\colon Z \to X$ be a $\mathbb{Q}$-factorialization, whose existence is established in [31, Cor. 1.37]. Recall that π is a small birational projective morphism and that Z is $\mathbb{Q}$-factorial with klt singularities. Then we have the decomposition

$$T_Z = \pi^{-1}\mathscr{F}_1 \oplus \pi^{-1}\mathscr{F}_2$$

into involutive subsheaves with algebraic leaves.

Suppose that there exist normal projective varieties W_1 and W_2 and a quasi-étale cover

$$g\colon W_1 \times W_2 \to Z$$

such that the decomposition $T_Z = \pi^{-1}\mathscr{F}_1 \oplus \pi^{-1}\mathscr{F}_2$ lifts to the canonical decomposition

$$T_{W_1\times W_2} = \mathrm{pr}_1^* \, T_{W_1} \oplus \mathrm{pr}_2^* \, T_{W_2}.$$

The Stein factorization

$$f\colon Y \to X$$

of $\pi\circ g$ is then a quasi-étale cover, and the natural map

$$W_1 \times W_2 \to Y$$

is a small birational morphism. Moreover, by [30, Prop. 3.16], Y has klt singularities. In particular, it has rational singularities. Lemma 26 below applied to $Y \dashrightarrow W_1 \times W_2$ then implies that X satisfies the conclusion of Theorem 25.

Therefore, replacing X by Z, if necessary, we may assume without loss of generality that X is $\mathbb{Q}$-factorial with klt singularities.

Step 2. Covering construction. By Lemma 12, $\mathscr{F}_i$ is a weakly regular foliation. Therefore, by Theorem 17, $\mathscr{F}_i$ is induced by a surjective equidimensional morphism $p_i\colon X \to T_i$ onto a normal projective variety T_i. Moreover, p_i has irreducible fibers over a big open set contained in T_i. Let F_i be a general fiber of $p_{\tau(i)}$.

Let M_i denote the normalization of the product $F_i \times_{T_i} X$, and let $M_i \to N_i \to X$ denote the Stein factorization of the natural morphism $M_i \to X$. We will show that $N_i \to X$ is a quasi-étale cover. Notice that for any prime P on T_i, $p_i^* P$ is well-defined (see [16, §2.7]) and has irreducible support.

Write $p_i^* P = mQ$ for some prime divisor Q on X and some integer $m \geq 1$. Set $n := \dim X$, and $s := \dim T_i$. By general choice of F_i, we may assume that $F_i \setminus X_{\text{reg}}$ has codimension at least 2 in F_i. In particular, $F_i \cap Q \cap X_{\text{reg}} \neq \emptyset$. Let $x \in F_i \cap Q \cap X_{\text{reg}}$ be a general point. Since $\mathscr{F}_1$ and $\mathscr{F}_2$ are regular foliations at x and $T_X = \mathscr{F}_1 \oplus \mathscr{F}_2$, there exist local analytic coordinates centered at x and $p_i(x)$ respectively such that p_i is given by

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1^m, x_2 \ldots, x_s),$$

and such that F_i is given by the equations

$$x_{s+1} = \cdots = x_n = 0.$$

A straightforward local computation then shows that $N_i \to X$ is a quasi-étale cover over the generic point of $p_i^{-1}(P)$. This immediately implies that $N_i \to X$ is a quasi-étale cover.

Let Y be the normalization of X in the compositum of the function fields $\mathbb{C}(N_i)$, and let $f\colon Y \to X$ be the natural morphism. Set $\mathscr{G}_i := f^{-1}\mathscr{F}_i$. By construction, f is a quasi-étale cover, and $\mathscr{G}_i$ is induced by a surjective equidimensional morphism $q_i\colon Y \to R_i$ with reduced fibers over a big open set contained in R_i. Moreover, there exists a subvariety $G_i \subseteq f^{-1}(F_i)$ such that the restriction $G_i \to R_i$ of q_i to G_i is a birational morphism.

Step 3. End of proof. Let R_i° denote the smooth locus of R_i, and set $Y_i^\circ := q_i^{-1}(R_i^\circ)$. Let $Z_i^\circ \subseteq Y_i^\circ$ be the open set where $q_i|_{Y_i^\circ}$ is smooth. Notice that Z_i° has complement of codimension at least 2 in Y_i° since q_i has reduced fibers over a big open set contained in R_i.

The restriction of the tangent map

$$Tq_i|_{Y_i^\circ}\colon T_{Y_i^\circ} \to \big(q_i|_{Y_i^\circ}\big)^* T_{R_i^\circ}$$

to $\mathscr{G}_{\tau(i)}|_{Z_i^\circ} \subseteq T_{Z_i^\circ}$ then induces an isomorphism $\mathscr{G}_{\tau(i)}|_{Z_i^\circ} \simeq \big(q_i|_{Z_i^\circ}\big)^* T_{R_i^\circ}$. Since $\mathscr{G}_{\tau(i)}|_{Y_i^\circ}$ and $\big(q_i|_{Y_i^\circ}\big)^* T_{R_i^\circ}$ are both reflexive sheaves, we finally obtain an isomorphism

$$\mathscr{G}_{\tau(i)}|_{Y_i^\circ} \simeq \big(q_i|_{Y_i^\circ}\big)^* T_{R_i^\circ}.$$

A classical result of complex analysis says that complex flows of vector fields on analytic spaces exist (see [27]). It follows that $q_i|_{Y_i^\circ}$ is a locally trivial analytic fibration for the analytic topology.

The morphism $q_1 \times q_2\colon Y \to R_1 \times R_2$ then induces an isomorphism

$$q_1^{-1}(R_1^\circ) \cap q_2^{-1}(R_2^\circ) \simeq R_1^\circ \times R_2^\circ$$

since $G_1 \cdot G_2 = 1$ and q_i is locally trivial over R_i°. In particular, $q_1 \times q_2$ is a small birational morphism. By [30, Prop. 3.16] again, Y has klt singularities. Hence, it has rational singularities. Lemma 26 below applied to $q_1 \times q_1$ then implies that X satisfies the conclusion of Theorem 25, completing the proof of the theorem. ☐

Lemma 26 ([32, Prop. 18]). *Let X, Y_1 and Y_2 be normal projective varieties, and let $\pi\colon X \dashrightarrow Y_1 \times Y_2$ be a birational map that does not contract any divisor. Suppose in addition that X has rational singularities. Then X decomposes as a product $X \simeq X_1 \times X_2$ and there exist birational maps $\pi_i\colon X_i \dashrightarrow Y_i$ such that $\pi = \pi_1 \times \pi_2$.*

5. Proof of Theorem A

The present section is devoted to the proof of Theorem A.

Proof of Theorem A. We have seen in Theorem 6 that the tangent sheaf of X is polystable. By definition it means that we have a decomposition

$$T_X = \bigoplus_{i \in I} \mathscr{F}_i$$

where the $\mathscr{F}_i$ are stable with respect to $c_1(X)$ and have the same slope. Moreover, each subsheaf $\mathscr{F}_i$ defines on X_{reg} a parallel subbundle of $T_{X_{\mathrm{reg}}}$ with respect to the Kähler–Einstein metric $\omega_{\mathrm{KE}}|_{X_{\mathrm{reg}}}$. This immediately implies that $\mathscr{F}_i|_{X_{\mathrm{reg}}}$ is involutive.

Claim 27. Each foliation $\mathscr{F}_i$ has algebraic leaves.

Proof. Let m be a positive integer such that $-mK_X$ is very ample, and let $C \subset X$ be a general complete intersection curve of elements in $|-mK_X|$. By general choice of C, we may assume that $C \subset X_{\mathrm{reg}}$ and that $\mathscr{F}_i$ is locally free in a neighborhood of C. If m is large enough, then the vector bundle $\mathscr{F}_i|_C$ is semistable by [19, Thm. 1.2]). We conclude that it is ample since it has positive slope. Then [5, Fact 2.1.1] says that $\mathscr{F}_i$ has algebraic leaves. Alternatively, one can apply [7, Thm. 1.1] to the foliation $\widehat{\mathscr{F}_i}$ on the resolution $\widehat{X}$ (cf. Notation 5) induced by $\mathscr{F}_i$ by pullback over X_{reg} and saturation inside $T_{\widehat{X}}$. □

Let $f : Y \to X$ be the quasi-étale cover and $Y = \prod_{i \in I} Y_i$ be the splitting that are both provided by Theorem 25. The decomposition

$$T_Y = \bigoplus_{i \in I} \mathrm{pr}_i^* T_{Y_i} \tag{37}$$

is a decomposition of T_Y into summands of maximal slope. If there exists $i \in I$ such that T_{Y_i} is not stable with respect to $c_1(Y_i)$, then it means that the polystable decomposition of T_Y provided by Theorem 6 via $f^*\omega_{\mathrm{KE}}$ refines strictly the decomposition (37). By applying Theorem 25 again, we can find another quasi-étale cover $Y' \to Y$ which splits according to the polystable decomposition of T_Y and one can then compare again the polystable decomposition of $T_{Y'}$ to the one coming from T_Y. After finitely many such steps, one can find a quasi-étale cover $g : Z \to X$ such that

(i) There exists a splitting $Z = \prod_{k \in K} Z_k$ into a product of $\mathbb{Q}$-Fano varieties.
(ii) For any $k \in K$, the tangent sheaf T_{Z_k} is stable with respect to $c_1(Z_k)$.
(iii) The variety Z admits a Kähler–Einstein metric given by $g^*\omega_{\mathrm{KE}}$.

Theorem A is a consequence of the Claim below.

Claim 28. There exists a Kähler–Einstein metric ω_k on each variety Z_k such that $g^*\omega = \sum_{k \in K} \mathrm{pr}_k^* \omega_k$.

Proof of Claim 28. We set $n_k := \dim Z_k$. As the saturated subsheaf $\mathscr{F}_k := \mathrm{pr}_k^* T_{Z_k} \subset T_Z$ is stable with maximal slope with respect to $c_1(Z)$, it has to coincide with one of the factors in the decomposition of T_Z provided by Theorem 6 (one can see that by looking at the projections on each factor and use stability). In particular, the $\mathscr{F}_k|_{Z_{\mathrm{reg}}}$ are mutually orthogonal with respect to $g^*\omega_{\mathrm{KE}}$, which enables one to define a smooth hermitian metric ω_k on Z_k^{reg} such that $g^*\omega_{\mathrm{KE}} = \sum_{k \in K} \mathrm{pr}_k^* \omega_k$ on Z_{reg}. Since $g^*\omega_{\mathrm{KE}}$ is closed and d commutes with pr_k^*, it follows that ω_k is a Kähler metric on Z_{reg}.

Clearly, one has $\mathrm{Ric}\,\omega_k = \omega_k$ on Z_k^{reg}. In order to check that ω_k defines a Kähler–Einstein metric on Z_k in the sense of Definition 2, it is sufficient to check that $\int_{Z_k^{\mathrm{reg}}} \omega_k^{n_k} = c_1(Z_k)^{n_k}$ by Remark 3. By [2, Prop. 3.8] we always have the inequality $\int_{Z_k^{\mathrm{reg}}} \omega_k^{n_k} \le c_1(Z_k)^{n_k}$ and therefore

$$c_1(Z)^n = \int_{Z_{\mathrm{reg}}} g^*\omega_{\mathrm{KE}}^n = \prod_{k \in K} \int_{Z_k^{\mathrm{reg}}} \omega_k^{n_k} \le \prod_{k \in K} c_1(Z_k)^{n_k}.$$

Since $c_1(Z)^n = \prod_{k \in K} c_1(Z_k)^{n_k}$, one must have $\int_{Z_k^{\mathrm{reg}}} \omega_k^{n_k} = c_1(Z_k)^{n_k}$ for all $k \in K$. □

Theorem A is now proved. □

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Complex algebraic geometry, in memory of Jean-Pierre Demailly /
Géométrie algébrique complexe, en mémoire de Jean-Pierre Demailly

Families of jets of arc type and higher (co)dimensional Du Val singularities

Familles de jets de type arc et singularités de Du Val de (co)dimension supérieure

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In memory of Jean-Pierre Demailly

Abstract. Families of jets through singularities of algebraic varieties are here studied in relation to the families of arcs originally studied by Nash. After proving a general result relating them, we look at normal locally complete intersection varieties with rational singularities and focus on a class of singularities we call *higher Du Val singularities*, a higher dimensional (and codimensional) version of Du Val singularities that is closely related to Arnold singularities. More generally, we introduce the notion of *higher compound Du Val singularities*, whose definition parallels that of compound Du Val singularities. For such singularities, we prove that there exists a one-to-one correspondence between families of arcs and families of jets of sufficiently high order through the singularities. In dimension two, the result partially recovers a theorem of Mourtada on the jet schemes of Du Val singularities. As an application, we give a solution of the Nash problem for higher Du Val singularities.

Résumé. Les familles de jets à travers les singularités des variétés algébriques sont étudiées ici en relation avec les familles d'arcs initialement étudiées par Nash. Après avoir démontré un résultat général les concernant, nous examinons les variétés d'intersection localement complètes normales avec des singularités rationnelles et nous concentrons sur une classe de singularités que nous appelons « singularités de Du Val supérieures », une version de dimension (et codimension) supérieure des singularités de Du Val étroitement liée aux singularités d'Arnold. Plus généralement, nous introduisons la notion de « singularités de Du Val composées supérieures », dont la définition est parallèle à celle des singularités de Du Val composées. Pour de telles singularités, nous démontrons qu'il existe une correspondance bijective entre les familles d'arcs et les familles de jets d'ordre suffisamment élevé à travers les singularités. En dimension deux, le résultat récupère partiellement un théorème de Mourtada sur les schémas de jets des singularités de Du Val. En tant qu'application, nous proposons une solution au problème de Nash pour les singularités de Du Val supérieures.

Keywords. Jet scheme, arc space, Nash problem, rational singularity.

Mots-clés. Schémas de jet, espace d'arc, problème de Nash, singularité rationnelle.

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1. Introduction

The space of arcs through the singular locus of a complex variety decomposes into a finite union of irreducible components, each defining a distinct divisorial valuation, that is, a prime divisor on some resolution of singularities. These components were studied by Nash [38]; we will refer to them as *Nash families of arcs*, and to the valuations they define as *Nash valuations*. The problem of characterizing Nash families of arcs in terms of resolutions of singularities fits within the *Nash problem*, which was motivated by the desire of understanding what different resolutions would have in common.

It is natural to ask whether a similar picture holds for families of jets through the singular locus, at least when one looks at jets of sufficiently high order. (For clarity of exposition, in this introduction we restrict the discussion to the case where families of arcs and families of jets all stem from the singular locus of the variety; we refer to the main body of the paper for a more general formulation of the question.) As jets are parametrized by schemes of finite type, the fact that there are finitely many irreducible components of the set of jets of fixed order through the singular locus is clear. The question is how the families of jets defined by such components relate to the families of arcs identified by Nash.

Even though families of jets are introduced similarly to families of arcs, at the core there is an essential difference between the two: Nash families of arcs lift to resolutions of singularities and are naturally related to divisorial valuations; by contrast, families of jets through singularities do not lift to resolutions and cannot be related to valuations in any obvious way. In particular, the approach followed by Nash to study families of arcs using resolution of singularities does not apply to finite order jets.

Families of jets have been computed in several concrete examples, see, e.g., the works on plane curves and surface singularities [6, 28, 32–36]; in many of these works, the computation is carried out through a direct analysis of the defining equations. The problem of understanding families of jets is closely related to the *embedded Nash problem*, which aims to describe the irreducible components of contact loci of effective divisors on smooth ambient varieties in terms of embedded resolutions. A breakthrough in this direction was recently made in [3], where the problem was solved for unibranched plane curves; see also, e.g., [11, 21] for earlier work on this problem.

The purpose of this paper is to unveil a natural correspondence between families of arcs and certain families of jets of sufficiently high order. Our starting point is the following general property.

Theorem A (Theorem 4). *Among all families of jets of sufficiently high order stemming the singular locus of a variety, there is a selection of them that is in natural one-to-one correspondence with the Nash families of arcs.*

The correspondence is obtained by defining, in a geometric meaningful way, an injective map from the set of Nash families of arcs to the set of families of jets through the singular locus. We say that a family of jets is *of arc type* if it is in the image of this map.

We then address the question whether all families of jets of sufficiently high order through the singular locus are of arc type. Although in general there are more families of jets compared to families of arcs (see, e.g., the case of toric surface singularities [33, 35]), we will show that there is a one-to-one correspondence for certain rational singularities of arbitrary dimensions. One case we already understand, thanks to [34], is that of Du Val singularities, where there is a one-to-one correspondence. Here we extend the existence of such correspondence to a large class of locally complete intersection rational singularities of arbitrary dimensions which include isolated compound Du Val singularities.

For every normal locally complete intersection variety X there is a bound on embedding codimension in terms of minimal log discrepancy. The bound, which is proved in Proposition 16, is given by

$$\operatorname{ecodim}(\mathscr{O}_{X,x}) \leq \dim(\mathscr{O}_{X,x}) - \operatorname{mld}_x(X)$$

for every $x \in X$. We say that X has *maximal embedding codimension* at x if the bound is achieved. Within this class of singularities, we have those for which

$$\operatorname{mld}_x(X) = \dim(\mathscr{O}_{X,x}) - \operatorname{ecodim}(\mathscr{O}_{X,x}) = 1.$$

It is easy to see that these are isolated singularities. We will see in a moment that these singularities have many properties that are natural higher dimensional analogues of properties characterizing Du Val singularities in dimension two (the analogy is also manifest in the examples provided in Proposition 25). For this reason, we call these singularities *higher Du Val singularities.* In dimension two, this class of singularities coincides with Du Val singularities.

We then look at rational singularities of maximal embedding dimension that reduce to higher Du Val singularities under generic hyperplane sections. One should think of this condition as an analogue of the definition of compound Du Val singularity. We call these singularities *higher compound Du Val singularities.* We have the following result.

Theorem B (Theorem 34). *On an isolated higher compound Du Val singularity $x \in X$, all families of jets of sufficiently high order stemming from x are of arc type.*

As a special case, we see that all families of jets of sufficiently high order stemming from an isolated compound Du Val singularities are of arc type. Theorem B addresses our motivating question on families of jets. Combined with Theorem A, the theorem relates to and partially recover a result of Mourtada on families of jets on Du Val singularities [34] (see Corollary 36). Mourtada asked whether for any locally complete intersection variety with rational singularities the number of families of jets of sufficiently high order stemming from the singular locus is the same as the number of Nash families of arcs [34, Question 4.5]. Our result provides evidence in this direction.

For higher Du Val singularities, we have a more precise result (see Theorem 28) which we use to solve the Nash problem for this class of singularities. In our solution, Nash valuations are characterized in terms of certain partial resolutions of the variety (the terminal models) that originate from the minimal model program. Valuations defined by the exceptional divisors on these models are called *terminal valuations.*

Theorem C (Corollary 29). *For a divisorial valuation* ord_E *on a variety X with higher Du Val singularities, the following are equivalent:*

(1) ord_E *is a Nash valuation.*
(2) ord_E *is a terminal valuation.*
(3) *E is a crepant exceptional divisor over X.*

This result is in line with the point of view proposed in [15]. It can be viewed as a higher dimensional generalization of one of the properties characterizing Du Val singularities among normal surface singularities.

In dimension two, there are four proofs of the Nash problem for Du Val singularities [10, 15, 39, 40]. While the proof given here follows a different path, relying on inversion of adjunction and the minimal model program, it also uses on the main theorem of [15] and therefore it should not be considered as providing a new proof in dimension two for Du Val singularities. In higher dimensions, however, Theorem C does not follow directly from [15].

Throughout the paper, we work over an algebraically closed field k of characteristic zero.

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2. Arc spaces and jet schemes

For a scheme X over k, we denote by X_∞ the *arc space* of X over k and by X_m the *m-th jet scheme* of X. We refer to [9, 13, 44] for general references on the subject. An arc $\alpha \in X_\infty$ is a morphism $\alpha\colon \operatorname{Spec} k_\alpha[\![t]\!] \to X$ and a jet $\beta \in X_m$ a morphism $\beta\colon \operatorname{Spec} k_\beta[t]/(t^{m+1}) \to X$. We denote by $\alpha(0)$ and $\beta(0)$ the images of the respective closed points, and by $\alpha(\eta)$ the image of the generic point of $\operatorname{Spec} k_\alpha[\![t]\!]$. There are truncation maps $\pi\colon X_\infty \to X$ and $\pi_m\colon X_m \to X$ sending an arc α (respectively, an m-jet β) to its special point $\alpha(0)$ (respectively, $\beta(0)$), as well as $\psi_m\colon X_\infty \to X_m$ and $\pi_{n,m}\colon X_n \to X_m$ for $n > m$. We denote these maps by π^X, π^X_m, ψ^X_m, and $\pi^X_{n,m}$ whenever there is a need to specify the underlying scheme X.

Let now X be a variety. Constructibility in X_∞ is defined as in [19] (see also [42, Tag 005G]): a subset $C \subset X_\infty$ is *constructible* if and only if it is a finite union of finite intersections of retrocompact open sets and their complements; equivalently, C is constructible if and only $C = \psi_m^{-1}(S)$ for some m and some constructible set $S \subset X_m$. An irreducible subset $C \subset X_\infty$ is *non-degenerate* if $C \not\subset (\operatorname{Sing} X)_\infty$.

When X is smooth, constructible sets are also called *cylinders*. Their *codimension* is defined by $\operatorname{codim}(C, X_\infty) := \operatorname{codim}(S, X_m)$ where, as before, $C = \psi_m^{-1}(S)$. Using the simple structure of the truncation maps $\pi_{n,m}$, it is easy to check that this is well defined. The codimension of C defined above agrees with topological codimension of the closure of C in X_∞; if C is irreducible and $\alpha \in C$ is the generic point, then this is the same as $\dim(\mathscr{O}_{X_\infty,\alpha})$.

When X is singular, one defines the *jet codimension* of a constructible set $C \subset X_m$ by setting $\operatorname{jet-codim}(C, X_\infty) := (m+1)\dim(X) - \dim(S)$ where, again, $C = \psi_m^{-1}(S)$ (cf. [16]). If C is irreducible and $\alpha \in C$ is the generic point, then this agrees with $\operatorname{edim}(\mathscr{O}_{X_\infty,\alpha})$.

Every arc $\alpha \in X_\infty$ defines a semi-valuation $\operatorname{ord}_\alpha\colon \mathscr{O}_{X,\alpha(0)} \to \mathbb{Z} \cup \{\infty\}$, given by $\operatorname{ord}_\alpha(h) = \operatorname{ord}_t(\alpha^\sharp(h))$, which extends to a valuation of the function field of X if and only if the generic point $\alpha(\eta)$ of the arc is the generic point of X. In a similar fashion, every jet $\beta \in X_m$ defines a function $\operatorname{ord}_\beta\colon \mathscr{O}_{X,\beta(0)} \to \{0, 1, \dots, m\} \cup \{\infty\}$ given by $\operatorname{ord}_\beta(h) = \operatorname{ord}_t(\beta^\sharp(h))$, where we set $\operatorname{ord}_t(0) = \infty$.

A *prime divisor* over X is, by definition, a prime divisor E on a normal birational model $Y \to X$. Any such divisor E defines a valuation ord_E on X. A valuation on X of the form $v = q \operatorname{ord}_E$ where E is a prime divisor over X and q is a positive integer is called a *divisorial valuation*. The image in X of the generic point of E is called the *center* of the valuation (or of E), and is denoted by $c_X(v)$ or $c_X(E)$. For a divisorial valuation $v = q \operatorname{ord}_E$, the closure $C_X(v) \subset X_\infty$ of the set of arcs α such that $\operatorname{ord}_\alpha = v$ is called the *maximal divisorial set* associated to v. This is an irreducible closed constructible subset of X_∞. When $v = \operatorname{ord}_E$, we also denote this set by $C_X(E)$.

Let now X be a variety. As shown in [38] (see also, e.g., [13, 22]), the set $\pi^{-1}(\operatorname{Sing} X)$ decomposes as a finite union of irreducible components, and each component defines a divisorial valuation on X. These are called *Nash valuations* and the problem is to characterize them. Nash conjectured that, in dimension two, Nash valuations are precisely those defined by the exceptional divisors on the minimal resolution, and proposed the notion of *essential divisor* as a possible higher dimensional generalization which he speculated may characterize Nash valuations in all dimensions. These questions, which are generally referred to as the *Nash problem*, have generated a lot of activity.

Culminating the work of many people, the complete solution of the Nash problem in dimension two was eventually found by Fernandez de Bobadilla and Pe Pereira in [10], and before that, in the toric case by Ishii and Kollár [22]. A new, algebraic proof in the surface case was later found in [15], where it was proved that, in any dimension, all valuations defined by exceptional divisors on terminal models over X are Nash valuations; we call the valuations arising in this way *terminal valuations*. Nash's original guess of what the picture should be in dimension ≥ 3, however, turned out to be incorrect [12, 22, 23]. In view of this, one can reinterpret the Nash problem as asking for a characterization of Nash valuations in terms of resolution of singularities of a variety X and, more generally, its birational geometry.

3. Minimal log discrepancies

Let X be a normal variety, and assume that its canonical class K_X is $\mathbb{Q}$-Cartier. For every prime divisor E over X, if $f\colon Y \to X$ is the normal birational model where E lies, then we define the *log discrepancy* of E over X by $a_E(X) := \mathrm{ord}_E(K_{Y/X}) + 1$, and the *Mather log discrepancy* of E over X by $\widehat{a}_E(X) := \mathrm{ord}_E(\mathrm{Jac}_f) + 1$. These invariants of E only depends on the valuation ord_E, and they agree if X is smooth at the center of E.

An *effective* $\mathbb{R}$*-subscheme* Z of X is an expression $Z = \sum_{j=1}^s c_j Z_j$ where $Z_j \subset X$ is a proper closed subscheme and $c_j > 0$ for every j. Its *support* is the union of the support of the Z_j. For any effective $\mathbb{R}$-subscheme Z, we define the *log discrepancy* of E over the pair (X, Z) to be $a_E(X, Z) := a_E(X) - \sum c_j \,\mathrm{ord}_E(\mathscr{I}_{Z_j})$ where $\mathscr{I}_{Z_j} \subset \mathscr{O}_X$ is the ideal sheaf of Z_j. The *minimal log discrepancy* of (X, Z) at a point x is defined by

$$\mathrm{mld}_x(X, Z) := \inf_{c_X(E)=x} a_E(X, Z)$$

where the infimum is taken over all prime divisors E with center x. When there is no Z, we just write $\mathrm{mld}_x(X)$. We set $\mathrm{mld}_x(X, Z) = 0$ if x is the generic point of X. If $\dim X \geq 2$, then $\mathrm{mld}_x(X, Z) \in \{-\infty\} \cup [0, \infty)$. For sake of uniformity, it is convenient to declare that $\mathrm{mld}_x(X, Z) = -\infty$ whenever it is negative when $\dim X = 1$ as well.

The following is a slightly more general reformulation of the main theorem of [8]. The proof is essentially contained in [9]. We review the key part of the argument for completeness. A similar argument will also be used later in the paper, so it is useful to review it here anyway.

Theorem 1 (Inversion of adjunction [8]). *Let X be a smooth variety, $Y = H_1 \cap \cdots \cap H_e \subset X$ a normal positive dimensional subvariety defined by the complete intersection of e hypersurfaces $H_i \subset X$, and $Z = \sum c_j Z_j$ an effective $\mathbb{R}$-subscheme of X not containing Y in its support. Then for every $x \in Y$ we have*

$$\mathrm{mld}_x(Y, Z|_Y) = \mathrm{mld}_x(X, Z + eY) = \mathrm{mld}_x\Big(X, Z + \sum_{i=1}^e H_i\Big),$$

where $Z|_Y := \sum c_j (Z_j \cap Y)$.

Proof. We may assume that x is not the generic point of Y, the statement being elementary in that case. The proofs of the inequalities $\mathrm{mld}_x(Y, Z|_Y) \geq \mathrm{mld}_x(X, Z + eY) \geq \mathrm{mld}_x(X, Z + \sum H_i)$ are fairly standard and are omitted. We review the proof of the inequality

$$\mathrm{mld}_x(Y, Z|_Y) \leq \mathrm{mld}_x\Big(X, Z + \sum H_i\Big),$$

which is the hard part of the theorem. To this end, it suffices to show that for every divisorial valuation $v = \mathrm{ord}_F$ on X with center $c_X(v) = x$, there is a divisorial valuation $w = q\,\mathrm{ord}_E$ over Y with center $c_Y(w) = x$ such that

$$q\, a_E(Y, Z|_Y) \leq a_F\Big(X, Z + \sum H_i\Big).$$

We denote by Y_∞^x the reduced inverse image of x under the projection $\pi^Y\colon Y_\infty \to Y$. By definition, Y_∞^x is the set of arcs in Y stemming from x.

Let $C_X(v) \subset X_\infty$ be the maximal divisorial set associated to v. Note that $\pi^X(C_X(v))$ is an irreducible constructible set with generic point x. Consider the intersection

$$C_X(v) \cap Y_\infty.$$

As v is centered at x and $C_X(v)$ is closed under the action of the morphism $\Phi_\infty \colon \mathbb{A}^1 \times X_\infty \to X_\infty$ given by $(a, \alpha(t)) \mapsto \alpha(at)$ (cf. [8, Section 3]), we see that $C_X(v)$ contains the constant arc at x, hence $C_X(v) \cap Y_\infty^x \neq \emptyset$. It follows that x is the generic point of $\pi^X(C_X(v) \cap Y_\infty)$. Therefore we can pick an irreducible component W of $C_X(v) \cap Y_\infty$ such that $\pi^Y(W)$ has x as its generic point. Note that [9, Lemma 8.3] applies to $C_X(v) \cap Y_\infty^x$ since both $C_X(v)$ and Y_∞^x are closed under the action of the morphism Φ_∞, hence $C_X(v) \cap Y_\infty^x \not\subset (\operatorname{Sing} Y)_\infty$. Therefore we can assume that W is not contained in $(\operatorname{Sing} Y)_\infty$. By construction W is the closure of an irreducible constructible set in Y_∞, hence, by [16], its generic point $\gamma \in W$ defines a divisorial valuation $w = q \operatorname{ord}_E$ on Y, and [9, Lemma 8.4] (its proof, to be precise) gives

$$\operatorname{jet-codim}(W, Y_\infty) \leq \operatorname{codim}(C_X(v), X_\infty) + q \operatorname{ord}_E(\operatorname{Jac}_Y) - \sum \operatorname{ord}_F(\mathscr{I}_{H_i}).$$

Since $W \subset C_Y(w)$, [16, Theorem 3.8] implies that $\operatorname{jet-codim}(W, Y_\infty) \geq q \cdot \widehat{a}_E(Y)$ where $\widehat{a}_E(Y)$ is the Mather log discrepancy. As Y is normal and locally complete intersection, we have $\widehat{a}_E(Y) = a_E(Y) + \operatorname{ord}_E(\operatorname{Jac}_Y)$ (see, e.g., [14, Corollary 3.5]), hence

$$\operatorname{jet-codim}(W, Y_\infty) \geq q(a_E(Y) + \operatorname{ord}_E(\operatorname{Jac}_Y)).$$

On the other hand, as X is smooth, we have

$$\operatorname{codim}(C_X(v), X_\infty) = a_F(X).$$

Finally, by the semicontinuity of order of contact function induced by $\mathscr{I}_{Z_j}$ on X_∞, we have

$$q \operatorname{ord}_E(\mathscr{I}_{Z_j} \cdot \mathscr{O}_Y) \geq \operatorname{ord}_F(\mathscr{I}_{Z_j}).$$

By combining the above formulas, we conclude that $q\, a_E(Y, Z|_Y) \leq a_F(X, Z + \sum H_i)$. □

Remark 2. Going through the above proof (with $Z = 0$), suppose that $a_F(X, \sum H_i) = \operatorname{mld}_x(X, \sum H_i) \geq 0$. Then we necessarily have $q\, a_E(Y) = a_F(X, \sum H_i)$, since $q\, a_E(Y) \geq a_E(Y) \geq \operatorname{mld}_x(Y)$, hence $q\, a_E(Y) = a_E(Y) = \operatorname{mld}_x(Y)$. In particular, if $\operatorname{mld}_x(Y) > 0$ then $q = 1$. Furthermore, the inequalities in the formulas displayed in the proof must all be equalities, hence $W = C_Y(w)$.

Corollary 3. *Let X be a normal locally complete intersection variety, $Y = H_1 \cap \cdots \cap H_e \subset X$ a normal positive dimensional subvariety defined by the complete intersection of e hypersurfaces $H_i \subset X$, and $Z = \sum c_j Z_j$ an effective $\mathbb{R}$-subscheme of X not containing Y in its support. Then for every $x \in Y$ we have*

$$\operatorname{mld}_x(Y, Z|_Y) = \operatorname{mld}_x(X, Z + eY) = \operatorname{mld}_x\left(X, Z + \sum_{i=1}^{e} H_i\right).$$

Proof. Again, it suffices to prove that $\operatorname{mld}_x(Y, Z_Y) = \operatorname{mld}_x(X, Z + \sum H_i)$. Working locally near x, we can fix a closed embedding $X \subset A$ where A is a smooth variety, and hypersurfaces $D_1, \dots, D_r \subset A$ where $r = \operatorname{codim}(Y, A)$, such that $H_i = D_i \cap X$ for $i = 1, \dots, e$ and $X = D_{e+1} \cap \cdots \cap D_r$. Note that $Y = D_1 \cap \cdots \cap D_r$. By Theorem 1 (applied twice, to $Y \subset A$ and $X \subset A$), we have

$$\operatorname{mld}_x(Y, Z|_Y) = \operatorname{mld}_x\left(A, Z + \sum_{i=1}^{r} D_i\right) = \operatorname{mld}_x\left(X, Z + \sum_{i=1}^{e} H_i\right).$$

This completes the proof. □

4. Families of jets of arc type

Let X be a positive dimensional variety. For any subset $\Sigma \subset X$, we consider the sets

$$X_\infty^\Sigma := \pi^{-1}(\Sigma)_{\mathrm{red}} = \{\alpha \in X_\infty \mid \alpha(0) \in \Sigma\}$$

and

$$X_m^\Sigma := \pi_m^{-1}(\Sigma)_{\mathrm{red}} = \{\beta \in X_m \mid \beta(0) \in \Sigma\}.$$

By definition, X_∞^Σ is the set of arcs on X through Σ, and X_m^Σ is the set of m-jets through Σ.

Assume that $\Sigma \subset X$ is a closed subset. Since X_m is a scheme of finite type, each X_m^Σ decomposes into a finite union of irreducible components, and a generalization of Nash's theorem [38] tells us that the same happens for X_∞^Σ.

In the following, we denote by $\Gamma \subset X_\infty^\Sigma \setminus (\operatorname{Sing} X)_\infty$ the set of generic points; that is, $\alpha \in \Gamma$ if and only if α is the generic point of a non-degenerate irreducible component of X_∞^Σ. Let

$$\mu := \max_{\alpha \in \Gamma} \operatorname{ord}_\alpha(\operatorname{Jac}_X).$$

Note that $\mu < \infty$ since Γ is finite and each $\alpha \in \Gamma$ is non-degenerate.

We fix an integer $\nu \geq \mu$ such that the images $\psi_\nu(\alpha) \in X_\nu$, for $\alpha \in \Gamma$, are all distinct and there are no specializations within the set $\psi_\nu(\Gamma) \subset X_\nu$ (meaning that $\psi_\nu(\Gamma)$, with the induced topology, is discrete). The existence of such integer follows from the definition of X_∞ as inverse image of the jet schemes under the truncation maps.

Theorem 4. *Let X be a variety and $\Sigma \subset X$ a closed subset. Then for every $m \geq \mu + \nu$ there is a naturally defined injective map*

$$\Psi_m^\Sigma : \{\textit{non-degenerate irreducible components of } X_\infty^\Sigma\} \to \{\textit{irreducible components of } X_m^\Sigma\}$$

sending a non-degenerate irreducible component C of X_∞^Σ to the unique irreducible component D of X_m^Σ containing the image of C in X_m.

Definition 5. *We say that an irreducible component of X_m^Σ is* of arc type *if it is in the image of Ψ_m^Σ.*

Remark 6. There are two special cases about Theorem 4. The first is when we take $\Sigma = \operatorname{Sing} X$. In this case every irreducible component of $X_\infty^{\operatorname{Sing} X}$ is non-degenerate and the domain of $\Psi_m^{\operatorname{Sing} X}$ is the set of Nash families of arcs. The second special case is when $\Sigma = X$. In this case, the domain of Ψ_m^X is a singleton and the image of Ψ_m^X is the irreducible component of X_m dominating X, namely, the closure of $(X_{\mathrm{reg}})_m$.

We will break the proof of Theorem 4 into two steps: proving that Ψ_m^Σ is well-defined, and showing that it is injective. We may assume that Σ is nonempty, the statement being trivial otherwise.

We start with the basic observation that

$$\psi_m(X_\infty^\Sigma) \subset X_m^\Sigma.$$

This implies that for every non-degenerate irreducible component C of X_∞^Σ there exists an irreducible component D of X_m^Σ such that $\psi_m(C) \subset D$. Our goal is to prove that if $m \geq \mu + \nu$ then such component D is unique (proving well-definedness), and that a different component D of X_m^Σ occurs for each non-degenerate component C of X_∞^Σ (proving injectivity).

These properties follow by standard facts about the structure of the truncation maps, specifically from Greenberg's theorem on liftable jets [18] and from a result of Looijenga on the fibers of the truncation maps between jet schemes [29]. For convenience, we will cite these results from [9].

We start with the first assertion.

Lemma 7. *If $m \geq \mu+\nu$, then for every non-degenerate irreducible component C of X_∞^Σ there exists a unique irreducible component D of X_m^Σ such that $\psi_m(C) \subset D$.*

Proof. We proceed by contradiction and assume that there exists an integer $m \geq \mu+\nu$ and a non-degenerate irreducible component C of X_∞^Σ such that $\psi_m(C)$ is contained in the intersection of two distinct irreducible components D and D' of X_m^Σ. Whatever the value of m, we can find another integer n such that

(1) $n \geq \nu$ and
(2) $2n \geq m \geq \mu + n$.

A choice of n can be made by setting $n = \nu + k$ where k is defined by $m = \mu + \nu + k$.

Let $\alpha \in C$, $\beta \in D$ and $\beta' \in D'$ denote the respective generic points, and let $\alpha_n = \psi_n(\alpha)$, $\beta_n = \pi_{m,n}(\beta)$, and $\beta'_n = \pi_{m,n}(\beta')$ be their images in X_n. Note that both β_n and β'_n specialize to α_n. Since $\mathrm{ord}_\alpha(\mathrm{Jac}_X) \leq \mu \leq n$, we have

$$\mathrm{ord}_{\beta_n}(\mathrm{Jac}_X) \leq \mathrm{ord}_{\alpha_n}(\mathrm{Jac}_X) = \mathrm{ord}_\alpha(\mathrm{Jac}_X) \leq \mu \leq n,$$

hence [9, Proposition 4.1 (i)] implies that $\beta_n = \psi_n(\gamma)$ for some arc $\gamma \in X_\infty$. Similarly, we have $\beta'_n = \psi_n(\gamma')$ for some $\gamma' \in X_\infty$.

Note that $\gamma, \gamma' \in X_\infty^\Sigma$. In fact, as $n \geq \nu$, we see that $\gamma, \gamma' \in C$ since, by the definition of ν, no other irreducible component of X_∞^Σ contains a point whose image in X_m specializes to α_m. In particular, γ and γ' are specializations of α, hence β_n and β'_n are both generalizations and specializations of α_n, meaning that

$$\beta_n = \alpha_n = \beta'_n,$$

This means that β and β' belong to the same fiber of $X_m \to X_n$, namely, $\pi_{m,n}^{-1}(\alpha_n)$.

As $\alpha_n \in X_n^\Sigma$, the fiber $\pi_{m,n}^{-1}(\alpha_n)$ is contained in X_m^Σ, and since it contains the generic points β and β' of the irreducible components D and D' of X_m^Σ, it follows that D and D' are irreducible components of $\pi_{m,n}^{-1}(\alpha_n)$. This contradicts the fact that, by [9, Proposition 4.4 (ii)], this fiber is irreducible. □

We now turn to the second assertion.

Lemma 8. *If $m \geq \mu + \nu$, then for every irreducible component D of X_m^Σ there exists at most one non-degenerate irreducible component C of X_∞^Σ such that $\psi_m(C) \subset D$.*

Proof. We need to prove that if $m \geq \mu + \nu$ and $\alpha, \alpha' \in \Gamma$ are such that their images α_m and α'_m in X_m belongs to the same irreducible component D of X_m^Σ, then $\alpha = \alpha'$.

To prove this, let $\beta \in D$ be the generic point. Then β specializes to both α_m and α'_m, hence its image $\beta_{m-\mu} := \pi_{m,m-\mu}(\beta) \in X_{m-\mu}$ specializes to both images $\alpha_{m-\mu}$ and $\alpha'_{m-\mu}$ of α and α' in $X_{m-\mu}$. Note that $m - \mu \geq \nu \geq \mu$. By semicontinuity,

$$\mathrm{ord}_{\beta_{m-\mu}}(\mathrm{Jac}_X) \leq \mu$$

Then, by [9, Proposition 4.1 (i)], we see that $\beta_{m-\mu}$ lifts to an arc; that is, there exists $\gamma \in X_\infty$ such that $\psi_{m-\mu}(\gamma) = \beta_{m-\mu}$. By construction, $\gamma \in \pi^{-1}(\Sigma)$, hence there exists $\alpha'' \in \Gamma$ specializing to γ. It follows that the image of α'' in $X_{m-\mu}$ specializes to both $\alpha_{m-\mu}$ and $\alpha'_{m-\mu}$. As $m - \mu \geq \nu$, we conclude that $\alpha = \alpha'' = \alpha'$. □

Proof of Theorem 4. Lemma 7 implies that Ψ_m^Σ is well-defined for $m \geq \mu + \nu$, and Lemma 8 that this map is injective. □

Remark 9. The definition of the function Ψ_m^Σ constructed in Theorem 4 can be extended to all $m \geq 0$ as long as one is willing to regard them as multivalued function, sending each C to all components D containing the image of C.

5. The question of surjectivity

Given Theorem 4, it is natural to ask under which conditions on singularities one can guarantee that the maps Ψ_m^{Σ} are surjective. These functions are well-defined for $m \gg 1$, but if we are willing to regard them as a multivalued functions, then we can remove the constrain on m. The question of surjectivity still makes sense for multivalued functions.

Before we move to discuss the case we will be focusing on, it may be instructive to point out that there is already an interesting answer to the problem (a sufficient condition for surjectivity) in the special case where $\Sigma = X$. This comes from Mustaţă's theorem on locally complete intersection canonical singularities.

Theorem 10 ([37]). *Let X be a locally complete intersection variety with canonical singularities. Then Ψ_m^X is well defined and surjective for every m.*

Proof. As X_∞ has only one non-degenerate irreducible component (and in fact only one irreducible component since it is irreducible by Kolchin's theorem [24]), this is just a restatement of Mustaţă's theorem on the irreducibility of the jet schemes, since any additional irreducible component of X_m would lie over the singular locus of X and therefore would not contain the image of X_∞. □

Like in Mustaţă's theorem, we will be focusing on locally complete intersection canonical singularities. Our goal is to find a class of singularities for which $\Psi_m^{\operatorname{Sing} X}$ is surjective.

To get a sense of what one can expect, we start by reviewing some cases that are already understood.

Example 11 (Nodal curve). The case where X is a nodal curve already shows that one cannot expect $\Psi_m^{\operatorname{Sing} X}$ to be always surjective. Indeed, if $x \in X$ is a node, then for $m \geq 3$ the set X_m^x has $m-1$ irreducible components, and only two of them are in the image of Ψ_m^x.

Example 12 (Affine cones). Let $V \subset \mathbb{P}^{N-1}$ be a smooth complete intersection variety defined by equations of degree r, let $X \subset \mathbb{A}^N$ be the affine cone over V, and let $x \in X$ be the vertex. As the blow-up of x gives a resolution of X with a single exceptional divisor, one easily see that X_∞^x is irreducible. On the other hand, for every $m \geq r$ we have

$$\pi_m^{-1}(x) \cong X_{m-r} \times \mathbb{A}^{N(r-1)},$$

see, e.g., the proof of [17, Theorem 3.5]. By [37, Theorem 0.1], we know that if X is canonical then X_m is irreducible for all m, and conversely, using also [37, Proposition 1.6], if X is not canonical at x then X_m is reducible for all $m \gg 1$. It follows that X_m^x is irreducible (hence Ψ_m^x is surjective) for all $m \geq r$ if X is canonical, and is reducible (hence Ψ_m^x fails to be surjective) for all $m \gg 1$ if X is not canonical.

Mourtada, in part in collaboration with Plénat and Cobo, has studied the irreducible decomposition of $X_m^{\operatorname{Sing} X}$ in many explicit situations where X is a surface [6, 34–36]; see also [26] for related work. While in some cases these results indicate that the number of components continues to grow with m, there are also cases where the number of components stabilizes and matches the number of Nash families.

Example 13 (Toric surface singularities). The irreducible decomposition of $X_m^{\operatorname{Sing} X}$ was computed for toric surfaces by Mourtada [35], and the only case where we have the same number of components as Nash families is when X has A_n-singularities.

Example 14 (Du Val singularities). It is proved in [34] that, for $m \gg 1$, the number of families of m-jets through a Du Val singularity coincides with the number of exceptional divisors on the minimal resolution, hence with the number of Nash families of arcs. It follows in particular that in this case $\Psi_m^{\operatorname{Sing} X}$ is a bijection.

Example 15 (cA-type singularities). Another case where we can check directly that $\Psi_m^{\mathrm{Sing}\,X}$ is a bijection is that of cA-type singularities. Nash families of arcs on these singularities were described in [23], and the deformation argument used in their proof can be adapted to show that, for $m \gg 1$, there is the same number of families of m-jets through the singularity, proving that $\Psi_m^{\mathrm{Sing}\,X}$ is a bijection in this case as well. More specifically, suppose X is defined by an equation

$$xy = f(z_1, \dots, z_{d-1})$$

in $A = \mathbb{A}^{d+1}$, where $\mu := \mathrm{mult}_0(f) \geq 2$. The proof in [23] begins by identifying $\mu - 1$ irreducible open sets $U_i \subset X_\infty^0$, for $1 \leq i \leq \mu - 1$, given by

$$U_i = \left\{\alpha \in X_\infty^0 \;\middle|\; \mathrm{ord}_\alpha(x) = i, \mathrm{ord}_\alpha(y) = \mu - i, \mathrm{ord}_\alpha(f) = \mu\right\}.$$

The proof then goes by showing that every arc $\alpha \in X_\infty^0$ can be deformed (in X_∞^0) to an arc α^* with $\mathrm{ord}_{\alpha^*}(f) = \mu$. Clearly such arc must belong to one of the U_i, hence proving that the closures of these sets give all irreducible components of X_∞^0. The deformation is done in several steps: first, one deforms α to an arc α' with $\mathrm{ord}_{\alpha'}(f) < \infty$, and if $\mathrm{ord}_{\alpha'} > \mu$, then one deforms α' to an arc α'' with $\mathrm{ord}_{\alpha''}(f) < \mathrm{ord}_{\alpha'}(f)$. After a finite number of steps, this process produces the desired arc α^*.

This argument can be adapted to characterize the irreducible components of X_m^0, for any given $m \geq \mu$, as follows. For $1 \leq i \leq \mu - 1$, we consider the irreducible open sets

$$V_i = \left\{\beta \in X_m^0 \;\middle|\; \mathrm{ord}_\beta(x) = i, \mathrm{ord}_\beta(y) = \mu - i, \mathrm{ord}_\beta(f) = \mu\right\}.$$

Given any $\beta \in X_m^0$, we take any lift $\alpha \in A_\infty^0$ (i.e., any arc α on A such that $\psi_m^A(\alpha) = \beta$) and apply the same deformation argument as in [23] to produce a new arc $\alpha^* \in A_\infty^0$ such that $\mathrm{ord}_{\alpha^*}(f) = \mu$. In fact, without loss of generality we can pick α so that $\mathrm{ord}_\alpha(f) < \infty$, hence skip the first deformation and just deform to reduce $\mathrm{ord}_\alpha(f)$ if the order of contact is larger than μ. The key observation here is that, just like in [23] the deformation keeps the arc on X, in this setting the deformation maintains the order of contact of the arc with X, hence the corresponding deformation at level m stays on X_m.

The above examples are mainly understood through their equations. Our goal is to identify a new class of examples of arbitrary dimensions where $\Psi_m^{\mathrm{Sing}\,X}$ is surjective, without having to rely on explicit equations. This will be done in the next two sections.

6. Singularities of maximal embedding codimension

For a local ring $(R, \mathfrak{m})$ we denote by $\dim(R)$ the Krull dimension, by $\mathrm{edim}(R)$ the embedding dimension (the dimension of the Zariski tangent space) and by $\mathrm{ecodim}(R)$ the embedding codimension (the codimension of the tangent cone in the Zaristi tangent space). When R is Noetherian, the latter is also known as the regularity defect [27] and is equal to $\mathrm{edim}(R) - \dim(R)$.

We start by establishing the following bound on embedding codimension for normal locally complete intersection singularities. The bound is likely known to experts.

Proposition 16. *Let X be a normal locally complete intersection variety. Then*

$$\mathrm{ecodim}(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{X,x}) - \mathrm{mld}_x(X)$$

for every $x \in X$.

Proof. The assertion being trivial if $\mathrm{mld}_x(X) = -\infty$, we assume that $\mathrm{mld}_x(X) \geq 0$. Working locally in X, we may assume that X is embedded in an affine space $A := \mathbb{A}^N$. Let $d = \dim(X)$, $r = \dim(\mathcal{O}_{X,x})$, $e = \mathrm{ecodim}(\mathcal{O}_{X,x})$ and $c = \mathrm{codim}(X, A)$. By inversion of adjunction (see Theorem 1),

$$\mathrm{mld}_x(X) = \mathrm{mld}_x(A, cX).$$

Let $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ be the maximal ideal. By applying [31, Theorem 25.2] to the sequence $k \to \mathcal{O}_{X,x} \to k_x$, we get the exact sequence

$$0 \to \mathfrak{m}_x/\mathfrak{m}_x^2 \to \Omega_{X/k} \otimes k_x \to \Omega_{k_x/k} \to 0.$$

This gives

$$\dim_{k_x}(\Omega_{X/k} \otimes k_x) = \operatorname{edim}(\mathcal{O}_{X,x}) + d - r = d + e.$$

By the isomorphism $X_1 \cong \operatorname{Spec}(\operatorname{Sym}(\Omega_{X/k}))$ (see [9, Example 2.5] or [44, (1.4)]), we have $X_1^x \cong \operatorname{Spec}(\operatorname{Sym}(\Omega_{X/k} \otimes k_x))$, hence

$$\dim_k(\overline{X_1^x}) = \dim_{k_x}(X_1^x) + d - r = 2d + e - r.$$

The reduced inverse image $V \subset A_\infty$ of the closure $\overline{X_1^x} \subset A_1$ of X_1^x is a closed irreducible cylinder. Let v be the valuation defined by V (namely, $v = \operatorname{ord}_\alpha$ where $\alpha \in V$ is the generic point). By [7, Theorem C], v is a divisorial valuation, i.e., $v = p \operatorname{ord}_F$ where F is a prime divisor over A and p is a positive integer. Note that, by construction, we have $v(\mathcal{I}_X) \geq 2$. If $C_A(v) \subset A_\infty$ is the maximal divisorial set associated to the valuation, then we have $V \subset C(v)$, hence

$$\operatorname{codim}(V, A_\infty) \geq \operatorname{codim}(C_X(v), A_\infty) = p\, a_F(A)$$

(the last formula is implicit in [7]; for a direct reference, see [16, Theorem 3.8]). On the other hand,

$$\begin{aligned}\operatorname{codim}(V, A_\infty) &= \operatorname{codim}(\overline{X_1^x}, A_1)\\ &= \dim(A_1) - \dim(\overline{X_1^x})\\ &= 2(d+c) - (2d+e-r)\\ &= r - e + 2c.\end{aligned}$$

It follows that

$$\operatorname{mld}_x(A, cX) \leq a_F(A, cX) \leq \frac{1}{p}\Big(\operatorname{codim}(V, A_\infty) - 2c\Big) \leq r - e,$$

where we use in the last inequality that $\operatorname{mld}_x(X) \geq 0$ to ensure that the inequality is preserved when we clear the denominator p. □

Definition 17. *In accordance with Proposition 16, we say that a normal locally complete intersection variety X has* maximal embedding codimension singularities *if*

$$\operatorname{ecodim}(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X,x}) - \operatorname{mld}_x(X)$$

for every $x \in X$.

Remark 18. Smooth varieties have maximal embedding codimension singularities.

Remark 19. Every locally complete intersection variety with maximal embedding codimension singularities has log canonical singularities, since the condition implies that $\operatorname{mld}_x(X) \neq -\infty$ hence $\operatorname{mld}_x(X) \geq 0$ for all $x \in X$. Note that if X is a curve then normality already implies that X is smooth.

Example 20 (Hypersurface singularities). A normal hypersurface singularity $x \in X$ has maximal embedding codimension if and only if $\operatorname{mld}_x(X) = \dim(\mathcal{O}_{X,x}) - 1$. In particular Du Val singularities in dimension 2 and isolated cDV singularities of dimension 3 are all the examples in these dimensions of isolated hypersurface singularities of maximal embedding codimension (cf. [41]).

7. Higher Du Val singularities

We now identify a particular subclass of locally complete intersection varieties with maximal embedding codimension singularities which can be thought as a higher dimensional version of Du Val singularities.

Definition 21. *Let X be a normal locally complete intersection variety of dimension $d \geq 2$. We say that a point $x \in X$ is a* higher Du Val *(hDV)* singularity *if*

$$\mathrm{mld}_x(X) = \dim(\mathscr{O}_{X,x}) - \mathrm{ecodim}(\mathscr{O}_{X,x}) = 1.$$

By definition, hDV singularities are canonical but not terminal. They can be locally embedded as complete intersection singularities of codimension $d-1$ in $\mathbb{A}^{2d-1}$ (cf. [5, Theorem 3.15]) but not in any smaller affine space. In dimension two, these are the same as the Du Val singularities.

Remark 22. It is useful to compare the above definition with another classical way of generalizing Du Val singularities, namely, compound Du Val singularities. Compound Du Val singularities preserve two properties of Du Val singularities: being hypersurface singularities, and having minimal log discrepancy $\mathrm{mld}_x(X) = \dim(X) - 1$. By contrast, the definition of hDV singularities preserves the condition that $\mathrm{mld}_x(X) = 1$ and requires maximal embedding codimension. The attribute "higher" in hDV singularity reflects at the same time that these are higher dimensional and higher codimensional generalizations of Du Val singularities.

Remark 23. If we extended Definition 21 to the case $d = 1$, then in dimension one the definition would characterize smooth points on curves. This says something meaningful about the behavior of this notion as a function of dimension. We prefer to assume $d \geq 2$ as we want to regard this as defining a class of actual singular points.

Example 24 (Intersections of quadric cones). In higher codimensions, the simplest example of a hDV singularity is the cone $X \subset \mathbb{A}^{2e+1}$ over the transversal intersection of e smooth quadrics in $\mathbb{P}^{2e}$. The blow-up of the vertex x of the cone gives a log resolution of $(\mathbb{A}^{2e+1}, X)$, and

$$\mathrm{mld}_x(X) = \mathrm{mld}_x(\mathbb{A}^{2e+1}, eX) = 1$$

where the minimal log discrepancy is computed by the exceptional divisor of the blow-up.

More generally, we have the following set of examples, which shows the clear analogy with Du Val singularities.

Proposition 25. *Let $e \geq 1$, let $(u_1, \dots, u_{2e-2}, x, y, z)$ denote affine coordinates of $\mathbb{A}^{2e+1}$, and let $X \subset \mathbb{A}^{2e+1}$ be the subvariety defined by the vanishing of e general linear combinations of any finite set of generators of the ideal*

$$\mathfrak{a} = (u_1, \dots, u_{2e-2})^2 + \mathfrak{b}$$

of $k[u_1, \dots, u_{2e-2}, x, y, z]$, where $\mathfrak{b}$ is one of the following:

$$\mathfrak{b} = \begin{cases} (x^2, y^2, z^{n+1}) \quad (n \geq 1) & A_n\text{-type} \\ (z^2, x^2 y, y^{n-2}) \quad (n \geq 4) & D_n\text{-type} \\ (z^2, x^3, y^4) & E_6\text{-type} \\ (z^2, x^3, xy^3) & E_7\text{-type} \\ (z^2, x^3, y^5) & E_8\text{-type} \end{cases}$$

Then X has a hDV singularity at the origin $0 \in \mathbb{A}^{2e+1}$.

Proof. Clearly, X is a complete intersection variety with an isolated singularity at the origin, and $\operatorname{ecodim}(\mathscr{O}_{X,0}) = e$. What is left to show is that $\operatorname{mld}_0(X) = 1$. Note that $\operatorname{mld}_0(X) = \operatorname{mld}_0(\mathbb{A}^{2e+1}, eX)$. By looking at the exceptional divisor of the blow-up of $\mathbb{A}^{2e+1}$ at the origin, we see that $\operatorname{mld}_0(\mathbb{A}^{2e+1}, eX) \le 1$. On the other hand, a special case of the Thom–Sebastiani theorem (see [25, Proposition 8.21]) gives us the following formula for the log canonical thresholds of $\mathfrak{a}$:

$$\operatorname{lct}(\mathfrak{a}) = \operatorname{lct}((u_1, \dots, u_{2e-2})^2) + \operatorname{lct}(\mathfrak{b}) = e - 1 + \operatorname{lct}(\mathfrak{b}).$$

What we know about Du Val singularities already tells us that $\operatorname{lct}(\mathfrak{b}) > 1$; this can also be checked directly using Howald's formula for the log canonical threshold of monomial ideals [20]. Therefore $\operatorname{lct}(\mathfrak{a}) > e$, hence $\operatorname{mld}_0(\mathbb{A}^{2e+1}, eX) > 0$. We conclude that $\operatorname{mld}_0(\mathbb{A}^{2e+1}, eX) = 1$, as required. □

Remark 26. Assuming $k = \mathbb{C}$, hDV singularities are closely related certain hypersurface singularities studied by Arnol'd [1]. These are isolated hypersurface singularities characterized by the property that their versal deformations only contain finitely many analytically inequivalent singularities, and are known as *simple singularities*. They were classified in [1]; see also [4, Example (3.4)]. In the notation of Proposition 25, for any $\mathfrak{a}$ (which, according to the proposition, corresponds to an example of a hDV singularity) the vanishing of a general element $h \in \mathfrak{a}$ defines a simple singularity, and all simple singularities arise in this way. Conversely, the examples of hDV singularities provided by Proposition 25 are complete intersections of simple singularities of the same type.

Proposition 27. *Let X be a variety with hDV singularities. Then X has isolated singularities.*

Proof. Let $f\colon Y \to X$ be a log resolution that is an isomorphism over X_{reg}, and let E be the reduced exceptional locus. Note that $K_{Y/X} \ge 0$.

If $\dim(\operatorname{Sing} X) \ge 1$, then we can find a closed point $x \in \operatorname{Sing} X$ such that x is not the center of any component of E. On the other hand, $x \in f(E)$. Now, let F be an arbitrary prime divisor over X with center $c_X(F) = x$. We may assume that F lies on a nonsingular model $g\colon Z \to Y$. Since $f^{-1}(x)$ has codimension at least 2 in Y and contains the center of F in Y, we have $\operatorname{ord}_F(K_{Z/Y}) \ge 1$. It follows that $\operatorname{ord}_F(K_{Z/X}) \ge 1$, hence $a_F(X) \ge 2$. This contradicts the fact that, by hypothesis, $\operatorname{mld}_x(X) = 1$. □

Theorem 28. *Let $x \in X$ be a hDV singularity.*

(1) *The multivalued function Ψ^x_m is surjective for all m.*
(2) *An irreducible set $C \subset X^x_\infty$ is a non-degenerate irreducible component if and only if $C = C_X(E)$ for some prime divisor E over X with center $c_X(E) = x$ and log discrepancy $a_E(X) = 1$.*

Proof. By Proposition 27, $x \in X$ is an isolated singularity.

Let $d = \dim(X) = \dim(\mathscr{O}_{X,x})$ and $e = \operatorname{ecodim}(\mathscr{O}_{X,x})$. Note that, by our assumption, $e = d - 1$. Though not strictly necessary, to simplify the notation we apply [5, Theorem 3.15] to reduce to the case where X is embedded in $A := \mathbb{A}^{d+e}$.

Let $f_1, \dots, f_e \in k[x_1, \dots, x_{d+e}]$ be local generators of the ideal of X in A at the point x. For every $j \ge 1$, we denote by $f_i^{(j)}$ the j-th Hasse–Schmidt derivative of f_i. As $X^x_1 = A^x_1$ (by our choice of embedding), the polynomials f_i and f_i' vanish identically on A^x_1, hence on A^x_m. Therefore, the ideal of X^x_m in A^x_m is generated by the elements $f_i^{(j)}$ for $1 \le i \le e$ and $2 \le j \le m$. In particular, if D is any irreducible component of X^x_m, then

$$\operatorname{codim}(D, A^x_m) \le e(m-1).$$

Noticing that $\operatorname{codim}(A^x_m, A_m) = d + e = 2e + 1$, it follows that

$$\operatorname{codim}(D, A_m) \le e(m+1) + 1$$

Let $V \subset A_\infty$ be the cylinder over $D \subset A_m$. This is a closed irreducible cylinder of codimension

$$\operatorname{codim}(V, A_\infty) = \operatorname{codim}(D, A_m) \le e(m+1) + 1.$$

If $v = p \operatorname{ord}_F$ is the divisorial valuation defined by the generic point of V, then $V \subset C(v)$, hence

$$\operatorname{codim}(V, A_\infty) \ge \operatorname{codim}(C_X(v), A_\infty) = p\, a_F(A).$$

Note that $v(\mathscr{I}_X) \ge m+1$. Then

$$\operatorname{mld}_x(A, eX) \le \frac{1}{p}\big(\operatorname{codim}(V, A_\infty) - e(m+1)\big) \le 1.$$

Since by our assumption on the singularity we have $\operatorname{mld}_x(X) = 1$, and $\operatorname{mld}_x(X) = \operatorname{mld}_x(A, eX)$ by inversion of adjunction, it follows that all inequalities in the above formula are equalities, and in particular $V = C_A(v)$.

We see from the proof of Theorem 1 (see also Remark 2) that there is a non-degenerate irreducible component W of $V \cap X_\infty$. Furthermore, any such component W is equal to $C_X(E)$ for some prime divisor E over X with center $c_X(E) = x$ and log discrepancy $a_E(X) = 1$. Note that $W \subset X_\infty^x$.

We may assume that E is an exceptional divisor on a log resolution $f\colon X' \to X$ of X. We apply [2, Corollary 1.4.3] to X and f, with $\Delta = \Delta_0 = 0$ and $\mathfrak{E}$ equal to the set of exceptional divisors with log discrepancy at most 1. The output of this operation is a terminal model Y over X where the center of val_E has codimension 1. This implies that val_E is a terminal valuation, hence, by [15, Theorem 1.1], a Nash valuation.

The fact that W is the maximal divisorial set of a Nash valuation implies that W is an irreducible component of X_∞^x. By construction, the image of W in X_m^x is contained in D, showing that D is in the image of Ψ_m^x. This proves (1).

To conclude, we use what we just proved and the injectivity of Ψ_m^x established in Theorem 4 for $m \gg 1$ to infer that every non-degenerate irreducible component of X_∞^x is of the form $C_X(E)$ for some prime divisor E over X with center $c_X(E) = x$ and log discrepancy $a_E(X) = 1$. Conversely, as explained above, [15, Theorem 1.1] implies that for every prime divisor E over X with center $c_X(E) = x$ and log discrepancy $a_E(X) = 1$, the set $C_X(E)$ is an irreducible component of X_∞^x. This gives (2). □

We apply this result to give a solution of the Nash problem for varieties with hDV singularities.

Corollary 29. *Let X be a variety with hDV singularities. For a divisorial valuation ord_E on X, the following are equivalent:*

(1) ord_E *is a Nash valuation.*
(2) ord_E *is a terminal valuation.*
(3) *E is exceptional over X and $a_E(X) = 1$.*

Proof. The implication (3) ⇒ (2) follows by [2, Corollary 1.4.3], the implication (2) ⇒ (1) follows by [15, Theorem 1.1], and the implication (1) ⇒ (3) follows by Theorem 28. □

This result illustrates how this class of singularities preserves some of the properties that characterize Du Val singularities. By [2, Corollary 1.4.3], there is a terminal model $Y \to X$ whose exceptional locus consists exactly of the divisors with log discrepancy 1 over X; from this perspective, this model should be regarded as the analogue of the minimal resolution of a Du Val singularity. Needless to say, it would be interesting to further study the structure of these higher dimensional singularities.

8. Higher compound Du Val singularities

In this section, we look again at rational singularities of maximal embedding codimension. We recall that these are normal, isolated, locally complete intersection singularities. A particular example of such singularities is given by isolated compound Du Val singularities. Compound Du Val singularities were originally introduced in dimension three in [41]. In general, they are defined as follows.

Definition 30. *We say that $x \in X$ is a* compound Du Val *(cDV)* singularity *if the surface $S \subset X$ cut out by* $\dim(X) - 2$ *general hyperplane sections through x has a Du Val singularity at x.*

The following property characterizes isolated cDV singularities (cf. [30] for an earlier result in this direction in dimension three).

Proposition 31. *Let $x \in X$ be an isolated hypersurface singularity of dimension $d \geq 3$. Then the following are equivalent:*

(1) *$x \in X$ is a cDV singularity.*
(2) $\mathrm{mld}_x(X) = d - 1$*, and for every divisor E over X computing* $\mathrm{mld}_x(X)$ *we have* $\mathrm{ord}_E(\mathfrak{m}_x) = 1$ *and E computes* $\mathrm{mld}_x(X, (d-2)\{x\})$.

In particular, isolated cDV singularities are normal locally complete intersection singularities of maximal embedding codimension, according to Definition 17.

Proof. First note that if $x \in X$ is a normal locally complete intersection singularity, then, by Proposition 16, we have $\mathrm{mld}_x(X) \leq d-1$ and $\mathrm{ord}_E(\mathfrak{m}_x) \geq 1$ for any divisor E over X with center x. On the other hand, if S is cut out by $d-2$ general hyperplane sections through x, then $\mathrm{mld}_x(S) \leq 1$, and $x \in S$ is a Du Val singularity if and only if $\mathrm{mld}_x(S) = 1$.

Assume (1) holds. If S is cut out by general hyperplane sections as in Definition 30, then $\mathrm{ord}_E(\mathscr{I}_S) = \mathrm{ord}_E(\mathfrak{m}_x)$ for any E computing $\mathrm{mld}_x(X)$ and

$$1 = \mathrm{mld}_x(S) = \mathrm{mld}_x(X, (d-2)S) \leq a_E(X, (d-2)S) = \mathrm{mld}_x(X) - (d-2)\,\mathrm{ord}_E(\mathfrak{m}_x)$$

by inversion of adjunction (Corollary 3). The properties listed in (2) follows easily from this inequality.

Conversely, if (2) holds and E is any divisor computing $\mathrm{mld}_x(X)$, then we have

$$\mathrm{mld}_x(S) = \mathrm{mld}_x(X, (d-2)S) = a_E(X, (d-2)S) = a_E(X, (d-2)\{x\}) = 1,$$

hence S is a Du Val singularity. Here we used again that S is cut out by general hyperplane sections through x, hence $\mathrm{ord}_E(\mathscr{I}_S) = \mathrm{ord}_E(\mathfrak{m}_x)$. □

Proposition 31 implies in particular that cDV singularities are examples of rational singularities of maximal embedding codimension. However, they satisfy an additional property, namely, the condition that for every divisor E over X computing $\mathrm{mld}_x(X)$ we have $\mathrm{ord}_E(\mathfrak{m}_x) = 1$ and E computes $\mathrm{mld}_x(X, (d-2)\{x\})$. It is not clear to us whether this condition might follow from the definition of singularity of maximal embedding codimension.

By regarding hDV singularities as a higher dimensional version of Du Val singularities, we extend the notion of cDV singularity in the following way.

Definition 32. *We say that $x \in X$ is a* higher compound Du Val *(hcDV)* singularity *if, for some $r \geq 0$, the variety $Y \subset X$ cut out by r general hyperplane sections through x has a hDV singularity at x. (Alternatively, one could call these singularities* compound higher Du Val singularities.*)*

A straightforward adaptation of Proposition 31 gives the following property.

Proposition 33. *Let $x \in X$ be an isolated locally complete intersection singularity of dimension $d \ge 3$ and embedding codimension e. Then the following are equivalent:*

(1) *$x \in X$ is a hcDV singularity.*
(2) *$\mathrm{mld}_x(X) = d - e$, and for every divisor E over X computing $\mathrm{mld}_x(X)$ we have $\mathrm{ord}_E(\mathfrak{m}_x) = 1$ and E computes $\mathrm{mld}_x(X, (d-e-1)\{x\})$.*

In particular, isolated hcDV singularities are normal locally complete intersection singularities of maximal embedding codimension, according to Definition 17.

Theorem 34. *Let $x \in X$ be an isolated hcDV singularity. Then the function Ψ^x_m is surjective, hence a bijection, for all $m \gg 1$.*

Proof. With the case of hDV singularities already settled in Theorem 28, we may assume that $\mathrm{mld}_x(X) > 1$. Let $d = \dim(X)$ and $e = \mathrm{ecodim}(\mathscr{O}_{X,x})$. Note that $\mathrm{mld}_x(X) = d - e$. As in the proof of Theorem 28, for simplicity we reduce to the case where X is embedded in $A := \mathbb{A}^{d+e}$. Let $H := \mathbb{A}^{2e+1} \subset A$ a general linear subspace of codimension $d - e - 1$ through x, so that $Y := X \cap H$ is a variety with a hDV singularity at x.

Let m be any positive integer such that:

(1) Theorem 4 holds for Y (with $\Sigma = \{x\}$), and
(2) for every divisor E over X computing $\mathrm{mld}_x(X)$, we have
$$d(m+1) - \dim(\psi^X_m(C_X(E))) = \text{jet-codim}(C_X(E), X_\infty).$$

Note that these conditions hold for all $m \gg 1$. We can guarantee (1) because there are only finitely many divisorial valuations computing $\mathrm{mld}_x(X)$ since the minimal log discrepancy is positive.

Let D be an irreducible component of X^x_m, and pick an irreducible component D' of $D \cap Y^x_m$. If $h_1, \dots, h_{d-e-1}$ are linear forms on A cutting out H, then $D \cap Y^x_m$ is cut out off D by the equations $h_i^{(j)} = 0$ for $1 \le i \le d-e-1$ and $1 \le j \le m$, hence
$$\mathrm{codim}(D', D) \le (d-e-1)m.$$

If $f_1 = \dots = f_e = 0$ are local equations of X at x in A, then X^x_m is cut out in A^x_m by the equations $f_i^{(j)} = 0$ for $1 \le i \le e$ and $2 \le j \le m$. Here we are using that X is singular at x hence, for all i, both f_i and f_i' vanish identically on A^x_m. This implies that
$$\mathrm{codim}(D, A^x_m) \le e(m-1).$$

Since $\mathrm{codim}(H^x_m, A^x_m) = (d-e-1)m$, we obtain
$$\mathrm{codim}(D', H^x_m) \le e(m-1),$$

hence
$$\mathrm{codim}(D', H_m) \le e(m+1) + 1.$$

Let $V' \subset H_\infty$ the cylinder over D'. We have
$$\mathrm{codim}(V', H_\infty) \le e(m+1) + 1.$$

Write $\mathrm{ord}_{V'} = p' \mathrm{ord}_{F'}$ for some divisor F' over H and some positive integer p'. The same argument as in the proof of Theorem 28 implies
$$1 = \mathrm{mld}_x(Y) = \mathrm{mld}_x(H, eY) \le \frac{1}{p}\big(\mathrm{codim}(V', H_\infty) - e(m+1)\big) \le 1.$$

This implies that $p' = 1$, $V' = C_H(F')$, and F' computes $\mathrm{mld}_x(H, eY)$. If $W' \subset Y_\infty$ is any non-degenerate irreducible component of $V' \cap Y_\infty$, then the argument also shows that W' is an irreducible component of Y^x_∞ and it is equal to $C_Y(E')$ for some divisor E' over Y with $a_{E'}(Y) = 1$. Furthermore, the argument implies that all inequalities above are equalities.

In particular, if $V \subset A_\infty$ is the cylinder over D then

$$\operatorname{codim}(V, A_\infty) = em + d.$$

Writing $\operatorname{ord}_V = p \operatorname{ord}_F$ for some divisor F over A and arguing again as in the proof of Theorem 28 (using now that, by Proposition 33, $\operatorname{mld}_x(A, eX) = d - e$), we conclude that $V = C_X(F)$ where F is a divisor over A computing $\operatorname{mld}_x(A, X)$. Moreover, there is an irreducible component W of $V \cap X_\infty$ that is not contained in $(\operatorname{Sing} X)_\infty$, and this component is of the form $W = C_X(E)$ for a divisor E over X computing $\operatorname{mld}_x(X)$.

By construction,

$$\psi^X_m(W) \subset D.$$

We do not know, however, that W is an irreducible component of X^x_∞. Note that we cannot apply [15] as we did in the proof of Theorem 28 (and, above, for W') since now E does not define a terminal valuation over X. The claim is that $Z \subset X^x_\infty$ is any irreducible component containing W, then

$$\psi^X_m(Z) \subset D.$$

This is all we need to conclude that D is in the image of Ψ^x_m.

To prove the claim, we proceed as follows. First, note that $W' \subset W \cap Y_\infty$. As discussed above, we have $W = C_X(E)$ and $W' = C_S(E')$ where E and E' are divisors over X and S, respectively, with center x and log discrepancies $a_E(X) = d - e$ and $a_{E'}(X) = 1$. In particular,

$$a_{E'}(X) = a_E(X) - (d - e - 1).$$

Since X and S are locally complete intersections at x, we have

$$a_E(X) = \widehat{a}_E(X) - \operatorname{ord}_E(\operatorname{Jac}_X),$$
$$a_{E'}(Y) = \widehat{a}_{E'}(Y) - \operatorname{ord}_{E'}(\operatorname{Jac}_Y)$$

by [14, Corollary 3.5]. By Teissier's Idealistic Bertini Theorem [43, 2.15 Corollary 3], we have $\overline{\operatorname{Jac}_Y} = \overline{\operatorname{Jac}_X|_Y}$ (the bar denoting integral closure), hence it follows by the inclusion $W' \subset W \cap Y_\infty$ that

$$\operatorname{ord}_{E'}(\operatorname{Jac}_Y) \geq \operatorname{ord}_E(\operatorname{Jac}_X).$$

Combining these formulas, we see that

$$\widehat{a}_{E'}(Y) \geq \widehat{a}_E(X) - (d - e - 1).$$

By [16] and the assumption (2) on our choice of m, we have

$$\widehat{a}_E(X) = d(m+1) - \dim\big(\psi^X_m(W)\big),$$
$$\widehat{a}_{E'}(Y) \leq (e+1)(m+1) - \dim\big(\psi^Y_m(W')\big).$$

Using the previous inequality, we get

$$\dim\big(\psi^Y_m(W')\big) \leq \dim\big(\psi^X_m(W)\big) - (d - e - 1)n.$$

Observe that $\psi^Y_m(W')$ is contained in $\psi^X_m(W) \cap Y^x_m$, which is cut out from $\psi^X_m(W)$ by the equations $h^{(j)}_i = 0$ for $1 \leq i \leq d - e - 1$ and $1 \leq j \leq m$. Here we are using that the polynomials h_i already vanish on X^x_m, hence on $\psi^X_m(W)$. It follows that

$$\dim\big(\psi^Y_m(W')\big) = \dim\big(\psi^X_m(W)\big) - (d - e - 1)m,$$

and the $h^{(j)}_i$ form a regular sequence at the generic point of $\psi^Y_m(W')$.

Now, let Z be an irreducible component of X^x_∞ containing W, and assume by contradiction that $\psi^X_m(Z) \not\subset D$. Then $\psi^X_m(Z)$ must be contained in another irreducible component of X^x_m. In particular, if $\widetilde{D}$ denote the union of all irreducible components of X^x_m containing $\psi^Y_m(W')$ and different from D, then

$$\psi^Y_m(W') \subset D \cap \widetilde{D}.$$

Note that $(D \cup \widetilde{D}) \cap Y_m^x$ is the union of the irreducible components of Y_m^x containing $\psi_m^Y(W')$. Since the elements $h_i^{(j)}$ form a regular sequence at each generic point of $D \cap \widetilde{D}$ and cut out Y_m^x on X_m^x, it follows that $(D \cup \widetilde{D}) \cap Y_m^x$ must be reducible. This means that $\psi_m^Y(W')$ is contained in more than one irreducible component of Y_m^x, contradicting Theorem 4, which is supposed to holds for Y by our assumption (1) on m.

We conclude that $\psi_m^X(Z) \subset D$, as claimed. This finishes the proof of the theorem. □

9. The graph generated by families of jets

Following [6, 34, 35], to any variety X we associate a directed graph Γ_X as follows.

Definition 35. *Given a variety X, let Γ_X be the directed graph whose vertices corresponds to the irreducible components of $X_m^{\operatorname{Sing} X}$ for $m \geq 0$; an edge is drawn from a vertex v to a vertex v' whenever v and v' correspond, respectively, to irreducible components $D \subset X_m^{\operatorname{Sing} X}$ and $D' \subset X_{m+1}^{\operatorname{Sing} X}$ with $\pi_{m+1,m}(D') \subset D$. We say that a vertex v has* order m, *and write* $\operatorname{ord}(v) = m$, *if v corresponds to an irreducible component of $X_m^{\operatorname{Sing} X}$. The orientation is defined by the order of the vertices. For every m, we denote by $\Gamma_X^{\geq m}$ and $\Gamma_X^{\leq m}$ the subgraphs of Γ_X obtained by removing all vertices of order $< m$, respectively, $> m$. We call the* root *of Γ_X the set of vertices of order zero. For any vertex v of Γ_X, the* branch *of Γ_X stemming from v is the subgraph $\Gamma_X^{\geq v}$ obtained by removing all vertices that are not reachable by v.*

By construction Γ_X is a directed acyclic graph, that is, a directed graph with no directed cycles. Due to the finiteness of the irreducible components of $X_m^{\operatorname{Sing} X}$, this graph has finitely many vertices of any given order. In particular, $\Gamma_X^{\leq m}$ is finite for every m.

Corollary 36. *Let X be a variety with isolated hcDV singularities, and let Γ_X be the associated graph.*

(1) (Root). *The root of Γ_X is in natural bijection with the singular points of X. Each root in contained in a distinct connected component of Γ_X.*
(2) (Finite branches). *There are no finite branches in Γ_X beyond a certain order. That is, there is an integer m_0 such that for every vertex v of Γ_X of order* $\operatorname{ord}(v) \geq m_0$ *and every* $m \geq \operatorname{ord}(v)$, *there exists a vertex u of order m that is reachable by v.*
(3) (Infinite branches). *The infinite branches of Γ_X are in bijection with the Nash valuations on X. More precisely, for $m \gg 1$, the subgraph $\Gamma_X^{\geq m} \subset \Gamma_X$ is a disjoint union of infinite chains whose vertices have increasing orders $m, m+1, m+2, \ldots$. The number of chains is the number of Nash valuations on X, and each chain is in natural correspondence with a distinct Nash valuation.*

In particular, for $m \geq 1$ the number of irreducible components of $X_m^{\operatorname{Sing} X}$ is equal to the number of irreducible components of $X_\infty^{\operatorname{Sing} X}$, and the function $\Psi_m^{\operatorname{Sing} X}$ is a bijection.

Proof. Property (1) is clear since the vertices in the root of Γ_X corresponds to the singular points of X, viewed as 0-jets on X. Properties (2) and (3) follow from Theorems 4 and 34, which establish that $\Psi_m^{\operatorname{Sing} X}$ is a bijection for $m \gg 1$. The correspondence is defined by associating to each chain of $\Gamma_X^{\geq m}$ the unique irreducible component C of $X_\infty^{\operatorname{Sing} X}$ such that for $n \geq m$ its image $\psi_n(C)$ is contained in the irreducible component of $X_n^{\operatorname{Sing} X}$ corresponding to the vertex of order n in the given chain.

Implicit in these arguments is the compatibility of the functions $\Psi_m^{\operatorname{Sing} X}$ as m varies. Specifically, in the range of application of Theorem 4, if $D = \Phi_m^{\operatorname{Sing} X}(C)$ and $D' = \Phi_{m+1}^{\operatorname{Sing} X}(C)$, then it follows by the geometric definition of these functions and their injectivity that $\pi_{m+1,m}(D') \subset D$, hence the corresponding vertices v and v' are joined by an edge. □

Remark 37. Regarding part (2) of Corollary 36, we should remark that bounded branching of arbitrary large order does occur for other singularities (e.g., see [6, 35]). As for (3), one can visualize the correspondence as attaching one vertex at the end of each chain, with such vertex corresponding to the Nash component. Thinking of the chain as consisting of the integers on $[m,\infty)$, with the intervals $[n,n+1]$ representing the edges, this is the same as adding ∞ to get $[m,\infty]$. Note that this extension of Γ_X is not a graph, since we want to see its geometric realization as a connected set but there is no edge ending at ∞.

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Complex algebraic geometry, in memory of Jean-Pierre Demailly /
Géométrie algébrique complexe, en mémoire de Jean-Pierre Demailly

Miyaoka–Yau inequalities and the topological characterization of certain klt varieties

Inégalités de Miyaoka–Yau et caractérisation topologique de certaines variétés klt

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Abstract. Ball quotients, hyperelliptic varieties, and projective spaces are characterized by their Chern classes, as the varieties where the Miyaoka–Yau inequality becomes an equality. Ball quotients, Abelian varieties, and projective spaces are also characterized topologically: if a complex, projective manifold X is homeomorphic to a variety of this type, then X is itself of this type. In this paper, similar results are established for projective varieties with klt singularities that are homeomorphic to singular ball quotients, quotients of Abelian varieties, or projective spaces.

Résumé. Les quotients de boules, les variétés hyperelliptiques et les espaces projectifs sont caractérisés par leurs classes de Chern, comme les variétés pour lesquelles l'inégalité de Miyaoka–Yau devient une égalité. Les quotients de boules, les variétés abéliennes et les espaces projectifs sont aussi caractérisés topologiquement : si une variété projective complexe X est homéomorphe à une variété de ce type, alors X est elle-même de ce type. Dans cet article, des résultats similaires sont établis pour les variétés projectives avec des singularités klt qui sont homéomorphes à des quotients de boules singulières, à des quotients de variétés abéliennes, ou à des espaces projectifs.

Keywords. Miyaoka–Yau inequality, klt singularities, uniformisation, homeomorphisms.

Mots-clés. Inégalité de Miyaoka–Yau, singularités klt, uniformisation, homéomorphismes.

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1. Introduction

1.1. *The Miyaoka–Yau inequality for projective manifolds*

Let X be an n-dimensional complex-projective manifold and let D be any divisor on X. Recall that X is said to "satisfy the Miyaoka–Yau inequality for D" if the following Chern class inequality holds,

$$\big(2(n+1)\cdot c_2(X) - n\cdot c_1(X)^2\big)\cdot [D]^{n-2} \geq 0.$$

It is a classic fact that n-dimensional projective manifolds X whose canonical bundles are ample or trivial satisfy Miyaoka–Yau inequalities. In case of equality, the universal covers are of particularly simple form.

Theorem 1 (Ball quotients and hyperelliptic varieties). *Let X be an n-dimensional complex projective manifold.*

- *If K_X is ample, then X satisfies the Miyaoka–Yau inequality for K_X. In case of equality, the universal cover of X is the unit the ball $\mathbb{B}^n$.*
- *If K_X is trivial and D is any ample divisor, then X satisfies the Miyaoka–Yau inequality for D. In case of equality, the universal cover of X is the affine space $\mathbb{C}^n$.*

We refer the reader to [24] for a full discussion and references to the original literature.

In the Fano case, where $-K_X$ is ample, the situation is more complicated, due to the fact that the tangent bundle $\mathscr{T}_X$ and the canonical extension $\mathscr{E}_X$ need not be semistable[1]. If $\mathscr{E}_X$ is semistable, then analogous results hold, see [21, Thm. 1.3], as well as further references given there.

Theorem 2 (Projective space). *Let X be an n-dimensional projective manifold. If $-K_X$ is ample and if the canonical extension is semistable with respect to $-K_X$, then X satisfies the Miyaoka–Yau inequality for $-K_X$. In case of equality, X is isomorphic to the projective space $\mathbb{P}^n$.*

In each of the three settings, the equality cases are characterized topologically: if M is any projective manifold homeomorphic to a ball quotient, a finite étale quotient of an Abelian variety or the projective space, then M itself is biholomorphic to a ball quotient, to a finite étale quotient of an Abelian variety, or to the projective space. For ball quotients, this is a theorem of Siu [40]. The torus case is due to Catanese [6], whereas the Fano case is due to Hirzebruch–Kodaira [27] and Yau [44].

1.2. *Spaces with MMP singularities*

In general, it is rarely the case that the canonical bundle of a projective variety has a definite "sign". Minimal model theory offers a solution to this problem, at the expense of introducing singularities. It is therefore natural to extend our study from projective manifolds to projective varieties with Kawamata log terminal (= klt) singularities. For klt varieties whose canonical sheaves are ample, trivial or negative, analogues of Theorems 1 and 2 have been found in the last few years. We refer the reader to [23, Thm. 1.5] for a characterization of singular ball quotients among projective varieties with klt singularities (see Definition 8 for the notion of singular ball quotients). Characterizations of torus quotients and quotients of the projective space can be found in [33], [20, Thm. 1.2] and [21, Thm. 1.3]. In each case, we find it striking that the Chern class equalities imply that the underlying space has no worse than quotient singularities.

[1] Recall that the canonical extension $\mathscr{E}_X$ is defined as the middle term of the exact sequence $0 \to \mathscr{O}_X \to \mathscr{E}_X \to \mathscr{T}_X \to 0$ whose extension class equals $c_1(X) \in H^1\big(X, \Omega^1_X\big)$.

1.3. *Main results of this paper*

This paper asks whether the topological characterizations of ball quotients, Abelian varieties and the projective spaces have analogues in the klt settings. Section 2 establishes a topological characterization of singular ball quotients. The main result of this section, Theorem 10, can be seen as a direct analogue of Siu's rigidity theorems.

Theorem 3 (Rigidity in the klt setting, see Theorem 10). *Let X be a singular quotient of an irreducible bounded symmetric domain and let M be a normal projective variety that is homeomorphic to X. If $\dim X \geq 2$, then, M is biholomorphic or conjugate-biholomorphic to X.*

Using somewhat different methods, Section 3 generalizes Catanese's result to the klt setting.

Theorem 4 (Varieties homeomorphic to torus quotients, see Theorem 18). *Let M be a compact complex space with klt singularities. Assume that M is bimeromorphic to a Kähler manifold. If M is homeomorphic to a singular torus quotient, then M is a singular torus quotient.*

In both cases, we find that certain Chern classes equalities are invariant under homeomorphisms.

Varieties homeomorphic to projective spaces are harder to investigate. Section 4 gives a full topological characterization of $\mathbb{P}^3$, but cannot fully solve the characterization problem in higher dimensions.

Theorem 5 (Topological $\mathbb{P}^3$, see Theorem 40). *Let X be a projective klt variety that is homeomorphic to $\mathbb{P}^3$. Then, $X \cong \mathbb{P}^3$.*

However, we present some partial results that severely restrict the geometry of potential exotic varieties homeomorphic to $\mathbb{P}^n$. These allow us to show the following.

Theorem 6 ($\mathbb{Q}$-Fanos in dimension 4 and 5, see Theorem 41). *Let X be a projective klt variety that is homeomorphic to $\mathbb{P}^n$ with $n = 4$ or $n = 5$. Then, $X \cong \mathbb{P}^n$, unless K_X is ample.*

Dedication

We dedicate this paper to the memory of Jean-Pierre Demailly. His passing is a tremendous loss to the mathematical community and to all who knew him.

Greb

When I was a PhD student, Jean-Pierre's book "Complex Analytic and Differential Geometry" was a revelation for me, as it connected the classical concepts of Complex Analysis with those of modern Complex Differential Geometry and Algebraic Geometry. This greatly shaped my mathematical interests and still influences me today. When I later got to know him during several "Komplexe Analysis" Oberwolfach meetings, I was deeply impressed by his vast knowledge of the field that he shared generously and in his kind and gentle manner, especially with younger people.

Kebekus

I first met Jean-Pierre in the late 90s, when he graciously invited me to Grenoble for my first extended research stay abroad. From the moment I arrived, I was struck by his relaxed and positive air, and by his can-do attitude towards the hardest problems. Over the years, I tried and tested his legendary patience, when he generously shared his vast knowledge with newcomers to the field, myself included. Jean-Pierre's unparalleled clarity made even the most challenging mathematical concepts accessible, and I cherished our discussions on a wide range of topics, from free software to the intricacies of French labour laws[2].

Peternell

Since the late 1980s I had an invaluable close scientific and personal contact with Jean-Pierre, with various mutual joint visits in Bayreuth and Grenoble. I will always commemorate Jean-Pierre's scientific wisdom and his great personality.

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After finishing the paper we were informed by Haidong Liu that, using the recent preprint "Kawamata–Miyaoka type inequality for canonical $\mathbb{Q}$-Fano varieties" [31], instead of Ou's result cited in Proposition 4.19, Theorem 4.21 can be shown to hold also in dimensions 6 and 7.

2. Mostow Rigidity for singular quotients of symmetric domains

Consider a compact Kähler manifold X whose universal cover is a bounded symmetric domain. Siu has shown in [40, Thm. 4] and [41, Main Theorem] that any compact Kähler manifold M which is homotopy equivalent to X is biholomorphic or conjugate-biholomorphic[3] to X. We show an analogous result for homeomorphisms between *singular* varieties M and X. The following notion will be used.

Definition 7 (Quasi-étale cover). *A finite, surjective morphism between normal, irreducible complex spaces is called* quasi-étale cover *if it is unbranched in codimension one.*

Definition 8 (Singular quotient of bounded symmetric domtain). *Let Ω be an irreducible bounded symmetric domain. A normal projective variety X is called a* singular quotient of Ω *if there exists a quasi-étale cover $\widehat{X} \to X$, where $\widehat{X}$ is a smooth variety whose universal cover is Ω.*

Remark 9 (Singular quotients are quotients). Let X be a singular quotient of an irreducible bounded symmetric domain Ω. Passing to a suitable Galois closure, one finds a quasi-étale *Galois* cover $\widehat{X} \to X$, where $\widehat{X}$ is a smooth variety whose universal cover is Ω. In particular, it follows that X is a quotient variety and that it has quotient singularities. Moreover, it can be shown as in [22, §9] that X is actually a quotient of Ω by the fundamental group of X_{reg}, which acts properly discontinously on Ω. In addition, the action is free in codimension one.

Theorem 10 (Mostow rigidity in the klt setting). *Let X be a singular quotient of an irreducible bounded symmetric domain and let M be a normal projective variety that is homeomorphic to X. If $\dim X \geq 2$, then, M is biholomorphic or conjugate-biholomorphic to X.*

Remark 11 (Varieties conjugate-biholomorphic to ball quotients). We are particularly interested in the case where the bounded symmetric domain of Theorem 10 is the unit ball. For this, observe that the set of (singular) ball quotients is invariant under conjugation. It follows that if the variety M of Theorem 10 is biholomorphic or conjugate-biholomorphic to a (singular) ball quotient X, then M is itself a (singular) ball quotient.

[2] Solidarity strike = no food on campus because train drivers demand better working conditions

[3] See also [7, §7] and [1, Chapt. 5 and 6] as general references for the main ideas behind Siu's results and for related topics.

Before proving Theorem 10 in Sections 2.1–2.3 below, we note a first application: the Miyaoka–Yau Equality is a topological property. The symbols $\widehat{c}_\bullet(X)$ in Corollary 12 are the $\mathbb{Q}$-Chern classes of the klt space X, as defined and discussed for instance [22, §3.7].

Corollary 12 (Topological invariance of the Miyaoka–Yau equality). *Let X be a projective klt variety with K_X ample. Assume that the Miyaoka–Yau equality holds:*

$$\big(2(n+1)\cdot\widehat{c}_2(\mathscr{T}_X) - n\cdot\widehat{c}_1(\mathscr{T}_X)^2\big)\cdot[K_X]^{n-2} = 0.$$

Let M be a normal projective variety homeomorphic to X. Then M is klt, K_M is ample and

$$\big(2(n+1)\cdot\widehat{c}_2(\mathscr{T}_M) - n\cdot\widehat{c}_1(\mathscr{T}_M)^2\big)\cdot[K_M]^{n-2} = 0.$$

Proof. Since the Miyaoka–Yau Equality holds on X, there is a quasi-étale cover $\widetilde{X}\to X$ such that the universal cover of $\widetilde{X}$ is the ball, [22]. By Theorem 10, there is a quasi-étale cover $\widetilde{M}\to M$ such that $\widetilde{M}\cong\widetilde{X}$ biholomorphically or conjugate-biholomorphically. Hence, the universal cover of $\widetilde{M}$ is the ball. It follows that M is klt, K_M is ample, and that the Miyaoka–Yau Equality holds on M. □

2.1. *Preparation for the proof of Theorem 10*

The following lemma of independent interest might be well-known. We include a full proof for lack of a good reference.

Lemma 13. *Let X be a normal complex space. Then, the set $X_{\mathrm{sing,top}}\subset X$ of topological singularities is a complex-analytic set.*

Proof. Recall from [16, Thm. on p. 43] that X admits a Whitney stratification where all strata are locally closed complex-analytic submanifolds of X. Recall from [32, Chapt. IV.8] that the closures of the strata are complex-analytic subsets of X. Since Whitney stratifications are locally topologically trivial along the strata[4], it follows that $X_{\mathrm{sing,top}}$ is locally the union of finitely many strata. The additional observation that the set of topologically smooth points, $X\setminus X_{\mathrm{sing,top}}$, is open in the Euclidean topology implies that $X_{\mathrm{sing,top}}$ is locally the union of the closures of finitely many strata, hence analytic. □

2.2. *Proof of Theorem 10 if X is smooth*

We maintain the notation of Theorem 10 in this section and assume additionally that X is smooth. To begin, fix a homeomorphism $f: M\to X$ and choose a resolution of singularities, say $\pi:\widetilde{M}\to M$. The composed map $g = f\circ\pi$ is continuous and induces an isomorphism

$$g_*: H_{2n}\big(\widetilde{M},\mathbb{Z}\big)\to H_{2n}\big(X,\mathbb{Z}\big). \tag{1}$$

Hence, by Siu's general rigidity result [40, Thm. 6] in combination with the curvature computations for the classical, respectively exceptional Hermitian symmetric domains done in [40, 41], the continuous map g is homotopic to a holomorphic or conjugate-holomorphic map $\widetilde{g}:\widetilde{M}\to X$. Replacing the complex structure on X by the conjugate complex structure, if necessary, we may assume without loss of generality that $\widetilde{g}$ is holomorphic and hence in particular algebraic. The isomorphism (1) maps the fundamental class of $\widetilde{M}$ to the fundamental class of X, and $\widetilde{g}$ is hence birational.

We claim that the bimeromorphic morphism $\widetilde{g}$ factors via π. To begin, observe that since g contracts the fibres of π and since $\widetilde{g}$ is homotopic to g, the map $\widetilde{g}$ contracts the fibres of π as well.

[4] See [16, Part I, §1.4] for a detailed discussion.

In fact, given any curve $\widetilde{C} \subset \widetilde{M}$ with $\pi(\widetilde{C})$ a point, consider its fundamental class $[\widetilde{C}] \in H_2(\widetilde{M}, \mathbb{R})$. By assumption, we find that

$$\widetilde{g}_*([\widetilde{C}]) = g_*([\widetilde{C}]) = 0 \in H_2(X, \mathbb{R}).$$

Given that X is projective, this is only possible if $\widetilde{g}(\widetilde{C})$ is a point. Since M is normal and since $\widetilde{g}$ contracts the (connected) fibres of the resolution map π, we obtain the desired factorisation of $\widetilde{g}$, as follows

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\pi"'] \arrow[rr, bend left, "\widetilde{g}"] & M \arrow[r, "\exists! \widetilde{f}"'] & X. \end{tikzcd}$$

We claim that the birational map $\widetilde{f}$ is biholomorphic.[5] By Zariski's Main Theorem, [25, V Thm. 5.2], it suffices to verify that it does not contract any curve $C \subset M$. Aiming for a contradiction, assume that there exists a curve $\widetilde{C} \subset \widetilde{M}$ whose image $C := \pi(\widetilde{C})$ is a curve in M, while $\widetilde{g}(\widetilde{C}) = \widetilde{f}(C) = (*)$ is a point in X. Let $d > 0$ be the degree of the restricted map $\pi|_{\widetilde{C}} : \widetilde{C} \to C$. Then, on the one hand,

$$f_*(d \cdot [C]) = f_*(\pi_*[\widetilde{C}]) = g_*[\widetilde{C}] = \widetilde{g}_*[\widetilde{C}] = 0 \in H_2(X, \mathbb{R}).$$

On the other hand, projectivity of M implies that $d \cdot [C]$ is a non-trivial element of $H_2(M, \mathbb{R})$, which therefore must be mapped to a non-trivial element of $H_2(X, \mathbb{R})$, since f is assumed to be a homeomorphism. This finishes the proof of Theorem 10 in the case where X is smooth.

2.3. *Proof of Theorem 10 in general*

Maintain the setting of Theorem 10.

Step 1: Setup

By assumption, there exists a bounded symmetric domain Ω and a quasi-étale cover $\tau_X : \widehat{X} \to X$ such that the universal cover of $\widehat{X}$ is Ω. Choose a homeomorphism $f : M \to X$ and let $\widehat{M} := \widehat{X} \times_X M$ be the topological fibre product. The situation is summarized in the following commutative diagram,

$$\begin{tikzcd} \widehat{M} \arrow[r, two heads, "\tau_M"] \arrow[d, "\simeq"'] & M \arrow[d, "\simeq"', "f"] \\ \widehat{X} \arrow[r, two heads, "\tau_X\text{, quasi-étale}"'] & X, \end{tikzcd} \tag{2}$$

in which the vertical maps are homeomorphisms and the horizontal maps are surjective with finite fibres.

Step 2: A complex structure on $\widehat{M}$

The spaces M, $\widehat{X}$ and X all carry complex structures. We aim to equip $\widehat{M}$ with a structure so that all horizontal arrows in (2) become holomorphic.

Claim 14. There exists a normal complex structure on $\widehat{M}$ that makes τ_M a finite, holomorphic, and quasi-étale cover.

Proof of Claim 14. Let X_0 be the smooth locus of X, set $M_0 := f^{-1}(X_0)$ and $\widehat{M}_0 := \tau_M^{-1}(M_0)$. The map $\tau_M|_{\widehat{M}_0}$ being a local homeomorphism, there is a uniquely determined complex structure on $\widehat{M}_0$ such that $\tau_M|_{\widehat{M}_0} : \widehat{M}_0 \to M_0$ is a finite holomorphic cover. Since X has quotient singularities,

[5] Cf. [7, Rem. 86(2)]

the topological and holomorphic singularities agree, $X_{\text{sing,top}} = X_{\text{sing}}$. Hence, f being a homeomorphism, we note that

$$M_{\text{sing,top}} = f^{-1}\left(X_{\text{sing}}\right) \quad \text{and} \quad M \setminus M_0 = M_{\text{sing,top}}.$$

We have seen in Lemma 13 that $M_{\text{sing,top}}$ is an analytic set. Therefore, by [9, Thm. 3.4] and [42, Satz 1], the complex structure on $\widehat{M}_0$ uniquely extends to a normal complex structure on the topological manifold $\widehat{M}$, making τ_M holomorphic and finite. The branch locus of τ_M has the same topological dimension as the branch locus of τ_X, so that τ_M is quasi-étale, as claimed. □

Note that as a finite cover of the projective variety M, the normal complex space $\widehat{M}$ is again projective.

Step 3: $\widehat{M}$ as a quotient of Ω

The homeomorphic varieties $\widehat{X}$ and $\widehat{M}$ reproduce the assumptions of Theorem 10. The partial results of Section 2.2 therefore apply to show that the complex spaces $\widehat{M}$ and $\widehat{X}$ are biholomorphic or conjugate-biholomorphic. Replacing the complex structures on M and $\widehat{M}$ by their conjugates, if necessary, we assume without loss of generality for the remainder of this proof that $\widehat{M}$ and $\widehat{X}$ are biholomorphic. This has two consequences.

(1) The projective variety $\widehat{M}$ is smooth. The universal cover of $\widehat{M}$ is biholomorphic to Ω.
(2) Its quotient M is a singular quotient of Ω and has only quotient singularities.

Recalling that quotient singularities are not topologically smooth, Item (2) implies that the homeomorphism $f : M \to X$ restricts to a homeomorphism between the smooth loci, X_{reg} and M_{reg}. The situation is summarized in the following commutative diagram,

$$\begin{tikzcd}
\Omega \arrow[r, two heads, "u_X\text{, univ. cover}"] \arrow[d, "\simeq"'] & \widehat{M} \arrow[r, two heads, "\tau_M\text{, quasi-étale}"] \arrow[d, "\simeq"'] & M \arrow[d, "\simeq"', "f"] & M_{\text{reg}} \arrow[l, hook', "\text{inclusion}"'] \arrow[d, "\simeq"', "f|_{X_{\text{reg}}}"] \\
\Omega \arrow[r, two heads, "u_M\text{, univ. cover}"'] & \widehat{X} \arrow[r, two heads, "\tau_X\text{, quasi-étale}"'] & X & X_{\text{reg}}, \arrow[l, hook', "\text{inclusion}"]
\end{tikzcd}$$

where all horizontal maps are holomorphic, and all vertical maps are homeomorphic.

The description of M as a singular quotient of Ω can be made precise. The argument in [22, §9.1] shows that the fundamental group $\pi_1(M_{\text{reg}})$ acts properly discontinously on Ω with quotient M. In particular, we have an injective homomorphism from $\pi_1(M_{\text{reg}})$ into the holomorphic automorphism group $\operatorname{Aut}(\Omega)$ of Ω, with image a discrete cocompact subgroup $\Gamma_M \subseteq \operatorname{Aut}(\Omega)$.

The same reasoning also applies to X and presents X as a quotient $X = \Omega/\pi_1(X_{\text{reg}})$, where $\pi_1(X_{\text{reg}})$ again acts via an injective homomorphism $\pi_1(X_{\text{reg}}) \hookrightarrow \operatorname{Aut}(\Omega)$, with image a cocompact, discrete subgroup Γ_X of $\operatorname{Aut}(\Omega)$.

As we have seen above, f induces a homeomorphism from M_{reg} to X_{red}, from which we obtain an abstract group isomorphism $\theta : \Gamma_M \to \Gamma_X$.

Step 4: End of proof

In the remainder of the proof we will show that not only $\widehat{M}$ and $\widehat{X}$ are (conjugate-)-biholomorphic, but that this actually holds for M and X. This will be a consequence of Mostow's rigidity theorem for lattices in connected semisimple real Lie groups. As the groups appearing in our situation are not necessarily connected, we have to do some work to reduce to the connected case[6].

Given that Ω is an irreducible Hermitian symmetric domain of dimension greater than one, the identity component $\operatorname{Aut}^\circ(\Omega) \subseteq \operatorname{Aut}(\Omega)$ coincides with the identity component $I^\circ(\Omega)$ of the

[6]Alternatively, one could trace the finite group actions through the proof of the results used in Section 2.2.

isometry group $I(\Omega)$ of the Riemannian symmetric space Ω, [26, VIII.Lem. 4.3][7], which is a non-compact simple Lie group without non-trivial proper compact normal subgroups and with trivial centre, [10, Prop. 2.1.1 and bottom of p. 379]. We also note that a Bergman-metric argument shows that $\mathrm{Aut}(\Omega)$ is contained in $I(\Omega)$. Furthermore, both Lie groups have only finitely many connected components.

Claim 15. There exists an isometry $F \in I(\Omega)$ such that

$$F \circ \gamma = \theta(\gamma) \circ F, \quad \text{for every } \gamma \in \Gamma_M. \tag{3}$$

Proof of Claim 15. If the rank of Ω is equal to one, then $\Omega \cong \mathbb{B}_n$, the unit ball in $\mathbb{B}^n$, see [26, §X.6.3/4]. Consequently, the group $\mathrm{Aut}(\Omega)$ is connected, and we may apply [34, Thm. A$'$ on p. 4] to obtain an automorphism of real Lie groups, $\Theta : \mathrm{Aut}(\Omega) \to \mathrm{Aut}(\Omega)$ such that $\Theta|_{\Gamma_M} = \theta$. The desired isometry is then produced by an application of [10, Prop. 3.9.11].

We consider the case $\mathrm{rank}(\Omega) \geq 2$ for the remainder of the present proof, where the automorphism group may be non-connected. To deal with this slight difficulty, we proceed as in [10, p. 379]: as $\mathrm{Aut}(\Omega)$ has finitely many connected components, we may assume that the subgroups $\Gamma_{\widehat{M}} \subseteq \Gamma_M$ and $\Gamma_{\widehat{X}} \subseteq \Gamma_X$ corresponding to the deck transformation groups of u_M and u_X, respectively, are contained in the identity component $I^\circ(\Omega) = \mathrm{Aut}^\circ(\Omega)$. Again, apply [34, Thm. A$'$ on p. 4] to obtain an automorphism of real Lie groups $\Theta : I^\circ(\Omega) \to I^\circ(\Omega)$ such that $\Theta|_{\Gamma_{\widehat{M}}} = \theta|_{\Gamma_{\widehat{M}}}$ and then [10, Prop. 3.9.11] to obtain an isometry $F \in I(\Omega)$ such that

$$F \circ g = \Theta(g) \circ F, \quad \text{for every } g \in I^\circ(\Omega).$$

This in particular yields (3) for all γ contained in the finite index subgroup $\Gamma_{\widehat{M}}$ of Γ_M. This is not yet enough.

However, noticing that for any finite index subgroup $\Gamma'_M < \Gamma_M$, every Γ'_M-periodic vector in the sense of [10, Def. 4.5.13] by definition is also Γ_M-periodic, we see with the argument given in [10, p. 379], which uses essentially the same notation as we have introduced here, that the set of Γ_M-periodic vectors is dense in the unit sphere bundle $S\Omega$ of Ω. The subsequent argument in [10, bottom of p. 379 and upper part of p. 380] then applies verbatim to yield the desired relation (3) for all $\gamma \in \Gamma_M$; this is [10, equation (5) on p. 380]. □

Now, since the Hermitian symmetric domain Ω is assumed to be irreducible, the Γ-equivariant isometry $F \in I(\Omega)$ is either holomorphic or conjugate-holomorphic, as follows for example from [5] together with [26, VIII.Prop. 4.2]. By the universal property of the quotient map π with respect to Γ-invariant holomorphic maps, F hence descends to a holomorphic or conjugate-holomorphic isomorphism from M to X. This completes the proof of Theorem 10.

3. Topological characterization of torus quotients

In line with the results of Section 2, we show that a Kähler space with klt singularities is a singular torus quotient if and only if it is homeomorphic to a singular torus quotient. In the smooth case, this was shown by Catanese [6], but see also [3, Thm. 2.2]. The following notion is a direct analogue of Definition 8 above.

Definition 16 (Singular torus quotient). *A normal complex space X is called a* singular torus quotient *if there exists a quasi-étale cover $\widehat{X} \to X$, where $\widehat{X}$ is a compact complex torus.*

Remark 17 (Singular torus quotients are quotients). Let X be a singular torus quotient. Passing to a suitable Galois closure, one finds a quasi-étale *Galois* cover $\widehat{X} \to X$, where $\widehat{X}$ is a compact torus.

[7] As Ω is irreducible, the compatible Riemannian metric on Ω is unique up to a positive real multiple that does not change the isometry group.

Theorem 18 (Varieties homeomorphic to torus quotients). *Let M be a compact complex space with klt singularities. Assume that M is bimeromorphic to a Kähler manifold. If M is homeomorphic to a singular torus quotient, then M is a singular torus quotient.*

Theorem 18 will be shown in Sections 3.1–3.2 below. In analogy to Corollary 12 above, we note that vanishing of $\mathbb{Q}$-Chern classes is a topological property among compact Kähler spaces with klt singularities.

Corollary 19 (Topological invariance of vanishing Chern classes). *Let X be a compact Kähler space with klt singularities. Assume that the canonical class vanishes numerically, $K_X \equiv 0$, and that the second $\mathbb{Q}$-Chern class of $\mathscr{T}_X$ satisfies*

$$\widehat{c}_2(\mathscr{T}_X) \cdot \alpha_1 \dots \alpha_{\dim X-2} = 0,$$

for every $(\dim X-2)$-tuple of Kähler classes on X. If M is any compact Kähler space with klt singularities that is homeomorphic to X, then $K_M \equiv 0$, and the second $\mathbb{Q}$-Chern class of $\mathscr{T}_M$ satisfies

$$\widehat{c}_2(\mathscr{T}_M) \cdot \beta_1 \dots \beta_{\dim M-2} = 0,$$

for every $(\dim X-2)$-tuple of Kähler classes on M.

Proof. The characterization of singular torus quotients in terms of Chern classes by Claudon, Graf and Guenancia, [8, Cor. 1.7], guarantees that X is a torus quotient[8]. By Theorem 18, then so is M. □

3.1. *Proof of Theorem 18 if M is homeomorphic to a torus*

As before, we prove Theorem 18 first in case where the (potentially singular) space M is homeomorphic to a torus. Recalling that klt singularities are rational, see [30, Thm. 5.22] for the algebraic case and [11, Thm. 3.12] (together with the vanishing theorems proven in [12]) for the analytic case, we show the following, slightly stronger statement.

Proposition 20. *Let M be a compact complex space with rational singularities. Assume that M is bimeromorphic to a Kähler manifold. If M is homotopy equivalent to a compact torus, then M is a compact torus.*

Proof. We follow the arguments of Catanese, [6, Thm. 4.8], and choose a resolution of singularities, $\pi : \widetilde{M} \to M$, which owing to the assumptions on M we may assume to be a compact Kähler manifold. Using the assumption that M has rational singularities together with the push-forward of the exponential sequence, we observe that the pull-back map $H^1(M,\mathbb{Z}) \to H^1(\widetilde{M},\mathbb{Z})$ is an isomorphism. In particular, first Betti numbers of M and $\widetilde{M}$ agree. As a next step, consider the Albanese map of $\widetilde{M}$, observing that $\widetilde{M}$ is bimeromorphic to a Kähler manifold since M is. Again using that M has rational singularities, recall from [38, Prop. 2.3] that the Albanese factors via M,

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\mathrm{alb}"] \arrow[d, two heads, "\pi,\ \text{resolution}"'] & \mathrm{Alb}. \\ M \arrow[ur, bend right, "\alpha"'] & \end{tikzcd}$$

Since the pull-back morphisms

$$\begin{aligned} \mathrm{alb}^* = \pi^* \circ \alpha^* &: H^1(\mathrm{Alb},\mathbb{Z}) \to H^1(\widetilde{M},\mathbb{Z}) \\ \pi^* &: H^1(M,\mathbb{Z}) \to H^1(\widetilde{M},\mathbb{Z}) \end{aligned}$$

[8]See [33, Thm. 1.2] for the projective case and see [19, Thm. 1.17] for the case where X is projective and smooth in codimension two.

are both isomorphic, we find that $\alpha^* : H^1(\mathrm{Alb}, \mathbb{Z}) \to H^1(M, \mathbb{Z})$ must likewise be an isomorphism. There is more that we can say. Since the topological cohomology ring of a torus is an exterior algebra,

$$H^*(\mathrm{Alb}, \mathbb{Z}) = \wedge^* H^1(\mathrm{Alb}, \mathbb{Z}) \quad \text{and} \quad H^*(M, \mathbb{Z}) = \wedge^* H^1(M, \mathbb{Z}),$$

we find that all pull-back morphisms are isomorphisms,

$$\alpha^* : H^q(\mathrm{Alb}, \mathbb{Z}) \xrightarrow{\cong} H^q(M, \mathbb{Z}), \quad \text{for every } 0 \le q \le 2 \cdot \dim M.$$

Applying this to $q = 2 \cdot \dim M$, we see α is surjective of degree one, hence birational. Again, more is true: if α failed to be isomorphic, Zariski's Main Theorem would guarantee that α contracts a positive-dimensional subvariety, so $b_2(M) > b_2(\mathrm{Alb})$. But we have seen above that equality holds and hence reached a contradiction. □

3.2. *Proof of Theorem 18 in general*

By assumption, there exists a homeomorphism $f : M \to X$, where X is a singular torus quotient. Choose a quasi-étale cover $\tau_X : \widehat{X} \to X$, where $\widehat{X}$ is a complex torus, and proceed as in the proof of Theorem 10, in order to construct a diagram of continuous mappings between normal complex spaces,

$$\begin{tikzcd} \widehat{M} \arrow[r, two heads, "\tau_M\text{, quasi-étale}"] \arrow[d, "\cong"'] & M \arrow[d, "f", "\cong"'] \\ \widehat{X} \arrow[r, two heads, "\tau_X\text{, quasi-étale}"'] & X, \end{tikzcd}$$

where

- the vertical maps are homeomorphisms, and
- the horizontal maps are holomorphic, surjective, and finite.

Since M is bimeromorphic to a Kähler manifold, so is $\widehat{M}$. Recalling from [30, Prop. 5.20] that also $\widehat{M}$ has no worse than klt singularities, Proposition 20 will then guarantee that $\widehat{M}$ is a complex torus, as claimed.

4. Rigidity results for projective spaces

Recall the classical theorem of Hirzebruch–Kodaira, which asserts that the projective space carries a unique structure as a Kähler manifold.

Theorem 21 (Rigidity of the projective space, [27, p. 367]). *Let X be a compact Kähler manifold. If X is homeomorphic to $\mathbb{P}^n$, then X is biholomorphic to $\mathbb{P}^n$.*

Remark 22. Strictly speaking, Hirzebruch–Kodaira proved a somewhat weaker result: X is biholomorphic to $\mathbb{P}^n$ if either n is odd, or if n is even and $c_1(X) \neq -(n+1) \cdot g$, where g is a generator of $H^2(X, \mathbb{Z})$ and the fundamental class of a Kähler metric on X. The second case was later ruled out by Yau's solution to the Calabi conjecture, which implies that then the universal cover of X is the ball, contradicting $\pi_1(X) = 0$.

Since the topological invariance of the Pontrjagin classes, [35], was not known at that time, Hirzebruch–Kodaira also had to assume that X is diffeomorphic to $\mathbb{P}^n$ rather than merely homeomorphic.

We ask whether an analogue of Hirzebruch–Kodaira's theorem remains true in the context of minimal model theory.

Question 23. *Let X be a projective variety with klt singularities. Assume that X is homeomorphic to $\mathbb{P}^n$. Is X then biholomorphic to $\mathbb{P}^n$?*

4.1. *Varieties homeomorphic to projective space*

We do not have a full answer to Question 23. The following proposition will, however, restrict the geometry of potential varieties substantially. It will later be used to answer Question 23 in a number of special settings.

Proposition 24 (Varieties homeomorphic to $\mathbb{P}^n$). *Let X be a projective klt variety. If X is homeomorphic to $\mathbb{P}^n$, then the following holds.*

(1) *We have $H^q(X, \mathscr{O}_X) = 0$ for every $1 \le q$.*
(2) *The Chern class map $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism.*
(3) *The variety X is smooth in codimension two.*
(4) *The maps $r_q : H^q(X, \mathbb{Z}) \to H^q(X_{\mathrm{reg}}, \mathbb{Z})$ are isomorphic, for every $0 \le q \le 4$. The same statement holds for $\mathbb{Z}_2$ coefficients.*
(5) *Every Weil divisor on X is Cartier, i.e., X is factorial. In particular, X is Gorenstein.*
(6) *The canonical divisor K_X is ample or anti-ample.*

Proof. We prove the items of Proposition 24 separately.

Item (1). This is a consequence of the rationality of the singularities of X and the isomorphisms $H^q(X, \mathbb{C}) \simeq H^q(\mathbb{P}^n, \mathbb{C})$. In fact, since X has rational singularities, the morphisms

$$\varphi_q : H^q(X, \mathbb{C}) \to H^q(X, \mathscr{O}_X)$$

induced by the canonical inclusion $\mathbb{C} \to \mathscr{O}_X$, are surjective, [29, Thm. 12.3]. If q is odd, this already implies that $H^q(X, \mathscr{O}_X) = 0$. If q is even, it suffices to note that φ_q has a non-trivial kernel. For this, choose an ample line bundle $\mathscr{L} \in \mathrm{Pic}(X)$ and observe that

$$\varphi_q\big(c_1(\mathscr{L})^{q/2}\big) = 0 \in H^q(X, \mathscr{O}_X).$$

To prove the observation, pass to a desingularisation and use the Hodge decomposition there.

Item (2). The description of c_1 follows from (1) and the exponential sequence.

Item (3). Recall that klt varieties have quotient singularities in codimension two, [18, Prop. 9.3]. Smoothness follows because quotient singularities have non-trivial local fundamental groups and are hence not topologically smooth.

Item (4). We describe the relevant cohomology groups in terms of Borel-Moore homology, [4], and also refer to the reader to [15, §19.1] for a summary of the relevant facts (over $\mathbb{Z}$). The assumption that X is homeomorphic to an oriented, connected, real manifold implies that singular cohomology and Borel-Moore homology agree, [4, Thm. 7.6] and [15, p. 371]. The same holds for the non-compact manifold X_{reg}, i.e., for $R = \mathbb{Z}, \mathbb{Z}_2$ we have

$$H^q(X, R) = H^{BM}_{2\cdot n-q}(X, R) \quad \text{and} \quad H^q(X_{\mathrm{reg}}, R) = H^{BM}_{2\cdot n-q}(X_{\mathrm{reg}}, R), \quad \text{for every } q.$$

The isomorphisms identify the restriction maps r_q with the pull-back maps for Borel-Moore homology. These feature in the localization sequence for Borel-Moore homology, [4, Thm.3.8],

$$\cdots \to H^{BM}_{2\cdot n-q}(X_{\mathrm{sing}}, R) \to H^{BM}_{2\cdot n-q}(X, R) \xrightarrow{r_q} H^{BM}_{2\cdot n-q}(X_{\mathrm{reg}}, R) \to H^{BM}_{2\cdot n-q-1}(X_{\mathrm{sing}}, R) \to \dots$$

Recalling from [15, Lem. 19.1.1] that $H^{BM}_i(X_{\mathrm{sing}}, \mathbb{Z}) = 0$ for every $i > 2 \cdot \dim_{\mathbb{C}} X_{\mathrm{sing}}$ and noticing that the inductive argument employed in the proof also works for $\mathbb{Z}_2$-coefficients, the claim of Item (4) thus follows from smoothness in codimension two, Item (3).

Item (5). Remaining in the analytic category, writing down the exponential sequences for X and X_{reg},

$$\begin{tikzcd}
H^1\big(X, \mathbb{Z}\big) \arrow[r] \arrow[d, "r_1"] & H^1\big(X, \mathscr{O}_X\big) \arrow[r] \arrow[d] & \operatorname{Pic}(X) \arrow[r, "c_1"] \arrow[d, hook] & H^2\big(X, \mathbb{Z}\big) \arrow[r] \arrow[d, "r_2"] & H^2\big(X, \mathscr{O}_X\big) \arrow[d] \\
H^1\big(X_{\mathrm{reg}}, \mathbb{Z}\big) \arrow[r] & H^1\big(X_{\mathrm{reg}}, \mathscr{O}_{X_{\mathrm{reg}}}\big) \arrow[r] & \operatorname{Pic}\big(X_{\mathrm{reg}}\big) \arrow[r, "c_1"'] & H^2\big(X_{\mathrm{reg}}, \mathbb{Z}\big) \arrow[r] & H^2\big(X_{\mathrm{reg}}, \mathscr{O}_{X_{\mathrm{reg}}}\big),
\end{tikzcd}$$

and filling in what we already know, we find a commutative diagram with exact rows, as follows,

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \operatorname{Pic}(X) \arrow[r, hook, two heads, "c_1\text{, iso.}"] \arrow[d, hook] & H^2\big(X, \mathbb{Z}\big) \arrow[r] \arrow[d, hook, two heads, "r_2\text{, iso.}"] & 0 \\
0 \arrow[r] & H^1\big(X_{\mathrm{reg}}, \mathscr{O}_{X_{\mathrm{reg}}}\big) \arrow[r] & \operatorname{Pic}(X_{\mathrm{reg}}) \arrow[r, two heads, "c_1"'] & H^2\big(X_{\mathrm{reg}}, \mathbb{Z}\big) \arrow[r] & 0.
\end{tikzcd}$$

The snake lemma now asserts that

$$H^1\big(X_{\mathrm{reg}}, \mathscr{O}_{X_{\mathrm{reg}}}\big) \cong {}^{\operatorname{Pic}(X_{\mathrm{reg}})}\!/_{\operatorname{Pic}(X)}. \tag{4}$$

We claim that $H^1\big(X_{\mathrm{reg}}, \mathscr{O}_{X_{\mathrm{reg}}}\big)$ vanishes. For this, recall that the singularities of X are rational, so every local ring $\mathscr{O}_{X,x}$ of the (holomorphic) structure sheaf has depth equal to n. Since the singular set of X has codimension at least 3 in X by Item (3), we may apply [39, §5, Korollar after Satz III] or alternatively [2, Chap. II, Cor. 3.9 and Thm. 3.6] to see that the restriction homomorphism

$$H^1\big(X, \mathscr{O}_X\big) \to H^1\big(X_{\mathrm{reg}}, \mathscr{O}_{X_{\mathrm{reg}}}\big)$$

is bijective. However, the cohomology group on the left side was shown to vanish in Item (1) above.

In summary, we find that every invertible sheaf on X_{reg} extends to an invertible sheaf on X. If $D \in \operatorname{Div}(X)$ is any Weil divisor, the invertible sheaf $\mathscr{O}_{X_{\mathrm{reg}}}(D)$ will therefore extend to an invertible sheaf on X, which necessarily equals the (reflexive) Weil divisorial sheaf $\mathscr{O}_X(D)$. It follows that D is Cartier. This applies in particular to the canonical divisor, so X is $\mathbb{Q}$-Gorenstein of index one. Since X is Cohen–Macaulay, we conclude that X is Gorenstein.

Item (6). Given that $\operatorname{Pic}(X) = \mathbb{Z}$, every line bundle is ample, anti-ample, or trivial; we need to exclude the case that K_X is trivial. But if K_X were trivial, use that X is Gorenstein and apply Serre duality to find

$$h^n\big(X, \mathscr{O}_X\big) = h^0\big(X, \omega_X\big) = h^0\big(X, \mathscr{O}_X\big) = 1.$$

This contradicts Item (1) above. □

Notation 25 (Line bundles on varieties homeomorphic to $\mathbb{P}^n$). If X is a projective klt variety that is homeomorphic to $\mathbb{P}^n$, Item (2) shows the existence of a unique ample line bundle that generates $\operatorname{Pic}(X) \cong \mathbb{Z}$. We refer to this line bundle as $\mathscr{O}_X(1)$. Item (5) equips us with a unique number $r \in \mathbb{N}$ and such that $\omega_X \cong \mathscr{O}_X(r)$. Item (6) guarantees that $r \neq 0$.

Remark 26 (Pull-back of line bundles). The cohomology rings of X and $\mathbb{P}^n$ are isomorphic. If $\phi : X \to \mathbb{P}^n$ is any homeomorphism, then $\phi^* c_1\big(\mathscr{O}_{\mathbb{P}^n}(1)\big) = c_1\big(\mathscr{O}_X(\pm 1)\big)$. The cup products $c_1\big(\mathscr{O}_X(1)\big)^q$ generate the groups $H^{2q}\big(X, \mathbb{Z}\big) \cong \mathbb{Z}$.

4.2. *Characteristic classes*

We have seen in Proposition 24 that X is smooth away from a closed set of codimension ≥ 3. This allows defining a number of characteristic classes.

Notation 27 (Chern classes on varieties homeomorphic to $\mathbb{P}^n$). If X is a projective klt variety that is homeomorphic to $\mathbb{P}^n$, Item (4) allows defining first and second Chern classes, as well as a first Pontrjagin class and a second Stiefel–Whitney class

$$\begin{aligned} c_1(X) &= r_2^{-1} c_1(X_{\mathrm{reg}}) \in H^2\big(X, \mathbb{Z}\big) \\ c_2(X) &= r_4^{-1} c_2(X_{\mathrm{reg}}) \in H^4\big(X, \mathbb{Z}\big) \\ p_1(X) &= r_4^{-1} p_1(X_{\mathrm{reg}}) \in H^4\big(X, \mathbb{Z}\big) \\ w_2(X) &= r_2^{-1} w_2(X_{\mathrm{reg}}) \in H^2\big(X, \mathbb{Z}_2\big). \end{aligned}$$

Remark 28 (Pontrjagin and Chern classes). If X be a projective klt variety that is homeomorphic to $\mathbb{P}^n$, the restriction maps $r_\bullet : H^\bullet\big(X, \mathbb{Z}\big) \to H^\bullet\big(X_{\mathrm{reg}}, \mathbb{Z}\big)$ commute with the cup products on X and X_{reg}, which implies in particular that

$$p_1(X) = r_4^{-1} p_1(X_{\mathrm{reg}}) = r_4^{-1}\Big(c_1(X_{\mathrm{reg}})^2 - 2\cdot c_2(X_{\mathrm{reg}})\Big) = c_1(X)^2 - 2\cdot c_2(X) \in H^4\big(X, \mathbb{Z}\big).$$

Remark 29 (Stiefel–Whitney class and first Chern class). By definition and the well-known relation in the smooth case, we have

$$w_2(X) = c_1(X) \bmod 2.$$

Novikov's result on the topological invariance of Pontrjagin classes extends to the generalized Pontrjagin class defined in Notation 27.

Proposition 30 (Topological invariance of Pontrjagin classes). *Let X be a projective klt variety. If $\phi : X \to \mathbb{P}^n$ is any homeomorphism, then $\phi^* p_1(\mathbb{P}^n) = p_1(X)$ in $H^4\big(X, \mathbb{Z}\big)$.*

Proof. Consider the open set $\mathbb{P}^n_{\mathrm{reg}} := \phi(X_{\mathrm{reg}})$ and the restricted homeomorphism $\phi_{\mathrm{reg}} : X_{\mathrm{reg}} \to \mathbb{P}^n_{\mathrm{reg}}$. Recalling from Item (4) of Propositions 24 that the restriction maps

$$r_4 : H^4\big(X, \mathbb{Z}\big) \to H^4\big(X_{\mathrm{reg}}, \mathbb{Z}\big) \quad \text{and} \quad r_4 : H^4\big(\mathbb{P}^n, \mathbb{Z}\big) \to H^4\big(\mathbb{P}^n_{\mathrm{reg}}, \mathbb{Z}\big)$$

are isomorphic, it suffices to show that the restricted classes in rational cohomology agree. More precisely,

$$\begin{aligned} & \phi^* p_1(\mathbb{P}^n) = p_1(X) && \text{in } H^4\big(X, \mathbb{Z}\big) \\ \iff\ & r_4 \phi^* p_1(\mathbb{P}^n) = r_4 p_1(X) && \text{in } H^4\big(X_{\mathrm{reg}}, \mathbb{Z}\big), \text{ since } r_4\text{'s are iso.} \\ \iff\ & \phi^*_{\mathrm{reg}} p_1(\mathbb{P}^n_{\mathrm{reg}}) = p_1(X_{\mathrm{reg}}) && \text{in } H^4\big(X_{\mathrm{reg}}, \mathbb{Z}\big), \text{ definition, functoriality} \\ \iff\ & \phi^*_{\mathrm{reg}} p_1(\mathbb{P}^n_{\mathrm{reg}}) = p_1(X_{\mathrm{reg}}) && \text{in } H^4\big(X_{\mathrm{reg}}, \mathbb{Q}\big), \text{ since } H^4\big(X_{\mathrm{reg}}, \mathbb{Z}\big) = \mathbb{Z} \end{aligned}$$

The last equation is Novikov's famous topological invariance of Pontrjagin classes, [35][9]. □

Corollary 31 (Relation between Chern classes on varieties homeomorphic to $\mathbb{P}^n$). *If X is a projective klt variety that is homeomorphic to $\mathbb{P}^n$, then*

$$2\cdot c_2(X) = \big[r^2 - (n+1)\big]\cdot c_1\big(\mathscr{O}_X(1)\big)^2 \quad \text{in } H^4\big(X, \mathbb{Z}\big).$$

Proof. Choose a homeomorphism $\phi : X \to \mathbb{P}^n$, in order to compare the Pontrjagin class of $\mathbb{P}^n$ with that of X.

$$\begin{aligned} & p_1(\mathbb{P}^n) = (n+1)\cdot c_1\big(\mathscr{O}_{\mathbb{P}^n}(1)\big)^2 && \text{in } H^4\big(\mathbb{P}^n, \mathbb{Z}\big) \\ \iff\ & \phi^* p_1(\mathbb{P}^n) = (n+1)\cdot \phi^* c_1\big(\mathscr{O}_{\mathbb{P}^n}(1)\big)^2 && \text{in } H^4\big(X, \mathbb{Z}\big) \\ \iff\ & p_1(X) = (n+1)\cdot c_1\big(\mathscr{O}_X(\pm 1)\big)^2 && \text{Prop. 30 and Rem. 26} \\ \iff\ & c_1\big(\mathscr{O}_X(r)\big)^2 - 2\cdot c_2(X) = (n+1)\cdot c_1\big(\mathscr{O}_X(1)\big)^2 && \text{Rem. 28} \end{aligned}$$

The claim thus follows. □

[9] See [17, Thm. 0] for the precise result used here and see [37, Appendix] for a history of the result. Igor Belegradek explains on MathOverflow (https://mathoverflow.net/q/442025) why compactness assumptions are not required.

Corollary 31 allows reformulating the $\mathbb{Q}$-Miyaoka–Yau inequality and $\mathbb{Q}$-Bogomolov–Gieseker inequality as inequalities between the index r and the dimension n. The first remark will be relevant for varieties of general type, whereas the second one will be used for Fano varieties.

Remark 32 (Reformulation of the $\mathbb{Q}$-Miyaoka–Yau inequality). Let X be a projective klt variety that is homeomorphic to $\mathbb{P}^n$. Since X is smooth in codimension two, the Miyaoka–Yau inequality for $\mathbb{Q}$-Chern classes,

$$\big(2(n+1)\cdot\widehat{c}_2(X)-n\cdot\widehat{c}_1(X)^2\big)\cdot[H]^{n-2}\geq 0,\quad\text{for one ample } H,$$

is equivalent to the assertion that there exists a non-negative constant $c\in\mathbb{R}^{\geq 0}$ such that

$$\begin{aligned}
&\big(2(n+1)\cdot c_2(X)-n\cdot c_1(X)^2\big)\geq c\cdot c_1\big(\mathscr{O}_X(1)\big)^2 && \text{in } H^4\big(X,\mathbb{Z}\big)\\
\iff\quad &\big((n+1)(r^2-(n+1))-n\cdot r^2\big)\cdot c_1\big(\mathscr{O}_X(1)\big)^2\geq c\cdot c_1\big(\mathscr{O}_X(1)\big)^2 && \text{Cor. 31}\\
\iff\quad &\big(r^2-(n+1)^2\big)\cdot c_1\big(\mathscr{O}_X(1)\big)^2\geq c\cdot c_1\big(\mathscr{O}_X(1)\big)^2\\
\iff\quad &|r|\geq n+1.
\end{aligned}$$

The Miyaoka–Yau inequality is an equality if and only if $|r|=n+1$.

Remark 33 (Reformulation of the $\mathbb{Q}$-Bogomolov–Gieseker inequality). Let X be a projective klt variety that is homeomorphic to $\mathbb{P}^n$. Since X is smooth in codimension two, the Bogomolov–Gieseker inequality for $\mathbb{Q}$-Chern classes,

$$\big(2n\cdot\widehat{c}_2(X)-(n-1)\cdot\widehat{c}_1(X)^2\big)\cdot[H]^{n-2}\geq 0,\quad\text{for one (equiv. every) ample } H,$$

is equivalent to the assertion that $|r|>n$.

We will also need the topological invariance of the second Stiefel–Whitney class w_2.

Proposition 34 (Topological invariance of the second Stiefel–Whitney class). *Let X be a projective klt variety. If $\phi: X\to\mathbb{P}^n$ is any homeomorphism, then $\phi^* w_2(\mathbb{P}^n)=w_2(X)$ in $H^2\big(X,\mathbb{Z}/2\mathbb{Z}\big)$.*

Proof. We can argue as in the proof of Proposition 30, replacing Novikov's Theorem by the corresponding invariance result for Stiefel–Whitney classes due to Thom, [43, Thm. III.8]. ☐

Corollary 35 (Parity of the first Chern class of varieties homeomorphic to $\mathbb{P}^n$). *If X is a projective klt variety that is homeomorphic to $\mathbb{P}^n$, then $r-(n+1)$ is even.*

Proof. This follows from the topological invariance established just above together with Remark 29 and the relation $\varphi^*(c_1(\mathscr{O}_{\mathbb{P}^n}(1)))=c_1(\mathscr{O}_X(\pm 1))$. ☐

4.3. *Partial answers to Question 23*

We conclude the present Section 4 with three partial answers to Question 23: for threefolds, we answer Question 23 in the affirmative. In dimension four and five, we give an affirmative answer for Fano manifolds. In higher dimensions, we can at least describe and restrict the geometry of potential exotic klt varieties homeomorphic to $\mathbb{P}^n$.

Proposition 36 (Topological $\mathbb{P}^n$ with ample canonical bundle). *Let X be a projective klt variety that is homeomorphic to $\mathbb{P}^n$. If K_X is ample, then $r>n+1$.*

Proof. Recall from [22, Thm. 1.1] that X satisfies the $\mathbb{Q}$-Miyaoka–Yau inequality. We have seen in Remark 32 that this implies $r=|r|\geq n+1$, with $r=n+1$ if and only if equality holds in $\mathbb{Q}$-Miyaoka–Yau inequality. In the latter case, recall from [22, Thm. 1.2] that X has no worse than quotient singularities. Since quotient singularities are not topologically smooth, it turns out that X cannot have any singularities at all. By Yau's theorem (or again by [22, Thm. 1.2]), X must then be a smooth ball quotient, contradicting $\pi_1(X)=\pi_1(\mathbb{P}_n)=\{1\}$. ☐

Proposition 37 (Topological $\mathbb{P}^n$ with ample anti-canonical bundle). *Let X be a projective klt variety that is homeomorphic to $\mathbb{P}^n$. If $-K_X$ is ample, then either $X \cong \mathbb{P}^n$ or $\mathscr{T}_X$ is unstable.*

Remark 38. Recall from [28, Cor. 32] that Fano varieties with unstable tangent bundles admit natural sequences of rationally connected foliations. These might be used to study their geometry further. If in the situation of Proposition 37 we additionally assume that the index is one, i.e., that $r = -1$, then $\Omega_X^{[1]}$ is always semistable: if $\mathscr{S} \subsetneq \Omega_X^{[1]}$ was destabilizing, then $\det \mathscr{S} \subseteq \Omega_X^{[\operatorname{rank} \mathscr{S}]}$ is either trivial (hence violating the non-existence of reflexive forms, [45, Thm. 1] and [18, Thm. 5.1]) or ample (hence violating the Bogomolov–Sommese vanishing theorem for klt varieties, [18, Thm. 7.2]).

Proof of Proposition 37. If $\mathscr{T}_X$ is semistable, then the $\mathbb{Q}$-Bogomolov–Gieseker inequality holds, and we have seen in Remark 33 that $-r = |r| > n$. Fujita's singular version of the Kobayashi–Ochiai theorem, [13, Thm. 1], will then apply to show that $X \cong \mathbb{P}^n$. □

While the Bogomolov–Gieseker inequality does not necessarily hold on a Fano variety with unstable tangent sheaf, we still get some restriction on the index from the following result.

Proposition 39. *Let X be a projective klt variety that is homeomorphic to $\mathbb{P}^n$. If $-K_X$ is ample, then $r^2 \geq n+1$. In particular, if $n \geq 4$, then $r \geq 3$.*

Proof. Since X is factorial by Proposition 24 (5) and non-singular in codimension two by Proposition 24 (3), we may apply [36, Cor. 1.5] to obtain the bound $c_2(X) \cdot c_1(\mathscr{O}_X(1))^{n-2} \geq 0$. Then, we conclude by Corollary 31. □

In dimension three we can now fully answer Question 23.

Theorem 40 (Topological $\mathbb{P}^3$). *Let X be a projective klt variety that is homeomorphic to $\mathbb{P}^3$. Then, $X \cong \mathbb{P}^3$.*

Proof. Since X is a threefold with isolated, rational Gorenstein singularities, Riemann–Roch takes a particularly simple form:

$$1 \overset{\text{Prop. 24 (1)}}{=} \chi(\mathscr{O}_X) = \frac{1}{24} \cdot [-K_X] \cdot c_2(X).$$

With Corollary 31, this reads

$$-48 = r \cdot (r^2 - 4).$$

This equation has only one real solution: $r = -4$; in particular, $-K_X$ is ample. As before, Fujita's theorem [13, Thm. 1] applies to show that $X \cong \mathbb{P}^n$. □

Finally, in dimensions four and five we show the following.

Theorem 41 ($\mathbb{Q}$-Fano 4- and 5-folds homeomorphic to projective spaces). *Let X be a projective klt variety homeomorphic to $\mathbb{P}^n$, with $n = 4$ or 5. Assume that K_X is not ample. Then, $X \cong \mathbb{P}^n$.*

Proof. Recall that X is a Gorenstein Fano variety of index $i = -r$, with canonical singularities, smooth in codimension two. By [13, Thm. 1 and 2], we may assume that $i \leq \dim X - 1$. Further, from Proposition 39, we see that $i \geq 3$. These cases have to be excluded.

If $i = \dim X - 1$, then by [14], X is a hypersurface of weighted degree 6 embedded in the smooth part of the weighted projective space $\mathbb{P}(3, 2, 1^n)$. Smooth such hypersurfaces have semistable tangent bundle by [21, Prop. 6.15]; in particular, they satisfy the Bogomolov–Gieseker inequality. Since X is smooth in codimension two, the "principle of conservation of numbers", [15, Thm. 10.2], implies that X satisfies the Bogomolov–Gieseker inequality as well, which in turn contradicts Remark 33.

The remaining case, $n = 5$ and $i = 3$, is ruled out by Corollary 35, which implies that $i = -r$ has to be even. □

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Complex algebraic geometry, in memory of Jean-Pierre Demailly /
Géométrie algébrique complexe, en mémoire de Jean-Pierre Demailly

Deformations over non-commutative base

Déformations sur une base non-commutative

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Dedicated to the Memory of Professor Jean-Pierre Demailly

Abstract. We make some remarks on deformation theory over non-commutative base. We describe the base algebra of semi-universal non-commutative deformations using vector spaces T^1 and T^2.

Résumé. Nous faisons quelques remarques sur la théorie des déformations sur des bases non commutatives. Nous décrivons l'algèbre de base des déformations non commutatives semi-universelles à l'aide des espaces vectoriels T^1 et T^2.

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We will consider deformation theory over non-commutative (NC) base algebras. Such a theory is interesting because there are more deformations than the usual deformations over commutative bases. The deformations over commutative base can possibly be regarded as the "first order" approximation of more general "higher order" deformations. The formal theories of deformations over commutative and non-commutative bases are parallel and the extension to the non-commutative case is simple, but some new phenomena and invariants appear.

We make some remarks on NC deformations. The first remark is that the deformations over NC base is natural. This is because the differential graded algebras (DGA) which govern the deformations of sheaves are naturally non-commutative. Hence it is natural to consider deformations parametrized by NC base algebras. We will also consider the problem of convergence of formal NC deformations and the moduli space. The second remark is that we obtain "higher order invariants" because there are more NC deformations than commutative ones by slightly generalizing results of [12] and [4]. The last remark is that a description of the base algebra using the tangent space T^1 and the obstruction space T^2 is possible.

We use the abbreviation NC for "not necessarily commutative". In Section 1, we recall the definition of NC deformations, and explain how the base algebra of semi-universal NC deformations is described by a minimal A^∞-algebra arising from DGA in the case of deformations of coherent sheaves. In Section 2, we consider the problem of convergence and the existence of moduli space by taking an example of deformations of linear subspaces in a linear space. In

Section 3, we consider another example of flopping contractions of 3-dimensional manifolds, and show how invariants appear beyond those obtained by commutative deformations. We will give a description of the base algebra of the semi-universal NC deformation by using the tangent space and the obstruction space in Section 4.

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1. Multi-pointed non-commutative deformations

We recall the non-commutative deformation theory developed by [9] (see also [3], [6]). We use NC as “not necessarily commutative”. This is a generalization of the formal commutative deformation theory of [10] to the case where the base algebras are allowed to be NC.

Let k^r be the direct product ring of a field k, and let (Art_r) be the category of *augmented associative* k^r-algebras R which are finite dimensional as k-modules and such that the two-sided ideal $M = \mathrm{Ker}(R \to k^r)$ is nilpotent. We assume that the composition of the structure homomorphisms $k^r \to R \to k^r$ is the identity. (Art_r) is the category of the base spaces for r-*pointed* NC deformations.

Let $k_i \cong k$ be the i-th direct factor of the product ring k^r for $1 \le i \le r$. k_i is generated by $e_i = (0,\dots,1,\dots,0) \in k^r$, where 1 is placed at the i-th entry. A left k^r-module F has a direct sum decomposition $F = \bigoplus_{i=1}^r F_i$ as k-modules by $F_i = e_i F$, and k^r-bimodule has a further decomposition $F = \bigoplus_{i,j=1}^r F_{ij}$ by $F_{ij} = e_i F e_j$.

$R \in (\mathrm{Art}_r)$ is an NC Artin semi-local algebra with maximal two-sided ideals $M_i = \mathrm{Ker}(R \to k_i)$. NC deformation is *multi-pointed* because an NC semi-local algebra is not necessarily a direct product of local algebras unlike the case of a commutative algebra.

The model case is a deformation of a direct sum of coherent sheaves $F = \bigoplus_{i=1}^r F_i$ (r-pointed sheaf). The sheaves F_i interact each other and there are more NC deformations of F than those of the individual sheaves F_i.

Let F be something defined over k^r which will be deformed over $R \in (\mathrm{Art}_r)$. An *NC deformation* of F over R is a pair $(\tilde{F}, \phi)$ where $\tilde{F}$ is “flat” over R and $\phi : F \to R/M \otimes_R \tilde{F}$ is an isomorphism. The definition depends on the cases what kind of F we are considering. The set of isomorphism classes of deformations of F over R gives an *NC deformation functor* $\Phi = \mathrm{Def}_F : (\mathrm{Art}_r) \to (\mathrm{Set})$.

More concretely, an r-*pointed NC deformation functor* $\Phi : (\mathrm{Art}_r) \to (\mathrm{Set})$ in this paper is a covariant functor which satisfies the conditions $(H_0), (H_f), (H_e), (\tilde{H})$ stated below following [10] (see also [11, Chapter 2]).

We define an object $R_e \in (\mathrm{Art}_r)$ as a generalization of the ring of dual numbers $k[\epsilon]/(\epsilon^2)$. Let R_e be the trivial extension $k^r \oplus \mathrm{End}(k^r)$, where $\mathrm{End}(k^r)$ is a square zero two-sided ideal, and the multiplication of k^r and $\mathrm{End}(k^r)$ is induced from the embedding to diagonal matrices $k^r \to \mathrm{End}(k^r)$. As a k-module,

$$R_e = k^r \oplus \bigoplus_{i,j=1}^r k e_{ij}.$$

The multiplication is defined by $e_i e_{jk} = \delta_{ij} e_{jk}$, $e_{ij} e_k = \delta_{jk} e_{ij}$ and $e_{ij} e_{kl} = 0$ for all i,j,k,l. The augmentation $R_e \to k^r$ is given by $e_{ij} \mapsto 0$.

Now we state the conditions $(H_0),(H_f),(H_e),(\widetilde{H})$. For ring homomorphisms $R' \to R$ and $R'' \to R$ in (Art_r), let $\alpha : \Phi(R' \times_R R'') \to \Phi(R') \times_{\Phi(R)} \Phi(R'')$ be the map naturally defined by Φ.

(H_0) $\Phi(k^r)$ consists of one element.
(H_f) $\Phi(R_e)$ is finite dimensional as a k-module.
(H_e) The natural map α is bijective if $R = k^r$ and $R'' = R_e$.
$(\widetilde{H})$ The natural map α is surjective if $R'' \to R$ is surjective.

The *tangent space* T^1 of the functor Φ is defined by $T^1 = \Phi(R_e)$. The k^r-bimodule structure of the ideal $\mathrm{End}(k^r) \subset R_e$ induces a k^r-bimodule structure on T^1, so we can write $T^1 = \bigoplus_{i,j=1}^r T^1_{ij}$. We have $T^1_{ij} = \Phi(k^r \oplus ke_{ij})$. Indeed $T^1_{ij} = e_i T^1 e_j = \Phi(e_i R_e e_j) = \Phi(k^r \oplus ke_{ij})$.

An element $\xi \in \Phi(R)$ for $R \in (\mathrm{Art}_r)$ is called an *r-pointed NC deformation* over R of the unique element of $\Phi(k^r)$.

Let $T_R = (M/M^2)^*$ be the Zariski tangent space of R. It is a k^r-bimodule. The *Kodaira–Spencer map* $KS_\xi : T_R \to T^1$ associated to the deformation ξ is defined as follows. A tangent vector $v \in (T_R)_{ij} = (M/M^2)^*_{ij}$ induces a ring homomorphism $v_* : R \to k^r \oplus ke_{ij}$, hence $\Phi(v_*) : \Phi(R) \to \Phi(k^r \oplus ke_{ij}) = T^1_{ij}$. Then we define $KS_\xi(v) = \Phi(v_*)(\xi)$.

Let $\widehat{R} := \varprojlim R_i \in (\widehat{\mathrm{Art}}_r)$ be a pro-object of (Art_r), and let $\widehat{\xi} := \varprojlim \xi_i \in \widehat{\Phi}(\widehat{R}) := \varprojlim \Phi(R_i)$ be an element of a projective limit. Then $\widehat{\xi}$ is called a *formal r-pointed NC deformation* over $\widehat{R}$. The Kodaira–Spencer map $KS_{\widehat{\xi}} : T_{\widehat{R}} \to T^1$ is similarly defined.

A formal deformation $\widehat{\xi} \in \widehat{\Phi}(\widehat{R})$ is called a *versal* NC deformation if the following holds: for any NC deformation $\xi' \in \Phi(R')$, there exists a morphism $h : \widehat{R} \to R'$ such that $\xi' = \widehat{\Phi}(h)(\widehat{\xi})$.

In this case, the Kodaira–Spencer map $KS_{\widehat{\xi}} : T_{\widehat{R}} \to T^1$ is surjective. Indeed, let $v' \in T^1_{ij} = \Phi(k^r \oplus ke_{ij})$ be any element. Then there is a morphism $h : \widehat{R} \to k^r \oplus ke_{ij}$ such that $v' = \widehat{\Phi}(h)(\widehat{\xi})$. Let $v : (\widehat{M}/\widehat{M}^2)_{ij} \to ke_{ij}$ be the homomorphism induced from h. Then $v_* = h$ and $KS_{\widehat{\xi}}(v) = v'$.

A versal NC deformation is said to be *semi-universal* if the Kodaira–Spencer map is bijective. In this case, we have $\widehat{M}/\widehat{M}^2 \cong (T^1)^*$. We note that it is called “versal” in some literatures. The existence of the semi-universal NC deformation is proved in a similar way to [10] from the conditions $(H_0),(H_f),(H_e),(\widetilde{H})$.

In the case $r = 1$, if we take the abelianization $\widehat{R}^{ab} = \widehat{R}/[\widehat{R},\widehat{R}]$ of the base ring of the semi-universal deformation, then we obtain a usual semi-universal commutative deformation $\widehat{\xi}^{ab}$ over $\widehat{R}^{ab}$ given by $\widehat{\xi}^{ab} = \Phi(q)(\widehat{\xi})$, where $q : \widehat{R} \to \widehat{R}^{ab}$ is the quotient map.

We recall a description of the semi-universal NC deformation in the case of deformations of a coherent sheaf using an A^∞-algebra formalism ([8]). Let X be an algebraic variety over k and let $F = \bigoplus_{i=1}^r F_i$ be a coherent sheaf with proper support. Then the infinitesimal deformations of F are controlled by a *differential graded algebra (DGA)* $\mathrm{RHom}_X(F,F)$. The tangent space and the obstruction space are given by k^r-bimodules $T^i = \mathrm{Ext}^i_X(F,F)$ for $i = 1,2$ (cf. Section 4).

It is also controlled by an A^∞-algebra structure $\{m_d\}_{d\ge 2}$ of the cohomology group $A = \bigoplus_{p\ge 0} A_p := \bigoplus_{p\ge 0} \mathrm{Ext}^p(F,F) = \bigoplus_{p,i,j} \mathrm{Ext}^p(F_i,F_j)$;

$$m_d : T^d_{k^r} A := A \otimes_{k^r} \cdots \otimes_{k^r} A \longrightarrow A(2-d)$$

are the higher multiplications of degree $2-d$, where the left hand side is a tensor product with d factors over k^r and the right hand side has degree shift $2-d$. In particular, we have

$$m_d : T^d_{k^r} A_1 := A_1 \otimes_{k^r} \cdots \otimes_{k^r} A_1 \longrightarrow A_2$$

for $d \ge 2$.

In general, for a k^r-bimodule E, we have $E = \bigoplus_{i,j=1}^r E_{ij}$ with $E_{ij} = e_i E e_j$. We define a completed tensor algebra $\widehat{T}_{k^r} E = \prod_{d\ge 0} T^d_{k^r} E$ by

$$T^d_{k^r} E = E \otimes_{k^r} E \otimes_{k^r} \cdots \otimes_{k^r} E$$

where there are d-times E on the right hand side. We apply this construction to $E = (T^1)^*$. If $\{x^s_{ij}\}_s$ is a basis of E_{ij}, then we have

$$\widehat{T}_{k^r} E = k^r \langle\langle x^s_{ij} \rangle\rangle / (e_i x^s_{i'j}, x^s_{ij} e_{j'}, x^s_{i'j'} x^{s'}_{i''j''} \mid i \neq i', j \neq j', j' \neq i'').$$

Thus the set of monomials

$$x^{s_1}_{i_0 i_1} x^{s_2}_{i_1 i_2} \dots x^{s_d}_{i_{d-1} i_d}$$

with $i = i_0$ and $j = i_d$ is a k-basis of $(\widehat{T}_{k^r} E)_{ij}$.

Let

$$m^* = \sum_{d \geq 2} m^*_d : \mathrm{Ext}^2(F,F)^* \longrightarrow \widehat{T}_{k^r}(\mathrm{Ext}^1(F,F)^*)$$

be the formal sum of dual maps of m_d. Then the base algebra $\widehat{R}$ of the semi-universal NC deformation $\widehat{F}$ is determined as an augmented k^r-algebra to be

$$\widehat{R} = \widehat{T}_{k^r}(\mathrm{Ext}^1(X,X)^*)/(m^*(\mathrm{Ext}^2(X,X)^*))$$

([8]). Thus the Taylor coefficients of the equations of the formal NC moduli space are determined by A^∞-multiplications.

There is another way of describing a semi-universal r-pointed NC deformation of a direct sum of coherent sheaves with proper support $F = \bigoplus_{i=1}^r F_i$. The semi-universal NC deformation $\widehat{F}$ of F is given by a tower $\{F^{(n)}\}$ of universal extensions (cf. [6]):

$$0 \longrightarrow \mathrm{Ext}^1(F^{(n)},F)^* \otimes_{k^r} F \longrightarrow F^{(n+1)} \longrightarrow F^{(n)} \longrightarrow 0$$

with $F^{(0)} = F$ and $\widehat{F} = \varprojlim F^{(n)}$. We have direct sum decompositions $F^{(n)} = \bigoplus_i F^{(n)}_i$, and we can write

$$0 \longrightarrow \bigoplus_{i,j} \mathrm{Ext}^1(F^{(n)}_i, F_j)^* \otimes_k F_j \longrightarrow \bigoplus_i F^{(n+1)}_i \longrightarrow \bigoplus_i F^{(n)}_i \longrightarrow 0.$$

If $\mathrm{End}(F) \cong k^r$, i.e., if $\mathrm{End}(F_i) \cong k$ and $\mathrm{Hom}(F_i,F_j) \cong 0$ for $i \neq j$, then F is called a *simple collection* ([6]). The deformation theory of a simple collection is particularly nice. In this case, $F^{(n)}$ is flat over $R^{(n)} = \mathrm{End}(F^{(n)})$, and the parameter algebra $\widehat{R}$ of the semi-universal deformation $\widehat{F}$ is given by $\widehat{R} = \varprojlim R^{(n)}$ ([6, Theorem 4.8]).

Remarks 1.

(1) We do not consider deformation theory of *varieties* over non-commutative base in this paper, because such a theory seems to be difficult by the following reason. Suppose that there is an infinitesimal deformation X_R of a variety X over an NC ring R. Then the structure sheaf $\mathcal{O}_{X_R}$ should be NC too. When we consider a base change over a ring homomorphism $R \to R'$, it seems necessary that the base rings should be commutative in order for the tensor product $\mathcal{O}_{X_R} \otimes_R R'$ to have a ring structure. Indeed the DGLie algebra which controls the deformations of X is NC but its non-commutativity is restricted.

But when X is a subvariety of an ambient variety Y, then we can consider a deformation of X inside Y over an NC base as a deformation of the structure sheaf $\mathcal{O}_X$ as a sheaf on Y (see Section 2).

(2) The deformation functor is pro-representable when there is a universal deformation. But a universal deformation does not exist in general (see [6, Remark 4.10]).

2. Convergence and moduli

The above described semi-universal NC deformation is a formal deformation, and the question on the convergence is important. We will make some remarks on the convergence of the formal NC deformations and the relationship with the moduli space of commutative deformations. We consider only 1-pointed NC deformations, and we take an example of the moduli space of linear subspaces in a fixed linear space. We consider NC deformations of the structure sheaves of linear subspaces.

We would like to say that the formal semi-universal NC deformation is convergent if the corresponding semi-universal commutative deformation is convergent. This is because the numbers of commutative monomials and non-commutative ones on n variables of degree d grow similarly to n^d. Maybe we should require that the growth of the Taylor coefficients of the non-commutative power series are bounded in a similar way as the commutative power series.

Any k-algebra homomorphism $R \to k$ for any associative k-algebra R factors through the abelianization $R \to R^{ab}$. Therefore we can think that the set of closed points of the moduli spaces are the same for commutative and NC deformation problems. In other words, when we observe points, then the moduli space of NC deformations is reduced to the usual moduli space. We can say that the NC deformations give an additional infinitesimal or formal structure at each point of the commutative moduli space. And the formal structure is usually convergent. However, a compactification is another problem, and it seems that it does not exists.

As an example, we consider NC deformations of linear subspaces in a finite dimensional vector space. As explained in Remark 1.1, we consider the NC deformations of the structure sheaf of the subspace instead of the subspace as a variety. The following is a slight generalization of [8, Example 7.8]. The commutative deformations are unobstructed and yield a compact moduli space, a Grassmann variety. But we will see that NC deformations are obstructed.

Let $V \cong k^n$ be an n-dimensional linear space with coordinate linear functions $x_1, \dots, x_n$, and let W be an m-dimensional linear subspace defined by an ideal $I = (x_{m+1}, \dots, x_n)$. The commutative moduli space $G(m,n)$ has an affine open subset $\mathrm{Hom}(W, V/W) \cong k^{m(n-m)}$ with coordinates $a_{i,j}$ ($1 \le i \le m$, $m+1 \le j \le n$). We consider NC deformations of W as a linear subspace of V, i.e., the NC deformations of the ideal sheaves generated by linear functions.

Proposition 2. *Let $V \cong k^n$ with coordinate linear functions $x_1, \dots, x_n$, and let $W \cong k^m$ be defined by $x_{m+1} = \cdots = x_n = 0$. Then the formal semi-universal NC deformation of W as a linear subspace of V has the parameter algebra $\hat{R}$ and the ideal $\hat{I}$ given as follows:*

$$\begin{aligned}
&\hat{R} = k\langle\langle a_{ij} \mid 1 \le i \le m < j \le n \rangle\rangle / \hat{J} \\
&\hat{J} = \big(a_{ij_1}a_{ij_2} - a_{ij_2}a_{ij_1},\ a_{i_1j_1}a_{i_2j_2} - a_{i_2j_2}a_{i_1j_1} + a_{i_1j_2}a_{i_2j_1} - a_{i_2j_1}a_{i_1j_2} \\
&\qquad\qquad \big|\, 0 \le i \le m, 1 \le i_1 < i_2 \le m < j_1 < j_2 \le n\big) \\
&\hat{I} = \left(x_j + \sum_{i=1}^m a_{ij}x_i \,\middle|\, m+1 \le j \le n\right).
\end{aligned}$$

Proof. This is almost the same as [8, Example 7.8]. Let $Y = \mathbf{P}(W^*) \subset X = \mathbf{P}(V^*)$ be the corresponding projective spaces. We consider NC deformations of a coherent sheaf $F = \mathcal{O}_Y$ on X. The normal bundle of Y in X is given by $N_{Y/X} \cong \mathcal{O}_Y(1)^{\oplus n-m}$. Hence $T^1 = \mathrm{Ext}^1(F,F) \cong H^0(Y, N_{Y/X}) \cong k^{\oplus m(n-m)}$ and $T^2 = \mathrm{Ext}^2(F,F) \cong H^0(Y, \bigwedge^2 N_{Y/X}) \cong k^{\oplus \binom{m+1}{2}\binom{n-m}{2}}$.

Let $I' = \mathcal{O}_X(-Y)$ be the ideal sheaf of $Y \subset X$ generated by the homogeneous coordinates $x_{m+1}, \dots, x_n$. By [8, Lemma 7.6], the semi-universal NC deformation of F is given in the form

$$\hat{F} = \varprojlim (R_n \otimes \mathcal{O}_X)/I'_n$$

where $(R_n, M_n) \in (\mathrm{Art}_1)$ such that $M_n^{n+1} = 0$. By the flatness, the ideal sheaf I'_n is generated by linear forms $x_j + \sum_{i=1}^m a_{ij}x_i$ for $m+1 \le j \le n$, where $a_{ij} \in M_n$.

Since the x_i are commutative variables in $R_n \otimes \mathcal{O}_X$, we have $x_j x_l = x_l x_j$ for $m+1 \le j, l \le n$. Hence equalities

$$\sum_{i,k=1}^m a_{ij}a_{kl}x_i x_k = \sum_{i,k=1}^m a_{kl}a_{ij}x_i x_k$$

hold in $F_n = (R_n \otimes \mathcal{O}_X)/I'_n$ for such j, l. It follows that

$$\begin{aligned} a_{ij}a_{il} - a_{il}a_{ij} &= 0 \quad (1 \le i \le m < j < l \le n), \\ a_{ij}a_{kl} - a_{kl}a_{ij} + a_{kj}a_{il} - a_{il}a_{kj} &= 0 \quad (1 \le i < k \le m < j < l \le n) \end{aligned}$$

in $\widehat{R} = \varprojlim R_n$. The above relations are non-commutative polynomials which are linearly independent quadratic forms, and their number is equal to

$$m\binom{n-m}{2} + \binom{m}{2}\binom{n-m}{2} = \binom{m+1}{2}\binom{n-m}{2}.$$

This is equal to the dimension of the obstruction space. Therefore there are no more independent relations contained in $\widehat{J}$. □

The above deformation is "algebraizable". There is an NC deformation of ideals $\widetilde{I}$ over a parameter algebra $\widetilde{R}$ which is a quotient algebra of an NC polynomial algebra:

$$\begin{aligned} &\widetilde{R} = k\langle a_{ij} \mid 1 \le i \le m < j \le n\rangle / \widetilde{J} \\ &\widetilde{J} = \big(a_{ij_1}a_{ij_2} - a_{ij_2}a_{ij_1},\ a_{i_1j_1}a_{i_2j_2} - a_{i_2j_2}a_{i_1j_1} + a_{i_1j_2}a_{i_2j_1} - a_{i_2j_1}a_{i_1j_2} \\ &\qquad\qquad \big| \ 1 \le i \le m, 1 \le i_1 < i_2 \le m < j_1 < j_2 \le n\big) \\ &\widetilde{I} = \left(x_j + \sum_{i=1}^m a_{ij}x_i \ \middle|\ m+1 \le j \le n\right) \end{aligned}$$

The meaning of this formula is that it induces a semi-universal NC deformation at every closed point of an affine open subset $\mathrm{Spec}(\widetilde{R}^{ab}) \subset G(m+1, n+1)$ with $\widetilde{R}^{ab} = k[a_{ij} \mid 0 \le i \le m < j \le n]$. Indeed we have

$$(a_{ij} - a^0_{ij})(b_{kl} - b^0_{kl}) - (b_{kl} - b^0_{kl})(a_{ij} - a^0_{ij}) = a_{ij}b_{kl} - b_{kl}a_{ij}$$

for NC variables a_{ij}, b_{kl} and $a^0_{ij}, b^0_{kl} \in k$.

Hilbert schemes and Quot schemes are constructed from Grassmann varieties. We wonder if their NC deformations are also semi-globalizable.

Examples 3.

(1) $n = 3$ and $m = 1$. We have $G(1,3) \cong \mathbf{P}^2$. Then $\widetilde{R} \cong k\langle a, b\rangle/(ab - ba) = k[a, b]$.

(2) $n = 3$ and $m = 2$. We have $G(2,3) \cong \mathbf{P}^2$. Then $\widetilde{R} = k\langle a, b\rangle$ is not Noetherian. Indeed a two-sided ideal $(ab^k a \mid k > 0)$ is not finitely generated.

$\widetilde{R}$ has a following quotient algebra, which corresponds to an NC deformation which is not semi-universal:

$$R_\epsilon = k\langle a, b\rangle/(ab - ba - \epsilon)$$

where $\epsilon \in k$. For example, if $\epsilon = 1$, then $R_1 \cong k[t, d/dt]$.

(3) $n = 4$ and $m = 2$. We have $G(2,4)$. Then we have

$$\widetilde{R} = k\langle a, b, c, d\rangle/(ab - ba,\ cd - dc,\ ad - da - bc + cb).$$

$\widetilde{R}$ has a following quotient algebra:

$$R_{\epsilon_1,\epsilon_2} = k\langle a, b, c, d\rangle/(ab - ba,\ cd - dc,\ ad - da - 1,\ bc - cb - 1,\ ac - ca - \epsilon_1,\ bd - db - \epsilon_2)$$

where $\epsilon_i \in k$. For example, if $\epsilon_i = 0$, then $R_1 \cong k[t_1, t_2, \partial/\partial t_1, \partial/\partial t_2]$.

3. Flopping contractions of 3-folds

As a typical example of multi-pointed NC deformations, we will consider NC deformations of exceptional curves of a flopping contraction from a smooth 3-fold $f : Y \to X$ over $k = \mathbf{C}$. [2] observed that there are more NC deformations than commutative ones, and the base algebra of NC deformations gives an important invariant of the flopping contraction called the contraction algebra. Indeed Donovan and Wemyss conjectured that the contraction algebra, which is a finite dimensional associative algebra, determines the complex analytic type of the singularity of X. [12] and [4] proved that the dimension count of the contraction algebra yields Gopakumar-Vafa invariants of rational curves defined in [5]. We will consider slight generalizations where there are more than one exceptional curves.

Let $f : Y \to X = \mathrm{Spec}(B)$ be a projective birational morphism defined over $k = \mathbf{C}$ from a smooth 3-dimensional variety Y whose exceptional locus C is 1-dimensional. Let $C = \bigcup_{i=1}^r C_i$ be a decomposition into irreducible components. We assume that f is crepant, i.e., $(K_Y, C_i) = 0$ for all i. It is known that $C_i \cong \mathbf{P}^1$, the dual graph of the C_i is a tree, and X has only isolated hypersurface singularities of multiplicity 2.

The *contraction algebra* R for f is defined to be the base algebra of the semi-universal r-pointed NC deformation of the sheaf $F = \bigoplus_{i=1}^r \mathscr{O}_{C_i}(-1)$.

We consider commutative one parameter deformation of the contraction morphism $f : Y \to X$, and investigate the behavior of the contraction algebras under deformation. Let $p : \mathscr{X} \to \Delta$ be a one parameter flat deformation of X over a disk Δ, and assume that there is a flat deformation $\tilde{f} : \mathscr{Y} \to \mathscr{X}$ of the flopping contraction $f : Y \to X$. We assume that there are Cartier divisors $\mathscr{L}_1, \dots, \mathscr{L}_r$ on $\mathscr{Y}$ such that $(\mathscr{L}_i, C_j) = \delta_{i,j}$. This is always achieved when we replace X by its complex analytic germ containing $f(C)$ and Δ by a smaller disk.

Let $C^t = \bigcup_{j=1}^{s_t} C_j^t$ be the exceptional curves with decomposition to irreducible components for the flopping contraction $f_t : Y_t \to X_t$ for $t \ne 0$, where $Y_t = (p\tilde{f})^{-1}(t)$ and $X_t = p^{-1}(t)$. It is not necessarily connected even if C is connected. We may assume that $s = s_t$ is constant on $t \ne 0$. We define integers $m_{j,i}$ by the degeneration of 1-cycles $C_j^t \to \sum m_{j,i} C_i$ when $t \to 0$. This means that $\mathscr{O}_{C_j^t}$ degenerates in a flat family to $\mathscr{O}_{\sum_i m_{j,i} C_i}$. We have $(\mathscr{L}_i, C_j^t) = m_{j,i}$.

If the deformation $\tilde{f}$ is generic, then C^t is a disjoint union of $(-1,-1)$-curves, i.e., smooth rational curves whose normal bundles are isomorphic to $\mathscr{O}_{\mathbf{P}^1}(-1)^{\oplus 2}$. In this case, we denote

$$m_j = \sum_i m_{j,i}, \quad n_d = \#\{j \mid m_j = d\}.$$

The numbers n_d should be called the *Gopakumar-Vafa invariants* ([5] for the case $r = 1$). In the case $r = 1$, [12] proved that n_1 is equal to the dimension of the abelianization of the contraction algebra $n_1 = \dim R^{ab}$, while higher terms n_d for $d \ge 2$ contribute to $\dim R$ (see Theorem 4 (3)).

We consider NC deformations of $F = \bigoplus_{i=1}^r F_i$ for $F_i = \mathscr{O}_{C_i}(-1)$ on Y and $\mathscr{Y}$. The set $\{F_i\}$ is called a *simple collection* on Y and $\mathscr{Y}$ in the terminology of [6] in the sense that $\mathrm{Hom}_Y(F,F) \cong \mathrm{Hom}_{\mathscr{Y}}(F,F) \cong k^r$. The NC deformations of a simple collection behave particularly nice.

Let $\hat{\Delta} = \mathrm{Spec}(k[\![t]\!])$ be the completion of Δ at the origin. By the flat base change $\hat{\Delta} \to \Delta$, we define $\widehat{\mathscr{X}} = \mathscr{X} \times_\Delta \hat{\Delta}$ and $\widehat{\mathscr{Y}} = \mathscr{Y} \times_\Delta \hat{\Delta}$. Let $\hat{f} : \widehat{\mathscr{Y}} \to \widehat{\mathscr{X}}$ and $\hat{p} : \widehat{\mathscr{X}} \to \hat{\Delta}$ be natural morphisms.

Let $\widehat{\mathscr{F}} = \bigoplus_{i=1}^r \mathscr{F}_i$ and $\tilde{F}^0 = \bigoplus_{i=1}^r \tilde{F}_i^0$ be the semi-universal NC deformations of F on $\widehat{\mathscr{Y}}$ and Y, respectively, and let $\widehat{\mathscr{R}}$ and R be the base algebras of these semi-universal deformations. We note that $\tilde{F}^0$ is obtained by finite number of extensions of the F_i while $\widehat{\mathscr{F}}$ may not. This is because C is isolated in Y while C may move inside $\mathscr{Y}$. Hence we have $\dim R < \infty$ as k-modules. We will see that $\dim \widehat{\mathscr{R}} = \infty$ (see Theorem 4 (1)).

$\widehat{\mathscr{F}}$ is also a semi-universal NC deformation of F on $\mathscr{Y}$. We will see that there is also a "convergent version" $\mathscr{F}$ on $\mathscr{Y}$, and $\widehat{\mathscr{F}}$ is its completion.

By [6, Theorem 4.8], the base algebras coincide with the endomorphism algebras:

$$\widehat{\mathscr{R}} = \mathrm{End}_{\hat{\mathscr{Y}}}(\widehat{\mathscr{F}}), \quad R = \mathrm{End}_Y(\widetilde{F}).$$

$\widehat{\mathscr{F}}$ and $\widetilde{F}^0$ can be described explicitly in the following way ([2, 6, 7]). In particular, there exists a sheaf $\mathscr{F}$ on $\mathscr{Y}$ such that

$$\widehat{\mathscr{F}} \cong \mathscr{F} \otimes_{\mathscr{O}_{\mathscr{Y}}} \mathscr{O}_{\hat{\mathscr{Y}}} \tag{1}$$

i.e., the semi-universal NC deformation $\widehat{\mathscr{F}}$ is convergent when we replace Δ by a smaller disk if necessary.

By [13], we construct extensions of locally free sheaves on $\mathscr{Y}$:

$$0 \longrightarrow \mathscr{O}_{\mathscr{Y}}^{s_i} \longrightarrow M_i \longrightarrow \mathscr{L}_i \longrightarrow 0$$

with some integers s_i such that $R^1\widetilde{f}_*M_i^* = 0$, where M_i^* is the dual sheaf. Let $M = \bigoplus_{i=1}^r M_i$ and $M^0 = M \otimes_{\mathscr{O}_{\mathscr{Y}}} \mathscr{O}_Y$. We also denote $\widehat{M} = M \otimes_{\mathscr{O}_{\mathscr{Y}}} \mathscr{O}_{\hat{\mathscr{Y}}}$. We have an exact sequence

$$0 \longrightarrow M^* \longrightarrow M^* \longrightarrow (M^0)^* \longrightarrow 0.$$

Since the dimensions of fibers of $\widetilde{f}$ are at most 1, we obtain $R^1 f_*(M^0)^* = 0$ from $R^1\widetilde{f}_*M_i^* = 0$. Then semi-universal NC deformations $\widehat{\mathscr{F}} = \bigoplus \widehat{\mathscr{F}}_i$ and $\widetilde{F}^0$ are given as the kernels of natural homomorphisms ([7, Theorem 1.2]):

$$0 \longrightarrow \widehat{\mathscr{F}} \longrightarrow \hat{f}^*\hat{f}_*\widehat{M} \longrightarrow \widehat{M} \longrightarrow 0,$$
$$0 \longrightarrow \widetilde{F}^0 \longrightarrow f^*f_*M^0 \longrightarrow M^0 \longrightarrow 0.$$

We define $\mathscr{F}$ by an exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \widetilde{f}^*\widetilde{f}_*M \longrightarrow M \longrightarrow 0$$

and let $\mathscr{R} = \mathrm{End}_{\mathscr{Y}}(\mathscr{F})$. By the flat base change, we obtain (1) and

$$\widehat{\mathscr{R}} \cong \mathscr{R} \otimes_{\mathscr{O}_{\mathscr{Y}}} \mathscr{O}_{\hat{\mathscr{Y}}}.$$

We denote $\widetilde{F}^t = \mathscr{F} \otimes_{\mathscr{O}_{\mathscr{Y}}} \mathscr{O}_{Y_t}$ and $R^t = \mathscr{R} \otimes_{\mathscr{O}_\Delta} k_t$, where $Y_t = (p\widetilde{f})^{-1}(t)$ and k_t is the residue field at $t \in \Delta$.

The following is a slight generalization of results in [4] and [12]:

Theorem 4.

(1) *$\mathscr{F}$ is flat over Δ, and $\widetilde{F}^0 = \mathscr{F} \otimes_{\mathscr{O}_{\mathscr{Y}}} \mathscr{O}_Y$.*
(2) *([4, Conjecture 4.3]). $\mathscr{R}$ is a flat $\mathscr{O}_\Delta$-module, and $R \cong \mathscr{R} \otimes_{\mathscr{O}_\Delta} k$, where k is the residue field of $\mathscr{O}_\Delta$ at 0.*
(3) *Assume in addition that C^t is a disjoint union of $(-1,-1)$-curves C_j^t for $t \neq 0$. Then*

$$\widetilde{F}^t \cong \bigoplus_j \mathscr{O}_{C_j^t}(-1)^{m_j},$$
$$R^t \cong \prod_j \mathrm{Mat}(m_j \times m_j),$$
$$\dim R = \sum_j m_j^2 = \sum_d n_d d^2.$$

Proof. (1). We have an exact sequence

$$0 \longrightarrow M \longrightarrow M \longrightarrow M^0 \longrightarrow 0$$

where the first arrow is the multiplication by t. Because $R^1\widetilde{f}_*M = 0$, there is an exact sequence

$$0 \longrightarrow \widetilde{f}_*M \longrightarrow \widetilde{f}_*M \longrightarrow \widetilde{f}_*M^0 \longrightarrow 0.$$

Because $L_1\widetilde{f}^*\widetilde{f}_*M^0 = 0$ by [1] Lemma 3.4, we obtain the first row of the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{f}^*\widetilde{f}_*M \arrow[r] \arrow[d] & \widetilde{f}^*\widetilde{f}_*M \arrow[r] \arrow[d] & \widetilde{f}^*\widetilde{f}_*M^0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r] & M \arrow[r] & M^0 \arrow[r] & 0. \end{tikzcd}$$

By snake lemma, we obtain

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \widetilde{F}^0 \longrightarrow 0$$

hence the flatness.

(2). Since $t : \mathscr{F} \to \mathscr{F}$ is injective, $\mathscr{R}$ has no t-torsion. Thus it is sufficient to prove that the natural homomorphism $\mathrm{Hom}_{\mathscr{Y}}(\mathscr{F},\mathscr{F}) \to \mathrm{Hom}_Y(\widetilde{F}^0,\widetilde{F}^0)$ is surjective. By the flat base change, it is also sufficient to prove that $\mathrm{Hom}_{\widehat{\mathscr{Y}}}(\widehat{\mathscr{F}},\widehat{\mathscr{F}}) \to \mathrm{Hom}_Y(\widetilde{F}^0,\widetilde{F}^0)$ is surjective, i.e., $\widehat{\mathscr{R}} \to R$ is surjective. Then the assertion follows from the fact that $\widehat{\mathscr{R}}$ and R are the base algebras of NC semi-universal deformations of the same sheaf F with $Y \subset \mathscr{Y}$.

(3). This is proved in [4] and [12] when $r = 1$. Let $x_j^t = \widetilde{f}(C_j^t) \in X_t = p^{-1}(t)$ for $t \ne 0$. Since C_j^t is a $(-1,-1)$-curve, x_j^t is an ordinary double point on a 3-fold. We take a small complex analytic neighborhood $x_j^t \in U_j^t \subset X_t$, and let $V_j^t = \widetilde{f}^{-1}(U_j^t)$.

Let L_j^t be a Cartier divisor on V_j^t such that $(L_j^t, C_j^t) = 1$. We know that $(\mathscr{L}_i, C_j^t) = m_{j,i}$ and $R^1\widetilde{f}_*M_i^* = 0$. Since $C_j^t \cong \mathbf{P}^1$ and M_i is relatively generated, $M_i|_{V_j^t}$ is a direct sum of line bundles whose degrees are non-negative but at most 1. Since the total degree is equal to $m_{j,i}$, it follows that $M_i|_{V_j^t} = (L_j^t)^{\oplus m_{j,i}} \oplus \mathscr{O}_{V_j^t}^{\oplus(\mathrm{rank}(M_i)-m_{j,i})}$.

We will prove that $\mathrm{Ker}(\widetilde{f}^*\widetilde{f}_*L_j^t \to L_j^t) \cong \mathscr{O}_{C_j^t}(-1)$. Indeed there is a commutative diagram

$$\begin{tikzcd} & \widetilde{f}^*\widetilde{f}_*(L_j^t)^* \arrow[r] \arrow[d, "h_1"'] & \mathscr{O}_{V_j^t}^{\oplus 2} \arrow[r] \arrow[d, "\cong"'] & \widetilde{f}^*\widetilde{f}_*L_j^t \arrow[r] \arrow[d, "h_2"'] & 0 \\ 0 \arrow[r] & (L_j^t)^* \arrow[r] & \mathscr{O}_{V_j^t}^{\oplus 2} \arrow[r] & L_j^t \arrow[r] & 0. \end{tikzcd}$$

Hence $\mathrm{Ker}(h_2) \cong \mathrm{Coker}(h_1)$. Since $(L_j^t)^* \otimes_{\mathscr{O}_{V_j^t}} I_{C_j^t}$ for the ideal sheaf $I_{C_j^t}$ of $C_j^t \subset V_j^t$ is generated by global sections, we have $\mathrm{Coker}(h_1) \cong (L_j^t)^* \otimes \mathscr{O}_{C_j^t} \cong \mathscr{O}_{C_j^t}(-1)$.

Therefore $\mathscr{F}_i|_{V_j^t} = \mathscr{O}_{C_j^t}(-1)^{\oplus m_{j,i}}$. Hence $\mathscr{F}|_{V_j^t} = \mathscr{O}_{C_j^t}(-1)^{\oplus m_j}$, and $\widetilde{F}^t \cong \bigoplus_j \mathscr{O}_{C_j^t}(-1)^{m_j}$. Thus $\mathrm{End}_{Y_t}(\widetilde{F}^t) \cong \prod_j \mathrm{Mat}(m_j \times m_j)$, and the assertion is proved. □

4. Abstract description using T^1 and T^2

We will describe the base algebra of the semi-universal NC deformation of a deformation functor Φ which has the tangent space T^1 and the obstruction space T^2, which is defined below.

Let $\Phi : (\mathrm{Art}_r) \to (\mathrm{Set})$ be an NC deformation functor which has a formal semi-universal deformation $\hat{\xi} \in \hat{\Phi}(\hat{R})$. A k^r-bimodule $T^2 = \bigoplus_{i,j=1}^r T_{ij}^2$ is said to be the *obstruction space* if the following condition is satisfied. Let $\xi \in \Phi(R)$ be an NC deformation over $(R,M) \in (\mathrm{Art}_r)$, and let $(R',M') \in (\mathrm{Art}_r)$ be an extension of R by a two-sided ideal J:

$$0 \longrightarrow J \longrightarrow R' \longrightarrow R \longrightarrow 0$$

such that $M'J = 0$, so that J is a left k^r-module. Then there is an obstruction class $o_\xi \in T^2 \otimes_{k^r} J$ such that ξ extends to an NC deformation $\xi' \in \Phi(R')$ if and only if $o_\xi = 0$.

We assume that the obstruction class is functorial in the following sense. Let

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d, "g"] & R' \arrow[r] \arrow[d, "f'"] & R \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & J_1 \arrow[r] & R'_1 \arrow[r] & R_1 \arrow[r] & 0 \end{tikzcd} \tag{2}$$

be a commutative diagram of such extensions. Let $\xi \in \Phi(R)$ be an NC deformation, and let $\xi_1 = \Phi(f)(\xi) \in \Phi(R_1)$. Let $o_\xi \in T^2 \otimes_{k^r} J$ and $o_{\xi_1} \in T^2 \otimes_{k^r} J_1$ be the obstructions classes of extending ξ and ξ_1 over R' and R'_1, respectively. Then $o_{\xi_1} = g(o_\xi)$.

Theorem 5. *Let $\Phi : (\mathrm{Art}_r) \to (\mathrm{Set})$ be an NC deformation functor. Assume that the obstruction space T^2 is finite dimensional. Then there is a k^r-linear map $m : (T^2)^* \to \widehat{T}_{k^r}(T^1)^*$ such that $\widehat{R} \cong \widehat{T}_{k^r}(T^1)^*/(m((T^2)^*))$, a quotient algebra of the completed tensor algebra by a two-sided ideal generated by the image of m.*

Proof. Denote $\widehat{A} = \widehat{T}_{k^r}(T^1)^* = k^r \oplus \widehat{M}$. Then the base algebra of the semi-universal NC deformation $\widehat{R}$ is a quotient algebra $\widehat{A}/\widehat{I}$ by some two-sided ideal $\widehat{I}$. Let $\{z_l\}_{l=1}^N$ be a k-basis of T^2.

Let $R_k = \widehat{A}/(\widehat{I} + \widehat{M}^{k+1})$. We define a sequence of two-sided ideals $I_k \subset \widehat{A}/\widehat{M}^{k+1}$ by $R_k = \widehat{A}/(I_k + \widehat{M}^{k+1})$. By definition of the semi-universal deformation, there is an NC deformation $\xi_k \in \Phi(R_k)$. We will prove that I_k is generated by elements $\{s_{k,l}\}_{l=1}^N \in \widehat{A}/\widehat{M}^{k+1}$ such that $s_{k+1,l} \mapsto s_{k,l}$ by the natural map $\widehat{A}/\widehat{M}^{k+2} \to \widehat{A}/\widehat{M}^{k+1}$ inductively as follows.

We set $s_{1,l} = 0$ for all l, because $I_1 = 0$ and $R_1 = \widehat{A}/\widehat{M}^2$.

Let k be an arbitrary integer, and let $R = R_k$, $R' = \widehat{A}/(\widehat{M}\widehat{I} + \widehat{M}^{k+1})$ and $J = (\widehat{I} + \widehat{M}^{k+1})/(\widehat{M}\widehat{I} + \widehat{M}^{k+1})$. Then $R = R'/J$ and $M'J = 0$ for $M' = \widehat{M}/(\widehat{M}\widehat{I} + \widehat{M}^{k+1})$. We write the obstruction of extending ξ_k to R' as $o_{\xi_k} = \sum_l z_l \otimes s_{k,l} \in T^2 \otimes_{k^r} J$, where $s_{k,l} \in J$.

We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d] & R' \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & J/(s_{k,l}) \arrow[r] & R'/(s_{k,l}) \arrow[r] & R \arrow[r] & 0 \end{tikzcd}$$

By the functoriality of the obstruction class, the obstruction class of the lower sequence vanishes, and ξ_k is extendible to $R'/(s_{k,l})$. By the semi-universality, it follows that

$$\widehat{I} + \widehat{M}^{k+1} = (s_{k,l}) + \widehat{M}\widehat{I} + \widehat{M}^{k+1}.$$

By Nakayama's lemma, we have $\widehat{I} + \widehat{M}^{k+1} = (s_{k,l}) + \widehat{M}^{k+1}$. Thus we can write $I_k = (s_{k,l})_{l=1}^N$ as a two-sided ideal in $\widehat{A}/\widehat{M}^{k+1}$.

Here we use a following version of Nakayama's lemma. Let $(A, M) \in (\mathrm{Art}_r)$ and I a two-sided ideal. Assume that there are elements $h_i \in I$ such that $I = MI + (h_i)$. Then $I = (h_i)$. Indeed let $\bar{I} = I/(h_i) \subset \bar{A} = A/(h_i)$. Then $\bar{I} = M\bar{I}$. Since M is nilpotent, $\bar{I} = M\bar{I} = \cdots = M^m\bar{I} = 0$ for some m.

Now we have a commutative diagram

$$\begin{tikzcd} \widehat{A}/(\widehat{M}\widehat{I} + \widehat{M}^{k+2}) \arrow[r] \arrow[d] & \widehat{A}/(\widehat{I} + \widehat{M}^{k+2}) \arrow[d] \\ \widehat{A}/(\widehat{M}\widehat{I} + \widehat{M}^{k+1}) \arrow[r] & \widehat{A}/(\widehat{I} + \widehat{M}^{k+1}) \end{tikzcd}$$

Then the obstruction for the extension on the first line $o_{\xi_{k+1}} = \sum z_l \otimes s_{k+1,l}$ for $s_{k+1,l} \in (\widehat{I} + \widehat{M}^{k+2})/(\widehat{M}\widehat{I} + \widehat{M}^{k+2})$ is mapped to $o_{\xi_k} = \sum z_l \otimes s_{k,l}$. Hence we have $s_{k+1,l} + \widehat{M}^{k+1} = s_{k,l} + \widehat{M}^{k+1}$. Thus we can define $s_l \in \widehat{I}$ such that $s_l + \widehat{M}^{k+1} = s_{k,l} + \widehat{M}^{k+1}$ for all k. Then the s_l generate $\widehat{I}$. $\square$

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Complex algebraic geometry, in memory of Jean-Pierre Demailly / *Géométrie algébrique complexe, en mémoire de Jean-Pierre Demailly*

Special Kähler geometry and holomorphic Lagrangian fibrations

Géométrie kählérienne spéciale et fibrations lagrangiennes holomorphes

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Dedicated to the memory of Jean-Pierre Demailly

Abstract. Given a holomorphic Lagrangian fibration of a compact hyperkähler manifold, we use the differential geometry of the special Kähler metric that exists on the base away from the discriminant locus, and show that the pullback of the tangent bundle of the base to the total space of a family of minimal rational curves admits a parallel splitting. The splitting is nontrivial when the base is not half-dimensional projective space. Combining this with results of Voisin, Hwang and Bakker–Schnell, we deduce that the base must be projective space, a result first proved by Hwang.

Résumé. Étant donné une fibration lagrangienne holomorphe d'une variété hyperkählérienne compacte, nous utilisons la géométrie différentielle de la métrique kählérienne spéciale qui existe sur la base au dehors du lieu discriminant, et montrons que l'image réciproque du fibré tangent de la base par le morphisme d'évaluation d'une famille de courbes rationnelles minimales admet une décomposition parallèle. La décomposition n'est pas triviale lorsque la base n'est pas un espace projectif demi-dimensionnel. En combinant cela avec des résultats de Voisin, Hwang et Bakker–Schnell, nous en déduisons que la base doit être un espace projectif, résultat prouvé pour la première fois par Hwang.

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1. Introduction

Let X^{2n} be a hyperkähler manifold, so X is a simply connected compact Kähler manifold with a holomorphic symplectic 2-form Ω such that $H^{2,0}(X) = \mathbb{C}\Omega$. By Yau's Theorem every Kähler class on X contains a unique Ricci-flat Kähler metric. It was later realized by Beauville [5] that these metrics are hyperkähler, which means that they have holonomy equal to $Sp(n)$.

Suppose that B is an irreducible normal complex analytic space with $0 < \dim B < 2n$, and $f : X \to B$ is a holomorphic surjective map with connected fibers. Then work of Matsushita [38] shows that necessarily $\dim B = n$, that all irreducible components of the fibers of f are Lagrangian with respect to Ω, and the smooth fibers are tori. We call such f a holomorphic Lagrangian fibration. The following basic conjecture is widely expected to hold:

Conjecture 1. *If X is a hyperkähler manifold and $f : X \to B$ is a holomorphic Lagrangian fibration, then $B \cong \mathbb{P}^n$.*

This conjecture is clearly true when $n = 1$. The most striking result about this Conjecture is due to Hwang [28]:

Theorem 2 (Hwang [28]). *Conjecture 1 holds if X is projective and B is smooth.*

Theorem 2 was later extended to X Kähler and B smooth by Greb–Lehn [17]. Assuming X projective and $n = 2$, it was proved by Ou [49] that either B is smooth (hence $\mathbb{P}^2$) or else it has just one very specific singular point. This case was later ruled out independently by Bogomolov–Kurnosov [6] and Huybrechts–Xu [26], so the conjecture is known in this case. It is also known for some families of hyperkähler manifolds [4, 10, 37, 42, 63], but it remains open in general.

There are also a number of partial results towards Conjecture 1 in general, see [25] for an excellent recent overview. It is known that B must be a Kähler space (see e.g. [17, Proposition 2.2]) and Moishezon [39, Section 2.3], and that B is $\mathbb{Q}$-factorial and has at worst klt singularities (by [39, Theorem 2.1]). It follows that B has at worst rational singularities, and hence it is projective by [47, Corollary 1.7]. Again thanks to [39, Theorem 2.1] we see that B is a Fano variety with Picard number one, and in particular it is uniruled [43] and simply connected [55]. The rational cohomology of B is isomorphic to the one of $\mathbb{P}^n$ [51]. It is also known that the map f is locally projective [9], so the smooth fibers are abelian varieties, and if B is smooth then the discriminant locus $D \subset B$ of f has pure codimension 1 by [30, Proposition 3.1].

Our main result forms part of a new proof of Hwang's theorem, as well as Greb–Lehn's extension. In order to describe this, suppose B is not $\mathbb{P}^n$. Then from a result of Cho–Miyaoka–Shepherd–Barron [10], which uses Mori theory, it follows that there is a rational curve in B (not contained in D) with anticanonical degree at most n. We show that such a curve is free, and together these imply that the Grothendieck decomposition of the pullback of TB to this rational curve has some degree zero factors. Taking such rational curves with minimal anticanonical degree, we can consider the universal family $\mathscr{U}$ with evaluation map $\mu : \mathscr{U} \to B$, which we may assume is a submersion over a Zariski open set $B^\circ \subset B$ (which we may assume is equal to $B \backslash D$ up to enlarging D), and the positive degree factors in the Grothendieck decomposition define a nontrivial holomorphic subbundle $\mathscr{V} \subset \mu^* TB^\circ$, whose rank is strictly less than n. At the same time, classical work of Freed [13] shows that on B° there is a "special Kähler metric" g_{SK}, whose Kähler form ω_{SK} is parallel with respect to a "special Kähler connection" ∇^{SK} on $T^{\mathbb{R}}B^\circ$, which is torsion-free, flat, and $d^{\nabla^{\mathrm{SK}}} J = 0$ (where J is the complex structure of B). Our main result is then:

Theorem 3. *In this setting, $\mathscr{V}$ is preserved by the pullback of the Chern connection of g_{SK}.*

We also show that the corresponding real subbundle $\mathscr{V}_{\mathbb{R}} \subset \mu^* T^{\mathbb{R}} B^\circ$ is also preserved by the pullback of the special Kähler connection ∇^{SK}. This in turn can be interpreted as giving a nontrivial splitting of a real variation of Hodge structures (which naturally exists on B°) when pulled back via μ. As we will discuss below, by combining Theorem 3 with work of Voisin [60], Hwang [27, 28] and Bakker–Schnell [2], one can deduce Hwang's Theorem 2.

Let us first give some intuition for our approach. One of the key features of the rich geometry of special Kähler metrics is that they have nonnegative bisectional curvature. Recall here the fundamental theorem of Mori [45] and Siu–Yau [53] which states that a *compact* Kähler manifold with positive bisectional curvature must be isomorphic to $\mathbb{P}^n$. This was generalized by Mok [44] to classify compact Kähler manifolds with nonnegative bisectional curvature: their universal cover splits as a product of a Euclidean factor, of projective space, and of compact Hermitian symmetric spaces of rank $\geqslant 2$. A large part of our arguments are motivated by trying to extend Mok's techniques to our *noncompact* manifold $B\backslash D$ with an incomplete metric with nonnegative bisectional curvature, making essential use of the special features of special Kähler metrics, which are summarized in Section 2.

To prove Theorem 3, thanks to a recent result of Bakker [1] we need to consider two cases: either f has maximal variation or f is isotrivial. In the first case, we prove in Section 4 a crucial rigidity result (Theorem 16) which shows that the bisectional curvature of ω_{SK} vanishes when evaluated on a vector in $\mathcal{V}$ and a vector in its orthogonal complement. For this, we use results of Zhang and the second-named author [59] on the asymptotic behavior of ω_{SK} near D, as well as a strictly positive lower bound for ω_{SK} near D obtained by Gross, Zhang and the second-named author in [18, 19, 57]. These are explained in Section 3. In Section 5 we then supplement the rigidity result by showing that the rough Laplacian of the bisectional curvature of ω_{SK} evaluated on the same vectors vanishes as well. This result is analogous to a statement in Mok [44], although our proof is quite different. Equipped with these rigidity results, in Section 6 we adapt an argument of Mok [44] and conclude. In the isotrivial case the rigidity results are trivial because ω_{SK} is flat, but this flatness can be effectively exploited to show again that $\mathcal{V}$ is preserved by the Chern connection of g_{SK}.

In Section 7 we sketch how Theorem 2 follows by combining Theorem 3 with a number of recent results in the literature. As mentioned above, we first show that the real subbundle $\mathcal{V}_{\mathbb{R}} \subset \mu^* T^{\mathbb{R}} B^\circ$ which corresponds to $\mathcal{V}$ is preserved also by the pullback of the special Kähler connection ∇^{SK}. This uses again our rigidity theorem. Then we invoke an important result of Hwang [27, 28], which also has a recent proof by Bakker–Schnell [2] (Theorem 27 below), which gives that the map μ must have connected fibers. Thus, our splitting descends to a parallel splitting of $T^{\mathbb{R}} B^\circ$, from which we obtain a parallel real $(1,1)$-form on B° which is not proportional to ω_{SK}, which is contradiction to a result of Voisin [60].

Lastly, in Section 8 we make some comments on the obstacles that we faced when trying to extend our approach to the case when B is singular.

Remark 4. In the first draft of our paper, our original argument in Section 7 to construct the parallel form on B° turned out to be incomplete. After our first draft was posted to arXiv, Bakker and Schnell sent us their paper [2] with a new proof of Hwang's theorem. As mentioned above, to deduce Theorem 2 from Theorem 3 we now rely on their paper. On the other hand, without using [2], what our arguments show is that B must be $\mathbb{P}^n$ provided that μ has connected fibers. As pointed out to us by Hwang, this result was implictly proved by Cho–Miyaoka–Shepherd–Barron [10, Section 7] using a different method (under the extra assumption that f has a section, which was removed by Nagai [46]).

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2. Special Kähler metrics

2.1. *Notation*

Let us first fix some notation. For a complex manifold B we will denote by $T^{\mathbb{R}}B$ its real tangent bundle, and with $TB \subset T^{\mathbb{C}}B = T^{\mathbb{R}}B \otimes \mathbb{C}$ its holomorphic tangent bundle (of complex tangent vectors of type $(1,0)$). The dual of TB will be denoted by Ω^1_B. The complex structure will be denoted by $J : T^{\mathbb{R}}B \to T^{\mathbb{R}}B$. We will also denote $B^\circ := B \backslash D$ and $X^\circ := f^{-1}(B^\circ)$.

2.2. *Existence of special Kähler metrics*

The paper by Freed [13], following work of Donagi–Witten [12], shows that the base of an algebraic integrable system (which in our case is B°) admits a geometric structure called "special Kähler metric", ω_{SK}. This means that $(B^\circ, J, \omega_{\mathrm{SK}})$ is a Kähler manifold and there is a torsion-free flat connection ∇^{SK} on $T^{\mathbb{R}}B^\circ$ which makes ω_{SK} parallel and $d^{\nabla^{\mathrm{SK}}}J = 0$ (however, in general $\nabla^{\mathrm{SK}}J \neq 0$), where $d^{\nabla^{\mathrm{SK}}} : \Omega^1(T^{\mathbb{R}}B^\circ) \to \Omega^2(T^{\mathbb{R}}B^\circ)$ is the usual extension of ∇^{SK} (cf. [13, p. 33]). The Riemannian metric associated to ω_{SK} will be denoted by g_{SK} and its Levi-Civita/Chern connection, which in general is different from ∇^{SK}, will be denoted simply by ∇ (see (56) below for an explicit formula relating ∇ and ∇^{SK}). On every sufficiently small open set $U \subset B^\circ$ we can find special holomorphic local coordinates $\{z_j\}_{j=1}^n$ (whose real parts are flat Darboux coordinates) and a holomorphic map $Z : U \to \mathfrak{H}_n$ into the Siegel upper half space

$$\mathfrak{H}_n = \{A \in \mathfrak{gl}(n,\mathbb{C}) \mid A = A^t,\ \mathrm{Im}\,A > 0\},$$

such that $Z(y)$ are the periods of the torus fiber $f^{-1}(y)$, and we can write

$$\omega_{\mathrm{SK}} = \frac{1}{2}\sum_{i,j} \mathrm{Im}\,Z_{ij}\,\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j.$$

It is also worth noting that special Kähler manifolds can only be complete if they are flat, by a result of Lu [36]. See [59] for a description of the metric completion of $(B^\circ, \omega_{\mathrm{SK}})$ and of its metric singularities.

Special Kähler metrics have a Hodge-theoretic origin (see [24, 40]): as mentioned earlier there is a natural weight-one polarized real variation of Hodge structures $R^1 f_* \mathbb{R}_{X^\circ}$ on B°, whose Hodge bundle of type $(1,0)$ is isomorphic to TB° (by contracting with the holomorphic symplectic form), and its Hodge metric is exactly the special Kähler metric.

In [18, 19, 57] it is also shown that ω_{SK} can be written as $\omega_B + i\partial\bar{\partial}\varphi$ for some Kähler metric ω_B on B and some function $\varphi \in C^\infty(B^\circ) \cap L^\infty(B)$. In fact, a priori there is a different special Kähler metric on B° for each chosen Kähler class $[\omega_B]$ on B, but since B is smooth Fano and of Picard number one, it follows that $b_2(B) = 1$ so there is a unique choice of Kähler class up to scaling. In the following, we fix one such ω_B once and for all. This way, we can unambiguously talk about "the" special Kähler metric ω_{SK} in the following.

2.3. *Curvature properties*

Following Freed [13], there is a holomorphic symmetric cubic form $\Xi \in H^0(B^\circ, \mathrm{Sym}^3 T^* B^\circ)$ such that, in any local holomorphic coordinate system, the curvature tensor of ω_{SK} can be written as

$$R_{i\bar{j}k\bar{\ell}} = g_{\mathrm{SK}}^{p\bar{q}} \Xi_{ikp} \overline{\Xi_{j\ell q}}, \tag{1}$$

and on any sufficiently small U as above we can find a holomorphic function $\mathscr{F} : U \to \mathbb{C}$ such that, in special holomorphic coordinates, the period matrix and the cubic form can be written as

$$Z_{ij} = \frac{\partial^2 \mathscr{F}}{\partial z_i \partial z_j}, \quad \Xi_{ipk} = \frac{\partial^3 \mathscr{F}}{\partial z_i \partial z_k \partial z_p}. \tag{2}$$

From the curvature formula (1) we see in particular that ω_{SK} has nonnegative bisectional curvature on B°: given any $v, w \in T^{1,0} B^\circ$ we have

$$\mathrm{Rm}(v, \overline{v}, w, \overline{w}) = R_{i\bar{j}k\bar{\ell}} v^i \overline{v^j} w^k \overline{w^\ell} = \sum_p \left|\Xi(v, w, e_p)\right|^2 \geqslant 0,$$

where $\{e_p\}$ is any g_{SK}-unitary frame.

We will also use the following dichotomy, which was conjectured by Matsushita, and after progress by van Geemen–Voisin [14] it was recently proved by Bakker [1]:

Theorem 5. *Either f is isotrivial, or else f has maximal variation.*

This dichotomy is then reflected in the curvature properties of ω_{SK}:

Corollary 6. *Either ω_{SK} is flat on B°, or else ω_{SK} has positive Ricci curvature on a Zariski open subset of B°.*

In the second case, up to replacing D with a larger closed analytic subvariety we will always assume that $\mathrm{Ric}_{g_{\mathrm{SK}}} > 0$ on B°.

Proof. We use Bakker's Theorem 5. If f is isotrivial, then the local period map Z is constant, so from (2) we see that $\Xi \equiv 0$ on B°, and (1) shows that ω_{SK} is flat. If f has maximal variation, then the period map Z is generically of maximal rank (equal to n), so Z is an immersion on a Zariski open subset of B° (which, up to enlarging D, we may assume is equal to B°). Given any $v \in T^{1,0}_x B^\circ$, the Ricci curvature of ω_{SK} in the direction of v is given by

$$\mathrm{Ric}(v, \overline{v}) = \sum_{p,q} \left|\Xi(v, e_p, e_q)\right|^2 \geqslant 0,$$

and if this vanishes for some $v \neq 0$ then in special holomorphic coordinates we have that for all p, q

$$0 = \Xi(v, e_p, e_q) = \frac{\partial^3 \mathscr{F}}{\partial v \partial e_p \partial e_q} = \frac{\partial}{\partial v} Z_{pq},$$

so the period map is not an immersion at x, a contradiction. $\square$

Remark 7. The holomorphic sectional curvature of ω_{SK} is given by

$$\mathrm{HSC}(v) = R_{i\bar{j}k\bar{\ell}} v^i \overline{v^j} v^k \overline{v^\ell} = \sum_p \left| \frac{\partial^3 \mathscr{F}}{\partial v \partial v \partial e_p} \right|^2 \geqslant 0,$$

where $v \in T^{1,0} B^\circ$ is a unit vector (and in the last equality we use special holomorphic coordinates). The condition that ω_{SK} has (strictly) positive holomorphic sectional curvature on B° thus means that none of the "diagonal" entries of the period matrix Z

$$Z_{ij} v^i v^j = \frac{\partial^2 \mathscr{F}}{\partial v \partial v}$$

is locally constant. We expect that this always holds (up to enlarging D) when f has maximal variation.

Remark 8. We are grateful to B. Bakker for the following observation. Let $f : S \to \mathbb{P}^1$ be an elliptic fibration of a $K3$ surface S. For $n \geqslant 2$ let $X = S^{[n]}$ be the Hilbert scheme parametrizing length n subschemes of S. We obtain an induced holomorphic Lagrangian fibration $\widetilde{f} : X \to (\mathbb{P}^1)^{[n]} = \mathbb{P}^n$ whose general fiber is isomorphic to the product of n general fibers of f, and if f has maximal variation then so does $\widetilde{f}$. Since the period matrix of such a torus is diagonal, we see that the period map Z of $\widetilde{f}$ has $Z_{ij} = 0$ for $i \neq j$. It follows that for these examples the special Kähler metric, which is not flat if f has maximal variation, nevertheless does not have strictly positive bisectional curvature on $\mathbb{P}^n \setminus D$, since in local special coordinates we have

$$\mathrm{Rm}(e_i, \overline{e_i}, e_j, \overline{e_j}) = R_{i\bar{i}j\bar{j}} = 0,$$

for all $i \neq j$. Thus, to prove our main theorem, it would not be sufficient to prove a suitable noncompact version of the Mori–Siu–Yau theorem [45, 53], but we must instead generalize the work of Mok [44].

3. Estimates on the special Kähler metric

We collect in this section two crucial estimates for the special Kähler metric ω_{SK}, which are contained or follow from earlier work of the second-named author and coauthors [57–59]. See also [8, 22] for a study of the asymptotics of special Kähler metrics on Riemann surfaces.

3.1. *Strict positivity*

The first estimate, taken from [18, 19, 57, 58], says that the positivity of ω_{SK} does not degenerate as we approach D. Since this statement is valid even if B is singular, we present it in this generality.

Proposition 9. *Let X be a hyperkähler manifold, $f : X \to B$ a holomorphic Lagrangian fibration with B a normal analytic variety. Let ω_B be a smooth Kähler metric on B (in the sense of analytic spaces) and ω_{SK} the special Kähler metric on B° cohomologous to ω_B. Then there is $C > 0$ such that on B° we have*

$$\omega_{\mathrm{SK}} \geqslant C^{-1}\omega_B. \tag{3}$$

Proof. Fix a Kähler metric ω_X on X and for $t \geqslant 0$ let ω_t be the hyperkähler metric on X cohomologous to $f^*\omega_B + e^{-t}\omega_X$. Then the Schwarz Lemma [57, Lemma 3.1] (using also [58, Proof of Theorem 3.2] in the case when B is singular) gives

$$\omega_t \geqslant C^{-1} f^*\omega_B,$$

on X° (with C independent of $t \geqslant 0$), and thanks to [18, Theorem 1.1], [23] and [19, Theorem 1.2] we know that as $t \to \infty$ we have

$$\omega_t \to f^*\omega_{\mathrm{SK}},$$

locally uniformly on X° (and even locally smoothly), so we conclude that

$$f^*\omega_{\mathrm{SK}} \geqslant C^{-1} f^*\omega_B,$$

on X°, and since f is a submersion over B° this is equivalent to

$$\omega_{\mathrm{SK}} \geqslant C^{-1}\omega_B,$$

on B°. □

Remark 10. If B has quotient singularities (which is expected to hold in general [25, Remark 1.11]) then we can replace ω_B with an orbifold Kähler metric ω_{orb}, and a similar argument gives the stronger bound

$$\omega_{\mathrm{SK}} \geqslant C^{-1}\omega_{\mathrm{orb}},$$

on B°.

3.2. *Ricci curvature bounds near D*

From now on, we return to our standing assumption that B is smooth. The second crucial estimate is a bound for the Ricci curvature of ω_{SK}. We have seen in the previous section that ω_{SK} has nonnegative Ricci curvature on B°. In fact, as shown in [54, 57] (see also [59, Proposition 4.1]), we have

$$\mathrm{Ric}_{g_{\mathrm{SK}}} = \omega_{\mathrm{WP}} \geqslant 0,$$

where ω_{WP} is the Weil–Petersson form of the family of abelian varieties $f : X^\circ \to B^\circ$ (pullback of the Weil–Petersson metric on the moduli space via the moduli map). Concretely, on B° we have

$$\omega_{\mathrm{SK}}^n = c(-1)^{\frac{n^2}{2}} f_*(\sigma^n \wedge \overline{\sigma^n}), \tag{4}$$

where $c > 0$ and σ is a holomorphic symplectic form on X, and to obtain ω_{WP} it suffices to take $-i\partial\bar\partial\log$ of the fiber integral in (4) divided by the local Euclidean volume form.

Recall that the discriminant locus $D \subset B$ is a closed analytic subvariety of pure codimension 1, see [30, Proposition 3.1]. Let $x \in D$ be any smooth point of D, and choose an open neighborhood U of x with local holomorphic coordinates centered at x such that $D \cap U = \{z_1 = 0\}$. Thus, at points of $D \cap U$, the vectors $\frac{\partial}{\partial z_2}, \dots, \frac{\partial}{\partial z_n}$ are tangent to D, while $\frac{\partial}{\partial z_1}$ is transversal. The main claim is the following:

Proposition 11. *On $\{z_1 \neq 0\}$ the Ricci curvature tensor $R_{i\bar j} = \mathrm{Ric}_{g_{\mathrm{SK}}}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_j}}\right)$ of ω_{SK} satisfies*

$$0 \leqslant R_{i\bar i} \leqslant C, \quad 2 \leqslant i \leqslant n, \tag{5}$$

$$0 \leqslant R_{1\bar 1} \leqslant \frac{C}{|z_1|^2}, \tag{6}$$

for some constant $C > 0$.

Proof. We will use freely the arguments in [59, Section 4.3] (these are stated for X projective hyperkähler, but all arguments there go through for general X hyperkähler using that Lagrangian fibrations are locally projective [9]). By the Monodromy Theorem, there is $m \in \mathbb{N}_{>0}$ such that the eigenvalues of the monodromy operator T (acting on $H^1(f^{-1}(y), \mathbb{Z})$ for some fixed basepoint $y \in U\backslash D$) are mth roots of unity. We may assume without loss that in our coordinates U is the unit polydisc, and letting $\widetilde{U}$ be the unit polydisc with coordinates $(t_1, \dots, t_n)$, we define the branched covering

$$q : \widetilde{U} \to U, \quad q(t_1, \dots, t_n) = (t_1^m, t_2, \dots, t_n).$$

Then after pulling back to $\widetilde{U}$, the monodromy operator T becomes unipotent, with

$$(T - \mathrm{Id})^2 = 0.$$

Thanks to the argument in [59, p. 774], we can find holomorphic functions $w_1, \dots, w_n$ on $\widetilde{U}$, which are special holomorphic coordinates on $\widetilde{U} \cap \{t_1 \neq 0\}$ (but need not form a coordinate system at points on $\{t_1 = 0\}$, and they may even vanish there), such that, on $\widetilde{U} \cap \{t_1 \neq 0\}$, we can write

$$q^*\omega_{\mathrm{SK}} = \frac{i}{2}\sum_{j,k} \mathrm{Im}\, Z_{jk}(t) \mathrm{d}w_j \wedge \mathrm{d}\overline{w_k} = \frac{i}{2}\sum_{j,k,p,q} \mathrm{Im}\, Z_{jk}(t) \frac{\partial w_j}{\partial t_p}\overline{\frac{\partial w_k}{\partial t_q}} \mathrm{d}t_p \wedge \mathrm{d}\overline{t_q},$$

where $Z_{jk}(t)$ is the local period map pulled back to $\widetilde{U}$. Thus, if we denote by $\mathrm{d}V_E$ the Euclidean volume form on $\widetilde{U}$ given by the coordinates $t_1, \dots t_n$, we have

$$\log \frac{q^*\omega_{\mathrm{SK}}^n}{\mathrm{d}V_E} = \log\det \mathrm{Im}\, Z + \log\left|\det\left(\frac{\partial w_j}{\partial t_p}\right)\right|^2,$$

and since $\det\left(\frac{\partial w_j}{\partial t_p}\right)$ is holomorphic and nonzero on $\widetilde{U} \cap \{t_1 \neq 0\}$, we get

$$\mathrm{Ric}_{q^* g_{\mathrm{SK}}}\left(\frac{\partial}{\partial t_j}, \frac{\partial}{\partial \overline{t_k}}\right) = -\frac{\partial}{\partial t_j}\frac{\partial}{\partial \overline{t_k}} \log \frac{q^*\omega_{\mathrm{SK}}^n}{dV_E} = -\frac{\partial}{\partial t_j}\frac{\partial}{\partial \overline{t_k}} \log\det \mathrm{Im}\, Z. \tag{7}$$

To estimate this, following [59, Lemma 4.3] we use Schmid's Nilpotent Orbit Theorem [50] and see there are $b_{jk} \in \mathbb{Q}$ and a holomorphic map Q from $\widetilde{U}$ to the space of symmetric $n \times n$ complex matrices, such that on $\widetilde{U} \cap \{t_1 \neq 0\}$ we have

$$Z_{jk}(t) = Q_{jk}(t) + \frac{\log t_1}{2\pi i} b_{jk}, \quad 1 \leqslant j,k \leqslant n,$$

for some branch of log. Thus,

$$\operatorname{Im} Z_{jk}(t) = \operatorname{Im} Q_{jk}(t) - \frac{b_{jk}}{2\pi} \log|t_1|, \tag{8}$$

and furthermore (see [59, Lemma 4.3]) there is $C > 0$ such that on $\widetilde{U} \cap \{t_1 \neq 0\}$ we have

$$\operatorname{Im} Z(t) \geqslant C^{-1} \operatorname{Id}, \tag{9}$$

and so the inverse matrix of $\operatorname{Im} Z(t)$, whose entries will be denoted by $(\operatorname{Im} Z(t))^{pq}$, satisfies

$$0 < (\operatorname{Im} Z(t))^{-1} \leqslant C \operatorname{Id}. \tag{10}$$

Differentiating the determinant gives

$$\begin{aligned}-\frac{\partial}{\partial t_j}\frac{\partial}{\partial \overline{t_k}} \log\det\operatorname{Im} Z(t) = -(\operatorname{Im} Z(t))^{pq} \frac{\partial}{\partial t_j}\frac{\partial}{\partial \overline{t_k}} \operatorname{Im} Z_{pq}(t)&\\ + (\operatorname{Im} Z(t))^{pq}(\operatorname{Im} Z(t))^{rs} \frac{\partial}{\partial t_j} \operatorname{Im} Z_{pr}(t) \frac{\partial}{\partial \overline{t_k}} \operatorname{Im} Z_{qs}(t).&\end{aligned}$$

First we take $j \geqslant 2$, and differentiating (8) gives

$$-\frac{\partial}{\partial t_j}\frac{\partial}{\partial \overline{t_j}} \log\det\operatorname{Im} Z(t) = (\operatorname{Im} Z(t))^{pq}(\operatorname{Im} Z(t))^{rs} \frac{\partial}{\partial t_j} \operatorname{Im} Q_{pr}(t) \frac{\partial}{\partial \overline{t_j}} \operatorname{Im} Q_{qs}(t) \leqslant C,$$

using (10) and the fact that Q is holomorphic on all of $\widetilde{U}$. As for the t_1 direction, differentiating (8) we have

$$\begin{aligned}-\frac{\partial}{\partial t_1}\frac{\partial}{\partial \overline{t_1}} &\log\det\operatorname{Im} Z(t)\\ &= (\operatorname{Im} Z(t))^{pq}(\operatorname{Im} Z(t))^{rs} \frac{\partial}{\partial t_1}\left(\operatorname{Im} Q_{pr}(t) - \frac{b_{pr}}{2\pi}\log|t_1|\right) \frac{\partial}{\partial \overline{t_1}}\left(\operatorname{Im} Q_{qs}(t) - \frac{b_{qs}}{2\pi}\log|t_1|\right)\\ &\leqslant C\sum_{p,r}\left|\frac{\partial}{\partial t_1}\left(\operatorname{Im} Q_{pr}(t) - \frac{b_{pr}}{2\pi}\log|t_1|\right)\right|^2\\ &\leqslant C + C\left|\frac{\partial}{\partial t_1}\log|t_1|\right|^2\\ &\leqslant \frac{C}{|t_1|^2}.\end{aligned}$$

Going back to (7), this shows that on $\widetilde{U} \cap \{t_1 \neq 0\}$) we have

$$0 \leqslant \operatorname{Ric}_{q^* g_{\mathrm{SK}}}\left(\frac{\partial}{\partial t_j}, \frac{\partial}{\partial \overline{t_j}}\right) \leqslant C, \quad j \geqslant 2,$$

$$0 \leqslant \operatorname{Ric}_{q^* g_{\mathrm{SK}}}\left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial \overline{t_1}}\right) \leqslant \frac{C}{|t_1|^2},$$

and so on $U \cap \{z_1 \neq 0\}$ we have for $j \geqslant 2$,

$$0 \leqslant R_{j\bar{j}} = \operatorname{Ric}_{g_{\mathrm{SK}}}\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z_j}}\right) = \operatorname{Ric}_{q^* g_{\mathrm{SK}}}\left(\frac{\partial}{\partial t_j}, \frac{\partial}{\partial \overline{t_j}}\right) \leqslant C,$$

and

$$0 \leqslant R_{1\bar{1}} = \mathrm{Ric}_{g_{\mathrm{SK}}}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \overline{z_1}}\right) = \frac{1}{m^2|t_1|^{2m-2}}\,\mathrm{Ric}_{q^* g_{\mathrm{SK}}}\left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial \overline{t_1}}\right) \leqslant \frac{C}{m^2|t_1|^{2m}} \leqslant \frac{C}{|z_1|^2},$$

as desired. □

Remark 12. We expect that the sharp bound in (6) in general is of the form $\frac{C}{|z_1|^2 \log^2 |z_1|}$, cf. [62] when $\dim B = 1$. One may be able to show this by proving an asymptotic expansion for the fiber integral in (4) which can be differentiated term-by-term, as in [3, 56].

4. Rational curves and rigidity

Recall that B is a Fano manifold, hence uniruled. Let $\nu : \mathbb{P}^1 \to B$ be a rational curve (i.e. a nonconstant holomorphic map) whose image is not contained in D. Our first result of this section shows that ν is a *free* rational curve, in the terminology of Mori Theory, cf. [35].

4.1. *Freeness of the rational curve*

By Grothendieck's Theorem, the vector bundle $\nu^* TB$ splits and so we can write

$$\nu^* TB \cong \bigoplus_{i=1}^{n} \mathcal{O}(a_i), \tag{11}$$

for some integers a_i, which we order by $a_1 \geqslant \cdots \geqslant a_n$. Dualizing, we have

$$\nu^* \Omega^1_B \cong \bigoplus_{i=1}^{n} \mathcal{O}(-a_i), \tag{12}$$

and

$$q := -K_B \cdot \nu(\mathbb{P}^1) = \sum_{i=1}^{n} a_i > 0, \tag{13}$$

since B is Fano.

On B° we equip Ω^1_B with the Hermitian metric h_{SK} induced by the special Kähler metric ω_{SK}.

Lemma 13. *We have $a_n \geqslant 0$.*

Proof. This argument was suggested to us by M. Păun. Consider the nontrivial section $v \in H^0(\mathbb{P}^1, \nu^*\Omega^1_B \otimes \mathcal{O}(a_n))$ which corresponds to the quotient morphism $\nu^* TB \to \mathcal{O}(a_n)$. Equip $L := \mathcal{O}(a_n)$ with a smooth metric h_L on $\mathbb{P}^1$, and equip $\nu^*\Omega^1_B$ with the smooth metric $\nu^* h_{\mathrm{SK}}$ on $\mathbb{P}^1 \backslash \nu^{-1}(D)$ which is the pullback of the metric induced by ω_{SK}. Thus, the curvature of $\nu^* h_{\mathrm{SK}}$ is Griffiths nonpositive on $\mathbb{P}^1 \backslash \nu^{-1}(D)$, since ω_{SK} has nonnegative bisectional curvature on B° and dualization reverses the sign of Griffiths positivity (see e.g. [11, Section VII.6]). Equip then $\nu^*\Omega^1_B \otimes \mathcal{O}(a_n)$ with the metric $h = \nu^* h_{\mathrm{SK}} \otimes h_L$ on $\mathbb{P}^1 \backslash \nu^{-1}(D)$.

Differentiating $\log|v|^2_h$ on $\mathbb{P}^1 \backslash \nu^{-1}(D)$ we have the well-known identity of $(1,1)$-forms on $\mathbb{P}^1 \backslash \nu^{-1}(D)$

$$i\partial\bar{\partial}\log|v|^2_h = \frac{|\nabla v|^2_h}{|v|^2_h} - \frac{|\langle \nabla v, v\rangle_h|^2}{|v|^4_h} - R_{h_L} - \frac{\langle R_{\nu^* h_{\mathrm{SK}}}(v), v\rangle_h}{|v|^2_h},$$

where ∇v is an $\nu^*\Omega^1_B \otimes \mathcal{O}(a_n)$-valued $(1,0)$-form, so $|\nabla v|^2_h$ is a $(1,1)$-form, and similarly for the other terms. Using Cauchy–Schwarz we have

$$\frac{|\langle \nabla v, v\rangle_h|^2}{|v|^4_h} \leqslant \frac{|\nabla v|^2_h}{|v|^2_h},$$

and since on $\mathbb{P}^1\backslash\nu^{-1}(D)$ the curvature of ν^*h_{SK} is Griffiths nonpositive, we can estimate

$$\begin{aligned} i\partial\overline{\partial}\log|v|_h^2 &\geqslant -R_{h_L} - \frac{\langle R_{\nu^*h_{\mathrm{SK}}}(v), v\rangle_h}{|v|_h^2} \\ &\geqslant -R_{h_L}, \end{aligned} \tag{14}$$

Since R_{h_L} is a smooth form on $\mathbb{P}^1$, we see that $\log|v|_h^2$ is quasi-psh on $\mathbb{P}^1\backslash\nu^{-1}(D)$, and using (3) we see that

$$\sup_{\mathbb{P}^1\backslash\nu^{-1}(D)} \log|v|_h^2 \leqslant C + \sup_{\mathbb{P}^1\backslash\nu^{-1}(D)} \log|v|^2_{\nu^*h_B\otimes h_L} < \infty,$$

where h_B is the smooth metric on Ω^1_B induced by ω_B. Thus $\log|v|_h^2$ is bounded above, hence by the Grauert–Remmert extension theorem [15] the inequality $R_{h_L} + i\partial\overline{\partial}\log|v|_h^2 \geqslant 0$ extends over the singularities to all of $\mathbb{P}^1$ (in the weak sense). Integrating this over $\mathbb{P}^1$ and using Stokes thus gives

$$a_n = \int_{\mathbb{P}^1} R_{h_L} = \int_{\mathbb{P}^1} (R_{h_L} + i\partial\overline{\partial}\log|v|_h^2) \geqslant 0,$$

as desired. □

Lemma 13 says that every rational curve in B which is not contained in D is free, and by Mori Theory it deforms to cover a Zariski dense subset of B (see e.g. [35]).

The pullback morphism $\nu^*\Omega^1_B \to \Omega^1_{\mathbb{P}^1}$ dualizes to a nontrivial morphism $\mathscr{O}(2) \to \nu^*TB$, and hence $a_1 \geqslant 2$. Using this observation and Lemma 13 we can write the splittings in (11) and (12) as

$$\nu^*TB \cong \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_{n-\ell}) \oplus \mathscr{O}^{\oplus \ell}, \tag{15}$$

$$\nu^*\Omega^1_B \cong \mathscr{O}(-a_1) \oplus \cdots \oplus \mathscr{O}(-a_{n-\ell}) \oplus \mathscr{O}^{\oplus \ell}, \tag{16}$$

for some $0 \leqslant \ell \leqslant n-1$, where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_{n-\ell} \geqslant 1$, $a_1 \geqslant 2$, and

$$q = \sum_{i=1}^{n-\ell} a_i.$$

Recall now a result by Cho–Miyaoka–Shepherd–Barron [10, Corollary 0.4 (11)], which uses Mori theory:

Theorem 14. *Let B be a uniruled projective manifold, D an effective divisor, and suppose that, for any rational curve $\nu : \mathbb{P}^1 \to B$ which is not contained in D, we have the inequality*

$$-K_B \cdot \nu(\mathbb{P}^1) \geqslant n+1. \tag{17}$$

Then $B \cong \mathbb{P}^n$.

If, in our setting, for all rational curves $\nu : \mathbb{P}^1 \to B$ not contained in D we have $\ell = 0$, i.e. $a_i > 0$ for all i, then since $a_1 \geqslant 2$ it would follow that $-K_B \cdot \nu(\mathbb{P}^1) = \sum_{i=1}^n a_i \geqslant n+1$ and so B would be isomorphic to $\mathbb{P}^n$. In other words, if $B \ncong \mathbb{P}^n$ then there exists a rational curve $\nu_0 : \mathbb{P}^1 \to B$ not contained in D which has $\ell \geqslant 1$, i.e. there are some trivial factors $\mathscr{O}^{\oplus\ell}$ in the splitting (11). We may also assume that the anticanonical degree $q := -K_B \cdot \nu_0(\mathbb{P}^1)$ is as small as possible among all rational curves not contained in D (and satisfies $2 \leqslant q \leqslant n$), and we will call these *minimal degree rational curves*, which is consistent with the standard terminology, e.g. in [29]. By Lemma 13, this rational curve ν_0 is free and so it deforms to cover a Zariski dense subset of B. Let $\mathscr{K}$ be the irreducible component of the space of rational curves in B (see [35, Section II.2]) which contains ν_0, which we fix once and for all. From Mori Theory (see [35] and [29, Section 3]) we have that $\mathscr{K}$ is a quasiprojective variety equipped with a universal $\mathbb{P}^1$-bundle $\rho : \mathscr{U} \to \mathscr{K}$ and an evaluation map $\mu : \mathscr{U} \to B$. For any $t \in \mathscr{K}$ we will also write

$$\mathscr{U}_t := \rho^{-1}(t) \subset \mathscr{U},$$

so $\mathscr{U}_t \cong \mathbb{P}^1$ is the rational curve corresponding to t, and

$$\nu_t := \mu|_{\mathscr{U}_t} : \mathscr{U}_t \to B,$$

will denote the morphism to B.

Furthermore, the generic rational curve in $\mathscr{K}$ is free and not contained in D, $\mathscr{K}$ is smooth at such curves, and the integers a_i, ℓ in the decomposition (15) are the same for all generic such curves. Given $x \in B^\circ$ there is some minimal degree rational curve ν in $\mathscr{K}$ that passes through x and is smooth at x. Thanks to [35, Proposition II.3.7], we can also assume that $\nu(\mathbb{P}^1)$ intersects D only at the regular points of D (since the singularities of D have codimension at least 2 in B), and that these intersections are transverse. The evaluation morphism $\mu : \mathscr{U} \to B$ is a submersion over a Zariski open subset of B, which up to enlarging D we may assume equals B°. Thus, if we define $\mathscr{U}^\circ := \mu^{-1}(B^\circ)$, then $\mathscr{U}^\circ$ is smooth and $\mu : \mathscr{U}^\circ \to B^\circ$ is a submersion. The metric g_{SK} on TB° induces by pullback a metric $\mu^* TB^\circ$ over $\mathscr{U}^\circ$, which we will denote by the same symbol, and similarly for the connections ∇ and ∇^{SK}, which induce pullback connections denoted in the same way.

Lemma 15. *There is a locally free sheaf $\mathscr{V}^\sharp$ on $\mathscr{U}$ such that for every $t \in \mathscr{K}$, the restriction $\mathscr{V}^\sharp|_{\mathscr{U}_t}$ of $\mathscr{V}^\sharp$ to the rational curve $\mathscr{U}_t$ equals the factor $\mathscr{O}^{\oplus \ell}$ in the splitting* (16) *for $\nu_t^* \Omega_B^1$.*

Proof. For the sake of clarity, we first define the fiber $\mathscr{V}^\sharp$ at any point on $\mathscr{U}_t \cong \mathbb{P}^1$. For this, we consider $\nu_t^* \Omega_B^1$, which from the splitting (16) is isomorphic to $\mathscr{O}(-a_1) \oplus \cdots \oplus \mathscr{O}(-a_{n-\ell}) \oplus \mathscr{O}^{\oplus \ell}$. Its space of global sections $H^0(\mathscr{U}_t, \nu_t^* \Omega_B^1)$ is then ℓ-dimensional, and we can find a basis of such sections which are linearly independent at all points of $\mathbb{P}^1$. The fiber of $\mathscr{V}^\sharp$ at any point on $\mathscr{U}_t$ is then defined as the linear span of any given basis of $H^0(\mathscr{U}_t, \nu_t^* \Omega_B^1)$.

To prove that this collection of ℓ-dimensional vector spaces form a locally free sheaf, consider first the locally free sheaf $\mu^* \Omega_B^1$ on $\mathscr{U}$, and take its direct image sheaf $\rho_* \mu^* \Omega_B^1$. Since $h^0(\mathscr{U}_t, \mu^* \Omega_B^1|_{\mathscr{U}_t}) = \ell$ is independent of t, Grauert's Theorem on direct images [21, Corollary III.12.9] shows that $\rho_* \mu^* \Omega_B^1$ is a locally free sheaf on $\mathscr{K}$. We then set $\mathscr{V}^\sharp = \rho^* \rho_* \mu^* \Omega_B^1$, which is a locally free sheaf over $\mathscr{U}$ whose fibers agree with our previous description. □

Our main interest will be with the restriction of $\mathscr{V}^\sharp$ to $\mathscr{U}^\circ$, which will be denoted with the same notation. This is a holomorphic vector bundle over $\mathscr{U}^\circ$, which is naturally a subbundle of $\mu^* \Omega_{B^\circ}^1$. We then define a holomorphic subbundle $\mathscr{V} \subset \mu^* TB^\circ$ over $\mathscr{U}^\circ$ as the annihilator of $\mathscr{V}^\sharp$, namely

$$\mathscr{V} = \{ v \in \mu^* TB^\circ \mid \gamma(v) = 0, \text{ for all } \gamma \in \mathscr{V}^\sharp \}.$$

For any $t \in \mathscr{K}$ we have that the restriction of $\mathscr{V}$ to $\mathscr{U}_t$ equals the factor $\mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_{n-\ell})$ in the splitting (15) for $\nu_t^* TB$. Observe that since the pullback morphism $\nu_t^* \Omega_B^1 \to \Omega_{\mathbb{P}^1}^1$ dualizes to a nontrivial morphism $\mathscr{O}(2) \to \nu_t^* TB$, it follows that the tangent direction to the image of ν_t at any point on this curve (which is a line in TB°) when pulled back via μ lies in the fiber of $\mathscr{V}$ over $\mathscr{U}_t$.

We then define a smooth complex subbundle $\mathscr{N} \subset \mu^* TB^\circ$ over $\mathscr{U}^\circ$ as the g_{SK}-orthogonal complement of $\mathscr{V}$, and $\mathscr{N}^\sharp \subset \mu^* \Omega_{B^\circ}^1$ as its annihilator (or equivalently as the g_{SK}-orthogonal complement of $\mathscr{V}^\sharp$), so that over $\mathscr{U}^\circ$ we have the splittings

$$\mu^* TB^\circ = \mathscr{V} \oplus \mathscr{N}, \quad \mu^* \Omega_{B^\circ}^1 = \mathscr{V}^\sharp \oplus \mathscr{N}^\sharp. \tag{18}$$

The bundles $\mathscr{N}, \mathscr{N}^\sharp$ are not yet known to be holomorphic (we will prove this later on). Note also that the (complex antilinear) smooth isomorphism

$$\mu^* TB^\circ \to \mu^* \Omega_{B^\circ}^1, \tag{19}$$

defined by the metric g_{SK} (by "lowering the index" and conjugating) maps $\mathscr{N}$ isomorphically onto $\mathscr{V}^\sharp$.

4.2. *The rigidity theorem*

We have the following rigidity statement:

Theorem 16. *Given a rational curve $\mathscr{U}_t$ for some $t \in \mathscr{K}$, with morphism $\nu_t : \mathbb{P}^1 \to B$, and given a section $u \in H^0(\mathbb{P}^1, \mathcal{V}^\sharp|_{\mathscr{U}_t})$, let $\nu_t^* h_{\mathrm{SK}}$ be the smooth metric on $\nu_t^* \Omega_B^1$ over $\mathbb{P}^1 \setminus \nu_t^{-1}(D)$ induced by g_{SK}, and let $R_{\nu_t^* h_{\mathrm{SK}}}$ be its curvature. Then we have:*

(a) *On $\mathbb{P}^1 \setminus \nu_t^{-1}(D)$ we have*
$$\langle R_{\nu_t^* h_{\mathrm{SK}}}(u), u\rangle_{\nu_t^* h_{\mathrm{SK}}} = 0. \tag{20}$$

(b) *Let ζ be the smooth section of $\mathcal{N}|_{\mathscr{U}_t}$ over $\mathbb{P}^1 \setminus \nu_t^{-1}(D)$ which corresponds to u under* (19), *and let α be a tangent vector to $\nu_t(\mathbb{P}^1)$. Then at any point on $\nu_t(\mathbb{P}^1) \cap B^\circ$ the curvature tensor of g_{SK} satisfies*
$$R_{\alpha\bar{\alpha}\zeta\bar{\zeta}} = 0, \tag{21}$$
and hence
$$\Xi(\alpha, \zeta, \beta) = 0, \quad \text{for all } \beta \in TB^\circ. \tag{22}$$

(c) *For ζ as in (b), and for any section $v \in H^0(\mathbb{P}^1, \mathcal{V}|_{\mathscr{U}_t})$, at any point on $\nu_t(\mathbb{P}^1) \cap B^\circ$ we have*
$$R_{v\bar{v}\zeta\bar{\zeta}} = 0, \tag{23}$$
as well as
$$\Xi(v, \zeta, \beta) = 0, \quad \text{for all } \beta \in TB^\circ. \tag{24}$$

(d) *Every section $u \in H^0(\mathbb{P}^1, \mathcal{V}^\sharp|_{\mathscr{U}_t})$ is parallel on $\mathbb{P}^1 \setminus \nu_t^{-1}(D)$ with respect to the Chern connection ∇ induced by ω_{SK}.*

(e) *The splitting $\nu_t^* \Omega_B^1 = \mathcal{V}^\sharp|_{\mathscr{U}_t} \oplus \mathcal{N}^\sharp|_{\mathscr{U}_t}$ is preserved by ∇.*

Proof.

(a). Equip $\mathcal{V}^\sharp|_{\mathscr{U}_t}$ with the smooth metric h on $\mathbb{P}^1 \setminus \nu_t^{-1}(D)$ induced by ω_{SK} via $\mathcal{V}^\sharp|_{\mathscr{U}_t} \hookrightarrow \nu_t^* \Omega_B^1 \to \Omega_B^1$. Since ω_{SK} has nonnegative bisectional curvature, the induced metric on Ω_B^1 (and hence also the one on $\nu_t^* \Omega_B^1$) is Griffiths nonpositively curved, and since curvature decreases in subbundles, the metric h is also Griffiths nonpositively curved.

As in (14), on $\mathbb{P}^1 \setminus \nu_t^{-1}(D)$ we have
$$\begin{aligned} i\partial\bar{\partial} \log |u|_h^2 &= \frac{|\nabla u|_h^2}{|u|_h^2} - \frac{|\langle \nabla u, u\rangle_h|^2}{|u|_h^4} - \frac{\langle R_h(u), u\rangle_h}{|u|_h^2} \\ &\geqslant -\frac{\langle R_h(u), u\rangle_h}{|u|_h^2} \\ &\geqslant 0. \end{aligned} \tag{25}$$

Thus $\log |u|_h^2$ is psh on $\mathbb{P}^1 \setminus \nu_t^{-1}(D)$, and again using (3) we see that
$$\sup_{\mathbb{P}^1 \setminus \nu_t^{-1}(D)} \log |u|_h^2 \leqslant C + \sup_{\mathbb{P}^1 \setminus \nu_t^{-1}(D)} \log |u|_{\nu_t^* h_B}^2 < \infty,$$
where h_B is the smooth metric on Ω_B^1 induced by ω_B. Thus $\log |u|_h^2$ is bounded above, and by the Grauert–Remmert extension theorem [15] it extends to a global psh function on $\mathbb{P}^1$, which is therefore constant.

Thus $|u|_h^2$ is a nonzero constant, and from (25) we deduce that
$$\langle R_h(u), u\rangle_h = 0, \tag{26}$$
on $\mathbb{P}^1 \setminus \nu_t^{-1}(D)$. But using again the curvature decreasing property, we have
$$0 = \langle R_h(u), u\rangle_h \leqslant \langle R_{\nu_t^* h_{\mathrm{SK}}}(u), u\rangle_{\nu_t^* h_{\mathrm{SK}}} \leqslant 0,$$

and so

$$\langle R_{\nu_t^* h_{\mathrm{SK}}}(u), u\rangle_{\nu_t^* h_{\mathrm{SK}}} = 0, \tag{27}$$

on $\mathbb{P}^1\backslash\nu_t^{-1}(D)$, which proves (20).

(b). Since $\alpha \in TB^\circ$ is a tangent vector to $\nu_t(\mathbb{P}^1)$ and since u is equal to the image of ζ under (19), we have

$$0 = \langle R_{\nu_t^* h_{\mathrm{SK}}}(u), u\rangle_{\nu_t^* h_{\mathrm{SK}}} = -R_{\alpha\bar{\alpha}\zeta\bar{\zeta}},$$

which proves (21). The identity (22) is then a consequence of (1).

(c). Given $v \in H^0(\mathbb{P}^1, \mathcal{V}|_{\mathcal{U}_t})$ and a point $x \in \nu_t(\mathbb{P}^1) \cap B^\circ$, we can find a holomorphic family $\{\nu_s\}_{s\in\Delta}$ of rational curves in $\mathcal{K}$ that pass through x, with tangent vectors α_s at x (with $\Delta \subset \mathcal{K}$ a small disc in some chart centered at our original point $t \in \mathcal{K}$), and such that $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=t}\alpha_s = v(x)$. Let $w = \frac{\mathrm{d}}{\mathrm{d}s}\nu_s$ be the first-order deformation (holomorphic) vector field on this family. When restricted to each $\mathcal{U}_s$, w is a section of

$$\nu_s^* TB^\circ = \mathcal{V}\big|_{\mathcal{U}_s} \oplus \mathcal{N}\big|_{\mathcal{U}_s} \cong \bigoplus_i \mathcal{O}(a_i) \oplus \mathcal{O}^{\oplus \ell},$$

and since $w(x) = 0$, it must be a section of the $\bigoplus_i \mathcal{O}(a_i)$ factors, namely a section of $\mathcal{V}\big|_{\mathcal{U}_s}$. Pick a smooth family U of 1-forms on this family, i.e. a C^∞ section of the relative cotangent bundle, with $u_s := U|_{\mathcal{U}_s} \in H^0(\mathbb{P}^1, \mathcal{V}^\sharp\big|_{\mathcal{U}_s})$, and with $u_t = u$. Then by definition along ν_s we have

$$\iota_w U\big|_{\mathcal{U}_s} \equiv 0,$$

for all $s \in \Delta$, and so along ν_t we have

$$L_w U\big|_{\mathcal{U}_t} = (d\iota_w U)\big|_{\mathcal{U}_t} + (\iota_w dU)\big|_{\mathcal{U}_t} = (\iota_w dU)\big|_{\mathcal{U}_t},$$

which vanishes at x since $w(x) = 0$.

We now use this to prove (24), which by (1) implies (23). For this, let $\zeta_s, s \in \Delta$, be the smooth section of $\mathcal{N}\big|_{\mathcal{U}_s}$ over $\mathbb{P}^1\backslash\nu_s^{-1}(D)$ which maps to u_s under (19), and recall that from (22) at x we have

$$\Xi_x(\alpha_s, \zeta_s, \beta) = 0,$$

for all $s \in \Delta$. Taking $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=t}$ of this, we get

$$0 = \Xi_x(v, \zeta, \beta) + \Xi_x(\alpha, L_w\zeta, \beta). \tag{28}$$

Now at x we have that $L_w\zeta$ is the vector that maps to $L_w U$ under (19), since at x the metric g_{SK} does not get differentiated as it does not depend on s. Since we have shown that $(L_w U)(x) = 0$, we deduce that $(L_w\zeta)(x) = 0$, and so (24) follows from (28).

(d). Given a section $u \in H^0(\mathbb{P}^1, \nu_t^*\mathcal{V}^\sharp)$, an analogous computation as in (a) gives

$$0 = i\partial\bar{\partial}|u|_h^2 = |\nabla u|_h^2 - \langle R_h(u), u\rangle_h = |\nabla u|_h^2, \tag{29}$$

and so we conclude that $\nabla u = 0$ on $\mathbb{P}^1\backslash\nu_t^{-1}(D)$.

(e). This is a direct consequence of part (d) and [33, Proposition 1.4.18]. □

Given $x \in \mathcal{U}^\circ$ and $v \in \mathcal{V}_x, \zeta \in \mathcal{N}_x$, recall from (18) that

$$\mathcal{V}_x \oplus \mathcal{N}_x = T_{\mu(x)}B^\circ, \tag{30}$$

so we can view v and ζ also as tangent vectors in B°. With this in mind, we have the following useful corollary:

Corollary 17. *Let $x \in \mathscr{U}^\circ$, and let $v \in \mathcal{V}_x, \zeta \in \mathcal{N}_x$. Then at $\mu(x) \in B^\circ$ the curvature of the metric g_{SK} satisfies*

$$R_{v\bar{v}\zeta\bar{\zeta}} = 0, \tag{31}$$

as well as

$$\Xi(v,\zeta,\beta) = 0, \quad \textit{for all } \beta \in T_{\mu(x)}B^\circ. \tag{32}$$

Proof. Let $t \in \mathscr{K}$ be such that the corresponding rational curve $\mathscr{U}_t$ contains x, and as usual denoted by $\nu_t : \mathbb{P}^1 \to B$ the corresponding morphism. Since $\mathcal{V}|_{\mathscr{U}_t} \cong \oplus_i \mathscr{O}(a_i)$, $a_i > 0$, is a globally generated vector bundle, we can find a global section $V \in H^0(\mathbb{P}^1, \mathcal{V}|_{\mathscr{U}_t})$ such that $V(x) = v$. Let then $u \in \mathcal{V}^\sharp_x$ be the covector which is the image of ζ under (19). Since $\mathcal{V}^\sharp|_{\mathscr{U}_t} \cong \mathscr{O}^{\oplus \ell}$ is a trivial vector bundle, we can find a global section $U \in H^0(\mathbb{P}^1, \mathcal{V}^\sharp|_{\mathscr{U}_t})$ such that $U(x) = u$. Then Theorem 16 (c) applies to U and V, and (31), (32) follow from (23), (24). □

5. The Ricci curvature in the direction of $\mathcal{N}$

Given $x \in \mathscr{U}^\circ$ and vectors $v \in \mathcal{V}_x, \zeta \in \mathcal{N}_x$ (which we can also view as tangent vectors in $T_{\mu(x)}B^\circ$ using (30)), Corollary 17 shows that at $\mu(x)$ the Riemann curvature tensor of g_{SK} satisfies

$$R_{v\bar{v}\zeta\bar{\zeta}} = 0.$$

As customary, we define the "rough Laplacian" of the Riemann curvature tensor of g_{SK}, evaluated on v, ζ by

$$\Delta R_{v\bar{v}\zeta\bar{\zeta}} = \frac{1}{2}\left(\sum_i \nabla_i \nabla_{\bar{i}} R_{v\bar{v}\zeta\bar{\zeta}} + \sum_i \nabla_{\bar{i}} \nabla_i R_{v\bar{v}\zeta\bar{\zeta}}\right),$$

where $\{e_i\}$ is a local unitary frame.

The following is the main result of this section:

Theorem 18. *Given $x \in \mathscr{U}^\circ$ and $v \in \mathcal{V}_x, \zeta \in \mathcal{N}_x$, then at $\mu(x)$ we have*

$$\Delta R_{v\bar{v}\zeta\bar{\zeta}} = 0, \quad R_{v\bar{\zeta}\beta\bar{\gamma}} = 0, \quad \textit{for all } \beta, \gamma \in T_{\mu(x)}B^\circ. \tag{33}$$

Let $t \in \mathscr{K}$ be such that the corresponding rational curve $\mathscr{U}_t$ contains x, and as usual denoted by $\nu_t : \mathbb{P}^1 \to B$ the corresponding morphism. As in the proof of Corollary 17, we can extend v to a section $v \in H^0(\mathbb{P}^1, \mathcal{V}|_{\mathscr{U}_t})$ and we can find a section $u \in H^0(\mathbb{P}^1, \mathcal{V}^\sharp|_{\mathscr{U}_t})$ such that the image of u under (19) is a smooth section $\zeta \in \mathcal{N}|_{\mathscr{U}_t}$ over $\mathbb{P}^1 \backslash \nu_t^{-1}(D)$ which extends the given vector ζ. The Ricci curvature $R_{\zeta\bar{\zeta}}$ along this curve and evaluated at ζ will also be denoted by $\mathrm{Ric}_{g_{\mathrm{SK}}}(u, \bar{u})$, which is a smooth function on $\mathbb{P}^1 \backslash \nu_t^{-1}(D)$.

We wish to show that $\mathrm{Ric}_{g_{\mathrm{SK}}}(u, \bar{u})$ is a constant function on $\mathbb{P}^1 \backslash \nu^{-1}(D)$. We will proceed in steps.

5.1. *Subharmonicity of* $\mathrm{Ric}_{g_{\mathrm{SK}}}(u, \bar{u})$

To start, we prove the following:

Proposition 19. *The function $\mathrm{Ric}_{g_{\mathrm{SK}}}(u, \bar{u})$ on $\mathbb{P}^1 \backslash \nu_t^{-1}(D)$ is subharmonic.*

Proof. On B° define for $0 \leqslant s \ll 1$

$$g_s = g_{\mathrm{SK}} - s\,\mathrm{Ric}_{g_{\mathrm{SK}}}.$$

It is clear that given any compact $K \Subset B^\circ$ there is some $0 < s_K \ll 1$ such that g_s is a Kähler metric on K for $0 \leqslant s \leqslant s_K$.

Standard direct computations (cf. [44, p. 185]) show that given any $x \in B^\circ$ and two nonzero $(1,0)$ tangent vectors v,ζ at x, we have the evolution equation at x and $s=0$ for the bisectional curvature of g_s evaluated along v and ζ

$$\frac{\partial}{\partial s}\Big|_{s=0} R(g_s)_{v\bar{v}\zeta\bar{\zeta}} = \Delta R_{v\bar{v}\zeta\bar{\zeta}} + F(R)_{v\bar{v}\zeta\bar{\zeta}}, \tag{34}$$

where, as in Mok [44], we define

$$F(R)_{v\bar{v}\zeta\bar{\zeta}} = \sum_{\mu,\nu} R_{v\bar{v}\mu\bar{\nu}} R_{\zeta\bar{\zeta}\nu\bar{\mu}} - \sum_{\mu,\nu} |R_{v\bar{\mu}\zeta\bar{\nu}}|^2 + \sum_{\mu,\nu} |R_{v\bar{\zeta}\mu\bar{\nu}}|^2 - \mathrm{Re}\left(R_{v\bar{\mu}} R_{\mu\bar{v}\zeta\bar{\zeta}} + R_{\zeta\bar{\mu}} R_{v\bar{v}\mu\bar{\zeta}}\right).$$

Equation (34) is identical to the corresponding evolution of the bisectional curvature in the directions v,ζ along the Kähler–Ricci flow, see [44]. Thanks to the crucial Lemma 20 below, we see that

$$\frac{\partial}{\partial s}\Big|_{s=0} R(g_s)_{v\bar{v}\zeta\bar{\zeta}}(x) \geqslant 0. \tag{35}$$

Equip $\nu_t^*\Omega_B^1$ over the compact set $\nu_t^{-1}(K)$ with the Hermitian metric h_s induced by g_s. At any point $y \in \nu_t^{-1}(K)$ for $0 \leqslant s \leqslant s_K$, using the argument in (25), we have

$$i\partial\bar{\partial}\log|u|^2_{h_s} + \frac{\langle R_{h_s}(u), u\rangle_{h_s}}{|u|^2_{h_s}} \geqslant 0. \tag{36}$$

We know from Theorem 16 (d), that u is parallel with respect to h_0 (the metric induced by g_{SK}), hence (assuming without loss that u is nontrivial) we can scale and assume without loss that $|u|^2_{h_0} \equiv 1$ on $\mathbb{P}^1\setminus\nu_t^{-1}(D)$. On the other hand, from Theorem 16 (a), we know that (20) holds, and so

$$\langle R_{h_0}(u), u\rangle_{h_0} = 0.$$

Thus the LHS of (36) vanishes at y for $s=0$ and is nonnegative for $0 \leqslant s \leqslant s_K$, hence at y we have

$$\begin{aligned} 0 &\leqslant \frac{\partial}{\partial s}\Big|_{s=0}\left(i\partial\bar{\partial}\log|u|^2_{h_s} + \frac{\langle R_{h_s}(u), u\rangle_{h_s}}{|u|^2_{h_s}}\right) \\ &= i\partial\bar{\partial}\left(\frac{\partial}{\partial s}\Big|_{s=0}|u|^2_{h_s}\right) + \frac{\partial}{\partial t}\Big|_{t=0}\langle R_{h_s}(u), u\rangle_{h_s}, \end{aligned} \tag{37}$$

and writing $u = u_j dz^j$ and $|u|^2_{h_s} = u_i\overline{u_j}g_s^{i\bar{j}}$, observe that

$$\frac{\partial}{\partial s}\Big|_{s=0}\left(u_i\overline{u_j}g_s^{i\bar{j}}\right) = -u_i\overline{u_j}g_{\mathrm{SK}}^{i\bar{s}}g_{\mathrm{SK}}^{r\bar{j}}\frac{\partial}{\partial s}\Big|_{s=0}g_{s,r\bar{s}} = u_i\overline{u_j}g_{\mathrm{SK}}^{i\bar{s}}g_{\mathrm{SK}}^{r\bar{j}}R_{r\bar{s}} = \mathrm{Ric}_{g_{\mathrm{SK}}}(u,\bar{u}).$$

Furthermore, we can write

$$\langle R_{h_s}(u), u\rangle_{h_s} = -R(g_s)_{v\bar{v}i\bar{j}}g_s^{i\bar{q}}g_s^{p\bar{j}}u_p\overline{u_q},$$

so

$$\frac{\partial}{\partial s}\Big|_{s=0}\langle R_{h_s}(u), u\rangle_{h_s} = -\frac{\partial}{\partial s}\Big|_{s=0}R(g_s)_{v\bar{v}\zeta\bar{\zeta}} - R_{v\bar{v}i\bar{\zeta}}R_{\zeta\bar{i}} - R_{v\bar{v}\zeta\bar{i}}R_{\bar{\zeta}i},$$

but the last two terms vanish since using (1) and (24), we can write

$$R_{v\bar{v}i\bar{\zeta}} = \Xi_{viq}\overline{\Xi_{v\zeta q}} = 0, \quad R_{v\bar{v}\zeta\bar{i}} = \Xi_{v\zeta q}\overline{\Xi_{viq}} = 0,$$

and putting these all together gives

$$\begin{aligned} 0 &\leqslant i\partial\bar{\partial}\left(\mathrm{Ric}_{g_{\mathrm{SK}}}(u,\bar{u})\right) - \frac{\partial}{\partial s}\Big|_{s=0}R(g_s)_{v\bar{v}\zeta\bar{\zeta}} \\ &\leqslant i\partial\bar{\partial}\left(\mathrm{Ric}_{g_{\mathrm{SK}}}(u,\bar{u})\right), \end{aligned} \tag{38}$$

using (35). Since $K \Subset B^\circ$ is arbitrary, this shows that the function $\mathrm{Ric}_{g_{\mathrm{SK}}}(u,\bar{u})$ is subharmonic on $\mathbb{P}^1\setminus\nu_t^{-1}(D)$. □

We used the following lemma, which is the analog of "condition (♯)" in Mok, but the proof here is substantially easier:

Lemma 20. *In the setting of Theorem 18, at* $\mu(x)$ *we have*

$$\Delta R_{v\bar{v}\zeta\bar{\zeta}} \geqslant 0, \quad F(R)_{v\bar{v}\zeta\bar{\zeta}} \geqslant 0.$$

Proof. Recall from (1) that

$$R_{i\bar{j}k\bar{\ell}} = g^{p\bar{q}}_{\mathrm{SK}} \Xi_{ikp}\overline{\Xi_{j\ell q}}.$$

From (31) we then see that at $\mu(x)$ we have

$$0 = R_{v\bar{v}\zeta\bar{\zeta}} = \sum_{\beta} |\Xi_{v\zeta\beta}|^2,$$

and so $\Xi_{v\zeta\beta}(x) = 0$ for all $\beta \in T_{\mu(x)}B^{\circ}$, and furthermore for all $\mu, \nu \in T_{\mu(x)}B^{\circ}$

$$0 = \sum_{\beta} \Xi_{v\zeta\beta}\overline{\Xi_{\mu\nu\beta}} = R_{v\bar{\mu}\zeta\bar{\nu}}.$$

Now take the definition of ΔR and use (1) and the fact that Ξ is holomorphic to get

$$\Delta R_{v\bar{v}\zeta\bar{\zeta}} = \sum_{i,p} |\nabla_i \Xi_{v\zeta p}|^2 + \sum_{i,p} \mathrm{Re}(\overline{\Xi_{v\zeta p}} \nabla_{\bar{i}}\nabla_i \Xi_{v\zeta p}) = \sum_{i,p} |\nabla_i \Xi_{v\zeta p}|^2 \geqslant 0,$$

since $\Xi_{v\zeta p}(x) = 0$. For the $F(R)$ term, from its definition we see that at $\mu(x)$ we have

$$F(R)_{v\bar{v}\zeta\bar{\zeta}} = \sum_{\mu,\nu} R_{v\bar{v}\mu\bar{\nu}} R_{\zeta\bar{\zeta}\nu\bar{\mu}} + \sum_{\mu,\nu} |R_{v\bar{\zeta}\mu\bar{\nu}}|^2.$$

As in Mok [44, (7)], if we pick $\{e_{\mu}\}$ a unitary basis of eigenvectors of the Hermitian form $H_v(\mu,\nu) = R_{v\bar{v}\mu\bar{\nu}}$, then in this basis we see that

$$F(R)_{v\bar{v}\zeta\bar{\zeta}} = \sum_{\mu} R_{v\bar{v}\mu\bar{\mu}} R_{\zeta\bar{\zeta}\mu\bar{\mu}} + \sum_{\mu,\nu} |R_{v\bar{\zeta}\mu\bar{\nu}}|^2 \geqslant 0. \tag{39}$$

□

5.2. *Constancy of* $\mathrm{Ric}_{g_{\mathrm{SK}}}(u,\bar{u})$

The next step is the following:

Proposition 21. *The function* $\mathrm{Ric}_{g_{\mathrm{SK}}}(u,\bar{u})$ *on* $\mathbb{P}^1 \backslash \nu_t^{-1}(D)$ *is constant.*

Proof. Since the function $\mathrm{Ric}_{g_{\mathrm{SK}}}(u,\bar{u})$ on $\mathbb{P}^1 \backslash \nu_t^{-1}(D)$ is subharmonic by Proposition 19, it suffices to show that it is bounded.

Recall that, using (30), our sections v, ζ can be viewed as vector fields along $\nu_t(\mathbb{P}^1) \cap B^{\circ}$. Our first claim is that for every $y \in \nu_t(\mathbb{P}^1) \cap B^{\circ}$ and local sections v of $\mathcal{V}$ and ζ of $\mathcal{N}$ near y, we have

$$R_{v\bar{\zeta}} = 0. \tag{40}$$

Indeed, recall from (1) that

$$R_{v\bar{\zeta}} = \sum_{p,q} \Xi_{vpq}\overline{\Xi_{\zeta pq}},$$

where $\{e_p\}$ is a g_{SK}-unitary frame at our point y. Since $\mu^* TB^{\circ} = \mathcal{V} \oplus \mathcal{N}$, we may choose the frame so that $e_j \in \mathcal{V}$ for $1 \leqslant j \leqslant n-\ell$, and $e_j \in \mathcal{N}$ for $n-\ell+1 \leqslant j \leqslant n$. Recalling from (32) that $\Xi_{uvw} = 0$ whenever $u \in \mathcal{V}$ and $v \in \mathcal{N}$, we see that $\Xi_{vpq} = 0$ except possibly when $1 \leqslant p, q \leqslant n-\ell$, so that

$$R_{v\bar{\zeta}} = \sum_{p,q=1}^{n-\ell} \Xi_{vpq}\overline{\Xi_{\zeta pq}} = 0,$$

since $\Xi_{\zeta pq} = 0$ when $1 \leqslant p, q \leqslant n-\ell$, proving our claim.

Recall that, as explained earlier, we may assume that $\nu_t(\mathbb{P}^1)$ intersects D only at regular points of D and that these intersections are transverse. To prove the boundedness of $\mathrm{Ric}_{g_{\mathrm{SK}}}(u,\bar{u})$ it suffices to prove near any of the finitely many points in $\nu_t^{-1}(D)$. Let y be such a point, and choose an open neighborhood U of $z = \nu_t(y)$ in B with local holomorphic coordinates centered at z such that $D\cap U=\{z_1=0\}$ and $\nu_t(\mathbb{P}^1)\cap U=\{z_2=\cdots=z_n=0\}$, so that ∂_1 is tangent to the rational curve while $\partial_2,\dots,\partial_n$ are tangent to D. We will work on $\nu_t^{-1}(U\cap\{z_1\neq 0\})$ which in our chart is identified with $\{z_1\neq 0, z_2=\cdots=z_n=0\}=:V$.

Thanks to Proposition 11 we know that on V we have

$$0\leqslant R_{i\bar{i}}\leqslant C,\quad 2\leqslant i\leqslant n, \tag{41}$$

$$0\leqslant R_{1\bar{1}}\leqslant \frac{C}{|z_1|^2}. \tag{42}$$

Using (3), together with the fact that u is a holomorphic section on all of $\mathbb{P}^1$, we see that

$$\sup_V |\zeta|^2_{\nu_t^* g_B}\leqslant C\sup_V |u|^2_{\nu_t^* g_B}<\infty. \tag{43}$$

In our coordinates we can write

$$\zeta=\zeta^1\partial_1+\sum_{j\geqslant 2}\zeta^j\partial_j=:\zeta^1\partial_1+\zeta_D,$$

and the function ζ^1 is equal to $\langle \mathrm{d}z_1, u\rangle_{g_{\mathrm{SK}}}$. From (43) we see that

$$\sup_V |\zeta_D|^2_{\nu_t^* g_B}<\infty, \tag{44}$$

and from this and (41) we see that on V we have

$$0\leqslant R_{\zeta_D\overline{\zeta_D}}\leqslant C. \tag{45}$$

Since ∂_1 is the tangent vector to $\nu_t(\mathbb{P}^1)$, it belongs to $\mathcal{V}\big|_{\mathcal{U}_t}$. On the other hand ζ belongs to $\mathcal{N}\big|_{\mathcal{U}_t}$, hence (40) (restricted to V) gives

$$0=R_{1\bar{\zeta}}(z_1)=\overline{\zeta^1(z_1)}R_{1\bar{1}}(z_1)+R_{1\overline{\zeta_D}}(z_1),$$

and since $\mathrm{Ric}_{g_{\mathrm{SK}}}\geqslant 0$ on $\{z_1\neq 0\}$, Cauchy–Schwarz together with (41) and (45) give

$$|\zeta^1(z_1)|R_{1\bar{1}}(z_1)=|R_{1\overline{\zeta_D}}(z_1)|\leqslant R_{1\bar{1}}(z_1)^{\frac12}R_{\zeta_D\overline{\zeta_D}}(z_1)^{\frac12}\leqslant CR_{1\bar{1}}(z_1)^{\frac12},$$

i.e.

$$|\zeta^1(z_1)|R_{1\bar{1}}(z_1)^{\frac12}\leqslant C,$$

and using again that $\mathrm{Ric}_{g_{\mathrm{SK}}}\geqslant 0$, together with (45) we can estimate

$$0\leqslant R_{\zeta\bar{\zeta}}(z_1)\leqslant C|\zeta^1(z_1)|^2R_{1\bar{1}}(z_1)+CR_{\zeta_D\overline{\zeta_D}}(z_1)\leqslant C,$$

as desired. □

We can now conclude the proof of Theorem 18, by showing that at $\mu(x)$ we have

$$\Delta R_{\upsilon\bar{\upsilon}\zeta\bar{\zeta}}=0,\quad F(R)_{\upsilon\bar{\upsilon}\zeta\bar{\zeta}}=0. \tag{46}$$

Indeed, Proposition 21 shows that the function $\mathrm{Ric}_{g_{\mathrm{SK}}}(u,\bar{u})$ on $\mathbb{P}^1\backslash\nu_t^{-1}(D)$ is constant, hence going back to (38) and recalling (35) and (34) shows that

$$0=\frac{\partial}{\partial s}\Big|_{s=0}R(g_s)_{\upsilon\bar{\upsilon}\zeta\bar{\zeta}}=\Delta R_{\upsilon\bar{\upsilon}\zeta\bar{\zeta}}+F(R)_{\upsilon\bar{\upsilon}\zeta\bar{\zeta}}.$$

Recalling Lemma 20, we see that (46) holds. To finally deduce from (46) that the last equality in (33) holds, it suffices to plug in the fact that $F(R)_{\upsilon\bar{\upsilon}\zeta\bar{\zeta}}=0$ into (39), and see that

$$\sum_{\mu,\nu}|R_{\upsilon\bar{\zeta}\mu\bar{\nu}}|^2=0,$$

as desired.

6. Constructing a parallel subbundle of $\mu^* TB^\circ$

Recall that above we have constructed a decomposition $\mu^* TB^\circ = \mathscr{V} \oplus \mathscr{N}$ over $\mathscr{U}^\circ$, where $\mathscr{V}$ is a nontrivial holomorphic subbundle, which is not equal to $\mu^* TB^\circ$ whenever $B \not\cong \mathbb{P}^n$.

The following is then our main theorem (Theorem 3):

Theorem 22. *The holomorphic subbundle $\mathscr{V} \subset \mu^* TB^\circ$ over $\mathscr{U}^\circ$ is preserved by ∇, the pullback of the Levi-Civita connection of ω_{SK}.*

Recall that by Theorem 5 f is either of maximal variation or isotrivial. The proof of Theorem 22 will be quite different in these two cases.

Observe that after Theorem 22 is proved, it follows that the orthogonal complement $\mathscr{N} \subset \mu^* TB^\circ$ is also a holomorphic subbundle, preserved by ∇, see e.g. [33, Proposition 1.4.18], and the same holds for their duals $\mathscr{V}^\sharp, \mathscr{N}^\sharp \subset \mu^* \Omega^1_{B^\circ}$.

6.1. *Maximal Variation Case*

In this section we give the proof of Theorem 22 in the case when f has maximal variation. Recall from Corollary 6 that in this case g_{SK} has positive Ricci curvature on B°.

We work at a point $x \in \mathscr{U}^\circ$. Let v be a local holomorphic section of $\mathscr{V}$ near x, and let $\gamma : (-\varepsilon, \varepsilon) \to \mathscr{U}^\circ$ be a smooth curve with $\gamma(0) = x, \dot\gamma(0) = \eta \neq 0$. The goal of Theorem 22 is then to show that $\nabla_\eta v \in \mathscr{V}$. Using the decomposition $\mu^* TB^\circ = \mathscr{V} \oplus \mathscr{N}$, we can write

$$\nabla_\eta v = -\xi - \zeta, \quad \xi \in \mathscr{V}_x, \zeta \in \mathscr{N}_x,$$

(the minus sign is only to match the notation in Mok [44]), so we wish to show that $\zeta = 0$. The following argument is a modification of a result of Mok [44, Proposition 3.1′], specifically of equation (21) on p. 211:

Proposition 23. *At $\mu(x)$ we have*

$$R_{\zeta\bar\zeta\zeta'\overline{\zeta'}} = 0, \tag{47}$$

for all $\zeta' \in \mathscr{N}_x$.

Here and in the following we are again using (30) to view ζ, ζ' also as tangent vectors in $T_{\mu(x)} B^\circ$. Also, since ∇ is the pullback connection, when taking ∇_v for some $v \in T\mathscr{U}^\circ$ it is really only $\mu_*(v) \in TB^\circ$ that enters.

Proof. For $t \in (-\varepsilon, \varepsilon)$, let $\beta(t)$ be the parallel transport of $v(x)$ along γ, let $v(t) = v|_{\gamma(t)}$, and define $\xi(t), \zeta(t)$ by

$$\beta(t) = v(t) + t\xi(t) + t\zeta(t), \quad \xi(t) \in \mathscr{V}_{\gamma(t)}, \zeta(t) \in \mathscr{N}_{\gamma(t)},$$

so that

$$0 = \nabla_\eta \beta(0) = \nabla_\eta v + \xi(0) + \zeta(0),$$

and so we see that $\xi(0) = \xi, \zeta(0) = \zeta$. Given an arbitrary $\zeta' \in \mathscr{N}_x$, let $\chi(t)$ be the parallel transport of ζ' along γ, so that $\chi(0) = \zeta'$ and $\nabla_{\dot\gamma(t)} \chi(t) = 0$. We can also write

$$\chi(t) = \zeta'(t) + t\theta(t), \quad \zeta'(t) \in \mathscr{N}_{\gamma(t)}, \theta(t) \in \mathscr{V}_{\gamma(t)},$$

and $\zeta'(0) = \zeta'$. We can expand at the point $\gamma(t)$

$$\begin{aligned} R_{\beta(t)\overline{\beta(t)}\chi(t)\overline{\chi(t)}} = R_{v\bar v\zeta'\overline{\zeta'}} &+ t\Big(2\operatorname{Re} R_{v\bar v\zeta'\bar\theta} + 2\operatorname{Re} R_{v\bar\xi\zeta'\overline{\zeta'}} + 2\operatorname{Re} R_{v\bar\zeta\zeta'\overline{\zeta'}}\Big) \\ &+ t^2\Big(R_{v\bar v\theta\bar\theta} + 2\operatorname{Re} R_{v\bar\xi\zeta'\bar\theta} + 2\operatorname{Re} R_{v\bar\zeta\zeta'\bar\theta} + 2\operatorname{Re} R_{v\bar\xi\theta\overline{\zeta'}} + 2\operatorname{Re} R_{v\bar\zeta\theta\overline{\zeta'}} \\ &\qquad + R_{\xi\bar\xi\zeta'\overline{\zeta'}} + R_{\zeta\bar\zeta\zeta'\overline{\zeta'}} + 2\operatorname{Re} R_{\xi\bar\zeta\zeta'\overline{\zeta'}}\Big) + O(t^3), \end{aligned}$$

where $O(t^3)$ denotes a vector-valued function of length bounded above by Ct^3. Recalling (1), we can express the curvature tensor in terms of Ξ, and since $\Xi(v,\zeta',\beta)=\Xi(\xi,\zeta',\beta)=0$ for all $\zeta'\in\mathcal{N}_x$ and all $\beta\in T_{\mu(x)}B^\circ$ (by Corollary 17), many terms in this expansion vanish. Using furthermore that $R_{v\bar{\zeta}\beta\bar{\delta}}=0$ for all $\beta,\delta\in T_{\mu(x)}B^\circ$ (by Theorem 18), the expression finally reduces to

$$R_{\beta(t)\overline{\beta(t)}\chi(t)\overline{\chi(t)}}=t^2\Big(R_{v\bar{v}\theta\bar{\theta}}+R_{\zeta\bar{\zeta}\zeta'\overline{\zeta'}}\Big)+O(t^3).$$

Defining (similarly to Mok)

$$A=R_{v\bar{v}\theta\bar{\theta}}+R_{\zeta\bar{\zeta}\zeta'\overline{\zeta'}},$$

and since the bisectional curvature is nonnegative, we have $R_{v\bar{v}\theta\bar{\theta}}\geqslant 0$, and so

$$A\geqslant R_{\zeta\bar{\zeta}\zeta'\overline{\zeta'}}. \tag{48}$$

At this point notice that

$$A=\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0}R_{\beta(t)\overline{\beta(t)}\chi(t)\overline{\chi(t)}}=\nabla^2_{\eta\eta}R_{v\bar{v}\zeta'\overline{\zeta'}}, \tag{49}$$

using that $\beta(t),\chi(t)$ are parallel along γ and that ∇ is a pullback connection. On the other hand we claim that at x we have

$$\nabla^2_{ww}R_{v\bar{v}\zeta'\overline{\zeta'}}\geqslant 0,$$

for all real tangent vectors w at x. Indeed, pick a curve in $\mathscr{U}^\circ$ passing through x and tangent to w, and let $\tilde{v}(t),\widetilde{\zeta}'(t)$ be the parallel transport of v,ζ' along this curve, then $R_{\tilde{v}(t)\overline{\tilde{v}(t)}\tilde{\zeta}'(t)\overline{\tilde{\zeta}'(t)}}\geqslant 0$, and $R_{v\bar{v}\zeta'\overline{\zeta'}}=0$ by Corollary 17, and so

$$0\leqslant\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0}R_{\tilde{v}(t)\overline{\tilde{v}(t)}\tilde{\zeta}'(t)\overline{\tilde{\zeta}'(t)}}=\nabla^2_{ww}R_{v\bar{v}\zeta'\overline{\zeta'}},$$

as claimed. But recall that Theorem 18 showed that $\Delta R_{v\bar{v}\zeta'\zeta'}=0$, and since this is an average of terms of the form $\nabla^2_{ww}R_{v\bar{v}\zeta'\overline{\zeta'}}$ as $\mu_*(w)$ varies among all g_{SK}-unit tangent vectors at $\mu(x)$, we see that necessarily $\nabla^2_{ww}R_{v\bar{v}\zeta'\overline{\zeta'}}=0$ for all w. Using (48) and (49) we get

$$0=\nabla^2_{\eta\eta}R_{v\bar{v}\zeta'\overline{\zeta'}}=2A\geqslant R_{\zeta\bar{\zeta}\zeta'\overline{\zeta'}}\geqslant 0,$$

which proves (47). □

Now that (47) is established, we can show that $\zeta=0$ as follows: combining (47) with (1) gives

$$\Xi(\zeta,\zeta',\beta)=0,$$

for all $\beta\in T_{\mu(x)}B^\circ$ and all $\zeta'\in\mathcal{N}_x$. But thanks to Corollary 17 we also have

$$\Xi(\zeta,\mu,\beta)=0,$$

for all $\beta\in T_{\mu(x)}B^\circ$ and all $\mu\in\mathcal{V}_x$, and since $T_{\mu(x)}B^\circ\cong\mathcal{V}_x\oplus\mathcal{N}_x$, it follows that

$$\Xi(\zeta,\mu,\beta)=0,$$

for all $\mu,\beta\in T_{\mu(x)}B^\circ$. From the formula for the curvature tensor,

$$\mathrm{Ric}_{g_{\mathrm{SK}}}(\zeta,\bar{\zeta})=\sum_{p,q}|\Xi(\zeta,e_p,e_q)|^2=0.$$

Since we assume f of maximal variation, $\mathrm{Ric}_{g_{\mathrm{SK}}}>0$ on B°, and so $\zeta=0$. This concludes the proof of Theorem 22 when f has maximal variation.

6.2. *Isotrivial Case*

In this section we give the proof of Theorem 22 in the case when f is isotrivial, and so ω_{SK} is flat by Corollary 6. We wish to show that the subbundle $\mathcal{V} \subset \mu^* TB^\circ$ is parallel under ∇, and by duality this is equivalent to showing that $\mathcal{V}^\sharp \subset \mu^*\Omega^1_{B^\circ}$ is parallel under ∇. Recall that $\rho : \mathcal{U} \to \mathcal{K}$ is a $\mathbb{P}^1$-bundle. Thus, given $x \in \mathcal{U}^\circ$ and $v \in T_x\mathcal{U}^\circ$, we can decompose $T_x\mathcal{U}^\circ$ as the direct sum of the tangent line to the vertical $\mathbb{P}^1$ direction and a complementary subspace, and thus write $v = v_1 + v_2$, where v_1 is tangent to a rational curve $\mathcal{U}_t$ (for some $t \in \mathcal{K}$) that contains x (which on $\mathcal{U}_t$ corresponds to a point $y \in \mathbb{P}^1$) and v_2 is transverse to $\mathcal{U}_t$. The rational curve morphism will be as usual denoted by $\nu_t : \mathbb{P}^1 \to B$. We may also assume that $\nu_t(\mathbb{P}^1)$ intersects D only at regular points of D. Since ν_t is free, we can deform it in a 1-parameter family $\pi : \mathbb{P}^1 \times \Delta \to B$, with $s \in \Delta$ (where $\Delta \subset \mathcal{K}$ is a small disc in some chart centered at $t \in \mathcal{K}$), such that $\nu_s := \pi(\cdot, s) : \mathbb{P}^1 \to B$ are rational curves in $\mathcal{K}$ which are also not contained in D and such that the first order deformation vector $\frac{\partial}{\partial s}\big|_{s=t}\nu_s \in H^0(\mathbb{P}^1, \nu_t^* TB)$ agrees with v_2 at x. Up to shrinking Δ, we have a natural inclusion $\sigma : \mathbb{P}^1 \times \Delta \hookrightarrow \mathcal{U}$ such that $\mu \circ \sigma = \pi$. The intersection $\sigma(\mathbb{P}^1 \times \Delta) \cap \mathcal{U}^\circ$ is Zariski open in $\sigma(\mathbb{P}^1 \times \Delta)$ and contains the point $(y, 0)$.

We then choose a smooth $(1,0)$ vector field V on $\mathbb{P}^1 \times \Delta$ which restricted to $\mathbb{P}^1 \times \{0\}$ is the first order deformation vector, and so it satisfies $d\sigma_{(y,0)}(V) = v_2$. To prove that $\mathcal{V}^\sharp$ is preserved by ∇_v at x, it will suffice to construct a smooth frame $u_1, \ldots, u_\ell$ for $\sigma^*\mathcal{V}^\sharp$ over $\mathbb{P}^1 \times \Delta$ such that

$$(\nabla_V u_i)(y,0) \in \sigma^*\mathcal{V}^\sharp_x, \quad 1 \leqslant i \leqslant \ell, \tag{50}$$

where ∇ also denotes the pullback connection, since by Theorem 16 (d) we have that along ν_t

$$(\nabla_{v_1} u_i)(x) = 0.$$

For every $s \in \Delta$, $\sigma^*\mathcal{V}^\sharp|_{\mathbb{P}^1 \times \{s\}}$ is a trivial vector bundle of rank ℓ over ν_s, which over $\mathbb{P}^1 \backslash \nu_s^{-1}(D)$ is equipped with the metric induced by ω_{SK}. For each $s \in \Delta$ we can then choose a global holomorphic frame $u_1(s), \ldots, u_\ell(s) \in H^0(\mathbb{P}^1, \sigma^*\mathcal{V}^\sharp|_{\mathbb{P}^1 \times \{s\}})$, smoothly dependent on $s \in \Delta$. Thanks to Theorem 16 (d), each $u_i(s)$ is parallel (with respect to the connection induced by ω_{SK}) over $\mathbb{P}^1 \backslash \nu_s^{-1}(D)$. Varying s, these sections define a smooth frame $u_1, \ldots, u_\ell$ of $\sigma^*\mathcal{V}^\sharp$ over $\mathbb{P}^1 \times \Delta$, which is parallel when restricted to each $(\mathbb{P}^1 \times \{s\}) \cap \pi^{-1}(B^\circ)$. Fix now any $1 \leqslant i \leqslant \ell$, and recall that

$$\pi^*\Omega^1_{B^\circ} = \sigma^*\mathcal{V}^\sharp \oplus \sigma^*\mathcal{N}^\sharp, \tag{51}$$

where $\mathcal{N}^\sharp$ is the annihilator of $\mathcal{N} \subset \mu^* TB^\circ$. Let $\mathcal{P}$ be the g_{SK}-orthogonal projection onto the $\sigma^*\mathcal{N}^\sharp$ factor, which is defined on $\pi^{-1}(B^\circ)$ and consider

$$\mathcal{P}(\nabla_V u_i),$$

a smooth section of $\sigma^*\mathcal{N}^\sharp \subset \pi^*\Omega^1_{B^\circ}$ over $\pi^{-1}(B^\circ)$. Let also $\iota : \mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \times \Delta$ be the embedding $z \mapsto (z, 0)$, so $\pi \circ \iota = \nu_t$.

Lemma 24. *The pullback $\iota^*(\mathcal{P}(\nabla_V u_i))$ to $\mathbb{P}^1 \backslash \nu_t^{-1}(D)$ is a parallel section of $\mathcal{N}^\sharp|_{\mathcal{U}_t}$ over $\mathbb{P}^1 \backslash \nu_t^{-1}(D)$.*

Proof. We work at an arbitrary point in $\mathbb{P}^1 \backslash \nu_t^{-1}(D)$, let W be any local holomorphic vector field near our point which is tangent to the $\mathbb{P}^1$ factor. Since the splitting

$$\nu_s^*\Omega^1_{B^\circ} = \mathcal{V}^\sharp|_{\nu_s} \oplus \mathcal{N}^\sharp|_{\nu_s}$$

is preserved by ∇ (by Theorem 16 (e)), and since g_{SK} is flat, we have

$$\begin{aligned}
\nabla_W(\iota^*(\mathcal{P}(\nabla_V u_i))) &= \iota^*(\nabla_W(\mathcal{P}(\nabla_V u_i))) \\
&= \iota^*(\mathcal{P}(\nabla_W \nabla_V u_i)) \\
&= \iota^*(\mathcal{P}(\nabla_V \nabla_W u_i + \nabla_{[W,V]} u_i)).
\end{aligned}$$

Now, since u_i is parallel along the rational curve $\nu_s(\mathbb{P}^1)\backslash D$ for all $s \in \mathbb{C}$, we have that $\nabla_W u_i$ vanishes identically on $U \times \Delta$ and so

$$\nabla_V \nabla_W u_i = 0.$$

Furthermore, $[W, V] = -L_V W$ is also tangent to $\nu_t(\mathbb{P}^1)$, so $\nabla_{[W,V]} u_i = 0$ too. □

Since $\iota^*(\mathscr{P}(\nabla_V u_i))$ is parallel, it is in particular holomorphic over $\mathbb{P}^1\backslash\nu_t^{-1}(D)$. The following Lemma then implies that $\iota^*(\mathscr{P}(\nabla_V u_i))$ extends to a holomorphic section of $\nu_t^*\Omega_B^1$ over $\mathbb{P}^1$:

Proposition 25. *Let $w \in H^0(\mathbb{P}^1\backslash\nu_t^{-1}(D), \nu_t^*\Omega_B^1)$ be a holomorphic section which is parallel with respect to ∇ (the Chern connection induced by ω_{SK}). Then w extends to a holomorphic section of $\nu_t^*\Omega_B^1$ over all of $\mathbb{P}^1$.*

Proof. Since w is parallel, its pointwise length $|w|^2_{\nu_t^* g_{\mathrm{SK}}}$ is constant on $\mathbb{P}^1\backslash\nu_t^{-1}(D)$. Recall that from (3) we have that

$$\omega_{\mathrm{SK}} \geqslant C^{-1}\omega_B, \tag{52}$$

on B°. Since $\nu_t^{-1}(D)$ is a finite subset of $\mathbb{P}^1$, we consider the extension problem of w across each of these points, so let $y \in \nu_t^{-1}(D)$ be one of them. Recall that D is regular at the point $x = \nu_t(y)$, and we can choose local holomorphic coordinates $z_1, \dots z_n$ on a chart U centered at x such that $D \cap U = \{z_1 = 0\}$. The volume form ω_{SK}^n is given by a fiber integration as in (4), and its asymptotic behavior near D is studied in [20, Theorem 2.1] (see also [7, 19] for the case when $\dim B = 1$ and [32], [56] for $\dim B$ arbitrary) where it is shown that

$$\omega_{\mathrm{SK}}^n \leqslant \frac{C}{|z_1|^{2(1-\gamma)}}(-\log|z_1|)^C \omega_B^n, \tag{53}$$

on $U \cap \{z_1 \neq 0\}$, for some $C > 0$ and $\gamma \in (0, 1]$. Combining (52) and (53) gives the crude bound

$$\omega_{\mathrm{SK}} \leqslant \frac{C}{|z_1|^{2(1-\gamma)}}(-\log|z_1|)^C \omega_B, \tag{54}$$

see also [59, (2.1) and Theorem 3.4] and [20, Theorem 1.1] for sharper and more general such bounds. Passing to the dual metric on Ω_B^1 and pulling back via ν_t, (54) implies that on the punctured neighborhood $\nu_t^{-1}(U \cap \{z_1 \neq 0\})$ of y in $\mathbb{P}^1$ we have

$$|w|^2_{\nu_t^* g_B} \leqslant \frac{C}{|z_1|^{2(1-\gamma)}}(-\log|z_1|)^C |w|^2_{\nu_t^* g_{\mathrm{SK}}} = \frac{C'}{|z_1|^{2(1-\gamma)}}(-\log|z_1|)^C,$$

and from this we see that $|w|^2_{\nu_t^* g_B}$ is L^1 in $\nu_t^{-1}(U \cap \{z_1 \neq 0\})$. Since $\nu_t^*\Omega_B^1$ is a trivial bundle over $\nu_t^{-1}(U)$, we can represent w locally as an n-tuple of holomorphic functions on $\nu_t^{-1}(U \cap \{z_1 \neq 0\})$, and since these functions are in L^2, they extend holomorphically across the point y (see e.g. [48, Proposition 1.14]), which gives us the desired extension of w. □

At this point we have shown that $\iota^*(\mathscr{P}(\nabla_V u_i))$ gives a holomorphic section $w \in H^0(\mathbb{P}^1, \nu_t^*\Omega_B^1)$. Recalling the splitting (12), we see that w must be a section of the factor $\mathscr{O}^{\oplus \ell}$, i.e. a section of $\mathscr{V}^\sharp|_{\mathscr{U}_t}$. Since it is also a section of $\mathscr{N}^\sharp|_{\mathscr{U}_t}$, it must be identically zero. This shows that $\iota^*(\mathscr{P}(\nabla_V u_i)) = 0$, and so $\iota^*(\nabla_V u_i) \in \mathscr{V}^\sharp|_{\mathscr{U}_t}$, and so (50) is established. This concludes the proof of Theorem 22 when f is isotrivial.

7. Obtaining a parallel $(1,1)$-form and Hwang's Theorem

In this section we show how to combine our main theorem 3 with results of Voisin [60], Hwang [27, 28] and Bakker–Schnell [2] to deduce Theorem 2. The key step is the following:

Theorem 26. *Suppose that $B \not\cong \mathbb{P}^n$. Then there is a nontrivial real $(1,1)$-form ψ on B° with $\nabla^{\mathrm{SK}}\psi = 0$ and ψ not proportional to ω_{SK}.*

First, we show that Theorem 2 follows from this (we do not need to assume that X is projective):

Proof of Theorem 2. Suppose for a contradiction that $B \not\cong \mathbb{P}^n$. Then by Theorem 26 the 2-forms ω_{SK} and ψ on B° are both ∇^{SK}-parallel and not proportional, and thus they give us a 2-dimensional space of global sections of the local system $R^2 f_* \mathbb{R}_{X^\circ}$ over B°. However, as observed by Voisin [60, Lemma 5.5], a result of Matsushita [41] together with Deligne's invariant cycles theorem show that this space of sections is always 1-dimensional, a contradiction. □

Since $B \not\cong \mathbb{P}^n$, we know that $\mathcal{V} \subset \mu^* TB^\circ$ is a nontrivial proper holomorphic subbundle over $\mathcal{U}^\circ$, which by Theorem 22 is preserved by ∇. As mentioned after Theorem 22, the g_{SK}-orthogonal complement $\mathcal{N} \subset \mu^* TB^\circ$ of $\mathcal{V}$ is also a nontrivial proper holomorphic subbundle over $\mathcal{U}^\circ$ which is preserved by ∇. Define real subbundles $\mathcal{V}_{\mathbb{R}}, \mathcal{N}_{\mathbb{R}}$ of $\mu^* T^{\mathbb{R}} B^\circ$ over $\mathcal{U}^\circ$ by

$$\mathcal{V}_{\mathbb{R}} = \{v + \overline{v} \mid v \in \mathcal{V}\} \subset \mu^* T^{\mathbb{R}} B^\circ,$$

and analogously for $\mathcal{N}_{\mathbb{R}}$. The bundle $\mathcal{V}_{\mathbb{R}}$ is isomorphic to $\mathcal{V}$ via the usual inverse map $T^{\mathbb{R}} B \to TB$ given by $u \mapsto \frac{u - iJ(u)}{2}$ (and similarly for $\mathcal{N}_{\mathbb{R}}$), and on $\mathcal{U}^\circ$ we have a splitting

$$\mu^* T^{\mathbb{R}} B^\circ = \mathcal{V}_{\mathbb{R}} \oplus \mathcal{N}_{\mathbb{R}}. \tag{55}$$

Consider now the Stein factorization of $\mu : \mathcal{U} \to B$, given by

$$\mathcal{U} \to Z \xrightarrow{p} B,$$

where $\mathcal{U} \to Z$ has connected fibers and $p : Z \to B$ is finite. Define also $Z^\circ := p^{-1}(B^\circ)$. To complete the proof of Theorem 26, we will then need the following theorem which is implicit in the work of Hwang [28], and also appears in the recent work of Bakker–Schnell ([2, Proposition 3.2 and proof of Theorem 1.1]) relying on ideas of Hwang [27, 28]:

Theorem 27. *Suppose the splitting* (55) *is preserved by* ∇^{SK}*, then* $p : Z \to B$ *is an isomorphism.*

We can now give the proof of Theorem 26:

Proof of Theorem 26. Since $B \not\cong \mathbb{P}^n$, we have the nontrivial splitting (55). By definition, $\mathcal{V}_{\mathbb{R}}$ is preserved by J, and since $\mathcal{V}$ is preserved by ∇ (and $\nabla J = 0$), it follows that $\mathcal{V}_{\mathbb{R}}$ is also preserved by ∇.

We claim that $\mathcal{V}_{\mathbb{R}}$ is preserved by ∇^{SK}. To see this, recall that Freed shows in [13, (1.29)] that the special Kähler connection on $T^{\mathbb{R}} B$ is given by

$$\nabla^{\mathrm{SK}} = \nabla + A + \overline{A}, \tag{56}$$

where as usual ∇ is the Levi-Civita connection of g_{SK} and $A \in \Lambda^{1,0} \operatorname{Hom}(TB^\circ, \overline{TB^\circ})$ is given by

$$A_{ij}^{\overline{\ell}} = \sqrt{-1} g_{\mathrm{SK}}^{k\overline{\ell}} \Xi_{ijk}, \tag{57}$$

and the same holds for the pullback connection on $\mathcal{U}^\circ$. Given a local section α of $\mathcal{V}$ and a local $(1,0)$ vector field $v \in T\mathcal{U}$, we wish to show that

$$\nabla^{\mathrm{SK}}_{v+\overline{v}}(\alpha + \overline{\alpha}) \in \mathcal{V}_{\mathbb{R}}.$$

Since we know that $\nabla_{v+\overline{v}}(\alpha + \overline{\alpha}) \in \mathcal{V}_{\mathbb{R}}$, it suffices to check that

$$(A + \overline{A})_{v+\overline{v}}(\alpha + \overline{\alpha}) = A_v(\alpha) + \overline{A_v(\alpha)} \in \mathcal{V}_{\mathbb{R}},$$

and so it suffices to see that

$$A_v(\alpha) \in \overline{\mathcal{V}},$$

or equivalently that $g_{\mathrm{SK}}(A_v(\alpha), \overline{\zeta}) = 0$ for all local sections ζ of $\mathcal{N}$. But from (57) we see that

$$g_{\mathrm{SK}}(A_v(\alpha), \overline{\zeta}) = \sqrt{-1}\,\Xi(v, \alpha, \zeta),$$

which vanishes by Theorem 16 (c). This concludes the proof that $\mathcal{V}_{\mathbb{R}}$ is preserved by ∇^{SK}. An analogous argument shows that $\mathcal{N}_{\mathbb{R}}$ is also preserved by ∇^{SK}, and so the splitting (55) is preserved by ∇^{SK}. Applying Theorem 27 we see that $p : Z^\circ \to B^\circ$ is an isomorphism, so we may assume that $\mu : \mathcal{U}^\circ \to B^\circ$ has connected fibers. The vector bundle $\mu^* T^{\mathbb{R}} B^\circ$ is trivial when restricted to these fibers, and its subbundles $\mathcal{V}_{\mathbb{R}}, \mathcal{N}_{\mathbb{R}}$ restricted to a fiber are preserved by the pullback connection ∇^{SK} (which when restricted to the fiber is a trivial connection), and so $\mathcal{V}_{\mathbb{R}}$ and $\mathcal{N}_{\mathbb{R}}$ are pullbacks of vector bundles on B° (denoted by the same notation), which are subbundles of $T^{\mathbb{R}} B^\circ$ and are still preserved by ∇^{SK}.

We then define a $(1,1)$-form ψ on B° by projecting ω_{SK} onto $\mathcal{V}_{\mathbb{R}}$. Since $\nabla^{\mathrm{SK}} \omega_{\mathrm{SK}} = 0$ and $\mathcal{V}_{\mathbb{R}}$ is preserved by ∇^{SK}, it follows that $\nabla^{\mathrm{SK}} \psi = 0$ (and also $\nabla \psi = 0$ for the same reason), and since $\mathcal{V}_{\mathbb{R}}$ is a nontrivial proper subbundle of $T^{\mathbb{R}} B^\circ$, we see that ψ is nonzero and not proportional to ω_{SK}, and we are done. □

8. Comments about the case when B is singular

It is tempting to ask whether our method can be used to prove that $B \cong \mathbb{P}^n$ even when B is singular. As mentioned in the Introduction, this is currently known only for $n \leqslant 2$ [6, 26, 49]. In general, it is known that B is a normal projective variety, with at worst klt singularities, which is Fano with Picard number one. The natural generalization of our approach (following [52], who generalized Mori's Theorem [45] to the singular setting) would be to consider a functorial resolution of singularities $\pi : \widetilde{B} \to B$ and to show that we must have $\widetilde{B} \cong \mathbb{P}^n$, which forces $B \cong \mathbb{P}^n$ as well. In this setting, $\widetilde{B}$ is a uniruled projective manifold and $\widetilde{D} = \pi^{-1}(D)$ is a divisor, so many of our arguments above can be repeated on $\widetilde{B}^\circ := \widetilde{B} \backslash \widetilde{D}$, which carries a special Kähler metric ω_{SK}. The fact that π is functorial gives us a morphism $\mu : \pi^* TB \to T\widetilde{B}$ which is an isomorphism on $\widetilde{B}^\circ$, where $TB = \mathrm{Hom}(\Omega^1_B, \mathcal{O}_B)$ is the reflexive tangent sheaf. Given a rational curve $\nu : \mathbb{P}^1 \to \widetilde{B}$, which is not contained in $\widetilde{D}$, pulling back μ via ν we obtain a sheaf injection

$$\mathcal{A} := (\pi \circ \nu)^{[*]} TB \to \nu^* T\widetilde{B},$$

between these vector bundles on $\mathbb{P}^1$ (which both split as a direct sum of line bundles which should have nonnegative degrees). Here we use the standard reflexive pullback notation $(\pi \circ \nu)^{[*]} TB := (\nu^* \pi^* TB)^{**}$. Using Theorem 14, if $\widetilde{B} \ncong \mathbb{P}^n$ then $\nu^* T\widetilde{B}$ contains a nontrivial $\mathcal{O}$ factor, hence so does $\mathcal{A}$. To implement our strategy, one would need a rigidity statement like in Theorem 16 for either one of these trivial summands, and a crucial ingredient of the proof of the rigidity statement is that sections of the dual of the relevant bundle should have bounded norm (with respect to the pullback of ω_{SK}). The first fundamental issue is that it is not clear to us how to show that sections of $\nu^* \Omega^1_{\widetilde{B}}$ or of $\mathcal{A}^*$ have bounded norm. The key ingredient for this when B is smooth was the estimate (3), but when B is singular this by itself is not sufficient to prove boundedness.

What can be shown using results in [16] is rather that sections of the reflexive pullback $(\pi \circ \nu)^{[*]} \Omega^{[1]}_B$ have bounded norm, but in general the Grothendieck decomposition of this vector bundle is different from those of $\nu^* \Omega^1_{\widetilde{B}}$ and $\mathcal{A}^*$, and it may happen that these have some nontrivial $\mathcal{O}$ factor but $(\pi \circ \nu)^{[*]} \Omega^{[1]}_B$ does not, which invalidates our approach. This undesirable phenomenon can only happen when the generic rational curve (of the type that we are considering) when projected down to B always passes through some singular point of B. This however seems unavoidable in general, as finding low-degree rational curves in normal Fano varieties that can be deformed to avoid the singularities is a very delicate problem in algebraic geometry, see e.g. [31, 34, 61].

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