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# *Comptes Rendus*

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## *Mécanique*


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Volume 349, issue 1 (2021), p. 21-27

Published online: 5 February 2021

<https://doi.org/10.5802/crmeca.67>

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*Les Comptes Rendus. Mécanique* sont membres du  
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[www.centre-mersenne.org](http://www.centre-mersenne.org)  
e-ISSN : 1873-7234



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Short paper / Note

# On the Eringen model for nematic liquid crystals

## *Sur le modèle d'Eringen pour les cristaux liquides nématiques*

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**Abstract.** We introduce the three-dimensional Eringen system of equations for the nematodynamics of liquid crystals, announce the short time existence and uniqueness of strong solutions for the one-dimensional problem in the periodic case, and show the continuous dependence of the solution on the initial data.

**Résumé.** Nous présentons le système tridimensionnel d'équations d'Eringen pour la nématodynamique des cristaux liquides, annonçons l'existence en temps et l'unicité de solutions fortes pour le problème unidimensionnel dans le cas périodique et montrons la dépendance continue de la solution sur les données initiales.

**Keywords.** Liquid crystals, Eringen equations, Nematodynamics, Existence and uniqueness, Conservation laws, Micromomentum of molecules, Local solution.

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**Mots-clés.** Cristaux liquides, Équations d'Eringen, Nematodynamique, Existence et unicité, Lois de conservation, Micromomentum de molécules, Solution locale.

**2020 Mathematics Subject Classification.** 76A15, 35K61, 35A01, 35A02.

*Manuscript received 13th December 2020, revised and accepted 14th December 2020.*

### 1. Introduction

In papers [1–4] and [5], we considered the Ericksen–Leslie system of equations for nematodynamics and proved the existence and uniqueness theorems. But the Ericksen–Leslie equations do not take into account micromomentum of molecules. In this paper, we write down the right Eringen system of equations (see [6] and [7]), which takes into account the micromomentum of molecules, and study the same question of unique solvability.

In the present paper, we consider the simplest form of Eringen’s system for the nematodynamics of liquid crystals (see Figure 1 for example) and prove the local solvability of the corresponding initial–boundary value problem.

The existence of *global* solutions of the *general* Eringen system is still an open question. One can highlight two different cases: the existence of strong and weak solutions. For the strong solutions, it is natural to expect local solvability and for the weak solutions, global solvability.

### 2. The full system of Eringen equations

To describe the Eringen system for micropolar liquid crystals [6] in a domain  $\Omega \subset \mathbb{R}^3$  we define the unknowns:  $\rho : (0, t_0) \times \Omega \rightarrow \mathbb{R}_+$  is the mass density of the material;  $\mathbf{u} = (u^1, u^2, u^3) : (0, t_0) \times \Omega \rightarrow \mathbb{R}^3$  is the velocity;  $\mathbf{v} = (v^1, v^2, v^3) : (0, t_0) \times \Omega \rightarrow \mathbb{R}^3$  is the gyration vector, or, equivalently represented as a matrix  $v := \hat{\mathbf{v}} := \mathbf{v} \times \cdot = [-\varepsilon^k{}_{lm} v^m] : (0, t_0) \times \Omega \rightarrow \mathfrak{so}(3)$ , is the gyration tensor;  $j : (0, t_0) \times \Omega \rightarrow \mathfrak{gl}(3, \mathbb{R})$  is the microinertia tensor, a symmetric positive definite  $3 \times 3$  matrix;  $\boldsymbol{\gamma}_l = (\gamma_l^1, \gamma_l^2, \gamma_l^3) : (0, t_0) \times \Omega \rightarrow \mathbb{R}^3$ ,  $l = 1, 2, 3$ , where  $\gamma_l^a : (0, t_0) \times \Omega \rightarrow \mathbb{R}$ ,  $a = 1, 2, 3$ , or, equivalently represented as a matrix by  $\hat{\boldsymbol{\gamma}}_l := \boldsymbol{\gamma}_l \times \cdot = [-\varepsilon^k{}_{ra} \gamma_l^a] : (0, t_0) \times \Omega \rightarrow \mathfrak{so}(3)$ , is the wryness tensor (in our convention, taken from [8] which is based on geometrical considerations,  $\boldsymbol{\gamma}_l$  corresponds to  $-\boldsymbol{\gamma}_l$  in Eringen’s notation [6, 7]);  $T : (0, t_0) \times \Omega \rightarrow \mathbb{R}_+$  is the absolute temperature. The free energy is denoted by  $\Psi = \Psi(\rho^{-1}, j, \boldsymbol{\gamma}, T)$  and the dissipation potential by  $\Phi = \Phi(\rho^{-1}, j, \boldsymbol{\gamma}, T; a, b, \nabla T / T, \dot{T})$ , where

$$a_k^l := \partial_k u^l - \varepsilon^l{}_{ak} v^a, \quad b_l^a := \partial_l v^a, \quad \dot{T} := \partial_t T, \quad T_{x^k} := \partial_k T, \tag{1}$$

$\nabla T := (T_{x^1}, T_{x^2}, T_{x^3})$ , and  $\partial_k := \partial_{x_k} := \partial / \partial x_k$ . Explicit formulae for  $\Psi$  and  $\Phi$  are given below. Denote the internal energy density by  $\mathcal{E} = \Psi + T\eta$  and the entropy by  $\eta = -(\partial\Psi/\partial T) - (1/\rho)(\partial\Psi/\partial \dot{T})$ . Thus, in the absence of external forces, the total energy of the liquid crystal is

$$\frac{1}{2} \int_{\Omega} \rho \|\mathbf{u}\|^2 d^3x + \frac{1}{2} \int_{\Omega} \rho (j\mathbf{v}) \cdot \mathbf{v} d^3x + \int_{\Omega} \rho \mathcal{E} d^3x. \tag{2}$$

With these notations and conventions, Eringen’s equations for micropolar liquid crystals are given by the following system.

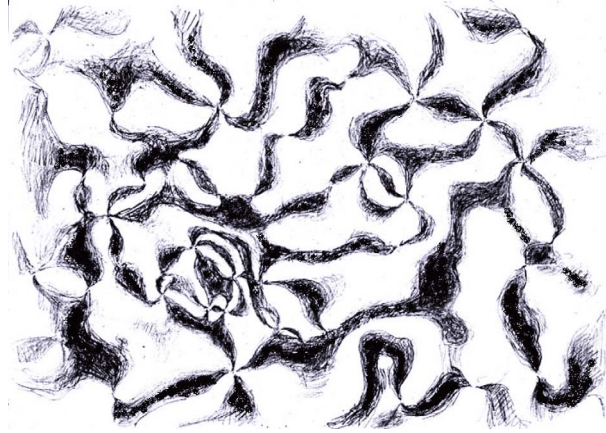
CONSERVATION OF MASS

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0. \tag{3}$$

BALANCE OF MOMENTUM FOR MACROMOTION

$$\rho \partial_t \mathbf{u} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla \left( \frac{\partial \Psi}{\partial \rho^{-1}} \right) - \partial_l \left( \rho \frac{\partial \Psi}{\partial \gamma_l^a} \boldsymbol{\gamma}^a \right) + \partial_l \frac{\partial \Phi}{\partial \mathbf{a}_l}, \tag{4}$$

where  $\mathbf{a}_l := (a_l^1, a_l^2, a_l^3) = \partial_l \mathbf{u} - (\varepsilon^1{}_{al} v^a, \varepsilon^2{}_{al} v^a, \varepsilon^3{}_{al} v^a)$ ,  $\boldsymbol{\gamma}^a := (\gamma_1^a, \gamma_2^a, \gamma_3^a)$ .



**Figure 1.** Schlieren texture of a nematic liquid crystal.

BALANCE OF MOMENT OF MOMENTUM FOR MICROMOTION

$$\begin{aligned} \rho \partial_t(j\mathbf{v}) + \rho(\mathbf{u} \cdot \nabla)(j\mathbf{v}) = & -\operatorname{div}\left(\rho \frac{\partial \Psi}{\partial \boldsymbol{\gamma}}\right) + \rho \boldsymbol{\gamma}^b \times \frac{\partial \Psi}{\partial \boldsymbol{\gamma}^b} + \operatorname{div}\left(\frac{\partial \Phi}{\partial \mathbf{b}}\right) \\ & + \left(\varepsilon_{1m}^n \frac{\partial \Phi}{\partial a_m^n}, \varepsilon_{2m}^n \frac{\partial \Phi}{\partial a_m^n}, \varepsilon_{3m}^n \frac{\partial \Phi}{\partial a_m^n}\right), \end{aligned} \quad (5)$$

where  $\mathbf{b}^a := (b_1^a, b_2^a, b_3^a) = (\partial_1 v^a, \partial_2 v^a, \partial_3 v^a)$ .

CONSERVATION OF THE MICROINERTIA TENSOR

$$\partial_t j_{ab} + u^i \partial_i j_{ab} + (\varepsilon_{ca}^p j_{bp} + \varepsilon_{cb}^p j_{ap}) v^c = 0 \iff \partial_t j + (\mathbf{u} \cdot \nabla) j + [j, \hat{\mathbf{v}}] = 0. \quad (6)$$

Equations (5) and (6) yield the equation for the gyration vector  $\mathbf{v}$ , namely

$$\begin{aligned} j \partial_t \mathbf{v} + j(\mathbf{u} \cdot \nabla) \mathbf{v} - (j\mathbf{v}) \times \mathbf{v} = & -\frac{1}{\rho} \operatorname{div}\left(\rho \frac{\partial \Psi}{\partial \boldsymbol{\gamma}}\right) + \boldsymbol{\gamma}^a \times \frac{\partial \Psi}{\partial \boldsymbol{\gamma}^a} + \frac{1}{\rho} \operatorname{div}\left(\frac{\partial \Phi}{\partial \mathbf{b}}\right) \\ & + \frac{1}{\rho} \left(\varepsilon_{1m}^n \frac{\partial \Phi}{\partial a_m^n}, \varepsilon_{2m}^n \frac{\partial \Phi}{\partial a_m^n}, \varepsilon_{3m}^n \frac{\partial \Phi}{\partial a_m^n}\right). \end{aligned} \quad (7)$$

Conversely, this equation and (6) implies (5).

WRYNESS TENSOR EQUATION

$$\partial_t \gamma_l^a + u^i \partial_i \gamma_l^a + \gamma_i^a \partial_l u^i + \partial_l v^a + \varepsilon_{bc}^a \gamma_l^b v^c = 0 \iff \partial_t \gamma + \mathcal{L}_{\mathbf{u}} \gamma + \mathbf{d}\hat{\mathbf{v}} + [\gamma, \hat{\mathbf{v}}] = 0, \quad (8)$$

where  $\gamma := \widehat{\boldsymbol{\gamma}}$  is a  $\mathfrak{so}(3)$ -valued one-form on  $\Omega$  for each  $t \in (0, t_0)$ , that is,  $\gamma = \widehat{\boldsymbol{\gamma}}_l dx^l$  and  $\widehat{\boldsymbol{\gamma}}_l : (0, t_0) \times \Omega \rightarrow \mathfrak{so}(3)$  are given by  $(\widehat{\boldsymbol{\gamma}}_l)_r^k = -\varepsilon_{ra}^k \gamma_l^a$ ,  $\boldsymbol{\gamma}_l = (\gamma_l^1, \gamma_l^2, \gamma_l^3) : (0, t_0) \times \Omega \rightarrow \mathbb{R}^3$ ;  $\mathcal{L}_{\mathbf{u}} \gamma$  is the Lie derivative of  $\gamma$  in the direction of the vector field  $\mathbf{u}$ , and  $\mathbf{d}\hat{\mathbf{v}}$  is the differential of the  $\mathfrak{so}(3)$ -valued function  $\hat{\mathbf{v}}$ . Thus, this equation is for  $\mathfrak{so}(3)$ -valued one-forms on  $\Omega$ ;  $\gamma$  is a connection one-form on the trivial bundle  $\Omega \times SO(3)$  and the last two terms are the  $\gamma$ -covariant derivative of  $\hat{\mathbf{v}}$ . See [8] for details.

BALANCE OF ENERGY

$$\begin{aligned} \rho \partial_t \mathcal{E} + \rho(\mathbf{u} \cdot \nabla) \mathcal{E} = & (\operatorname{div} \mathbf{u}) \frac{\partial \Psi}{\partial \rho^{-1}} - \rho \left( \left( \frac{\partial \Psi}{\partial \boldsymbol{\gamma}^a} \cdot \nabla \right) \mathbf{u} \right) \cdot \boldsymbol{\gamma}^a - \rho \mathbf{v} \cdot \left( \boldsymbol{\gamma}^a \times \frac{\partial \Psi}{\partial \boldsymbol{\gamma}^a} \right) \\ & - \rho \left( \frac{\partial \Psi}{\partial \boldsymbol{\gamma}^a} \cdot \nabla \right) \mathbf{v}^a + \partial_k \left( \frac{\partial \Phi}{\partial (T_{x_k} / T)} \right) + a_k^n \frac{\partial \Phi}{\partial a_k^n} + b_k^a \frac{\partial \Phi}{\partial b_k^a}. \end{aligned} \quad (9)$$

SECOND LAW OF THERMODYNAMICS

$$\rho \partial_t \eta + \rho (\mathbf{u} \cdot \nabla) \eta \geq \partial_k \left( T^{-1} \frac{\partial \Phi}{\partial (T_{x_k} / T)} \right). \tag{10}$$

Based on physical considerations for micropolar liquid crystals, Eringen [6, 7] proposed the following expressions for the free energy  $\Psi$  and the dissipation potential  $\Phi$ :

$$\Psi = \frac{1}{2\rho} A_{ab}^{km} \gamma_k^a \gamma_m^b, \tag{11}$$

$$\Phi = \frac{1}{2} \alpha_{ln}^{km} a_k^l a_m^n + \frac{1}{2} \beta_{ab}^{kl} b_k^a b_l^b + \frac{1}{T} d_a^{kl} b_l^a \partial_k T + \frac{1}{2T} \kappa^{kl} \partial_k T \partial_l T, \tag{12}$$

where

$$d_a^{kl} = (d_1 + d_2 \text{tr } j) \varepsilon_a^{kl} + d_3 \varepsilon_a^{km} j_{mn} \delta^{ln}, \quad \kappa^{kl} = (\kappa_1 + \kappa_2 \text{tr } j) \delta_{kl} + \kappa_3 j_{mn} \delta^{mk} \delta^{nl},$$

the coefficients  $A_{ab}^{km}$ ,  $\alpha_{ln}^{km}$ ,  $\beta_{ab}^{kl}$  are symmetric (i.e.,  $A_{ab}^{km} = A_{ba}^{mk}$ , etc.) and are represented as

$$\begin{aligned} A_{ab}^{kl} = & (A_1 + A_2 j_{cp} \delta^{cp}) \delta_a^k \delta_b^l + \frac{A_5}{2} (j_{bp} \delta^{pl} \delta_a^k + j_{ap} \delta^{pk} \delta_b^l) \\ & + \frac{1}{2} (A_4 + A_3 + (A_5 + A_8) j_{pq} \delta^{pq}) \delta_b^k \delta_a^l + \frac{1}{4} (A_6 + A_9) (j_{ap} \delta^{pl} \delta_b^k + j_{bp} \delta^{pk} \delta_a^l) \\ & + \frac{1}{2} (A_4 - A_7 + (A_5 - A_8) j_{cp} \delta^{cp}) \delta_{ab} \delta^{kl} + \frac{1}{4} (A_6 - A_9) (j_{pc} \delta^{pl} \delta^{ck} \delta_{ab} + j_{ab} \delta^{kl}) \\ & + \frac{A_{10}}{4} (j_{ab} \delta^{kl} - j_{pq} \delta^{pl} \delta^{qk} \delta_{ab}), \end{aligned} \tag{13}$$

with identical expressions for  $\alpha_{ln}^{km}$  and  $\beta_{ab}^{kl}$ , except that the coefficients  $A_i$  are replaced by different coefficients  $\alpha_i$  and  $\beta_i$ , respectively.  $A_i$ ,  $d_i$ ,  $\kappa_i$ ,  $\alpha_i$ ,  $\beta_i$  are some given functions, depending on  $j$ ,  $T$ , and  $\rho$ .

### 3. Initial-boundary value problem for the simplest Eringen system

#### 3.1. Assumptions and auxiliary propositions

From now on, we shall assume that the following conditions hold.

**Condition 1.**  $\Psi$ ,  $\partial \Phi / \partial a_m^n$ ,  $\partial \Phi / \partial b_m^a$  do not depend on temperature  $T$  and its gradient  $\nabla T$ . This means that the functions  $A_i$ ,  $\alpha_i$ ,  $\beta_i$  do not depend on  $T$  and that  $d_i \equiv 0$ .

Under this condition, system (3)–(8) has all coefficients independent of  $T$  and is therefore self-consistent; neither energy, nor temperature appear in this system. Hence, we can exclude (9) and (10) from consideration and we are left with system (3)–(8). Equations (9) and (10) can be solved separately.

**Condition 2.**  $j_{ab} = J(t, \mathbf{x}) \delta_{ab}$ .

In this case, Equation (6) states that the function  $J(t, \mathbf{x})$  is frozen in the flow. Since the microinertia tensor  $[j_{ab}]$  is a symmetric positive definite  $3 \times 3$  matrix, the function  $J$  is positive. Due to (6) it is sufficient to suppose that  $J > 0$  at the initial time.

**Condition 3.** All unknown variables depend only on time  $t$  and the first spatial coordinate  $x$ .

Even though this is a restrictive hypothesis requiring a one-dimensional spatial domain, compressibility of the liquid crystal ensures that the solutions of the system (3)–(8) are not necessarily trivial.

**Condition 4.**  $\mathbf{u} = (u, 0, 0)$ ,  $\mathbf{v} = (v, 0, 0)$ ,  $\boldsymbol{\gamma}_1 = (\gamma, 0, 0)$ ,  $\boldsymbol{\gamma}_2 = \boldsymbol{\gamma}_3 = 0$ .

This assumption puts conditions on the form of the solution. If it holds, Equation (8) is automatically satisfied for  $a \neq 1$  or  $k \neq 1$ .

**Condition 5.**  $A^{kl}, \beta^{kl}, \alpha^{mn}$  are constants.

Let us denote  $\alpha = \alpha_{11}^{11} > 0$ ,  $\lambda = \alpha_{13}^{12} - \alpha_{12}^{13}$ ,  $A = A_{11}^{11} > 0$ ,  $\beta = \beta_{11}^{11} > 0$ . Note that we assume the positivity of the constants  $\alpha, \beta, A$ . Then, under the five conditions stated above, Equations (3)–(8) can be written in the following form:

$$\rho_t + (\rho u)_x = 0, \tag{14}$$

$$\rho u_t + \rho u u_x = \partial_x \left( \alpha u_x + \lambda v - \frac{A}{2} \gamma^2 \right), \tag{15}$$

$$\rho J v_t + \rho J u v_x = \partial_x (\beta v_x - A \gamma), \tag{16}$$

$$(\rho J)_t + (u \rho J)_x = 0, \tag{17}$$

$$\gamma_t + \partial_x (u \gamma + v) = 0, \tag{18}$$

where the functions  $\rho > 0, J > 0$ .

Let  $\Omega = \mathbb{T}$ ,  $Q_{t_0} = \Omega \times (0, t_0)$ , where  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ , that is, all functions are 1-periodic with respect to  $x$ . In this paper, we study the system (14)–(18).

For simplicity, we use the notation  $V = (u, v, \rho, \rho J, \gamma)$ . The vector  $V$  is prescribed at  $t = 0$ , namely,

$$V|_{t=0} = V_0. \tag{19}$$

**Notation 6.** If  $\mu$  is a measure on  $\Omega$ , we denote by  $L_p(\Omega, \mu)$  the Banach space of real-valued functions on  $\Omega$  whose  $p$  power of the absolute value is integrable relative to  $\mu$ . If  $\mu$  is the Lebesgue measure, we omit  $\mu$  in the notation above. If  $E$  is a Banach space,  $L_p(0, T; E)$  denotes the Banach space of  $E$ -valued  $L_p$ -functions on  $(0, T)$  (relative to the Lebesgue measure).  $W_p^a(\Omega)$  is the Banach space of real-valued  $L_p$ -functions on  $\Omega$  that have all derivatives up to and including order  $a$  in  $L_p$ .  $W_p^{r;a}(Q_T)$  is the Banach space of all real-valued  $L_p$ -functions on  $Q_T$  that have all derivatives up to and including order  $r$  on  $(0, T)$  and up to and including order  $a$  on  $\Omega$  in  $L_p$ . If  $E$  is a Hilbert space and  $p = 2$ , all these spaces are Hilbert spaces.

**Definition 7.** The vector  $V$  is a solution to the problem (14)–(18), (19) if

- $V \in \mathcal{W}_{t_0}$ , where  $\mathcal{W}_{t_0} := W_2 \times W_2 \times W_1 \times W_1 \times W_1$ ,  $W_1 := W_2^{1;1}(Q_{t_0}) \cap L_\infty(0, t_0; L_2(\Omega))$ ,  $W_2 := W_2^{2;1}(Q_{t_0}) \cap L_\infty(0, t_0; W_2^1(\Omega))$ ;
- Equations (14)–(18) hold almost everywhere in  $Q_{t_0}$ ;
- $V_0 \in \mathcal{W}_0 := W_2^2(\Omega) \times W_2^2(\Omega) \times W_2^1(\Omega) \times W_2^1(\Omega) \times W_2^1(\Omega)$ ;
- $\|V(x, t) - V_0(x)\|_{L_2} \rightarrow 0$  if  $t \rightarrow 0$ .

Another useful space is  $\mathcal{W}_{t_0}^b := W_2^b \times W_2^b \times W_1^b \times W_1^b \times W_1^b$ , where  $W_1^b := W_2^{1;0}(Q_{t_0}) \cap L_\infty(0, t_0; L_2(\Omega))$ ,  $W_2^b = W_2^{2;0}(Q_{t_0}) \cap L_\infty(0, t_0; W_2^1(\Omega))$ .

In ensuing analysis we use the following results.

**Lemma 8.** Let  $V$  be a strong solution of the problem (14)–(18), (19). Then

$$\mathcal{F}_1(V; t) \leq \mathcal{F}_1(V; 0) \exp \left\{ \lambda^2 \alpha^{-1} \int_0^t \sup_{x \in \Omega} (\rho(t, x) J(t, x))^{-1} dt \right\}, \tag{20}$$

where

$$\begin{aligned} \mathcal{F}_1(V; t) = & \sup_{(0, t)} \left( \|\rho^{\frac{1}{2}} u\|_{L_2(\Omega)}^2 + \|(\rho J)^{\frac{1}{2}} v\|_{L_2(\Omega)}^2 + A \|\gamma\|_{L_2(\Omega)}^2 \right) \\ & + \int_{Q_t} (\alpha u_x^2 + \beta v_x^2) dx dt. \end{aligned}$$

**Lemma 9.** *Let  $V$  be a smooth strong solution to the problem (14)–(18), (19). Then there exist  $t_1 > 0$  and  $C > 0$  depending on the coefficients and the  $\mathcal{W}_0$ -norm of the initial data such that*

$$\|V\|_{\mathcal{W}_t} \leq C \quad \text{for all } t \in (0, t_1).$$

Let us consider a regularization of the system (14)–(18). Let  $\varepsilon$  be a small parameter. The regularized system is

$$\rho_t + (\rho u)_x = \varepsilon \rho_{xx}, \tag{21}$$

$$\rho u_t + \rho u u_x = \partial_x \left( \alpha u_x + \lambda v - \frac{A}{2} \gamma^2 \right), \tag{22}$$

$$\rho J v_t + \rho J u v_x = \partial_x (\beta v_x - A \gamma), \tag{23}$$

$$(\rho J)_t + (\rho u J)_x = \varepsilon (\rho J)_{xx}, \tag{24}$$

$$\gamma_t + \partial_x (u \gamma + v) = \varepsilon \gamma_{xx}. \tag{25}$$

Let  $V^\varepsilon$  be a solution of the regularized system satisfying the initial conditions

$$V^\varepsilon|_{t=0} = V_0^\varepsilon. \tag{26}$$

**Lemma 10.** *Let  $V_0^\varepsilon \in C^{2+2\delta}(\Omega)$ ,  $\delta > 0$ . Then for any  $\varepsilon > 0$  there exists a unique solution  $V^\varepsilon \in C^{2+\delta, 1+\delta}(\overline{Q_t})$ .*

The proof is based on [9, Ch. VII, Theorem 7.1].

**Lemma 11.** *Let  $\lim_{\varepsilon \rightarrow 0} V_0^\varepsilon = V_0 \in \mathcal{W}_0$ , the limit taken in the  $\mathcal{W}_0$ -norm. Then there exists  $t_0 > 0$  such that  $V^\varepsilon$  are uniformly bounded in  $\mathcal{W}_{t_0}$ -norm.*

### 3.2. Existence theorem

**Theorem 12.** *Let  $V_0 \in \mathcal{W}_0$ . Then for some  $t_0 > 0$  there exists a strong solution  $V \in \mathcal{W}_{t_0}$ .*

**Proof.** Let us consider a family of regularized problems (21)–(25), (26),  $V_0^\varepsilon \in C^{2+2\delta}(\Omega)$ ,  $\mathcal{W}_0$ - $\lim_{\varepsilon \rightarrow 0} V_0^\varepsilon = V_0$ . By Lemma 11, there exists  $t_0 > 0$  such that this family of solutions  $V^\varepsilon$  is uniformly bounded in the  $\mathcal{W}_{t_0}$ -norm. Hence, we can choose a sequence  $V^{\varepsilon_n}$  such that  $V^{\varepsilon_n} \rightarrow V^*$ -weakly in  $\mathcal{W}_{t_0}$  and, in particular,

- $(u^{\varepsilon_n}, v^{\varepsilon_n}) \rightharpoonup (u, v)$  weakly in  $W_2^{2,1}(Q_{t_0})$ ;
- $(\rho^{\varepsilon_n}, (\rho J)^{\varepsilon_n}, \gamma^{\varepsilon_n}) \rightharpoonup (\rho, \rho J, \gamma)$  weakly in  $W_2^{1,0}(Q_{t_0})$ ;
- $(u^{\varepsilon_n}, v^{\varepsilon_n}) \rightrightarrows (u, v)$ , that is, converges uniformly on  $\overline{Q_{t_0}}$ .

□

So, for any  $\Xi := (\psi, \eta, \varphi, \zeta, \theta) \in C_0^\infty(Q_{t_0})$ , we have

$$(\mathcal{A}(V^{\varepsilon_n}), \Xi) \rightarrow (\mathcal{A}(V), \Xi),$$

where

$$\begin{aligned} (\mathcal{A}(V), \Xi) &\equiv \int_{Q_T} \rho(\varphi_t + u\varphi_x) \, dx \, dt \\ &+ \int_{Q_T} \left[ \rho u(\psi_t + u\psi_x) - \alpha u_x \psi_x - \lambda v \psi_x + \frac{A}{2} \gamma^2 \psi_x \right] \, dx \, dt \\ &+ \int_{Q_T} [\rho J v(\eta_t + u\eta_x) - \beta v_x \eta_x + A \gamma \eta_x] \, dx \, dt \\ &+ \int_{Q_T} \rho J(\zeta_t + u\zeta_x) \, dx \, dt \\ &+ \int_{Q_T} \gamma \theta_t + (u\gamma + v)\theta_x \, dx \, dt = 0, \end{aligned} \tag{27}$$

where  $\Xi := (\psi, \eta, \varphi, \zeta, \theta) \in C_0^\infty(Q_{t_0})$  (smooth functions on  $Q_{t_0}$  with compact support) is a test vector. By the definition, we have

$$(\mathcal{A}(V^\varepsilon), \Xi) = -\varepsilon \int_{Q_t} (\rho_x^\varepsilon \varphi_x + (\rho J)_x^\varepsilon \zeta_x + \gamma_x^\varepsilon \theta_x) dx dt. \quad (28)$$

The right-hand side of (28) tends to be zero if  $\varepsilon \rightarrow 0$ , since the norm of  $(\rho_x^\varepsilon, (\rho J)_x^\varepsilon, \gamma_x^\varepsilon)$  is uniformly bounded in  $L_2(Q_{t_0})$  and  $\Xi$  is fixed smooth vector. Hence

$$(\mathcal{A}(V), \Xi) = 0,$$

which proves that  $V$  is a weak solution of the system (14)–(18) (in the sense of integral identity). Since  $V \in \mathcal{W}_{t_0}$  it is easy to prove, that it is also a strong solution. Let us check that the initial conditions hold. By Lemma 11 and the compact embedding  $\mathcal{W}_{t_0} \hookrightarrow C(0, T; L_2(\Omega))$ , the solution  $V(t)$  is a continuous map from  $[0, t_0]$  to  $L_2(\Omega)$  and

$$\max_{t \in [0, t_0]} \|V^\varepsilon(t) - V(t)\|_{L_2(\Omega)} \rightarrow 0.$$

In particular,  $V(0) = (L_2)$ - $\lim_{\varepsilon \rightarrow 0} V^\varepsilon$ . On the other hand,  $V^\varepsilon(0) \rightarrow V_0$ . The theorem is proved.

### 3.3. Continuous dependence on initial data and uniqueness

Define  $\mathcal{V}_0 := W_2^1(\Omega) \times W_2^1(\Omega) \times L_2(\Omega) \times L_2(\Omega) \times L_2(\Omega)$ ,  $\mathcal{V}_{t_0} := W_1^p \times W_1^p \times L_{\infty,2} \times L_{\infty,2} \times L_{\infty,2}$ , where  $L_{\infty,2} := L_\infty(0, t_0; L_2(\Omega))$ . The following theorem holds.

**Theorem 13.** *Let  $V_1, V_2$  be two strong solutions to the system (14)–(18) in the domain  $Q_{t_0}$ . Then, for any  $T > 0$ , there exists  $C > 0$  depending on  $\|V_i\|_{\mathcal{W}_{t_0}}$ ,  $\inf \rho_1 > 0$ ,  $\inf (J\rho)_1 > 0$ ,  $\alpha, \beta, A$ , and  $\lambda$  such that*

$$\|V_1 - V_2\|_{\mathcal{V}_T} \leq C \|V_1(0) - V_2(0)\|_{\mathcal{V}_0}.$$

*In particular, problem (14)–(18), (19) has unique strong solution.*

## Acknowledgements

Work of GAC and MSR was partially supported by the Russian Science Foundation (project 20-11-20272). TSR was partially supported by the National Natural Science Foundation of China grant number 11871334 and the NCCR SwissMAP grant of the Swiss National Science Foundation. We also express our thanks to the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan for the support (AP08855579).

## References

- [1] G. A. Chechkin, T. S. Ratiu, M. S. Romanov, V. N. Samokhin, “On the existence and uniqueness theorem in 2D nematodynamics. Finite speed of propagation of perturbations”, *Russ. Acad. Sci. Dokl. Math.* **91** (2015), no. 3, p. 354-358 (Translated from *Dokl. Akad. Nauk.* **462** (2015), no. 5, p. 519-523).
- [2] G. A. Chechkin, T. S. Ratiu, M. S. Romanov, V. N. Samokhin, “On unique solvability of the full three-dimensional Ericksen–Leslie system”, *C. R. Méc.* **344** (2016), no. 7, p. 459-463.
- [3] G. A. Chechkin, T. S. Ratiu, M. S. Romanov, V. N. Samokhin, “Existence and uniqueness theorems for the two-dimensional Ericksen–Leslie system”, *J. Math. Fluid Mech.* **18** (2016), p. 571-589.
- [4] G. A. Chechkin, T. P. Chechkina, T. S. Ratiu, M. S. Romanov, “Nematodynamics and random homogenization”, *Appl. Anal.* **95** (2016), no. 10, p. 2243-2253.
- [5] G. A. Chechkin, T. S. Ratiu, M. S. Romanov, V. N. Samokhin, “Existence and uniqueness theorems for the full three-dimensional Ericksen–Leslie system”, *Math. Models Methods Appl. Sci. (M<sup>3</sup>AS)* **27** (2017), no. 5, p. 807-843.
- [6] A. C. Eringen, “A unified continuum theory of liquid crystals”, *ARI - Int. J. Phys. Eng. Sci.* **50** (1997), p. 73-84.
- [7] A. C. Eringen, *Microcontinuum Field Theories. Volume II. Fluent Media*, Springer-Verlag, New York, 2001.
- [8] F. Gay-Balmaz, T. S. Ratiu, “The geometric structure of complex fluids”, *Adv. Appl. Math.* **42** (2009), p. 176-275.
- [9] O. A. Ladyzhenskaja, V. A. Solonnikov, N. N. Ural’ceva, in *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, vol. 23, American Mathematical Society, Providence, RI, 1968.