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Comptes Rendus
Mécanique

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Volume 349, issue 1 (2021), p. 29-41

Published online: 5 February 2021

https://doi.org/10.5802/crmeca.68

Les Comptes Rendus. Mécanique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1873-7234

Comptes Rendus
Mécanique
2021, 349, nº 1, p. 29-41
https://doi.org/10.5802/crmeca.68

Short paper / *Note*

Existence and uniqueness of global strong solutions for a class of non-Newtonian fluids with small initial energy and vacuum

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Abstract. In this article, we investigate an initial and boundary value problem for a class of compressible non-Newtonian fluids, provided the initial energy is small and the initial density containing the vacuum state is allowed. For $p > 2$, we obtain the existence and uniqueness of the global strong solution for this problem in a one-dimensional bounded interval.

Keywords. Non-Newtonian fluid, Global strong solution, A priori estimate, Existence and uniqueness, Vacuum.

2020 Mathematics Subject Classification. 35A01, 35B45, 35E15, 76A05.

Manuscript received 16th October 2020, accepted 6th January 2021.

1. Introduction and main result

The motion of an isentropic compressible viscous fluid can be expressed by Navier–Stokes equations in the following form in $\mathbb{R}^3$ [1]:

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho\mathbf{u})=0,\\ \partial_t(\rho\mathbf{u})+\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})+\nabla\pi=\operatorname{div}\mathbf{\Gamma},\end{cases}$$

where ρ, $\mathbf{u}=(u_1,u_2,u_3)$ and $\pi=a\rho^\gamma$ $(a>0,\gamma>1)$ represent the density, velocity and pressure, respectively. The parameter $\mathbf{\Gamma}$ is the viscous stress tensor depending on $E_{ij}(\nabla\mathbf{u})$:

$$E_{ij}(\nabla\mathbf{u})=\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}.$$

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ISSN (electronic) : 1873-7234

Considering non-Newtonian fluids, Ladyzhenskaya [2] first presented a new form $\mathbf{\Gamma}$ to investigate the incompressible model:

$$\mathbf{\Gamma}_{ij} = (\mu_0 + \mu_1 |E(\nabla \mathbf{u})|^{p-2}) E_{ij}(\nabla \mathbf{u}),$$

where the model can be divided into five types when the parameters μ_0, μ_1 and p are assigned different values.

Later, there emerged some literature on compressible non-Newtonian fluids. Mamontov [3] obtained the global existence of sufficiently regular solutions for compressible non-Newtonian fluid equations in two and three dimensions, provided the initial density is strictly positive. Additionally, taking into account the appearance of the initial vacuum, Choe and Kim [4] studied a strong solution for isentropic compressible fluids while the initial data satisfied a natural compatibility condition and some other conditions. Yuan and Xu [5] proved the existence and uniqueness of local strong solutions for one-dimensional non-Newtonian fluids with the initial value satisfying the compatibility condition. For more details about the local solutions for compressible non-Newtonian fluids, one can refer for instance to papers [6–9].

Recently, in the study of global solutions for compressible non-Newtonian equations, Fang, Zhu and Guo [10] established the global classical solution to the equation with large initial data and vacuum while the initial density satisfied some restrictions. Yuan, Si and Feng [11] showed existence and uniqueness of the global strong solution to the initial boundary value problem of the equation with small initial energy and vacuum. For more related results on Navier–Stokes equations and non-Newtonian fluids, the readers are referred to [12–19] and the references therein.

Inspired by these works, in this paper, we deal with the following compressible non-Newtonian fluid, which is called a shear thickening fluid, in one-dimensional bounded intervals:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \quad (x,t) \in \Omega \times \mathbb{R}^+, \\ (\rho u)_t + (\rho u^2)_x + \pi_x = [(u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x]_x, \end{cases} \tag{1.1}$$

subject to the initial and boundary conditions

$$\begin{cases} (\rho, u)|_{t=0} = (\rho_0, u_0), \quad x \in [0,1], \\ u|_{x=0} = u|_{x=1} = 0, \quad t \geq 0. \end{cases} \tag{1.2}$$

Here the unknown functions $\rho = \rho(x,t)$ and $u = u(x,t)$; the initial density $\rho_0 \geq 0$; $\Omega := (0,1)$; $p > 2$ and $\mu_0 > 0$ are given constants.

We now demonstrate our main result as follows.

Theorem 1. *Assume $p > 2$ and that the initial value (ρ_0, u_0) satisfies*

$$\rho_0 \in H^1(\Omega), \quad u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \quad 0 \leq \rho_0 \leq \bar{\rho}, \tag{1.3}$$

and the compatibility condition

$$-((u_{0x}^2 + \mu_0)^{\frac{p-2}{2}} u_{0x})_x + \pi_x(\rho_0) = \rho_0^{1/2} g, \quad \textit{for a.e. } x \in \Omega, \tag{1.4}$$

where $g \in L^2(\Omega)$. Then there exists a constant $\varepsilon = \varepsilon(a, \gamma, \bar{\rho}) > 0$ such that if the initial energy $E_0 := \int_0^1 ((1/2)\rho_0 u_0^2 + a\rho_0^\gamma/(\gamma-1))\,\mathrm{d}x$ satisfies $E_0 \leq \varepsilon$, then problem (1.1)–(1.2) has a unique global strong solution (ρ, u), which satisfies

$$\begin{cases} \rho \in C([0,T]; H^1(\Omega)), \quad \rho_t \in C([0,T]; L^2(\Omega)), \quad u_t \in L^2(0,T; H_0^1(\Omega)), \\ u \in C([0,T]; W_0^{1,p}(\Omega)) \cap L^\infty(0,T; H^2(\Omega)), \quad \sqrt{\rho} u_t \in L^\infty(0,T; L^2(\Omega)), \\ ((u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x)_x \in C([0,T]; L^2(\Omega)), \end{cases}$$

for $0 < T < \infty$.

Since the global existence of the solution (Theorem 1) is obtained mainly by the global estimates of the local solution, the result that contains the local strong solution for problem (1.1)–(1.2) is given as follows.

Proposition 2 ([20]). *Suppose that $p > 2$ and the initial value (ρ_0, u_0) satisfies the following conditions:*

$$0 \le \rho_0 \in H^1(\Omega), \quad u_0 \in H_0^1(\Omega) \cap H^2(\Omega).$$

If there exists a function $g \in L^2(\Omega)$ satisfying

$$-((u_{0x}^2 + \mu_0)^{\frac{p-2}{2}} u_{0x})_x + \pi_x(\rho_0) = \rho_0^{1/2} g, \quad \textit{for a.e. } x \in \Omega,$$

then there exists a time $T_* \in (0, +\infty)$ such that problem (1.1)–(1.2) has a unique local strong solution (ρ, u), which satisfies

$$\begin{cases} \rho \in C([0, T_*]; H^1(\Omega)), \quad \rho_t \in C([0, T_*]; L^2(\Omega)), \quad u_t \in L^2(0, T_*; H_0^1(\Omega)), \\ u \in C([0, T_*]; W_0^{1,p}(\Omega)) \cap L^\infty(0, T_*; H^2(\Omega)), \quad \sqrt{\rho} u_t \in L^\infty(0, T_*; L^2(\Omega)), \\ ((u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x)_x \in C([0, T_*]; L^2(\Omega)). \end{cases}$$

2. A priori estimates

In this section, we obtain a priori estimates of the smooth solution. Fix the time $T > 0$, and let (ρ, u) be a smooth solution of problem (1.1)–(1.2) on $\Omega \times (0, T]$, where the initial value (ρ_0, u_0) satisfies conditions (1.3)–(1.4).

As a matter of convenience, we denote

$$\dot{u} := u_t + u u_x, \quad \|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}, \quad \|\cdot\|_{H^k} := \|\cdot\|_{H^k(\Omega)}, \quad \int \cdot \, \mathrm{d}x := \int_\Omega \cdot \, \mathrm{d}x,$$

where $\Omega := (0, 1)$, $1 \le p \le \infty$ and $k \in \mathbb{N}$.

Next, we state the following important energy estimate.

Lemma 3. *Suppose (ρ, u) is smooth and solves problem (1.1)–(1.2) on $\Omega \times (0, T]$; then one has*

$$\sup_{0 \le t \le T} \int \left(\frac{1}{2} \rho u^2 + \frac{a\rho^\gamma}{\gamma - 1} \right) \mathrm{d}x + \int_0^T \int (u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x^2 \, \mathrm{d}x \, \mathrm{d}t = E_0. \tag{2.1}$$

Proof. Multiplying (1.1) and (1.2) by $a\gamma\rho^{\gamma-1}$ and u, respectively, and adding them together, one can conclude (2.1) after integrating the result equation by parts over $\Omega \times (0, T)$. □

For $p > 2$, considering the smooth solution of problem (1.1)–(1.2) on $\Omega \times (0, T]$, we have the following a priori estimates.

Proposition 4. *Under the condition described in Theorem 1, there exists a positive constant ε depending only on a, γ and $\bar{\rho}$ such that if (ρ, u), which is a smooth solution of problem (1.1)–(1.2) on $\Omega \times (0, T]$, satisfies the inequalities*

$$\sup_{0 \le t \le \sigma(T)} \|u_x\|_p^p + \int_0^{\sigma(T)} \|\sqrt{\rho} \dot{u}\|_2^2 \, \mathrm{d}t \le 2K, \tag{2.2}$$

$$\sup_{0 \le t \le T} \sigma^m \|u_x\|_p^p + \int_0^T \sigma^m \|\sqrt{\rho} \dot{u}\|_2^2 \, \mathrm{d}t \le 2E_0^{1/2}, \tag{2.3}$$

$$\sup_{0 \le t \le T} \sigma^m \|\sqrt{\rho} \dot{u}\|_2^2 + \int_0^T \sigma^m \int (u_x^2 + \mu_0)^{\frac{p-2}{2}} \dot{u}_x^2 \, \mathrm{d}x \, \mathrm{d}t \le 2E_0^{1/4}, \tag{2.4}$$

$$\sup_{0 \le t \le T} \|\rho\|_\infty \le 2\bar{\rho}, \tag{2.5}$$

where m and K (not necessarily small) are positive constants, then the solution makes the following estimates hold true:

$$\sup_{0\le t\le\sigma(T)}\|u_x\|_p^p+\int_0^{\sigma(T)}\|\sqrt{\rho}\dot{u}\|_2^2\,\mathrm{d}t\le K, \tag{2.6}$$

$$\sup_{0\le t\le T}\sigma^m\|u_x\|_p^p+\int_0^T\sigma^m\|\sqrt{\rho}\dot{u}\|_2^2\,\mathrm{d}t\le E_0^{1/2}, \tag{2.7}$$

$$\sup_{0\le t\le T}\sigma^m\|\sqrt{\rho}\dot{u}\|_2^2+\int_0^T\sigma^m\int(u_x^2+\mu_0)^{\frac{p-2}{2}}\dot{u}_x^2\,\mathrm{d}x\,\mathrm{d}t\le E_0^{1/4}, \tag{2.8}$$

$$\sup_{0\le t\le T}\|\rho\|_\infty\le\frac{7\bar{\rho}}{4}, \tag{2.9}$$

where we suppose that $E_0\le\varepsilon$ *and* $\sigma(t):=\min\{1,t\}$.

Proof. To make the proof process rigorous but not lengthy, this proof can be divided into four steps.

Step 1. Estimate for $\|u_x\|_p$.

First, from $(1.1)_1$, equation $(1.1)_2$ can be rewritten in the following form:

$$\rho\dot{u}+\pi_x=[(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x]_x. \tag{2.10}$$

Multiplying (2.10) by $\sigma^m\dot{u}$ and integrating the result over Ω, we can deduce after integrating by parts and using the boundary value condition $u|_{\partial\Omega}=0$ that

$$\sigma^m\int\rho\dot{u}^2\,\mathrm{d}x+\sigma^m\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_xu_{xt}\,\mathrm{d}x:=I_1+I_2, \tag{2.11}$$

where $I_1=\sigma^m\int[(u_x^2+\mu_0)^{(p-2)/2}]_xuu_x\,\mathrm{d}x$ and $I_2=-\sigma^m\int\pi_x\dot{u}\,\mathrm{d}x$.

Due to $u|_{\partial\Omega}=0$, integration by parts leads to

$$\begin{aligned}I_1&=\sigma^m\int[(u_x^2+\mu_0)^{\frac{p-2}{2}}]_xuu_x\,\mathrm{d}x=-\sigma^m\int(u_x^2+\mu_0)^{\frac{p-2}{2}}(uu_x)_x\,\mathrm{d}x\\&=-\sigma^m\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^3\,\mathrm{d}x-\sigma^m\int(u_x^2+\mu_0)^{\frac{p-2}{2}}uu_xu_{xx}\,\mathrm{d}x.\end{aligned} \tag{2.12}$$

Owing to $|[(u_x^2+\mu_0)^{(p-2)/2}u_x]_x|\ge(u_x^2+\mu_0)^{(p-2)/2}|u_{xx}|$, one obtains

$$\begin{aligned}I_1&=-\sigma^m\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^3\,\mathrm{d}x-\sigma^m\int(u_x^2+\mu_0)^{\frac{p-2}{2}}uu_xu_{xx}\,\mathrm{d}x\\&\le C\sigma^m\|u_x\|_\infty\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^2\,\mathrm{d}x\\&\quad+C\sigma^m\|u_x\|_\infty\int(u_x^2+\mu_0)^{\frac{p-2}{2}}|u||u_{xx}|\,\mathrm{d}x\\&\le C\sigma^m\|u_x\|_\infty\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^2\,\mathrm{d}x\\&\quad+C\sigma^m\|u_x\|_\infty\int|[(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x]_x||u|\,\mathrm{d}x\\&\le C\sigma^m\|u_x\|_\infty\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^2\,\mathrm{d}x.\end{aligned} \tag{2.13}$$

Second, integrating by parts again and using $(1.1)_1$ as well as $u_t|_{\partial\Omega}=0$ give

$$I_2=\frac{\mathrm{d}}{\mathrm{d}t}\left(\sigma^m\int\pi u_x\,\mathrm{d}x\right)-m\sigma^{m-1}\sigma'\int\pi u_x\,\mathrm{d}x+\gamma\sigma^m\int\pi u_x^2\,\mathrm{d}x. \tag{2.14}$$

Computing the left-hand-side terms of (2.11), one has

$$\begin{aligned}\sigma^m\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_xu_{xt}\,\mathrm{d}x &= \frac{1}{2}\sigma^m\int(u_x^2+\mu_0)^{\frac{p-2}{2}}(u_x^2)_t\,\mathrm{d}x\\ &= \frac{1}{2}\sigma^m\int\int_0^{u_x^2}(s+\mu_0)^{\frac{p-2}{2}}\,\mathrm{d}s\,\mathrm{d}x\\ &\geq \frac{1}{2}\sigma^m\int\int_0^{u_x^2}s^{\frac{p-2}{2}}\,\mathrm{d}s\,\mathrm{d}x\\ &= \frac{\sigma^m}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int|u_x|^p\,\mathrm{d}x. \end{aligned}\tag{2.15}$$

Finally, substituting (2.13)–(2.15) into (2.11) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sigma^m}{p}\|u_x\|_p^p-\sigma^m\int\pi u_x\,\mathrm{d}x\right)+\sigma^m\int\rho|\dot{u}|^2\,\mathrm{d}x\\ &\leq C(\bar{\rho})\sigma^m\|u_x\|_\infty\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^2\,\mathrm{d}x\\ &\quad+C\sigma^{m-1}\sigma'\|u_x\|_\infty\int\pi\,\mathrm{d}x+C\sigma^m\|u_x\|_\infty^2\int\pi\,\mathrm{d}x. \end{aligned}\tag{2.16}$$

Step 2. Estimate for $\|\sqrt{\rho}\dot{u}\|_2$.

To start with, multiplying (2.10) by u and differentiating the resulting equation with respect to x give

$$(\rho\dot{u}u)_x+(\pi_x u)_x=[((u_x^2+\mu_0)^{\frac{p-2}{2}}u_x)_x u]_x.\tag{2.17}$$

Then, differentiating (2.10) with respect to t leads to

$$(\rho\dot{u})_t+(\pi_x)_t=[((u_x^2+\mu_0)^{\frac{p-2}{2}}u_x)_x]_t,\tag{2.18}$$

which together with (2.17) implies

$$(\rho\dot{u})_t+(\rho u\dot{u})_x+\pi_{xt}+(u\pi_x)_x=[((u_x^2+\mu_0)^{\frac{p-2}{2}}u_x)_x u]_x+[((u_x^2+\mu_0)^{\frac{p-2}{2}}u_x)_x]_t.\tag{2.19}$$

A direct calculation on the right-hand side of (2.19) gives

$$\begin{aligned}&(\rho\dot{u})_t+(\rho u\dot{u})_x+\pi_{xt}+(u\pi_x)_x\\ &\quad=[(p-2)(u_x^2+\mu_0)^{\frac{p-4}{2}}u_x^2(u_{xt}+uu_{xx})+(u_x^2+\mu_0)^{\frac{p-2}{2}}(u_{xt}+uu_{xx})]_x.\end{aligned}\tag{2.20}$$

Next, multiplying (2.20) by $\dot{u}$ and dealing with the left-hand-side terms by $(1.1)_1$ yield

$$\begin{aligned}&\tfrac{1}{2}(\rho\dot{u}^2)_t+\tfrac{1}{2}(\rho u\dot{u}^2)_x+(\pi_t\dot{u}+(u\pi_x)\dot{u})_x+\gamma\pi u_x\dot{u}_x\\ &\quad=[(p-2)(u_x^2+\mu_0)^{\frac{p-4}{2}}u_x^2(u_{xt}+uu_{xx})+(u_x^2+\mu_0)^{\frac{p-2}{2}}(u_{xt}+uu_{xx})]_x\dot{u}.\end{aligned}\tag{2.21}$$

Finally, multiplying (2.21) by σ^m and integrating it over Ω, we can obtain after integrating by parts and using the Cauchy–Schwarz inequality that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sigma^m\int\rho|\dot{u}|^2\,\mathrm{d}x\right)+\sigma^m\int(u_x^2+\mu_0)^{\frac{p-2}{2}}\dot{u}_x^2\,\mathrm{d}x\\ &=-(p-2)\sigma^m\int(u_x^2+\mu_0)^{\frac{p-4}{2}}u_x^2\dot{u}_x^2\,\mathrm{d}x+(p-2)\sigma^m\int(u_x^2+\mu_0)^{\frac{p-4}{2}}u_x^4\dot{u}_x\,\mathrm{d}x\\ &\quad+\sigma^m\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^2\dot{u}_x\,\mathrm{d}x-\sigma^m\int\gamma\pi u_x\dot{u}_x\,\mathrm{d}x\\ &\leq(p-1)\sigma^m\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^2|\dot{u}_x|\,\mathrm{d}x+\sigma^m\int|\gamma\pi u_x\dot{u}_x|\,\mathrm{d}x.\end{aligned}\tag{2.22}$$

Step 3. Estimate for $\|u_x\|_\infty$ and proofs of (2.6)–(2.8).

First, set

$$F := (u_x^2+\mu_0)^{\frac{p-2}{2}} u_x - \pi. \tag{2.23}$$

It follows from $(1.1)_2$ that

$$F_x = \rho\dot{u}. \tag{2.24}$$

We can deduce from (2.23) and (2.24) that

$$\|F_x\|_2 \le C(\bar\rho)\|\sqrt{\rho}\dot{u}\|_2, \tag{2.25}$$

$$\|u_x\|_\infty \le C\|F\|_\infty + C(\bar\rho). \tag{2.26}$$

By the fundamental formula for integration, there exists $\xi\in(0,1)$ satisfying

$$F(\xi) = \int_0^\xi F_x\,\mathrm{d}x + F(0),$$

which combined with (2.24) and the Cauchy–Schwarz inequality yields

$$F(\xi) \le \int_0^1 |\rho\dot{u}|\,\mathrm{d}x + F(0) \le C(\bar\rho)\|\sqrt{\rho}\dot{u}\|_2 + C(\bar\rho). \tag{2.27}$$

The fact, together with (2.26) and (2.27), leads to

$$\|u_x\|_\infty \le C(\bar\rho)\|\sqrt{\rho}\dot{u}\|_2 + C(\bar\rho). \tag{2.28}$$

Owing to (2.28), one gets

$$\sigma^m\|u_x\|_\infty \le C(\bar\rho)(\sigma^m\|\sqrt{\rho}\dot{u}\|_2^2)^{1/2}\sigma^{m/2} + C(\bar\rho). \tag{2.29}$$

Then, putting (2.29) into (2.16) and integrating the result over $[0,t]$, we can deduce from conditions (2.1)–(2.4) and $m\ge 2$ that

$$\begin{aligned}
&\left(\frac{\sigma^m}{p}\|u_x\|_p^p - \sigma^m\int \pi u_x\,\mathrm{d}x\right)(t) + \int_0^t \sigma^m\|\sqrt{\rho}\dot{u}\|_2^2(s)\,\mathrm{d}s\\
&\le C(\bar\rho)(\sigma^m\|u_x\|_\infty+1)\int_0^t\int (u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad + C\int_0^t \sigma^{m-1}\sigma'\|u_x\|_\infty\,\mathrm{d}s\cdot\sup_{0\le s\le t}\int\pi\,\mathrm{d}x\\
&\quad + C\int_0^t \sigma^m\|u_x\|_\infty^2\,\mathrm{d}s\cdot\sup_{0\le s\le t}\int\pi\,\mathrm{d}x.
\end{aligned} \tag{2.30}$$

Due to

$$\begin{aligned}
\int_0^t \sigma^{m-1}\sigma'\|u_x\|_\infty\,\mathrm{d}s &\le \int_0^1 \sigma^{m-1}\|u_x\|_\infty\,\mathrm{d}s\\
&\le C(\bar\rho)\int_0^1\sigma^{m-1}\|\sqrt{\rho}\dot{u}\|_2\,\mathrm{d}s + C(\bar\rho)\\
&\le C(\bar\rho)\left(\int_0^1\sigma^m\|\sqrt{\rho}\dot{u}\|_2^2\,\mathrm{d}s\right)^{1/2} + C(\bar\rho)\\
&\le C(\bar\rho)(1+E_0^{1/4})
\end{aligned} \tag{2.31}$$

and

$$\begin{aligned}
\int_0^t\sigma^m\|u_x\|_\infty^2\,\mathrm{d}s &\le C(\bar\rho)\int_0^t\sigma^m\|\sqrt{\rho}\dot{u}\|_2^2\,\mathrm{d}s + C(\bar\rho)\int_0^t\sigma^m\|\sqrt{\rho}\dot{u}\|_2\,\mathrm{d}s + C(\bar\rho)\\
&\le C(\bar\rho)(1+E_0^{1/2}+E_0^{1/4}),
\end{aligned} \tag{2.32}$$

combining with (2.30) gives

$$\begin{aligned}&\sup_{0\le t\le T}\sigma^m\|u_x\|_p^p+\int_0^T\sigma^m\|\sqrt{\rho}\dot u\|_2^2\,\mathrm{d}t\\&\le C(\bar\rho)E_0(1+E_0^{1/4})+C(\bar\rho)E_0(1+E_0^{1/2}+E_0^{1/4})\\&\le C_1(\bar\rho)E_0\le E_0^{1/2},\end{aligned}$$

where $E_0\le\delta_1:=\min\{1,C_1^{-2}\}$.

Second, inequality (2.28) implies that

$$\begin{aligned}\sigma^m\|u_x\|_\infty^2&\le C(\bar\rho)\sigma^m\|\sqrt{\rho}\dot u\|_2^2\,\mathrm{d}s+C(\bar\rho)\sigma^m\|\sqrt{\rho}\dot u\|_2\,\mathrm{d}s+C(\bar\rho)\\&\le C(\bar\rho)(1+E_0^{1/4}+E_0^{1/8}).\end{aligned}\tag{2.33}$$

Substituting (2.33) into (2.22), integrating the obtained result over $[0,t]$ and using conditions (2.3)–(2.4) as well as the Cauchy–Schwarz inequality, one can conclude that

$$\begin{aligned}&\frac12\left(\sigma^m\int\rho|\dot u|^2\,\mathrm{d}x\right)+\int_0^t\sigma^m\int(u_x^2+\mu_0)^{\frac{p-2}{2}}\dot u_x^2\,\mathrm{d}x\,\mathrm{d}s\\&\le(p-1)\int_0^t\sigma^m\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^2|\dot u_x|\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\sigma^m\int|\gamma\pi u_x\dot u_x|\,\mathrm{d}x\,\mathrm{d}s\\&\le C\sup_{0\le s\le t}(\sigma^m\|u_x\|_\infty^2)^{\frac12}\left(\int_0^t\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^2\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac12}\cdot\left(\int_0^t\sigma^m\int(u_x^2+\mu_0)^{\frac{p-2}{2}}\dot u_x^2\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac12}\\&\quad+C\sup_{0\le s\le t}\left(\int\pi\,\mathrm{d}x\right)\left(\int_0^t\int(u_x^2+\mu_0)^{\frac{p-2}{2}}\dot u_x^2\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac12}\left(\int_0^t\sigma^m\|u_x\|_\infty^2\,\mathrm{d}s\right)^{\frac12}.\end{aligned}\tag{2.34}$$

Hence

$$\begin{aligned}&\sup_{0\le t\le T}\sigma^m\|\sqrt{\rho}\dot u\|_2^2+\int_0^T\sigma^m\int(u_x^2+\mu_0)^{\frac{p-2}{2}}\dot u_x^2\,\mathrm{d}x\,\mathrm{d}t\\&\le C(\bar\rho)E_0^{5/8}(1+E_0^{1/4}+E_0^{1/8})+C(\bar\rho)E_0^{9/8}(1+E_0^{1/2}+E_0^{1/4})\\&\le C_2(\bar\rho)E_0^{1/2}\le E_0^{1/4},\end{aligned}$$

where $E_0\le\delta_2:=\min\{\delta_1,C_2^{-4}\}$.

Finally, to prove (2.6), taking $m=0$ in (2.16) gives

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac1p\|u_x\|_p^p-\int\pi u_x\right)+\int\rho|\dot u|^2\\&\le C(\bar\rho)\|u_x\|_\infty\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^2\,\mathrm{d}x+C\|u_x\|_\infty^2\int\pi\,\mathrm{d}x.\end{aligned}\tag{2.35}$$

Integrating (2.35) over $[0,\sigma(T)]$ and combining with (2.2), (2.28) and $p>2$ as well as the Hölder inequality lead to

$$\begin{aligned}&\frac1p\|u_x\|_p^p+\int_0^{\sigma(T)}\|\rho\dot u\|_2^2\,\mathrm{d}s\\&\le C(\bar\rho)\int_0^{\sigma(T)}\|u_x\|_\infty\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^2\,\mathrm{d}x\,\mathrm{d}s+C\int_0^{\sigma(T)}\|u_x\|_\infty^2\int\pi\,\mathrm{d}x\,\mathrm{d}s+\int\pi|u_x|\,\mathrm{d}x\\&\le C(\bar\rho)\int_0^{\sigma(T)}\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^2\,\mathrm{d}x\,\mathrm{d}s+C(\bar\rho)\left(\sup_{0\le s\le\sigma(T)}\|u_x\|_p^p+\mu_0^{p/2}\right)^{\frac12}\\&\quad\times\left(\int_0^{\sigma(T)}\|\sqrt{\rho}\dot u\|_2^2\,\mathrm{d}s\right)^{\frac12}\left(\int_0^{\sigma(T)}\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^2\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac12}\\&\quad+C(\bar\rho)\sup_{0\le s\le\sigma(T)}\int\pi\,\mathrm{d}x\cdot\int_0^{\sigma(T)}(\|\sqrt{\rho}\dot u\|_2^2+1)\,\mathrm{d}s+C(\bar\rho)\int\rho^\gamma\,\mathrm{d}x\cdot\|u_x\|_p\end{aligned}$$

$$\le C(\bar\rho)E_0 + C(\bar\rho)(2K+\mu_0^{p/2})^{1/2}K^{1/2}E_0^{1/2} + C(\bar\rho)E_0(2K+1) + C(\bar\rho)E_0K^{1/p}$$
$$\le C_3(\bar\rho)(2K+\mu_0^{p/2}+1)E_0^{1/2}.$$

Consequently,

$$\sup_{0\le t\le\sigma(T)} \|u_x\|_p^p + \int_0^{\sigma(T)} \|\sqrt{\rho}\dot u\|_2^2\,\mathrm{d}t \le K,$$

where $E_0 \le \delta_3 := \min\{\delta_2, K^2C_3^{-2}(2K+\mu_0^{p/2}+1)^{-2}\}$.

Step 4. Estimate for $\|\rho\|_\infty$.

To obtain the estimate for $\|\rho\|_\infty$, we recall the following Zlotnik inequality.

Lemma 5 (Zlotnik inequality). *Suppose that a function y satisfies*

$$D_t y(t) \le f(y) + D_t b(t) \quad \text{on } [0,T],\ y(0)=y_0, \tag{2.36}$$

where $f\in C(\mathbb{R})$ *and* $y,b\in W^{1,1}(0,T)$. *If* $f(\infty)=-\infty$ *and*

$$|b(t_2)-b(t_1)| \le N_0 + N_1(t_2-t_1),$$

for all $0\le t_1<t_2\le T$ *with some* $N_0\ge 0$ *and* $N_1\ge 0$, *then the following inequality holds:*

$$y(t) \le \max\{y_0,\bar\xi\} + N_0 < \infty \quad \text{on } [0,T].$$

Here $\bar\xi$ *is a constant, which satisfies*

$$f(\xi) \le -N_1, \quad \text{for } \xi\ge\bar\xi.$$

This proof is similar to that of [21, Lemma 1.3]; so the details are omitted.

With the help of $(1.1)_1$, one gets

$$D_t\rho \le -\rho u_x \le \rho|u_x|, \tag{2.37}$$

where $D_t\rho = \rho_t + \rho_x u$. Using the definition of F gives

$$|u_x| = (u_x^2+\mu_0)^{-\frac{p-2}{2}}|F+\pi| \le {\mu_0}^{-\frac{p-2}{2}}|F| + {\mu_0}^{-\frac{p-2}{2}}|\pi|. \tag{2.38}$$

By virtue of (2.37), (2.38) and Lemma 5, one can obtain

$$\begin{aligned} D_t\rho &\le \mu_0^{-\frac{p-2}{2}}\rho\pi + \mu_0^{-\frac{p-2}{2}}\rho|F| \\ &\le -\mu_0^{-\frac{p-2}{2}}\rho\pi + 2\mu_0^{-\frac{p-2}{2}}\rho\pi + \mu_0^{-\frac{p-2}{2}}\rho|F| \\ &\le -a\mu_0^{-\frac{p-2}{2}}\rho^{\gamma+1} + D_t b(t), \end{aligned}$$

where $b(t) = C\int_0^t \|\rho F(\cdot,s)\|_\infty\,\mathrm{d}s + C(\bar\rho)\int_0^t \|\rho^2\|_\infty\,\mathrm{d}s$ and $\rho=\rho(x(t),t)$.

For $0\le t_1\le t_2\le T$, to evaluate $|b(t_2)-b(t_1)|$, we first calculate $\int_0^t \|\rho F(\cdot,s)\|_\infty\,\mathrm{d}s$ and $\int_0^t\|\rho^2\|_\infty\,\mathrm{d}s$, separately.

Using the definition of F, Equations (1.2) and (2.1), one can deduce from the Minkowski inequality and the Cauchy–Schwarz inequality that

$$\begin{aligned} \int_{t_1}^{t_2}\|F\|_1\,\mathrm{d}s &\le \int_{t_1}^{t_2}\left(\mu_0^{-\frac12}\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^2\,\mathrm{d}x + \int\pi\,\mathrm{d}x\right)\mathrm{d}s \\ &\le C\int_{t_1}^{t_2}\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^2\,\mathrm{d}x\,\mathrm{d}s + \int_{t_1}^{t_2}\int\pi\,\mathrm{d}x\,\mathrm{d}s \\ &\le CE_0 + CE_0(t_2-t_1) \end{aligned} \tag{2.39}$$

and

$$\begin{aligned}
&\int_{t_1}^{t_2} \|F\|_2 \,\mathrm{d}s \\
&\le \int_{t_1}^{t_2} (\|(u_x^2+\mu_0)^{\frac{p-2}{2}} u_x\|_2 + \|\pi\|_2)\,\mathrm{d}s \\
&\le \int_{t_1}^{t_2} \left((\|u_x\|_\infty^2+\mu_0)^{\frac{p-2}{2}} \cdot \int (u_x^2+\mu_0)^{\frac{p-2}{2}} u_x^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \mathrm{d}s + C(\bar\rho) E_0^{1/2}(t_2-t_1) \\
&\le \left(\int_{t_1}^{t_2} (\|u_x\|_\infty^2+\mu_0)^{\frac{p-2}{2}} \,\mathrm{d}s \right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \int (u_x^2+\mu_0)^{\frac{p-2}{2}} u_x^2 \,\mathrm{d}x\,\mathrm{d}s \right)^{\frac{1}{2}} + C(\bar\rho) E_0^{1/2}(t_2-t_1) \\
&\le \left(\int_{t_1}^{t_2} (\|u_x\|_\infty^2+\mu_0)^{\frac{p-2}{2}} \,\mathrm{d}s \right)^{\frac{1}{2}} E_0^{1/2} + C(\bar\rho) E_0^{1/2}(t_2-t_1). \qquad (2.40)
\end{aligned}$$

It follows from the fact $W^{1,1}(\Omega) \hookrightarrow L^\infty(\Omega)$ that

$$\|\rho F(\cdot,s)\|_\infty \le C\|(\rho F)_x\|_1 + C\|\rho F\|_1 \le C\|\rho_x\|_2 \cdot \|F\|_2 + C\|\rho\|_2 \cdot \|F_x\|_2 + C(\bar\rho)\|F\|_1 \qquad (2.41)$$

and

$$\begin{aligned}
\|\rho^2\|_\infty &\le C\|\rho\rho_x\|_1 + C\|\rho^2\|_1 \le C\|\rho\|_2 + C(\bar\rho)\|\rho\|_1 \\
&\le C(\bar\rho) \left(\int \rho^\gamma \,\mathrm{d}x \right)^{\frac{1}{2\gamma}} \left(\int 1 \,\mathrm{d}x \right)^{\frac{\gamma-1}{2\gamma}} + C(\bar\rho) \left(\int \rho^\gamma \,\mathrm{d}x \right)^{\frac{1}{2\gamma}} \left(\int 1 \,\mathrm{d}x \right)^{\frac{\gamma-1}{\gamma}} \\
&\le C(\bar\rho) E_0^{\frac{1}{2\gamma}} + C(\bar\rho) E_0^{\frac{1}{\gamma}} \le C(\bar\rho) E_0^{\frac{1}{2\gamma}}. \qquad (2.42)
\end{aligned}$$

At present, we need to take into account the following two cases.

Case 1. $0 < t_1 < t_2 < \sigma(T)$.

From (2.1), (2.25) and (2.39)–(2.42), we can infer from the Hölder inequality that

$$\begin{aligned}
&|b(t_2) - b(t_1)| \le C \int_{t_1}^{t_2} \|\rho F(\cdot,s)\|_\infty \,\mathrm{d}s + C(\bar\rho) \int_{t_1}^{t_2} \|\rho^2\|_\infty \,\mathrm{d}s \\
&\le C(\bar\rho) \int_{t_1}^{t_2} (\|F\|_2 + \|\rho\|_2 \cdot \|F_x\|_2 + \|F\|_1)\,\mathrm{d}s + C(\bar\rho) E_0^{\frac{1}{2\gamma}} (t_2-t_1) \\
&\le C(\bar\rho) \int_{t_1}^{t_2} \|F\|_2 \,\mathrm{d}s + C(\bar\rho) \int_{t_1}^{t_2} \|\rho\|_2 \cdot \|\sqrt{\rho}\dot u\|_2 \,\mathrm{d}s + C(\bar\rho) \int_{t_1}^{t_2} \|F\|_1 \,\mathrm{d}s + C(\bar\rho) E_0^{\frac{1}{2\gamma}} (t_2-t_1) \\
&\le C(\bar\rho) \left(\int_{t_1}^{t_2} (\|u_x\|_\infty^2+\mu_0)^{\frac{p-2}{2}} \,\mathrm{d}s \right)^{\frac{1}{2}} E_0^{1/2} + C(\bar\rho) E_0^{1/2}(t_2-t_1) \\
&\quad + C(\bar\rho) \sup_{t_1\le t\le t_2} \|\rho\|_2 \cdot \int_{t_1}^{t_2} \|\sqrt{\rho}\dot u\|_2 \,\mathrm{d}s + CE_0 + CE_0(t_2-t_1) + C(\bar\rho) E_0^{\frac{1}{2\gamma}} (t_2-t_1) \\
&\le C(\bar\rho) E_0^{1/2} + C(\bar\rho) E_0^{1/2}(t_2-t_1) + C(\bar\rho) E_0^{1/(2\gamma)} K^{1/2} + CE_0 + CE_0(t_2-t_1) + C(\bar\rho) E_0^{\frac{1}{2\gamma}} (t_2-t_1) \\
&\le C(\bar\rho, K) E_0^{\frac{1}{2\gamma}}.
\end{aligned}$$

In view of Lemma 5, for $t \in [0,\sigma(T)]$, N_1 and N_0 are assigned values as follows:

$$N_1 = 0 \quad \text{and} \quad N_0 = C_4(\bar\rho, K) E_0^{1/(2\gamma)};$$

then $f(\xi) = -a\mu_0^{-(p-2)/2} \xi^{\gamma+1} \le 0$ for all $\xi \ge \bar\xi = 0$. One deduces from Lemma 5 that

$$\sup_{t\in[0,\sigma(T)]} \|\rho\|_\infty \le \max\{\bar\rho, \bar\xi\} + C_4(\eta, \bar\rho, K) E_0^{1/(2\gamma)} \le \frac{3\bar\rho}{2},$$

where $E_0 \le \delta_4 := \min\{\delta_3, (C_4^{-1}\bar\rho/2)^{2\gamma}\}$.

Case 2. $\sigma(T) < t_1 < t_2 < T$.

Making use of (2.3), (2.25) and (2.39)–(2.42), we can conclude from the Cauchy–Schwarz inequality and the Young inequality with ε that

$$\begin{aligned}
|b(t_2)-b(t_1)| &\le C(\bar\rho)\int_{t_1}^{t_2}\|F\|_\infty\,\mathrm{d}s + C(\bar\rho)\int_{t_1}^{t_2}\|\rho^2\|_\infty\,\mathrm{d}s\\
&\le C(\bar\rho)\int_{t_1}^{t_2}(\|F\|_2+\|\rho\|_2\cdot\|F_x\|_2+\|F\|_1)\,\mathrm{d}s + C(\bar\rho)E_0^{\frac{1}{2\gamma}}(t_2-t_1)\\
&\le C(\bar\rho)\int_{t_1}^{t_2}\|F\|_2\,\mathrm{d}s + C(\bar\rho)\int_{t_1}^{t_2}\|\rho\|_2\cdot\|\sqrt{\rho}\dot u\|_2\,\mathrm{d}s + C(\bar\rho)\int_{t_1}^{t_2}\|F\|_1\,\mathrm{d}s + C(\bar\rho)E_0^{\frac{1}{2\gamma}}(t_2-t_1)\\
&\le C(\bar\rho)\left(\int_{t_1}^{t_2}(\sigma^m\|u_x\|_\infty^2\cdot\sigma^{-m}+\mu_0)^{\frac{p-2}{2}}\,\mathrm{d}s\right)^{\frac12}E_0^{1/2} + C(\bar\rho)E_0^{1/2}(t_2-t_1) + C(\bar\rho)\sup_{t_1\le t\le t_2}\|\rho\|_2\\
&\quad\times\left(\int_{t_1}^{t_2}\sigma^m\|\sqrt{\rho}\dot u\|_2^2\,\mathrm{d}s\right)^{1/2}\cdot\left(\int_{t_1}^{t_2}\sigma^{-m}\,\mathrm{d}s\right)^{1/2} + CE_0 + CE_0(t_2-t_1) + C(\bar\rho)E_0^{\frac{1}{2\gamma}}(t_2-t_1)\\
&\le C(\bar\rho)E_0^{1/2}(t_2-t_1)^{1/2} + C(\bar\rho)E_0^{1/2}(t_2-t_1) + C(\bar\rho)E_0^{1/(2\gamma)}E_0^{1/4}(t_2-t_1)^{1/2}\\
&\quad + CE_0 + CE_0(t_2-t_1) + C(\bar\rho)E_0^{\frac{1}{2\gamma}}(t_2-t_1)\\
&\le C(\bar\rho)\left(E_0^{\frac12}+E_0^{\frac{1}{2\gamma}+\frac14}\right)(t_2-t_1)^{1/2} + C(\bar\rho)(E_0^{\frac12}+E_0+E_0^{\frac{1}{2\gamma}})(t_2-t_1) + CE_0\\
&\le C(\bar\rho)E_0^{\alpha}(t_2-t_1)^{1/2} + C(\bar\rho)E_0^{\frac{1}{2\gamma}}(t_2-t_1) + CE_0\\
&\le C(\bar\rho,\eta)E_0^{2\alpha} + (\eta + C(\bar\rho)E_0^{\frac{1}{2\gamma}})(t_2-t_1),
\end{aligned}$$

where we have used $\sup_{t_1\le t\le t_2}\sigma(t)^{-m} < C$ for $\sigma(T) < t_1 < t_2 < T$ and $\alpha = \min\{1/2, 1/2\gamma + 1/4\}$.

Then, we assign values to N_1 and N_0 as follows:

$$N_1 = \eta + C_6(\bar\rho)E_0^{1/(2\gamma)} \quad \text{and} \quad N_0 = C_5(\bar\rho,\eta)E_0^{2\alpha}.$$

Noting that

$$f(\xi) = -a{\mu_0}^{-\frac{p-2}{2}}\xi^{\gamma+1} \le -N_1 = -[\eta + C_6(\bar\rho)E_0^{1/(2\gamma)}]$$

and setting

$$\bar\xi = (a^{-1}\mu_0^{\frac{p-2}{2}})^{\frac{1}{\gamma+1}},$$

one can obtain after using Lemma 5 that

$$\sup_{t\in[\sigma(T),T]}\|\rho\|_\infty \le \max\left\{\frac{3\bar\rho}{2},\bar\xi\right\} + N_0 \le \frac{3\bar\rho}{2} + C_5(\bar\rho,\eta)E_0^{2\alpha} \le \frac{7\bar\rho}{4},$$

where $E_0 \le \delta_5 := \min\{\delta_4, (C_5^{-1}\bar\rho/4)^{1/(2\alpha)}, [(1-\eta)C_6^{-1}]^{2\gamma}\}$ and $\bar\rho > (a^{-1}\mu_0^{(p-2)/2})^{1/(\gamma+1)}$.

Thus, the proof of Proposition 4 is finished. □

Now, we discuss the higher estimates of the smooth solution (ρ, u). Hereafter, C denotes a positive constant, which may depend on T, K, a, γ, $\bar\rho$ and the initial value. One deduces from (2.6) and (2.7) that

$$\sup_{0\le t\le T}\|u_x\|_p^p + \int_0^T\|\sqrt{\rho}\dot u\|_2^2\,\mathrm{d}t \le C. \tag{2.43}$$

Putting $m = 0$ into (2.22) shows that

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\left(\int\rho|\dot u|^2\,\mathrm{d}x\right) + \int(u_x^2+\mu_0)^{\frac{p-2}{2}}\dot u_x^2\,\mathrm{d}x\\
&\le (p-1)\int(u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^2|\dot u_x|\,\mathrm{d}x + \int|\gamma\pi u_x\dot u_x|\,\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}&\le \frac{1}{2}\int (u_x^2+\mu_0)^{\frac{p-2}{2}}\dot{u}_x^2\,\mathrm{d}x + C\int (u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^4\,\mathrm{d}x + C\int (u_x^2+\mu_0)^{-\frac{p-2}{2}}u_x^2\,\mathrm{d}x\\
&\le \frac{1}{2}\int (u_x^2+\mu_0)^{\frac{p-2}{2}}\dot{u}_x^2\,\mathrm{d}x + C\int (u_x^2+\mu_0)^{\frac{p-2}{2}}u_x^4\,\mathrm{d}x + C\|u_x\|_\infty^2\\
&\le \frac{1}{2}\int (u_x^2+\mu_0)^{\frac{p-2}{2}}\dot{u}_x^2\,\mathrm{d}x + C\|x_x\|_p^p\cdot\|u_x\|_\infty^2 + C\|u_x\|_\infty^2 + C\\
&\le \frac{1}{2}\int (u_x^2+\mu_0)^{\frac{p-2}{2}}\dot{u}_x^2\,\mathrm{d}x + C(\bar{\rho})(1+\|\sqrt{\rho}\dot{u}\|_2^2),\end{aligned}$$

which combined with (2.43) and (2.28) gives after using the Gronwall inequality that

$$\sup_{0\le t\le T}\|\sqrt{\rho}\dot{u}\|_2^2 + \int_0^T\int (u_x^2+\mu_0)^{\frac{p-2}{2}}\dot{u}_x^2\,\mathrm{d}x\,\mathrm{d}t \le C, \tag{2.44}$$

$$\sup_{0\le t\le T}\|u_x\|_\infty \le C. \tag{2.45}$$

Applying ∂_x to $(1.1)_1$ and taking the inner product on the resulting equation with $2\rho_x$ over Ω, we can obtain after integrating by parts that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_x\|_2^2 \le C\|\rho_x\|_2^2(\|u_x\|_\infty+1) + C\|u_{xx}\|_2^2. \tag{2.46}$$

Equation $(1.1)_2$ implies

$$|u_{xx}| \le (u_x^2+\mu_0)^{-\frac{p-2}{2}}(|\rho\dot{u}| + \gamma\rho^{\gamma-1}|\rho_x|) + C. \tag{2.47}$$

Combining with (2.47), (2.44) and (2.45) yields

$$\|u_{xx}\|_2 \le C\|\rho_x\|_2 + C. \tag{2.48}$$

Putting (2.48) into (2.46) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_x\|_2^2 \le C\|\rho_x\|_2^2(\|u_x\|_\infty+1).$$

Using (2.45), (2.43) and the Gronwall inequality, one infers

$$\sup_{0\le t\le T}\|\rho_x\|_2^2 \le C. \tag{2.49}$$

This fact together with (2.48) gives

$$\sup_{0\le t\le T}\|u_{xx}\|_2^2 \le C.$$

One deduces from $(1.1)_1$, (2.45), (2.49) and the Hölder inequality that

$$\sup_{0\le t\le T}\|\rho_t\|_2 \le C\sup_{0\le t\le T}\|u_x\|_\infty + C\|\rho_x\|_2 \le C.$$

Next, $(1.1)_2$, (2.44) and (2.49) lead to

$$\|((u_x^2+\mu_0)^{\frac{p-2}{2}}u_x)_x\|_2 \le C\|\sqrt{\rho}\dot{u}\|_2 + C\|\rho_x\|_2 \le C.$$

Using the Poincaré inequality and (2.45), one can infer that

$$\|u\|_{H^1} \le C\|u_x\|_2 \le C\|u_x\|_\infty \le C.$$

From (2.44), one gets

$$\int_0^T \|u_t\|_2^2\,\mathrm{d}t \le C.$$

Based on the above discussion, the following result can be derived directly.

Lemma 6. *The following inequality holds:*

$$\sup_{0\le t\le T}(\|u_{xx}\|_2 + \|\rho_t\|_2 + \|((u_x^2+\mu_0)^{\frac{p-2}{2}}u_x)_x\|_2 + \|(\rho,u)\|_{H^1}) + \int_0^T \|\dot{u},u_t\|_2^2\,\mathrm{d}t \le C.$$

3. Proof of Theorem 1

Under the condition that a priori estimates are given in the previous section, we are ready to give the proof of the main result in this paper.

Due to Proposition 2, there exists a positive time T_*, which satisfies that the initial and boundary value problem (1.1)–(1.2) has a local unique strong solution (ρ, u) on $\Omega \times (0, T_*]$. It follows from a priori estimates, Proposition 4 and Lemma 6 that the local strong solution (ρ, u) can be extended to the whole time.

Owing to (1.3), there exists a $T_1 \in (0, T_*]$ satisfying that (2.2)–(2.5) hold for $T = T_1$. Let

$$T^* = \sup\{T \mid (2.2)\text{–}(2.5) \text{ hold}\}. \tag{3.1}$$

Then, for any $0 < \tau < T \le T^*$ with T finite, we can conclude from Lemmas 3 and 6 that

$$\rho^{1/2} \dot{u} \in C([\tau, T]; L^2(\Omega)). \tag{3.2}$$

Further, we assert that $T^* = \infty$. Otherwise, we suppose $T^* < \infty$. According to Proposition 4, inequalities (2.7)–(2.9) hold for $T = T^*$. We can deduce from (3.2) and Lemma 6 that $(\rho(x, T^*), u(x, T^*))$ satisfies (1.3), where $g(x) := \sqrt{\rho}\dot{u}(x, T^*)$ for $x \in \Omega$. Noting Proposition 2, we find that there exists $T^{**} > T^*$ such that inequalities (2.2)–(2.5) hold true for $T = T^{**}$. This implies a contradiction to (3.1). Moreover, for any $0 < T < T^* = \infty$, the uniqueness of solution (ρ, u), which is defined on $\Omega \times (0, T]$, can be inferred from Proposition 2 and Lemma 6.

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