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Homogenization of two-phase flow: high contrast of phase permeability

Homogénéisation d'écoulement diphasique : grand contraste de perméabilité d'une phase

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Abstract

The steady-state two-phase flow non-linear equation is considered in the case when one of phases has low effective permeability in some periodic set, while on the complementary set it is high; the second phase has no contrast of permeabilities in different zones. A homogenization procedure gives the homogenized model with macroscopic effective permeability of the second phase depending on the gradient and on the second order derivatives of the macroscopic pressure of the first phase. This effect cannot be obtained by classical (one small parameter) homogenization. ***To cite this article: G.P. Panasenko, G. Virnovsky, C. R. Mecanique 331 (2003).***

Résumé

On considére l'équation d'écoulement diphasique stationnaire non-linéaire dans le cas où une phase est peu perméable dans une zone alors que la perméabilité efficace est haute dans l'ensemble complémentaire ; la seconde phase n'a pas de contraste de perméabilité dans les différentes zones. L'homogénéisation de ce problème conduit à un modèle homogénéisé où la perméabilité macroscopique efficace de la seconde phase dépend du gradient et des dérivées secondes de la pression macroscopique de la prémiere phase. Cet effet ne peut pas être obtenu par l'homogénéisation classique (i.e. avec un seul petit paramétre). ***Pour citer cet article : G.P. Panasenko, G. Virnovsky, C. R. Mecanique 331 (2003).***

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Mots-clés : Mécanique des fluides numérique ; Homogénéisation ; Écoulement diphasique stationnaire ; Contraste de perméabilité d'une phase

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1. Motivation

The steady-state two-phase flow is simulated by a non-linear elliptic system of Eqs. [1] that is a vector analogue of the non-linear thermal conductivity equation when the conductivity coefficient depends on 'temperature'. The general homogenization procedure proposed by Bakhvalov in [2] gives the homogenized equation in the case when the only small parameter of the problem is equal to the ratio ε of the period of the microstructure to the characteristic size of the problem.

This homogenized equation is of the same type as the initial one, i.e., if the steady-state flow equation for phase pressures $p_{i\varepsilon}$ takes the form

$$\operatorname{div}\left(\lambda_i\left(\frac{x}{\varepsilon}, p_{1\varepsilon} - p_{2\varepsilon}\right)\nabla p_{i\varepsilon}\right) = f_i(x), \quad i = 1, 2, \; x \in \mathbb{R}^s, \; s = 2, 3 \tag{1}$$

with 1-periodic in ξ coefficient $\lambda_i(\xi, P_c)$ and with f_i smooth enough, then the homogenized equation is $\operatorname{div}(\hat{\lambda}_{0i}(p_{10} - p_{20})\nabla p_{i0}) = f_i(x)$, $i = 1, 2$, where $p_{i0}(i = 1, 2)$ are the macroscopic pressures and $\hat{\lambda}_i$ are the macroscopic effective phase permeabilities calculated according to the standard [3] homogenization procedure. The justification of the homogenized two-phase flow model in the nonsteady-state case was made in [4]. This justification can be modified and applied to a steady-state flow.

Thus the macroscopic effective phase permeabilities depend on the difference of phase pressures but not on the gradients of these pressures. On the other hand some numerical experiments [5] show that $\hat{\lambda}_{i0}$ depend on these gradients, and the contribution of this dependency is of order of 1.

The present paper explains this effect in the case when the model contains a second small parameter: the ratio of microscopic effective permeabilities of some low-permeable *for one of phases* zones occupying the domain G_2 and of the high-permeable *for the same phase* zones occupying G_1. It means that the material occupying G_1 is much more permeable than G_2 for one of phases while for the other phase their permeabilities are comparable.

This situation is realistic. For example, in some capillary pressure intervals the non-wetting phase can have permeability constrast of several orders of magnitude whereas for the wetting phase the permeability is of the same order in both high-permeable and the low-permeable zones. Let us consider two examples.

1.1. Conceptual Example 1

This example is generated using the Corey-type relative permeabilities functions for wetting (w) and non-wetting (n) phases (S_w stands for saturation of the wetting phase), $K_{rw} = S_w^2$, $K_{rn} = (1 - S_w)^5$, and a Corey-type expression for the capillary-pressure functions from [6], the contrast in absolute permeability is 20. The capillary pressure-saturation relationship is shown in Fig. 1, and the effective phase permeabilities are presented in Fig. 2. It is clearly seen that the effective permeability contrast is of several orders of magnitude for the non-wetting phase, whereas for the wetting phase the effective phase permeability is of the same order in both high-permeable and the low-permeable zones.

1.2. Realistic Example 2

This is presented using realistic data for a gas-oil system. The absolute permeability contrast is 30. As one observes from Fig. 3 the effective permeability in the high and the low permeable zones is of the same order of magnitude for the wetting phase in the capillary pressure range of 0.05–0.1 atm. For the non-wetting phase however, the effective permeability ratio is 100–1000 times in the same capillary pressure interval.

The importance of wettability for the analysis of effective phase permeabilities was pointed out in [7], where it was also proved the dependency of the character of nonsteady-state flow on the speed of flow. Note that these effects have another nature than the effect described below.

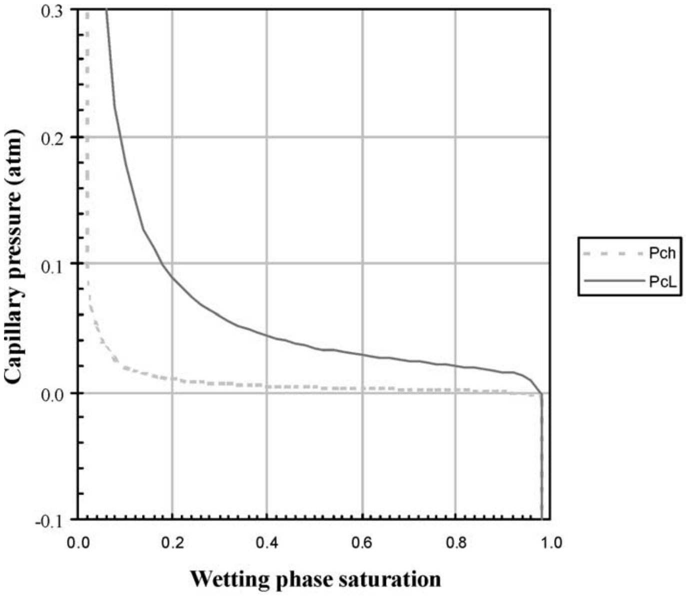

Fig. 1. Capillary pressure curves for high-permeable zones (dashed curve), and low-permeable zones (solid curve), Example 1.

Fig. 1. Courbes de la pression capillaire dans les zones de haute perméabilité (courbe pointillée) et dans les zones de basse perméabilité (courbe continue), Exemple 1.

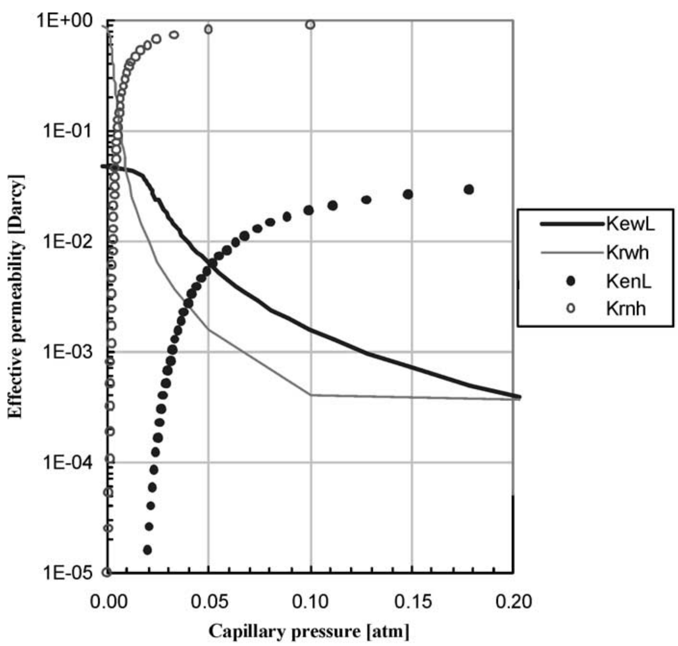

Fig. 2. Effective permeabilities for the wetting phase (curves), and for the non-wetting phase (points), Example 1.

Fig. 2. Perméabilités effectives pour la phase moillable (courbes), et pour la phase non-moillable (points), Exemple 1.

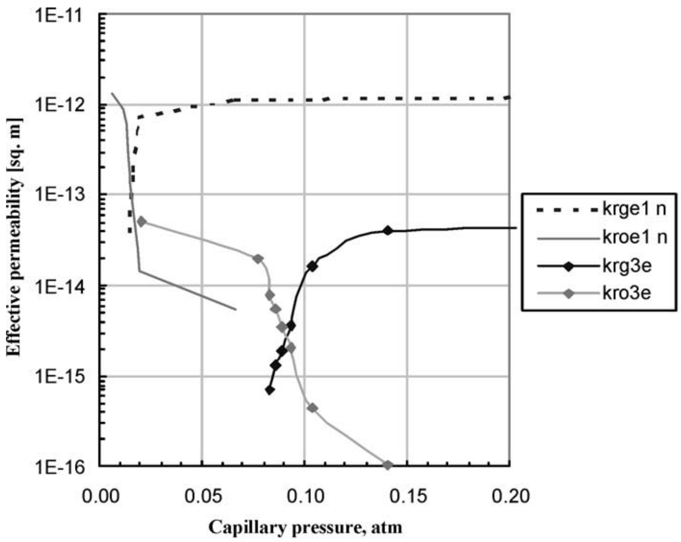

Fig. 3. Effective permeabilities for the wetting and the non-wetting phase in high-permeable zones (dashed curves), and in low-permeable zones (solid curve), Example 2.

Fig. 3. Perméabilités effectives pour les phases moillable et non-moillable dans les zones de haute perméabilité (courbes pointillées) et dans les zones de basse perméabilité (courbes continues), Exemple 2.

Another nonsteady state model was recently considered in case of one spacial dimension in [8], and it explains some new effects discussed in [9] analysing the influence of the capillary number that is taken as a small parameter related to ε.

Now we will introduce precisely the microscopic model.

2. Position of the problem

Consider the two-phase steady-state flow equations (1) in heterogeneous media; as mentioned above, $\lambda_i(\xi, P_c)$ are 1-periodic in $\xi \in \mathbb{R}^s$, $p_{i\varepsilon}$ stand for the phase pressures (the unknown functions), f_i stand for given sources, $s = 2$ or 3. We will call here λ_i effective permeabilities of the phases; λ_i are functions of the space variable x depending on it via the microscopic rapid variable $\xi = x/\varepsilon$. According [1] λ_i can be presented in a form of the ratio

$$\lambda_i(\xi, P_c) = \frac{k^*(\xi) K_{ri}(\mathcal{P}_c^{-1}(P_c, \xi), \xi)}{\mu_i}$$

where k^* stands for the absolute permeability, K_{ri} stand for relative permeabilities which are known functions of P_c and ξ (here $\mathcal{P}_c^{-1}$ dependency expresses the relation between the saturation of the first phase S_1 and the capillary pressure P_c: $S_1 = \mathcal{P}_c^{-1}(P_c, \xi)$); μ_i are the viscosities of the phases; the small parameter ε is the ratio between the characteristic microscopic and macroscopic sizes of the problem.

Remark 1. The gravitational terms were not written explicitely. These termes could be taken into account by introducing the new unknown functions $\bar{p}_{i\varepsilon} = p_{i\varepsilon} - \rho_i g x_3$, where x_3 is the vertical axis, ρ_i stand for the densities of the phases and g stands for the acceleration of gravity; $x = (x_1, \ldots, x_s)$.

Let us consider now the special case when the effective permeability of the first phase λ_1 is small on some periodic set:

$$\lambda_1(\xi, P_c) = \begin{cases} \varepsilon^2 \lambda_1^{(2)}(\xi, P_c) & \text{for } \xi \in G_2 \\ \lambda_1^{(1)}(\xi, P_c) & \text{for } \xi \in \mathbb{R}^S \backslash G_2 \end{cases} \tag{2}$$

where $\lambda_1^{(2)}$, $\lambda_1^{(1)}$ and λ_2 do not depend on ε (as functions of ξ and of P_c), G_2 is a 1-periodic open set such that $\partial G_2 \cap \partial Q = \emptyset$, $Q = (0,1)^s$; $\partial G_2 \in C^2$. So $\mathbb{R}^S \backslash G_2$ is a connected set, while G_2 is not connected.

Remark 2. On the interface $\{x/\varepsilon \in \partial G_2\}$ the following conditions hold:

$$[p_{i\varepsilon}] = 0, \qquad \left[\lambda_i\left(\frac{x}{\varepsilon}, p_{1\varepsilon} - p_{2\varepsilon}\right) \nabla p_{i\varepsilon} \cdot n\right] = 0$$

where $[\cdot]$ denotes the jump of a function on the interface, n is a normal vector to the interface.

We state Eqs. (1) in the whole space $\mathbb{R}^s$ assuming that f_i are T-periodic functions (T is a multiple of ε of order 1), $\int_{(0,T)^s} f_i(x)\,\mathrm{d}x = 0$; the solution $(p_{1\varepsilon}, p_{2\varepsilon})$ is sought in the class of functions T-periodic in all variables x_j except for x_1 having a linear growth $(\alpha_1 x_1, \alpha_2 x_1)$ with respect to x_1: it means that $(p_{1\varepsilon} - \alpha_1 x_1, p_{2\varepsilon} - \alpha_2 x_1)$ is T-periodic in all x_j; $\alpha_1, \alpha_2 \in \mathbb{R}$. This condition simulates a pressure drop at infinity for every phase; such a model is used normally to determine the macroscopic effective phase permeabilities. In applications, the right-hand side functions f_i often vanish.

3. Homogenization procedure

Under some assumptions on λ_i, $\mathcal{P}_c$ we obtain that the leading order term of an asymptotic solution to this problem has a form (up to terms of order $\mathrm{O}(\varepsilon)$)

$$p_{1\varepsilon}(x) \approx p_{10}(x) + N^2_{2,1,-1}\left(\frac{x}{\varepsilon}, p_{10} - p_{20}, \nabla p_{10}, \nabla(p_{10} - p_{20}), \nabla^2 p_{10}\right), \qquad p_{2\varepsilon}(x) \approx p_{20}(x) \tag{3}$$

where p_{10}, p_{20} stand for the macroscopic pressures satisfying the homogenized equations

$$\begin{cases} \operatorname{div}\bigl(\hat{\lambda}_{01}(p_{10} - p_{20}).\nabla p_{10}\bigr) = f_1(x) \\ \operatorname{div}\bigl(\hat{\lambda}_{02}\bigl(p_{10} - p_{20}, \nabla p_{10}, \nabla(p_{10} - p_{20}), \nabla^2 p_{10}\bigr)\nabla p_{20}\bigr) = f_2(x), \quad x \in \mathbb{R}^s \end{cases} \tag{4}$$

where p_{10}, p_{20} do not depend on ε; they are smooth functions of x, while $N^2_{2,1,-1}$ is a function of the variable $\xi \in \mathbb{R}^s$ and of parameters $(P_c, \nabla_1, \nabla_c, \nabla_1^{(2)}) \in \mathbb{R} \times \mathbb{R}^s \times \mathbb{R}^s \times \mathbb{R}^{s\times s}$ ($\mathbb{R}^s$ is the space of vector-columns); it is a solution to the problem

$$\begin{cases} \operatorname{div}_\xi\bigl(\lambda_1^{(2)}\bigl(\xi, P_c + N^2_{2,1,-1}\bigr)\nabla_\xi N^2_{2,1,-1}\bigr) = \hat{\lambda}_{00}\bigl(P_c, \nabla_1, \nabla_c, \nabla_1^{(2)}\bigr), \quad \xi \in G_2 \\ N^2_{2,1,-1}|_{\partial G_2} = 0 \end{cases} \tag{5}$$

Here $\hat{\lambda}_{00}$ is defined by the following two-steps algorithm:

- the first step is a search of an s-dimensional line $N^1_{110}(\xi, P_c)$, that is a 1-periodic in ξ solution of the boundary value problem ($P_c \in \mathbb{R}$ is a parameter)

  $$\operatorname{div}_\xi\bigl(\lambda_1^1(\xi, P_c)\bigl(\nabla_\xi N^1_{110} + I\bigr)\bigr) = 0, \quad \xi \in G_1, \qquad n_\xi^{\mathrm{T}}\lambda_1^1(\xi, P_c)\bigl(\nabla_\xi N^1_{110} + I\bigr) = 0, \quad \xi \in \partial G_1 \tag{6}$$

  where $G_1 = \mathbb{R}^s \backslash G_2$ and n_ξ is an outer normal vector to ∂G_1;

- the second step is a calculation

$$\hat{\lambda}_{00}\big(P_c,\nabla_1,\nabla_c,\nabla_1^{(2)}\big) = \int\limits_{G_1\cap Q} \nabla_c^{\mathrm{T}}\frac{\partial}{\partial P_c}\big\{\lambda_1^1(\xi,P_c)\big(\nabla_\xi N_{110}^1(\xi,P_c)+I\big)\big\}\nabla_1 + \lambda_1^1(\xi,P_c)\big(\nabla_\xi N_{110}^1(\xi,P_c)+I\big):\nabla_1^{(2)}\,\mathrm{d}\xi \tag{7}$$

where ':' denotes the tensor product.

The matrix-valued coefficients $\hat{\lambda}_{01}$ and $\hat{\lambda}_{02}$ of Eqs. (4) are calculated as follows:

$$\hat{\lambda}_{01}(P_c) = \int\limits_{G_1\cap Q} \lambda_1^1(\xi,P_c)\big(\nabla_\xi N_{110}^1(\xi,P_c)+I\big)\,\mathrm{d}\xi \tag{8}$$

where I is the identity matrix;

$$\hat{\lambda}_{02}\big(P_c,\nabla_1,\nabla_c,\nabla_1^{(2)}\big) = \int\limits_{G_1\cap Q} \lambda_1^2(\xi,P_c)\big(\nabla_\xi N_{120}^1\big(\xi,P_c,\nabla_1,\nabla_c,\nabla_1^{(2)}\big)+I\big)\,\mathrm{d}\xi$$
$$+\int\limits_{G_2\cap Q} \lambda_2^2\big(\xi,P_c+N_{2,2,-1}^2\big(\xi,P_c,\nabla_1,\nabla_c,\nabla_1^{(2)}\big)\big)\big(\nabla_\xi N_{120}^2\big(\xi,P_c,\nabla_1,\nabla_c,\nabla_1^{(2)}\big)+I\big)\,\mathrm{d}\xi \tag{9}$$

and the s-dimensional line N_{120}^1 and the s-dimensional line N_{120}^2 constitute a 1-periodic in ξ solution to the problem

$$\begin{aligned}
&\mathrm{div}_\xi\big(\lambda_2^1(\xi,P_c)\big(\nabla_\xi N_{120}^1+I\big)\big)=0, \quad \xi\in G_1\\
&\mathrm{div}_\xi\big(\lambda_2^2\big(\xi,P_c+N_{2,1,-1}^2\big(\xi,P_c,\nabla_1,\nabla_c,\nabla_1^{(2)}\big)\big)\big(\nabla_\xi N_{120}^2+I\big)\big)=0, \quad \xi\in G_2\\
&N_{120}^1|_{\partial G_2}=N_{120}^2|_{\partial G_2}\\
&n_\xi^T\lambda_2^1(\xi,P_c)\big(\nabla_\xi N_{120}^1+I\big)\big|_{\partial G_2}=n_\xi^T\lambda_2^2\big(\xi,P_c+N_{2,1,-1}^2\big(\xi,P_c,\nabla_1,\nabla_c,\nabla_1^{(2)}\big)\big)\big(\nabla_\xi N_{120}^2+I\big)\big|_{\partial G_2}
\end{aligned} \tag{10}$$

Thus the macroscopic pressures p_{10}, p_{20} satisfy the homogenized equation (4) with effective permeability of the second phase depending on ∇p_{10} and $\nabla^2 p_{10}$. This effect appears only if $\hat{\lambda}_{00}$ (7) is different from zero; it means that $\lambda_1^1(\xi,P_c)$ really depends on P_c or second derivatives of p_{10} are different from zero.

4. Asymptotic expansion and the estimate

Assume that for some natural n,

(1) $\lambda_i^j\in C^{n+4}(\mathbb{R};C^1(\overline{G}_j))$ as a function $P_c\mapsto\lambda_i^j(\cdot,P_c)$;
(2) $\exists\kappa_0,\kappa_1>0$ such that $\forall\xi\in\mathbb{R}^s$, $P_c\in\mathbb{R}$, $\kappa_0\leqslant\lambda_i^j(\xi,P_c)\leqslant\kappa_1$;
(3) there exist a unique solution $N_{2,1,-1}^2\in C^{n+2}(\mathbb{R}\times\mathbb{R}^s\times\mathbb{R}^s\times\mathbb{R}^{s\times s};C^2(G_2))$ to problem (5);
(4) the solution (N_{120}^1,N_{120}^2) to problem (10) belongs to $C^{n+3}(\mathbb{R}\times\mathbb{R}^s\times\mathbb{R}^s\times\mathbb{R}^{s\times s};(C^2(\overline{G}_1))^s\times(C^2(\overline{G}_2))^s)$;
(5) $f_i\in C^{n+2}(\mathbb{R}^s)$;
(6) there exist a unique solution $(p_{10},p_{20})\in(C^{n+4}(\mathbb{R}^s))^2$ such that $(p_{10}-\alpha_1x_1,p_{20}-\alpha_2x_1)$ is a T-periodic in all x_j pair;
(7) for any k, $0\leqslant k\leqslant n+1$, for any pair $(g_1,g_2)\in(C^k(\mathbb{R}^s))^2$ T-periodic in all variables, $\int_{(0,T)^s}g_i\,\mathrm{d}x=0$, there exist a T-periodic solution $(q_1,q_2)\in(C^{k+2}(\mathbb{R}^s))^2$ to equation

$$\begin{cases}\mathrm{div}(\hat{L}_1(q_1,q_2)+\hat{\lambda}_{01}(p_{10}-p_{20})\nabla q_1)=g_1(x)\\ \mathrm{div}(\hat{L}_2(q_1,q_2)+\hat{\lambda}_{02}(p_{10}-p_{20},\nabla p_{10},\nabla(p_{10}-p_{20}),\nabla^2p_{10})\nabla q_2)=g_2(x), \quad x\in\mathbb{R}^s\end{cases} \tag{11}$$

where $\hat{L}_1(q_1, q_2)$ and $\hat{L}_2(q_1, q_2)$ are the first differentials of the functions $\hat{\lambda}_{10}$ and $\hat{\lambda}_{20}$ respectively at the point $(P_c, \nabla_1, \nabla_c, \nabla_1^{(2)}) = (p_{10} - p_{20}, \nabla p_{10}, \nabla(p_{10} - p_{20}), \nabla^2 p_{10})$ restricted to $\mathrm{d}P_c = q_1 - q_2$, $\mathrm{d}\nabla_1 = \nabla q_1$, $\mathrm{d}\nabla_c = \nabla(q_1 - q_2)$, $\mathrm{d}\nabla_1^{(2)} = \nabla^2 q_1$.

Under these assumptions the n-th order asymptotic approximation can be constructed. It has a form

$$p_i^{(n)} = p_{i0}^{(n)} + \sum_{l=1}^{n+1} \varepsilon^l N_{l,i}\big(\xi, \nabla^{l+1} p^{(n)}\big)\big|_{\xi = x/\varepsilon} \tag{12}$$

where $\nabla^l p^{(n)}$ is the set of all derivatives of order less or equal to l for the vector-valued function $p^{(n)} = (p_{10}^{(n)}, p_{20}^{(n)})$; $N_{l,i}$ are the 1-periodic in ξ sums

$$N_{l,i}\big(\xi, \nabla^{(l+1)}\big) = \sum_{j=-\delta_{\sigma 2}\delta_{i1}(1-\delta_{l1})}^{n+1} \varepsilon^{2j} N_{l,i,j}^{\sigma}\big(\xi, \nabla^{(l+1)}\big) \tag{13}$$

for $\xi \in G_\sigma$, $\sigma = 1, 2$, $N_{l,i,j}^{\sigma}$ are also 1-periodic in ξ functions ([3], Chapter 7). The leading order term of this expansion (12), (13) coincides with (3). The asymptotic approximation (12), (13) is justified as follows:

$$\sup_{\varphi \in (H^1_{\mathrm{per}}((0,T)^s))^2} \frac{\big|\int_{(0,T)^s} \sum_{i=1}^{2} (\lambda_i(x/\varepsilon, p_1^{(n)} - p_2^{(n)}) \nabla p_i^{(n)} \cdot \nabla \varphi_i + f_i \varphi_i)\,\mathrm{d}x\big|}{T^s \sqrt{\sum_{i=1}^{2} \|\nabla \varphi_i\|^2_{L^2((0,T)^s)}}} = \mathrm{O}\big(\varepsilon^n\big) \tag{14}$$

where $H^1_{\mathrm{per}}((0,T)^s)$ is the closure of the space of T-periodic differentiable functions with respect to the norm $H^1((0,T)^s)$.

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